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# CLASS-NUMBER PROBLEMS FOR CUBIC NUMBER FIELDS

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## 1. Introduction

Let $\mathbf{M}$ be any number field. We let $D_{\mathrm{M}}$, $d_{\mathrm{M}}$, $h_{\mathrm{M}}$, $\zeta_{\mathrm{M}}$, $\mathbf{A}_{\mathrm{M}}$ and $\mathrm{Reg}_{\mathrm{M}}$ be the discriminant, the absolute value of the discriminant, the class-number, the Dedekind zeta-function, the ring of algebraic integers and the regulator of $\mathbf{M}$, respectively. We set $c = \dfrac{3+2\sqrt{2}}{2}$. If $q$ is any odd prime we let $(\cdot/q)$ denote the Legendre's symbol. We let $D_P$ and $d_P$ be the discriminant and the absolute value of the discriminant of a polynomial $P$.

LEMMA A (See [Sta, Lemma 3] and [Hof, Lemma 2]). *Let $\mathbf{M}$ be any number field. Then, $\zeta_{\mathrm{M}}$ has at most one real zero in*

$$\left[1 - \frac{1}{c \log d_{\mathrm{M}}}, 1\right[;$$

*if such a zero exists, it is simple and is called a Siegel zero.*

LEMMA B (See [Lou 2]). *Let $\mathbf{M}$ be a number field of degree $n = r_1 + 2r_2$ where $\mathbf{M}$ has $r_1$ real conjugate fields and $2r_2$ complex conjugate fields. Let $s_0 \in [(1/2), 1[$ be such that $\zeta_{\mathrm{M}}(s_0) \le 0$. Then,*

$$\mathrm{Res}_{s=1}(\zeta_{\mathrm{M}}) \ge (1 - s_0) d_{\mathrm{M}}^{(s_0-1)/2} \left(1 - \frac{2r_1}{d_{\mathrm{M}}^{s_0/2n}} - \frac{2\pi r_2}{d_{\mathrm{M}}^{s_0/n}}\right).$$

## 2. Lower bounds for class-numbers of cubic number fields

Let $\mathbf{K}$ be a cubic number field. If $\mathbf{K}/\mathbf{Q}$ is normal then $\mathbf{K}$ is a cyclic cubic number field. Let $f_{\mathbf{K}}$ be its conductor. Then $d_{\mathbf{K}} = f_{\mathbf{K}}^2$ and $\zeta_{\mathbf{K}}(s) = \zeta(s)L(s,\chi)L(s,\bar{\chi})$ where $\chi$ is a primitive cubic Dirichlet character modulo $f_{\mathbf{K}}$. Hence, we get

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$\zeta_{\mathbf{K}}(s) \leq 0$, $s \in [0,1[$. With $s_0 = 1 - (2/\log d_{\mathbf{K}})$ Lemma B provides the following lower bound which improves the one given in [Let]

$$\mathrm{Res}_{s=1}(\zeta_{\mathbf{K}}) \geq \frac{1}{3\log(f_{\mathbf{K}})}, f_{\mathbf{K}} \geq 10^6.$$

Lettl used his lower bound to determine all the simplest cubic number fields with small class-numbers. Here, the simplest cubic number fields are real cubic cyclic numbers fields defined as being the splitting fields of the polynomials $P(X) = X^3 - aX^2 - (a+3)X - 1$ $(a \geq 0)$ whose discriminants $d_P = (a^2 + 3a + 9)^2$ are square of a prime $p = a^2 + 3a + 9$, which implies $\mathbf{A_K} = \mathbf{Z}[\varepsilon]$ where $\varepsilon > 1$ is the only real root of $P(X)$ greater than one (we have $\varepsilon \in ]a+1, a+2[$). He found that there are 7 simplest cubic fields with class-number one, and none with class-number two or three.

