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THE h-VECTOR OF A GORENSTEIN CODIMENSION THREE DOMAIN

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Let k be an infinite field and A a *standard G-algebra*. This means that there exists a positive integer n such that $A = R/I$ where R is the polynomial ring $R := k[X_1, \ldots, X_n]$ and I is an homogeneous ideal of R. Thus the additive group of A has a direct sum decomposition $A = \bigoplus A_t$ where $A_i A_j \subseteq A_{i+j}$. Hence, for every $t \geq 0$, A_t is a finite-dimensional vector space over k. The *Hilbert Function* of A is defined by

$$H_A(t) := \dim_k(A_t), \quad t \geq 0.$$

The generating function of this numerical function is the formal power series

$$P_A(z) := \sum_{t \geq 0} H_A(t) z^t.$$

As a consequence of the Hilbert-Serre theorem we can write

$$P_A(z) = h_A(z)/(1 - z)^d$$

where $h_A(z) \in \mathbf{Z}[z]$ is a polynomial with integer coefficients such that $h_A(1) \neq 0$. Moreover d is the Krull dimension of the ring A.

The polynomial $h_A(z)$ is called the *h-polynomial* of A; if $h_A(z) = 1 + a_1 z + \cdots + a_s z^s$ with $a_s \neq 0$, then we say that the vector $(1, a_1, \ldots, a_s)$ is the *h-vector* of A. It is clear that the h-vector of A together with its Krull dimension determines the Hilbert Function of A and conversely.

A classical result of Macaulay gives an explicit numerical characterization of the *admissible* numerical functions, i.e. of the functions $H : \mathbf{N} \to \mathbf{N}$ which are the Hilbert Function of some standard G-algebra A. This result proved in [M] has been recently revisited by Stanley in [S]. One can easily find similar characterizations for reduced or Cohen-Macaulay G-algebras (see [GMR] and [S]).

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The problem is much more difficult if one deals with Cohen-Macaulay integral domains. Only in the codimension two case we have a complete answer given by Peskine and Gruson in [GP] using deep geometric methods.

If we come to the Gorenstein case, very little is known. In [S] Stanley used the structure theorem of Buchsbaum and Eisenbud for codimension three Gorenstein ideals in order to give a complete characterization of the corresponding h-vector. It is then natural to ask for other restrictions on the h-vector of a Gorenstein codimension three G-algebra A if we assume moreover that A is an integral domain.

In this paper we answer this question by using a lifting theorem recently proved in [HTV], which asserts that every codimension three homogeneous Gorenstein ideal with degree matrix verifying certain numerical conditions can be lifted to a codimension three Gorenstein prime ideal (see Lemma 3).

Let us fix some notations. If $h(z) \in \mathbf{Z}[z]$ we define its difference $\Delta h(z)$ by

$$\Delta h(z) := h(z)(1-z).$$

If $h(z)$ is a multiple of $1-z$ then we define its sum $\sum h(z)$ by

$$\sum h(z) := \frac{h(z)}{(1-z)}.$$

If we have $h(z) = \sum_{i=0}^{s} a_i z^i$, then it is clear that $\Delta h(z) = \sum_{i=0}^{s+1} b_i z^i$ where

$$b_i = a_i - a_{i-1}, \quad i = 0, \dots, s+1.$$

Moreover if $h(z)$ is a multiple of $1-z$ then $\sum h(z) := \sum_{i=0}^{s-1} c_i z^i$ where

$$c_i = \sum_{j=0}^{i} a_j, \quad i = 0, \dots, s-1.$$

We say that the polynomial $h(z) = \sum_{i=0}^{s} a_i z^i \in \mathbf{Z}[z]$ is *s-symmetric* if $a_i = a_{s-i}$ for every $i = 0, \dots, s$, while we say that it is *s-antisymmetric* if $a_i = -a_{s-i}$ for every $i = 0, \dots, s$.

It is easy to see that if $h(z)$ is s-symmetric then $\Delta h(z)$ is $(s+1)$-antisymmetric, while if $h(z)$ is a multiple of $1-z$ and is s-antisymmetric then $\sum h(z)$ is $(s-1)$-symmetric.

