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A QUESTION OF GROSS AND THE UNIQUENESS OF ENTIRE FUNCTIONS

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1. Introduction and main results

For any set S and any entire function f let

$$E_f(S) = \bigcup_{a \in S} \{z \mid f(z) - a = 0\},$$

where each zero of $f - a$ with multiplicity m is repeated m times in $E_f(S)$ (cf. [1]). It is assumed that the reader is familiar with the notations of the Nevanlinna Theory (see, for example, [2]). It will be convenient to let E denote any set of finite linear measure on $0 < r < \infty$, not necessarily the same at each occurrence. We denote by $S(r, f)$ any quantity satisfying $S(r, f) = o(T(r, f))$ $(r \to \infty, r \notin E)$.

In 1976 Gross proved [3] that there exist three finite sets S_j $(j = 1,2,3)$, such that any two entire functions f and g satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1,2,3$ must be identical. In the same paper Gross posed the following open question (Question 6): can one find two (or possible even one) finite set S_j $(j = 1,2)$ such that any two entire functions f and g satisfying $E_f(S_j) = E_g(S_j)$ $(j = 1,2)$ must be identical?

The present author [4] proved the following result which is partial answer of the above question.

THEOREM A. *Let* $S_1 = \{w \mid (w - a)^n - b^n = 0\}$, $S_2 = \{c\}$, *where* $n > 4$, a, b *and* c *are constants such that* $b \neq 0$, $c \neq a$ *and* $(c - a)^{2n} \neq b^{2n}$. *Suppose that* f *and* g *are nonconstant entire functions satisfying* $E_f(S_j) = E_g(S_j)$ *for* $j = 1,2$. *Then* $f \equiv g$.

The set S such that for any two nonconstant entire funstions f and g the condition $E_f(S) = E_g(S)$ implies $f \equiv g$ is called a unique range set (URS in brief) of

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entire functions (cf. [5]). In 1982, F. Gross and C. C. Yang proved the following result.

THEOREM B [5]. *The set* $S = \{w \mid e^w + w = 0\}$ *is a URS of entire functions.*

Note that the set $S = \{w \mid e^w + w = 0\}$ contains infinite number of elements and so Theorem B does not answer the question posed by Gross.

In this paper we give a positive answer to Gross's question. In fact, we prove more generally the following theorem.

THEOREM 1. *Let* n *and* m *be two positive integers such that* n *and* m *have no common factor and* $n > 2m + 4$. *Let* a *and* b *be two nonzero constants such that the algebraic equation* $w^n + aw^{n-m} + b = 0$ *has no multiple roots. Then the set* $S = \{w \mid w^n + aw^{n-m} + b = 0\}$ *is a URS of entire functions.*

EXAMPLE. The set $S = \{w \mid w^7 + w^6 + 1 = 0\}$ is a URS of entire functions with 7 elements.

Now it is natural to ask the following question:

Can one find a URS of entire functions with less than 7 elements?

Now we introduce the following notations:

$$U_E = \{S \mid S \text{ is a URS of entire functions}\},$$
$$C_E = \min\{n(S) \mid S \in U_E\},$$

where $n(S)$ denotes the cardinal number of the set S.

The above example shows that $C_E \leqslant 7$. In this paper we prove the following result.

THEOREM 2. $C_E \geqslant 4$.

2. Some lemmas

The following lemmas will be needed in the proof of Theorem 1.

LEMMA 1 (see [6]). *Let* f *and* g *be two nonconstant meromorphic functions, and let* c_1, c_2 *and* c_3 *be three nonzero constants. If*

$$c_1 f + c_2 g = c_3,$$

then

$$T(r,f) < \bar{N}\left(r,\frac{1}{f}\right) + \bar{N}\left(r,\frac{1}{g}\right) + \bar{N}(r,f) + S(r,f).$$

LEMMA 2 (see [7]). *Let $f_1, f_2, \ldots, f_n$ be linearly independent meromorphic functions satisfying*

$$\sum_{j=1}^{n} f_j = 1.$$

Then for $k = 1,2,\ldots,n$ we have

$$T(r,f_k) < \sum_{j=1}^{n} N\left(r,\frac{1}{f_j}\right) + N(r,f_k) + N(r,D) - \sum_{j=1}^{n} N(r,f_j)$$
$$- N\left(r,\frac{1}{D}\right) + o(T(r)) \quad (r \notin E),$$

where D denotes the Wronskian of the functions $f_1, f_2, \ldots, f_n$, and $T(r)$ denotes the maximum of $T(r,f_j)$, $j = 1,2,\ldots,n$.