In the same spirit, Lazarus determined all the simplest quartic fields with class-number 1 or 2. Here, the simplest quartic number fields are real quartic cyclic number fields defined as being the splitting fields of the polynomials $P(X) = X^4 - 2aX^3 - 6X^2 + 2aX + 1$ $(a \geqslant 0)$ whose discriminants $d_P = 256 d_a^3$ are such that $d_a = a^2 + 4$ is odd-square-free. He found that there are 6 simplest quartic fields with class-number one, and 3 with class-number two.

From now on, we assume that $\mathbf{K}$ is not normal. Thus, the normal closure $\mathbf{N}$ of $\mathbf{K}$ is a sextic number field with Galois group the symmetric group $\mathcal{S}_3$.

Lemma C. *Let $\mathbf{K}/\mathbf{Q}$ be a non-normal cubic extension with normal closure $\mathbf{N} = \mathbf{KL}$ where $\mathbf{L} = \mathbf{Q}(\sqrt{D_{\mathbf{K}}})$ is quadratic. It holds $\zeta_{\mathbf{N}}\zeta^2 = \zeta_{\mathbf{K}}^2\zeta_{\mathbf{L}}$. Hence, $d_{\mathbf{N}} = d_{\mathbf{K}}^2 d_{\mathbf{L}}$, and $d_{\mathbf{N}}$ divides $d_{\mathbf{K}}^3$. Finally, $\zeta_{\mathbf{K}}$ does not have any real zero in the range $[1 - (1/(3c\log d_{\mathbf{K}})), 1[$.*

*Proof.* The first point is proved page 227 of [Cas-Fro] using Artin $L$-series formalism. The second point follows from the first one using the functional equations satisfied by these zeta-functions (see [Lou 1]). According to the first point, any real zero in $]0,1[$ of $\zeta_{\mathbf{K}}$ is a multiple zero of $\zeta_{\mathbf{N}}$. Hence, the fourth point follows from Lemma A. It remains to prove the third point. According to Stickelberger's theorem, we have $D_{\mathbf{K}} \equiv 0,1 \pmod 4$. Hence, $D_{\mathbf{L}}$ divides $D_{\mathbf{K}}$, and we may define $f \geq 1$ by means of $D_{\mathbf{K}} = f^2 D_L$. □

According to Lemmata A, B and C, we get:

THEOREM 1. *Let* $\mathbf{K}$ *be a non-normal cubic number field. It holds*

$$h_{\mathbf{K}} \operatorname{Reg}_{\mathbf{K}} \geq \frac{1}{55} \frac{\sqrt{d_{\mathbf{K}}}}{\log d_{\mathbf{K}}}, \ d_{\mathbf{K}} \geq 4 \cdot 10^5.$$

## 3. Computation of the class-number of a non-abelian cubic number field K with negative discriminant

We make use of the results of [Bar-Lox-Wil] and [Bar-Wil-Ban]. Set $\Phi(s) = (\zeta_{\mathbf{K}}/\zeta)(s) = \sum_{j \geq 1} \alpha(j) j^{-s}$. Then,

$$h_{\mathbf{K}} \operatorname{Reg}_{\mathbf{K}} = \frac{\Phi(1)}{C} = \sum_{j \geq 1} \frac{\alpha(j)}{jC} e^{-jC} + \sum_{j \geq 1} \alpha(j) E(jC)$$

where

$$E(y) = \int_y^{\infty} \frac{e^{-x}}{x} dx = -\log x - \gamma - \sum_{j \geq 1} \frac{(-1)^j}{j(j!)} x^j,$$

where $\gamma = 0.577215664901\ldots$ is the Euler's constant, and where $C = 2\pi/\sqrt{d_{\mathbf{K}}}$.