Let now I be a codimension three homogeneous Gorenstein ideal of the polynomial ring $R := k[X_1, \dots, X_n]$. By the structure theorem of Buchsbaum and Eisenbud [BE], there exists an integer $g \geq 1$ such that I is minimally generated by the $2g$-pfaffians of a $(2g+1) \times (2g+1)$ skew-symmetric matrix (F_{ij}) with homogeneous entries. We denote by p_i the pfaffian of the skew-symmetric matrix which is obtained from (F_{ij}) by deleting the i-th row and the i-th column. Then $I = (p_1, \ldots, p_{2g+1})$. Let $a_1, \ldots, a_{2g+1}$ be the degrees of these pfaffians. Since R/I is Gorenstein, it has a self-dual free homogeneous resolution as an R-module:

$$0 \to R(-c) \to \bigoplus_{i=1}^{2g+1} R(-b_i) \to \bigoplus_{i=1}^{2g+1} R(-a_i) \to R \to R/I \to 0.$$

We may assume that

$$2 \leq a_1 \leq a_2 \leq \cdots \leq a_{2g+1}.$$

Since the resolution is self-dual we get

$$b_i = c - a_i, \quad i = 1, \ldots, 2g+1.$$

From the additivity of the Poincaré series, we can write

$$P_{R/I}(z) = P_R(z) - \sum_{i=1}^{2g+1} P_{R(-a_i)}(z) + \sum_{i=1}^{2g+1} P_{R(-b_i)}(z) - P_{R(-c)}(z) = \\ = \frac{1 - \sum_{i=1}^{2g+1} z^{a_i} + \sum_{i=1}^{2g+1} z^{b_i} - z^c}{(1-z)^n}.$$

Since $\dim(R/I) = n - 3$, we have

$$h_A(z) = \frac{f(z)}{(1-z)^3}$$

where

$$f(z) := 1 - \sum_{i=1}^{2g+1} z^{a_i} + \sum_{i=1}^{2g+1} z^{b_i} - z^c$$

is a multiple of $(1-z)^3$. This means that its derivative vanishes at 1 so that

$$-\sum_{i=1}^{2g+1} a_i + \sum_{i=1}^{2g+1} b_i - c = 0.$$

Using the fact that $b_i = c - a_i$, we get

$$c = \frac{1}{g} \sum_{i=1}^{2g+1} a_i.$$

This proves that the degrees of a minimal set of homogeneous generators of a Gorenstein codimension three ideal completely determine the numerical characters of the resolution.

We consider the matrix (u_{ij}) where we let

$$u_{ij} := b_i - a_j, \quad i, j = 1, \ldots, 2g+1.$$

This matrix is then uniquely determined by I and is called the *degree matrix* of I. It is clear that (u_{ij}) is a symmetric matrix and

$$\deg(F_{ij}) = b_i - a_j = u_{ij}, \quad i, j = 1, \ldots, 2g+1.$$

Since the resolution is minimal, this implies that $F_{ij} = 0$ if $u_{ij} \leq 0$.

The degree matrix of I verifies the following conditions:

(a) $u_{ij} \geq u_{st}$ for $i \leq s$ and $j \leq t$.

(b) $u_{ij} + u_{st} = u_{it} + u_{sj}$ for every i, j, s and t.

(c) $u_{ij} > 0$ for all i and j such that $i + j = 2g + 3$.

The first two conditions are obvious. As for (c), if $u_{r,2g+3-r} \leq 0$, then by (a) $u_{ij} \leq 0$ for every $i \geq r$ and $j \geq 2g + 3 - r$. This implies that $F_{ij} = 0$ for the same indexes. But then $p_1 = 0$, a contradiction to the minimality of the resolution.

We remark that condition a) above can be visualized by observing that it has the following meaning: the entries of the matrix do not decrease if we move up or left inside the matrix.

Further if $c \leq b_1 = c - a_1$ then $a_1 \leq 0$, a contradiction. Hence we certainly have

$$\begin{array}{ccccccccccc}
c > b_1 \geq & b_2 & \geq & b_3 & \geq \cdots \geq & b_{2g+1} & & \\
 & \vee & & \vee & & \vee & & \\
 & a_{2g+1} & \geq & a_{2g} & \geq \cdots \geq & a_2 & \geq a_1 \geq 2.
\end{array}$$

The converse of the above result is also true: we insert here a proof for the sake of completeness.