LEMMA 3 (see [8]). *Let f_1, $f_2(\not\equiv 0)$ and f_3 be three meromorphic functions satisfying $f_1 + f_2 + f_3 = 1$, and let $g_1 = -f_3/f_2$, $g_2 = 1/f_2$ and $g_3 = -f_1/f_2$. If f_1, f_2 and f_3 are linearly independent, then g_1, g_2 and g_3 are linearly independent.*

LEMMA 4 (see [9]). *Let f be a nonconstant meromorphic function, and let $P(f)$ be a polynomial in f of the form*

$$P(f) = a_0 f^n + a_1 f^{n-1} + \cdots + a_{n-1} f + a_n,$$

where $a_0(\neq 0)$, $a_1, \ldots, a_n$ are constants. Then

$$T(r,P(f)) = nT(r,f) + S(r,f).$$

3. Proof of Theorem 1

Let $w_1, w_2, \ldots, w_n$ be the roots of equation $w^n + aw^{n-m} + b = 0$. Suppose that f and g are nonconstant entire functions satisfying $E_f(S) = E_g(S)$. From Nevanlinna's second fundamental theorem, we have

$$(1) \qquad (n-1)T(r,g) < \sum_{j=1}^{n} N\left(r,\frac{1}{g-w_j}\right) + S(r,g)$$
$$= \sum_{j=1}^{n} N\left(r,\frac{1}{f-w_j}\right) + S(r,g)$$

$$< nT(r, f) + S(r, g).$$

Thus

$$T(r, g) = 0(T(r, f)) \quad (r \notin E). \tag{2}$$

Again by $E_f(S) = E_g(S)$, we obtain

$$\frac{f^n + af^{n-m} + b}{g^n + ag^{n-m} + b} = e^h, \tag{3}$$

where h is an entire function. From Lemma 4, (1) and (3), we have

$$\begin{aligned} T(r, e^h) &< T(r, f^n + af^{n-m} + b) + T(r, g^n + ag^{n-m} + b) + 0(1) \\ &= nT(r, f) + nT(r, g) + S(r, f) \\ &< \frac{n(2n-1)}{n-1} \cdot T(r, f) + S(r, f). \end{aligned}$$

Thus

$$T(r, e^h) = 0(T(r, f)) \quad (r \notin E). \tag{4}$$

Let us put

$$f_1 = -\frac{1}{b} f^{n-m}(f^m + a), \tag{5}$$

$$f_2 = e^h, \tag{6}$$

$$f_3 = \frac{1}{b} g^{n-m}(g^m + a) e^h, \tag{7}$$

and $T(r)$ denote the maximum of $T(r, f_j)$, $j = 1,2,3$. From (3), (5), (6) and (7), we obtain

$$f_1 + f_2 + f_3 = 1. \tag{8}$$

From (2), (4), (5), (6) *and* (7), we have

$$T(r) = 0(T(r, f)) \quad (r \notin E). \tag{9}$$

Suppose that f_1, f_2 and f_3 are linearly independent. Applying Lemma 2 to the functions f_j $(j = 1,2,3)$, from (8) and (9) we have

$$T(r, f_1) < \sum_{j=1}^{3} N\left(r, \frac{1}{f_j}\right) - N\left(r, \frac{1}{D}\right) + o(T(r, f)) \quad (r \notin E), \tag{10}$$

where

$$D = \begin{vmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{vmatrix}. \tag{11}$$