Now, $j \mapsto \alpha(j)$ is a multiplicative function such that $\alpha(p^n) = F(p^n) - F(p^{n-1})$ where $F(k)$ is the number of distinct ideals of $\mathbf{K}$ with norm $k \geq 1$. Moreover, if $p$ does not divide $d_{\mathbf{K}}$, then

$$(p) = \begin{cases} (p) & \text{which implies } (D_{\mathbf{K}}/p) = +1 \text{ (Type I)}, \\ \mathcal{P}_1\mathcal{P}_2 & \text{if and only if } (D_{\mathbf{K}}/p) = -1 \text{ (Type II)}, \\ \mathcal{P}_1\mathcal{P}_2\mathcal{P}_3 & \text{which implies } (D_{\mathbf{K}}/p) = +1 \text{ (Type III)}. \end{cases}$$

If $p$ divides $d_{\mathbf{K}}$, then with $f$ as in the proof of Lemma C we have

$$(p) = \begin{cases} \mathcal{P}_1^2\mathcal{P}_2 & \text{if } p \text{ does not divide } f \text{ (Type IV)}, \\ \mathcal{P}^3 & \text{if } p \text{ divides } f \text{ (Type V)}. \end{cases}$$

If the ring of algebraic integers of $\mathbf{K}$ is generated by an algebraic integer $x_{\mathbf{K}}$ and if $P_{\mathbf{K}}(X)$ is the minimum polynomial over $\mathbf{Q}$ of $x_{\mathbf{K}}$, then we are in the Type I or Type III cases according as $P_{\mathbf{K}}(X)$ does not have or has at least one root modulo $p$, and we are in the Type IV or Type V cases according as $P_{\mathbf{K}}(X)$ has a double or a triple root modulo $p$.

## 4. Explicit class-number problem for non-normal cubic number fields

**First example.** In the same way we got Theorem 1, we get

THEOREM 2. (a) (See [Fro-Tay, Chapter 5]) *Let* $l \geq 1$ *be an integer. Set* $P_l(X) = X^3 + lX - 1$. *Then* $P_l(X)$ *is irreducible in* $\mathbf{Q}[X]$, *has negative discriminant* $D_l = -d_l = -(4l^3 + 27)$, *and has exactly one real root* $x_l$. *Set* $\varepsilon_l = 1/x_l$. *Then,* $l < \varepsilon_l < l + 1$. *Set* $\mathbf{K}_l = \mathbf{Q}(x_l) = \mathbf{Q}(\varepsilon_l)$. *Then* $\mathbf{K}_l$ *is a real cubic number field with negative discriminant and* $\varepsilon_l$ *is the fundamental unit greater than one of the cubic order* $\mathbf{Z}[x_l] = \mathbf{Z}[\varepsilon_l]$.

(b) *Assume that the ring of algebraic integers* $\mathbf{A}_l$ *of* $\mathbf{K}_l$ *is equal to* $\mathbf{Z}[\varepsilon_l]$. *Then,*

$$h_l \geq \frac{\sqrt{d_l}}{20\log^2(d_l)} \text{ when } d_l \geq 2\cdot 10^5, \text{ and } h_l > 3 \text{ when } l > 400$$

(*here* $h_l$ *is the class-number of* $\mathbf{K}_l$). *Hence, there are* 5 *such* $\mathbf{K}_l$ *with class-number one, namely* $\mathbf{K}_1$, $\mathbf{K}_2$, $\mathbf{K}_3$, $\mathbf{K}_5$ *and* $\mathbf{K}_{11}$; *there are* 2 *such* $\mathbf{K}_l$ *with class-number two, namely* $\mathbf{K}_4$ *and* $\mathbf{K}_7$; *there are* 3 *such* $\mathbf{K}_l$ *with class-number three, namely* $\mathbf{K}_6$, $\mathbf{K}_{15}$ *and* $\mathbf{K}_{17}$.

LEMMA D. (a) *The ring of algebraic integers of* $\mathbf{K}_l$ *is equal to* $\mathbf{Z}[\varepsilon_l]$ *if and only if* $3^2$ *does not divide* $l$, *and* $p^2$ *does not divide* $d_l = 4l^3 + 27$ *for any prime* $p \geq 5$.