LEMMA 1. *Let* $2 \leq a_1 \leq \cdots \leq a_{2g+1}$ *be integers such that for some integer* c *we have* $cg = \sum_{i=1}^{2g+1} a_i$. *For every* $i = 1, \ldots, 2g+1$, *let* $b_i := c - a_i$. *If the matrix* $(u_{ij}) := (b_i - a_j)$, *which certainly verifies conditions* (a) *and* (b), *also verifies the above condition* (c), *then there exists a codimension three Gorenstein ideal* I *in* $R = k[X, Y, Z]$ *such that* R/I *has a minimal free resolution*

$$0 \to R(-c) \to \bigoplus_{i=1}^{2g+1} R(-b_i) \to \bigoplus_{i=1}^{2g+1} R(-a_i) \to R \to R/I \to 0.$$

In particular I *has degree matrix* (u_{ij}).

Proof. In the polynomial ring $k[X, Y, Z]$ let I be the ideal generated by the $2g \times 2g$ pfaffians of the skew-symmetric matrix (F_{ij}) where

$$\begin{cases} F_{i,2g+1-i} = X^{u_{i,2g+1-i}}, & i = 1, \dots, g, \\ F_{i,2g+1-i} = -X^{u_{i,2g+1-i}}, & i = g+1, \dots, 2g, \\ F_{i,2g+2-i} = Y^{u_{i,2g+2-i}}, & i = 1, \dots, g, \\ F_{i,2g+2-i} = -Y^{u_{i,2g+2-i}}, & i = g+2, \dots, 2g+1, \\ F_{i,2g+3-i} = Z^{u_{i,2g+3-i}}, & i = 2, \dots, g+1, \\ F_{i,2g+3-i} = -Z^{u_{i,2g+3-i}}, & i = g+2, \dots, 2g+1, \\ 0 & \text{otherwise.} \end{cases}$$

We remark that since the matrix (u_{ij}) verifies the condition c) all the exponents above are positive integers.

Furthermore, in order to get a homogeneous matrix, we assign degree u_{ij} to the zero on the i-th row and j-th column. It is easy to see that

$$p_1 = Z^{\sum_{i=2}^{g+1} u_{i,2g+3-i}}, \quad p_{2g+1} = X^{\sum_{i=1}^{g} u_{i,2g+1-i}}, \; p_{g+1} = Y^{\sum_{i=1}^{g} u_{i,2g+2-i}} + f(X, Y, Z)$$

where $f(X, Y, Z) \in (X, Z)$. This means that I is a codimension three ideal which is Gorenstein since it is generated by the pfaffians of a skew-symmetric matrix.

Moreover, since the determinant of a skew-symmetric matrix is the square of the pfaffian, we have

$$\deg(p_i) = \frac{\sum_{j \neq i} u_{jj}}{2} = \frac{\sum_{j \neq i} (c - 2a_j)}{2} = gc - \sum_{j \neq i} a_j = a_i.$$

The conclusion then follows since we have seen that the degrees of a minimal set of homogeneous generators of a codimension three Gorenstein ideal completely determine the other numerical characters of the resolution.

If we assume now that the codimension three Gorenstein ideal is prime, then we have a stronger condition on the degree matrix. This is the content of the following result proved in [HTV], Lemma 5.1.

LEMMA 2. *Let $I \subseteq R = k[X_1, \dots, X_n]$ be a codimension three homogeneous Gorenstein prime ideal with degree matrix (u_{ij}). If $g \geq 2$, then*

$$u_{i,2g+4-i} > 0, \quad i = 3, \dots, g+1.$$

Proof. If $u_{t,2g+4-t} \leq 0$ for some t such that $3 \leq t \leq g+1$, then $u_{ij} \leq 0$ for every $i \geq t$ and $j \geq 2g+4-t$, so that $F_{ij} = 0$ for the same indexes (here, as before, F_{ij} are the entries of the skew-symmetric matrix in the resolution of R/I). This implies that the $2(g+2-t)$-pfaffian obtained from the matrix (F_{ij}) by deleting the first $t-1$ and the last $t-2$ rows and columns, is a common factor of p_1 and p_2. A contradiction.