From (5), (6) and (7), we have

$$\begin{aligned} \sum_{j=1}^{3} N\left(r, \frac{1}{f_j}\right) = (n-m)N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f^m + a}\right) \\ + (n-m)N\left(r, \frac{1}{g}\right) + N\left(r, \frac{1}{g^m + a}\right). \end{aligned} \tag{12}$$

By looking at the zeros of f and g, from (5), (6), (7) and (11) we see that

$$N\left(r, \frac{1}{D}\right) \geqslant (n-m)N\left(r, \frac{1}{f}\right) - 2\bar{N}\left(r, \frac{1}{f}\right) + (n-m)N\left(r, \frac{1}{g}\right) - 2\bar{N}\left(r, \frac{1}{g}\right). \tag{13}$$

From Lemma 4, (5), (10), (12) and (13), we deduce

$$\begin{aligned} nT(r,f) &< 2\bar{N}\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f^m + a}\right) + 2\bar{N}\left(r, \frac{1}{g}\right) + N\left(r, \frac{1}{g^m + a}\right) + o(T(r,f)) \\ &< 2T(r,f) + T(r, f^m + a) + 2T(r,g) + T(r, g^m + a) + o(T(r,f)) \\ &= (2+m)T(r,f) + (2+m)T(r,g) + o(T(r,f)) \quad (r \notin E). \end{aligned} \tag{14}$$

Let $g_1 = -f_3/f_2 = -\dfrac{1}{b} g^{n-m}(g^m + a)$, $g_2 = 1/f_2 = e^{-h}$ and $g_3 = -f_1/f_2 = \dfrac{1}{b} f^{n-m}(f^m + a)e^{-h}$. From (8) we obtain

$$g_1 + g_2 + g_3 = 1.$$

By Lemma 3 we know that g_1, g_2 and g_3 are linearly independent. In the same manner as above, we have

$$nT(r,g) < (2+m)T(r,g) + (2+m)T(r,f) + o(T(r,f)) \quad (r \notin E). \tag{15}$$

Combining (14) and (15) we get

$$(n - 2m - 4)(T(r,f) + T(r,g)) < o(T(r,f)) \quad (r \notin E). \tag{16}$$

Since $n > 2m + 4$, (16) is absurd. Hence f_1, f_2 and f_3 are linearly dependent. Then, there exist three constants $(c_1, c_2, c_3) \neq (0,0,0)$ such that

$$c_1 f_1 + c_2 f_2 + c_3 f_3 = 0. \tag{17}$$

If $c_1 = 0$, from (17) we have $c_2 \neq 0$, $c_3 \neq 0$ and

$$f_3 = -\frac{c_2}{c_3} f_2.$$

Hence, from (6) and (7) we obtain

$$g^n + ag^{n-m} = -bc_2/c_3,$$

which is impossible. Thus $c_1 \neq 0$ and

$$f_1 = -\frac{c_2}{c_1} f_2 - \frac{c_3}{c_1} f_3. \tag{18}$$

Now combining (8) and (18) we get

$$\left(1 - \frac{c_2}{c_1}\right) f_2 + \left(1 - \frac{c_3}{c_1}\right) f_3 = 1. \tag{19}$$

We discuss the following three cases.

(a) Assume $c_1 \neq c_2$ and $c_1 \neq c_3$. From (6), (7) and (19) we have

$$-\frac{1}{b}\left(1 - \frac{c_3}{c_1}\right) g^{n-m}(g^m + a) + e^{-h} = 1 - \frac{c_2}{c_1}. \tag{20}$$

By Lemma 1, Lemma 4 and (20) we obtain

$$\begin{aligned} nT(r, g) &< \bar{N}\left(r, \frac{1}{g^{n-m}(g^m + a)}\right) + S(r, g) \\ &= \bar{N}\left(r, \frac{1}{g}\right) + \bar{N}\left(r, \frac{1}{g^m + a}\right) + S(r, g) \\ &< (1 + m)T(r, g) + S(r, g), \end{aligned}$$

which is impossible.