(b) *Under this assumption, we have*:

(i) *For any prime* $p \neq 3$ *that divides* $d_l$ *we have* $F(p^n) = n + 1$ *and* $\alpha(p^n) = 1$, $n \geq 1$.

(ii) *If* $p = 3$ *divides* $d_l$, *then* $F(3^n) = 1$ *and* $\alpha(3^n) = 0$, $n \geq 1$.

(iii) *If* $p$ *does not divide* $d_l$ *and* $(-d_l/p) = -1$, *then* $\alpha(p^n) = (1 + (-1)^n)/2$.

(iv) *If* $p$ *does not divide* $d_l$ *and* $(-d_l/p) = +1$, *then*

$$\alpha(p^n) = \begin{cases} n+1 & \text{if } P_l(X) \text{ has at least one root modulo } p, \\ (n+1/3) = \begin{cases} 1 & \text{if } n \equiv 0 \pmod 3, \\ -1 & \text{if } n \equiv 1 \pmod 3, \\ 0 & \text{if } n \equiv 2 \pmod 3, \end{cases} & \text{if } P_l(X) \text{ has no real root modulo } p. \end{cases}$$

*Proof.* Point (b) follows from Section 3 (see [Bar-Lox-Wil, Table 2]).

Now, we proceed with the proof of point (a). We note that the integral bases of all cubic fields are determined in [Alb]. Though, we give a simple proof for this special case. Let $t$ be an integer and set

$$y_{p,l}(t) = \frac{x_l^2 + tx_l + t^2 + l}{p}.$$

Then $y_{p,l}(t)$ is a root of the following polynomial

$$Y^3 - \frac{P_l'(t)}{p}Y^2 + \frac{3tP_l(t)}{p^2}Y - \frac{P_l(t)^2}{p^3}$$

(being the characteristic polynomial of the linear map $z \mapsto y_{p,l}(t)z$ of the three $\mathbf{Q}$-dimensional vector space $\mathbf{K}_l$, then $y_{p,l}(t)$ is indeed a root of this polynomial). Hence, $y_{p,l}(t)$ is an algebraic integer provided that $P_l(t) \equiv 0 \pmod{p^2}$ and $P_l'(t) \equiv 0 \pmod{p}$.

Now, assume that $\mathbf{Z}[x_l]$ is the ring of algebraic integers of $\mathbf{K}_l$ and let $p$ be any prime which divides $d_l = 4l^3 + 27$. Hence $p$ is odd.
First, assume that $p \neq 3$. Then $p$ does not divide $l$ and we take $t = t_{l,p}$ such that $2lt_{l,p} \equiv 3 \pmod{p}$ and where we hence write $2lt_{l,p} = 3 + ap$. Then,

$$4l^2P_l'(t_{p,l}) = 3(3+ap)^2 + 4l^3 \equiv d_l \equiv 0 \pmod{p},$$

$$8l^3P_l(t_{p,l}) = (3+ap)^2 + 4l^3(3+ap) - 8l^3 \equiv d_l + apd_l \equiv d_l \pmod{p^2}.$$

Hence, if $p^2$ would divide $d_l$ then $y_{p,l}(t_{p,l})$ would be an algebraic integer of $\mathbf{K}_l$ and $p$ would divide the index of $\mathbf{Z}[x_l]$ in $\mathbf{A}_l$. Contradiction.
Second, assume that $p = 3$. If $3^2$ would divide $l$, then $y_{3,l}(1)$ would be an algebraic integer. Hence, 3 would divide the index of $\mathbf{Z}[x_l]$ in $\mathbf{A}_l$. Contradiction.