We remark here that if we have $u_{i,2g+4-i} > 0$ for $i = 3, \ldots, g+1$ then, by the symmetry of the matrix (u_{ij}), we also have $u_{i,2g+4-i} > 0$ for $i = g+3, \ldots, 2g+1$. Thus on the diagonal where $i+j = 2g+4$ all the entries of the matrix (u_{ij}) are positive integers except, possibly, for $u_{g+2,g+2}$.

Further it is clear that, if $g \geq 2$, then a degree matrix such that $u_{i,2g+4-i} > 0$ for $i = 3, \ldots, g+1$ verifies also condition c) above, namely $u_{i,2g+3-i} > 0$ for every $i = 2, \ldots, 2g+1$. This because we can express this condition by saying that all the entries on the $(2g+4)$-diagonal are positive and remark that for every element of the $(2g+3)$-diagonal we can find an element on the $(2g+4)$-diagonal which is right or below the given element and is different from $u_{g+2,g+2}$.

The following less trivial result is the lifting theorem we referred to in the introduction.

Let $I \subseteq R = k[X_1, \ldots, X_n]$ be an homogeneous ideal. We say that the ideal I can be lifted to an ideal $J \subseteq S = k[X_1, \ldots, X_m]$, $m \geq n$, if there exist $r = m-n$ linear forms $l_1, \ldots, l_r \in S$ such that:

a) $l_1, \ldots, l_r$ form a regular sequence mod J.

b) In the canonical isomorphism

$$S/(l_1, \ldots, l_r)S \simeq R$$

the ideal $(J + (l_1, \ldots, l_r)S)/(l_1, \ldots, l_r)S$ corresponds to I.

It is clear that if the ideal I can be lifted to the ideal J, then

$$P_{R/I}(z) = (1-z)^{m-n} P_{S/J}(z).$$

In particular they share the same h-polynomial.

LEMMA 3. *Let $I \subseteq R = k[X_1, \ldots, X_n]$ be a codimension three homogeneous Gorenstein ideal. Let us assume that either $g = 1$ or $g \geq 2$ and the degree matrix (u_{ij}) of I satisfies the condition*

$$u_{i,2g+4-i} > 0$$

for every $i = 3, \ldots, g+1$. *Then* I *can be lifted to a codimension three Gorenstein prime ideal* $J \subseteq S = k[X_1, \ldots, X_m]$, *for some integer* $m \geq n$.

A proof of this crucial result can be found in [HTV], Lemma 5.5.

Now let $h(z) = 1 + 3z + h_2 z^2 + \cdots + h_s z^s$ be a polynomial in $\mathbf{Z}[z]$ such that $h_s \neq 0$. The integer

$$a := \min\left\{t \,\middle|\, h_t \neq \binom{t+2}{2}\right\}$$

is called the initial degree of $h(z)$. It is clear that $2 \leq a \leq s+1$. In the following, for a rational number q, we denote by $[q]$ its integer part.

LEMMA 4. *If the polynomial* $h(z)$ *is* s*-symmetric, then*

$$2 \leq a \leq \left[\frac{s}{2}\right] + 1.$$

Proof. If $2a - 2 > s$, then $s - a + 2 < a$. This implies $h_{s-a+2} = \binom{s-a+4}{2}$, hence, by the symmetry,

$$\binom{a}{2} = h_{a-2} = \binom{s-a+4}{2}.$$

It follows that $s - a + 4 = a$. If s is odd, this is a contradiction. If s is even, say $s = 2t$, then $a = t + 2$, hence

$$\binom{t+1}{2} = h_{t-1} = h_{t+1} = \binom{t+3}{2},$$

a contradiction. Hence $a \leq \frac{s}{2} + 1$ and the conclusion follows.

In the following we will often use the trivial inequalities:

$$s - 1 \leq 2\left[\frac{s}{2}\right] \leq s.$$

Given a polynomial $h(z) = 1 + 3z + h_2 z^2 + \cdots + h_s z^s \in \mathbf{Z}[z]$ such that $h_s \neq 0$, we denote by a its initial degree and also we let

$$\sum_{t=0}^{s+2} q_t z^t := h(z)(1-z)^2 = \Delta^2 h(z).$$

We can now prove the main result of this paper.