(b) Assume $c_1 = c_2$. From (19) we have $c_1 \neq c_3$ and

$$f_3 = \frac{c_1}{c_1 - c_3}. \tag{21}$$

From (7) and (21) we get

$$g^{n-m}(g^m + a) = \frac{bc_1}{c_1 - c_3} e^{-h}. \tag{22}$$

Let $a_1, a_2, \ldots, a_m$ be the roots of equation $w^m + a = 0$. From (22) we know that $0, a_1, a_2, \ldots, a_m$ are Picard exceptional values of g, which is impossible.

(c) Assume $c_1 = c_3$. From (19) we have $c_1 \neq c_2$ and

$$f_2 = \frac{c_1}{c_1 - c_2}$$

that is

$$e^h = \frac{c_1}{c_1 - c_2}. \tag{23}$$

From (5), (7), (8) and (23) we get

$$-\frac{1}{b} f^{n-m}(f^m + a) + \frac{c_1}{b(c_1 - c_2)} g^{n-m}(g^m + a) = \frac{c_2}{c_2 - c_1}. \tag{24}$$

If $c_2 \neq 0$, by Lemma 1 and Lemma 4, we have from (24),

$$\begin{aligned} nT(r, f) &< \bar{N}\left(r, \frac{1}{f^{n-m}(f^m + a)}\right) + \bar{N}\left(r, \frac{1}{g^{n-m}(g^m + a)}\right) + S(r, f) \\ &< \bar{N}\left(r, \frac{1}{f}\right) + \bar{N}\left(r, \frac{1}{f^m + a}\right) + \bar{N}\left(r, \frac{1}{g}\right) + \bar{N}\left(r, \frac{1}{g^m + a}\right) + S(r, f) \\ &< (1 + \mathrm{m}) T(r, f) + (1 + m) T(r, g) + S(r, f). \end{aligned}$$

In the same manner as above, we have

$$nT(r, g) < (1 + m) T(r, g) + (1 + m) T(r, f) + S(r, f).$$

Hence,

$$(n - 2m - 2) T(r, f) + (n - 2m - 2) T(r, g) < S(r, f),$$

which is impossible. Thus $c_2 = 0$. From (24) we deduce

$$f^n - g^n = -a(f^{n-m} - g^{n-m}). \tag{25}$$

If $f^n \not\equiv g^n$, from (25) we obtain

$$\frac{-a \prod_{k=1}^{n-m-1} \left(\frac{f}{g} - v^k\right)}{\prod_{j=1}^{n-1} \left(\frac{f}{g} - u^j\right)} = g^m, \tag{26}$$

where $u = \exp\left(\frac{2\pi i}{n}\right)$ and $v = \exp\left(\frac{2\pi i}{n - m}\right)$. From (26) we know that $\frac{f}{g}$ is a nonconstant meromorphic function. Since n and m have no common factors, again from (26) we know that u^j $(j = 1,2, \ldots, n - 1)$ are Picard exceptional values of $\frac{f}{g}$, which is impossible. Thus $f^n \equiv g^n$ and $f^{n-m} \equiv g^{n-m}$. However, since n and m have no common factors, we get $f \equiv g$. This completes the proof of Theorem 1.

4. Proof of Theorem 2

Let $S = \{a_1, a_2, a_3\}$, where a_j $(j = 1,2,3)$ are any three finite distinct complex numbers. If $a_2 + a_3 - 2a_1 = 0$, let

$$g(z) = 2a_1 - f(z),$$

where $f(z)$ is a nonconstant entire function. If $a_2 + a_3 - 2a_1 \neq 0$, let

$$f(z) = \frac{(a_2a_3 - a_1^2) + (a_2 - a_1)(a_3 - a_1)e^{h(z)}}{a_2 + a_3 - 2a_1},$$
$$g(z) = \frac{(a_2a_3 - a_1^2) + (a_2 - a_1)(a_3 - a_1)e^{-h(z)}}{a_2 + a_3 - 2a_1},$$

where $h(z)$ is a nonconstant entire function. It is easy to show that $E_f(S) = E_g(S)$, but $f \not\equiv g$. Hence $C_E \geqslant 4$, which proves Theorem 2.

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