Conversely, first assume that if $p \neq 3$ divides $d_l$ then $p^2$ does not divides $d_l$. Since $d_l = (\mathbf{A}_l : \mathbf{Z}[x_l])^2 d_{\mathbf{K}_l}$, then $p$ does not divides this index $(\mathbf{A}_l : \mathbf{Z}[x_l])$ which is thus a 3-power. Note that if 3 divides this index, then 3 divides $d_l$, i.e., 3 divides $l$.
Second, it is known that if $x$ is an algebraic integer whose minimum polynomial is $p$-Eisenstein for a prime $p$, then $p$ does not divides the index of $\mathbf{Z}[x]$ in the ring of algebraic integers of the number field $\mathbf{Q}(x)$ (see [Nar, Lemma 2.2, page 60]). Hence, if we can find an integer $a$ such that $P_l(a) \equiv P_l'(a) \equiv 0 \pmod{3}$ and $P_l(a) \not\equiv 0 \pmod{3^2}$, then the minimum polynomial $Q_a(Y) = P_l(Y+a) = Y^3 + 3aY^2 + P_l'(a)Y + P_l(a)$ of $y_l(a) = x_l - a$ is then 3-Eisenstein. Hence, 3 does not divide the index $(\mathbf{A}_l : \mathbf{Z}[x_l])$, which implies the desired result. Now, assume that 3 divides $l$ but that $3^2$ does not divide $l$. Then, $Q_1(Y)$ being 3-Eisenstein we get the desired result. □

The computational method of the third section and Lemma D provide the following table of class-numbers $h_l$ of these number fields $\mathbf{K}_l$ for the first values of $l \geq 1$ such that $\mathbf{Z}[\varepsilon_l]$ is the ring of algebraic integers of $\mathbf{K}_l$. According to the more extensive class-number computation of $h_l$ for $l \leq 400$, we easily get Theorem 2(b).

| $l$ | $d_l$ | $h_l$ | $l$ | $d_l$ | $h_l$ | $l$ | $d_l$ | $h_l$ | $l$ | $d_l$ | $h_l$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 31 | 1 | 13 | 8815 | 5 | 24 | 55323 | 18 | 37 | 202639 | 8 |
| 2 | 59 | 1 | 14 | 11003 | 8 | 25 | 62527 | 6 | 38 | 219515 | 23 |
| 3 | 135 | 1 | 15 | 13527 | 3 | 26 | 70331 | 8 | 39 | 237303 | 15 |
| 4 | 283 | 2 | 16 | 16411 | 8 | 28 | 87835 | 32 | 40 | 256027 | 28 |
| 5 | 527 | 1 | 17 | 19679 | 3 | 29 | 97583 | 6 | 41 | 275711 | 6 |
| 6 | 891 | 3 | 19 | 27463 | 7 | 30 | 108027 | 15 | 42 | 296379 | 39 |
| 7 | 1399 | 2 | 20 | 32027 | 10 | 31 | 119191 | 12 | 43 | 318055 | 16 |
| 10 | 4027 | 6 | 21 | 37071 | 6 | 32 | 131099 | 10 | 44 | 340763 | 18 |
| 11 | 5351 | 1 | 22 | 42619 | 12 | 34 | 157243 | 28 | 46 | 389371 | 27 |
| 12 | 6939 | 6 | 23 | 48695 | 4 | 35 | 171527 | 12 | 47 | 415319 | 8 |

$\square$

**Second example.** Let $\mathbf{K} = \mathbf{Q}(\sqrt[3]{d})$, $d \geq 2$, be a real pure cubic number field. Assume that $d$ is cube-free and define $a$ and $b$ by means of $(a, b) = 1$ and $d = ab^2$. Then, $D_{\mathbf{K}} = -3(ab)^2$ or $D_{\mathbf{K}} = -27(ab)^2$ according as $d \equiv \pm 1 \pmod 9$ or not. Now, $\mathbf{L} = \mathbf{Q}(\sqrt{-3})$ and $\zeta_{\mathbf{L}}$ does not have any real zero in $]0,1[$. Hence, we get $\zeta_{\mathbf{N}}(s) \leq 0$, $s \in ]0,1[$. According to Lemma C, we have $\mathrm{Res}_{s=1}(\zeta_{\mathbf{N}}) = (\mathrm{Res}_{s=1}(\zeta_{\mathbf{K}}))^2 \mathrm{Res}_{s=1}(\zeta_{\mathbf{L}})$ and we may apply Lemma B with $s_0 = 1 - (2/\log d_{\mathbf{N}})$, which provides an optimal lower bound on $\mathrm{Res}_{s=1}(\zeta_{\mathbf{N}})$. Since $\mathrm{Res}_{s=1}(\zeta_{\mathbf{L}}) = \pi/3\sqrt{3}$, we get the following lower bound that improves the one given in Theorem 1:

THEOREM 3 (See [Bar-Lou]). *Let $\mathbf{K}$ be a real pure cubic number field. Then,*

$$h_{\mathbf{K}} \mathrm{Reg}_{\mathbf{K}} \geq \frac{1}{9}\sqrt{\frac{d_{\mathbf{K}}}{\log d_{\mathbf{K}}}}, \; d_{\mathbf{K}} \geq 3 \cdot 10^4.$$

THEOREM 4. *When $m \geq 1$ is such that $m^3 \pm 1$ is cube-free, we set $\mathbf{K}_{\pm m} \overset{def}{=} \mathbf{Q}(\sqrt[3]{m^3 \pm 1})$ which is a pure cubic real number field. There are 2 such $\mathbf{K}_{\pm m}$ with class-number one, namely $\mathbf{K}_{+1}$ and $\mathbf{K}_{+2}$; there does not exist any such $\mathbf{K}_{\pm m}$ with class-number two; and there are 3 such $\mathbf{K}_{\pm m}$ with class-number three, namely $\mathbf{K}_{-2}$, $\mathbf{K}_{-3}$ and $\mathbf{K}_{+3}$.*

*Proof.* Set $d_{\pm m} = m^3 \pm 1$ and $\omega_{\pm m} = \sqrt[3]{d_{\pm m}}$. Then, $\varepsilon_{\pm m} = \pm/(\omega_{\pm m} - m) = \omega_{\pm m}^2 + m\omega_{\pm m} + m^2$ is a unit of $\mathbf{A}_{\mathbf{K}_{\pm m}}$, and we have $1 < \varepsilon_{\pm m} \leq 4\omega_{\pm m}^2 = (8d_{\pm m})^{2/3}$. Hence,

$$\mathrm{Reg}_{\mathbf{K}_{\pm m}} \leq \log \varepsilon_{\pm m} \leq \frac{2}{3}\log(8d_{\pm m}) \leq \frac{2}{3}\log(3d_{\mathbf{K}_{\pm m}}),$$

and according to Theorem 3 we get

$$h_{\mathbf{K}_{\pm m}} \geq \frac{1}{6}\sqrt{\frac{d_{\mathbf{K}_{\pm m}}}{\log^3(3d_{\mathbf{K}_{\pm m}})}},\ d_{\mathbf{K}_{\pm m}} \geq 3\cdot 10^4.$$

which implies $h_{\mathbf{K}_{\pm m}} > 3$ for $d_{\mathbf{K}_{\pm m}} > 1.1\cdot 10^6$. Note that $m > 72$ implies $d_{\mathbf{K}_{\pm m}} \geq 3d_m > 1.1\cdot 10^6$. Now, the computation of $a$ and $b$ for $d_{\pm m} = m^3 \pm 1$ and $1 \leq m \leq 72$ yields $d_{\mathbf{K}_{\pm m}} \leq 1.1\cdot 10^6$ if and only if $1 \leq m \leq 6$ when $d_{\pm m} = m^3 + 1$, and $2 \leq m \leq 7$ when $d_{\pm m} = m^3 - 1$ (note that $d_{\pm m}$ must be cube-free). The class-numbers of these number fields may be found in [Hos-Wad], and they provide the desired result. $\square$