THEOREM 5. *Given the polynomial* $h(z) = 1 + 3z + h_2 z^2 + \cdots + h_s z^s \in \mathbf{Z}[z]$ *with* $h_s \neq 0$, *there exists a codimension three Gorenstein G-domain which has* $h(z)$ *as h-polynomial if and only if the following conditions are satisfied*:

a) $h(z)$ *is s-symmetric.*

b) $q_t \leq 0$ *for every* t *such that* $a \leq t \leq \left[\frac{s}{2}\right] + 1$.

c) *It does not happen that* $q_t < 0$, $q_v = 0$ *and* $q_r < 0$ *with* $a \leq t < v < r \leq \left[\frac{s}{2}\right] + 1$.

Proof. Let us assume first that $h(z)$ is the h-polynomial of a Gorenstein G-domain A. Then it is well known that $h(z)$ is s-symmetric (see [S], Theorem 4.1). Let

$$0 \to R(-c) \to \bigoplus_{i=1}^{2g+1} R(-b_i) \to \bigoplus_{i=1}^{2g+1} R(-a_i) \to R \to R/I \to 0$$

be a graded free resolution of $A = R/I$, where we assume that

$$a_1 \leq a_2 \leq \cdots \leq a_{2g+1}.$$

As we have seen before we have

$$c = \frac{1}{g} \sum_{i=1}^{2g+1} a_i, \quad b_i = c - a_i,\ i = 1, \ldots, 2g+1$$

and

$$h(z) = \frac{f(z)}{(1-z)^3}$$

where we let

$$f(z) = 1 - \sum_{i=1}^{2g+1} z^{a_i} + \sum_{i=1}^{2g+1} z^{b_i} - z^c.$$

Since

$$h(z) = \left(1 - \sum_{i=1}^{2g+1} z^{a_i} + \sum_{i=1}^{2g+1} z^{b_i} - z^c\right)\left(\sum_{t\geq 0} \binom{t+2}{2} z^t\right),$$

and

$$a_1 = \min\{a_i, b_i\}_{i=1,\dots,2g+1},$$

we have

$$a = a_1.$$

Since, as we have seen before,

$$c > a_i,\ b_i, \quad i = 1, \dots, 2g+1,$$

we also have

$$c = s + 3.$$

Now

$$\sum_{t=0}^{s+2} q_t z^t := h(z)(1-z)^2 = \frac{1 - \sum_{i=1}^{2g+1} z^{a_i} + \sum_{i=1}^{2g+1} z^{b_i} - z^c}{1-z} = \sum f(z).$$

From this we get

$$(*) \qquad q_t = 1 + \#\{m \mid b_m \leq t\} - \#\{m \mid a_m \leq t\}, \quad t = 0\dots, s+2.$$

To better visualize our argument, we recall that, no matter I is prime or not, we have:

$$\begin{array}{ccccccccc} c > b_1 \geq & b_2 & \geq & b_3 & \geq \cdots \geq & b_{2g+1} & & \\ & \vee & & \vee & & \vee & & \\ & a_{2g+1} & \geq & a_{2g} & \geq \cdots \geq & a_2 & \geq a_1 \geq 2. \end{array}$$

We need also to remark that

$$c - b_1 = a_1 \leq a_2 < b_{2g+1} \leq b_1$$

so that

$$\left[\frac{s}{2}\right] + 1 = \left[\frac{s+2}{2}\right] = \left[\frac{c-1}{2}\right] \leq \left[\frac{c}{2}\right] \leq \frac{c}{2} < b_1.$$

We prove now that condition b) holds.

Let t be an integer such that

$$a \leq t \leq \left[\frac{s}{2}\right] + 1.$$

Then, by the above inequality, $a \leq t < b_1$. We have two possibilities: either

$t < b_{2g+1}$ or $b_{2g+1} \leq t$.

In the first case since $a = a_1 \leq t < b_{2g+1}$, we immediately get

$$\{m \mid b_m \leq t\} = \emptyset ,$$

and

$$\{m \mid a_m \leq t\} \supseteq \{1\}.$$

This implies by (∗)

$$q_t \leq 1 - 1 = 0,$$

as wanted.

In the second case we have $b_{2g+1} \leq t < b_1$, and we can find an integer r such that

$$2 \leq r \leq 2g + 1, \quad b_r \leq t < b_{r-1}.$$

Hence

$$a_{2g+3-r} < b_r \leq t < b_{r-1}$$

and we get

$$\{m \mid b_m \leq t\} = \{r, r + 1, \ldots, 2g + 1\},$$

$$\{m \mid a_m \leq t\} \supseteq \{1,2, \ldots, 2g + 3 - r\}.$$

From (∗) we get

$$q_t \leq 1 + (2g + 1 - r + 1) - (2g + 3 - r) = 0.$$

This proves b).