## 5. Another class-number problem

Let $\mathbf{K}$ be a real quadratic number field of discriminant $D > 0$. The ring of algebraic integers $\mathbf{A}_{\mathbf{K}} = \mathbf{Z}\left[\frac{D+\sqrt{D}}{2}\right]$ of $\mathbf{K}$ writes $\mathbf{A}_{\mathbf{K}} = \mathbf{Z}[\varepsilon]$ where $\varepsilon = \frac{u+v\sqrt{D}}{2} > 1$ is a unit of $\mathbf{A}_{\mathbf{K}}$ if and only if $v = 1$, hence if and only if $D = m^2 \pm 4$, $m \geq 1$. In that case, $\varepsilon_D \leq \varepsilon = \frac{m+\sqrt{D}}{2} \leq \sqrt{D+4}$ where $\varepsilon_D$ is the fundamental unit of $\mathbf{K}$. According to the Brauer-Siegel theorem, there are only finitely many real quadratic number fields with discriminants $D = m^2 \pm 4$ of given class-number. Up to now, no one knows how to make the Brauer-Siegel effective in the real quadratic case, without assuming a suitable generalized Riemann hypothesis (see [Mol-Wil]. In contrast to the real quadratic case, let $\mathbf{K}$ be a cubic number field with negative discriminant whose ring of algebraic integers $\mathbf{A}_{\mathbf{K}}$ is generated by a unit $\varepsilon$, i.e. such that $\mathbf{A}_{\mathbf{K}} = \mathbf{Z}[\varepsilon]$. This clearly amounts to saying that $\mathbf{A}_{\mathbf{K}} = \mathbf{Z}[\varepsilon_{\mathbf{K}}]$, where $\varepsilon_{\mathbf{K}} > 1$ is the fundamental unit of $\mathbf{K}$. According to Theorem 1 and the following Proposition 5, we get the effective Corollary 6 that would enable one to explicitly determine all the cubic number fields of negative discriminants whose have small class-numbers and whose rings of algebraic integers are generated by units.

PROPOSITION 5. *A polynomial $P(X)$ is said of type $(T)$ if it is a monic irreducible cubic polynomial with integral coefficients (say of the form $P(X) = X^3 - aX^2 + bX - 1$) with exactly one real root $\varepsilon_P$ (i.e. $D_P < 0$) which satisfies $\varepsilon_P > 1$ (i.e. we have*

$b \leq a-1$), *Then, there exists an effective constant* $c_1 > 0$ *such that for any polynomial of type* (*T*) *we have* $d_P = |D_P| \geq c_1 \varepsilon_P^{3/2}$.

COROLLARY 6. *Let* $h$ *be a positive integer. Then, there are only finitely many cubic number fields* $\mathbf{K}$ *of negative discriminants* $D_{\mathbf{K}}$ *which have class-number* $h$ *and whose rings of algebraic integers write* $\mathbf{A}_{\mathbf{K}} = \mathbf{Z}[\varepsilon_{\mathbf{K}}]$, *where* $\varepsilon_{\mathbf{K}} > 1$ *is the fundamental unit of* $\mathbf{K}$. *Moreover, there exists* $c_2 > 0$ *effective such that* $|D_{\mathrm{K}}| \leq c_2 h^2 \log^4 h$ *for these number fields.*

*Proo of Proposition* 5. Set $\varepsilon = \varepsilon_P$, $x_1 = \varepsilon$, and let $x_2 = \alpha + i\beta$ and $x_3 = \alpha - i\beta$ be the two complex conjugate roots of this polynomial. Then,

$$1 = x_1x_2x_3 = \varepsilon(\alpha^2+\beta^2),$$
$$b = x_1x_2 + x_1x_3 + x_2x_3 = 2\varepsilon\alpha + (\alpha^2+\beta^2) = 2\varepsilon\alpha + (1/\varepsilon),$$
$$a = x_1 + x_2 + x_3 = \varepsilon + 2\alpha.$$

Hence, we have $|\alpha| \leq 1/\sqrt{\varepsilon} < 1$ and $\varepsilon < \alpha + 2$, which implies $a \geq 0$. Moreover, we have $|b| \leq 2\sqrt{\varepsilon} + 1 \leq 2\sqrt{a+2} + 1$, which implies that there are only finitely many polynomials of type (T) with $0 \leq a \leq 17$. Hence, we may assume $a \geq 18$.

We have $-D_P = 4(a^3+b^3) - a^2b^2 - 18ab + 27$.

First, we assume $\alpha \geq 0$. Since $0 < 2\varepsilon\alpha + (1/\varepsilon) < 2\sqrt{\varepsilon + 2\alpha}$ (since $0 \leq \alpha \leq 1/\sqrt{\varepsilon}$ and $\varepsilon > 1$), we get $1 \leq b < 2\sqrt{a}$. Now $b \mapsto f(b) = -D_P = 4b^3 - a^2b^2 - 18ab + 4a^3 + 27$ is decreasing in the range $[1, 2\sqrt{a}]$ (since $a \geq 9$). So, we write $4a = m^2 + r$, with $m \geq 0$ and $0 \leq r \leq 2m$, which provides $16d_p \geq 16f(m)$ if $r \geq 1$, and $16d_P \geq 16f(m-1)$ if $r = 0$. Noticing that $16f(b) = (4a)^2(4a-b^2) - 72b(4a-b^2) - 8b^3 + 432$, we thus get

$$16d_P \geq \begin{cases} r(m^2+r)^2 - 72rm - 8m^3 + 432 \geq m^4 - 8m^3 + 2m^2 - 72m + 433 & \text{if } r \geq 1, \\ 2m^5 - m^4 - 8m^3 - 120m^2 + 192m + 368 & \text{if } r = 0. \end{cases}$$

Since $4a \leq m^2$ and $\varepsilon < a + 2$, we get the desired result.

Second, we assume $\alpha \leq 0$. Then, $b = 2\varepsilon\alpha + (1/\varepsilon) \leq 1/\varepsilon < 1$, i.e. $b \leq 0$. We set $B = -b$. Now $g(B) = -D_P = -4B^3 - a^2B^2 + 18aB + 4a^3 + 27$ is decreasing on $[1, +\infty[$ (since $a \geq 9$), and $g(\sqrt{4a+1}) < 0$ (since $a \geq 16$). Hence, we get $B < \sqrt{4a+1}$. So, we write $4a + 1 = m^2 + r$, with $m \geq 0$ and $0 \leq r \leq 2m$. Since $g(0) \geq g(1)$ (since $a \geq 18$), we have $16d_P \geq 16g(m)$ if $r \geq 1$, and $16d_P \geq 16g(m-1)$ if $r = 0$. We thus get

$$16d_P \geq \begin{cases} (r-1)(m^2+r-1)^2+72m(r-1)+8m^3+432 \geq 8m^3+432 & \text{if } r \geq 1, \\ 2m^5-2m^4+4m^3+124m^2-262m+566 & \text{if } r=0. \end{cases}$$

As in the first case, we get the desired result.

Let us note that when $P(X) = X^3 - M^2X^2 - 2MX - 1$ we have $d_P = 4M^3 + 27$ and $M^2 < \varepsilon_P < M^2 + 1$ $(M \geq 2)$ which implies $d_P \approx 4\varepsilon_P^{3/2}$. □

## 6. Conclusion

Let $h$ and $n$ be two given positive integers. Are there only finitely many number fields $\mathbf{K}$ of degree $n$ with class-number $h$ such that their rings of algebraic integers are generated by units? More precisely, let $n \geq 1$ be a given positive integer and let $\mathbf{K} = \mathbf{Q}(x)$ be a number field of degree $n$ where $x$ is an algebraic unit which is a root of any irreducible monic polynomial of the form $P(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_1X \pm 1$. If we assume that the ring of algebraic integers of $\mathbf{K}$ is equal to $\mathbf{Z}[x]$ and that $\mathbf{K}$ has a unit group of rank 1, is that true that the class-number of $\mathbf{K}$ tends to infinity with $d_{\mathrm{K}} = d_P$?

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