We remark that, up to this point, we did not use the primality assumption.

Let us come to the last statement. By contradiction, let

$$q_t < 0, \quad q_v = 0, \quad q_r < 0$$

with $a \leq t < v < r \leq \left[\frac{s}{2}\right] + 1$.

Under this assumption we claim that

$$b_{2g+1} \leq v \leq a_{2g+1} < b_2.$$

The first inequality comes from the fact that $q_t < 0$ and $q_v = 0$, hence we need at least one b_i's to get a positive contribution in the sum in ($*$). The second inequality follows from the same argument due to the fact that $q_v = 0$ and $q_r < 0$.

The claim implies that we can find an integer d such that

$$b_{d+1} \leq v < b_d.$$

In the case $g = 1$, we have

$$b_3 \leq v \leq a_3 < b_2,$$

hence $d = 2$ and either

$$a_{2g+3-d} = a_3 = v < b_2 = b_d,$$

or

$$b_{g+2} = b_3 \leq v < a_3 = a_{g+2}.$$

If $g \geq 2$, we use the full power of the primality assumption, which, after Lemma 2 and the subsequent remark, can be read in the following picture:

$$\begin{array}{ccccccccccccc} b_1 \geq b_2 \geq & b_3 & \geq \cdots \geq & b_{g+1} & \geq b_{g+2} \geq & b_{g+3} & \geq \cdots \geq & b_{2g+1} & \\ & \vee & & \vee & & \vee & & \vee & \\ & a_{2g+1} & \geq \cdots \geq & a_{g+3} & \geq a_{g+2} \geq & a_{g+1} & \geq \cdots \geq & a_3 & \geq a_2 \geq a_1. \end{array}$$

Looking at these inequalities, we see that:
if $d \neq g+1$, then

$$a_{2g+3-d} = a_{2g+4-(d+1)} < b_{d+1} \leq v \leq b_d,$$

if $d = g+1$ and $a_{g+2} \leq v$, then

$$a_{2g+3-d} = a_{g+2} \leq v < b_d,$$

if $d = g+1$ and $v < a_{g+2}$, then

$$b_{d+1} = b_{g+2} \leq v < a_{g+2}.$$

Hence we have to skip out these two possibilities:
i) $a_{2g+3-d} \leq v < b_d$.

In this case we have

$$\{m \mid b_m \leq v\} = \{d+1, \ldots, 2g+1\}$$

and

$$\{m \mid a_m \leq v\} \supseteq \{1,2,\ldots, 2g+3-d\}.$$

Hence by ($*$) we get

$$0 = q_v \leq 1 + (2g+1-(d+1)+1) - (2g+3-d) = -1,$$

a contradiction.

ii) $b_{g+2} \leq v < a_{g+2}$.

In this case $c - a_{g+2} < a_{g+2}$, hence $a_{g+2} > \frac{c}{2}$. From this we get

$$a_{g+1} < b_{g+2} \leq v < r \leq \left[\frac{s}{2}\right] + 1 \leq \frac{s+2}{2} < \frac{c}{2} < a_{g+2}.$$

This is absurd because we have no a_i's between v and r so that we cannot pass from $q_v = 0$ to $q_r < 0$.

We will prove now the converse. We have a polynomial

$$h(z) = 1 + 3z + h_2 z^2 + \cdots + h_s z^s \in \mathbf{Z}[z],$$

such that $h_s \neq 0$ and $h(z)$ verifies conditions a), b) and c) in the theorem. We let

$$c := s+3 \quad a := \min\left\{t \,\middle|\, h_t \neq \binom{t+2}{2}\right\}$$

and

$$\sum_{t=0}^{s+2} q_t z^t := h(z)(1-z)^2 = \Delta^2 h(z).$$

We have by b)

$$q_t \leq 0 \quad \text{if} \quad a \leq t \leq \left[\frac{s}{2}\right] + 1 = \left[\frac{c-3}{2}\right] + 1 = \left[\frac{c-1}{2}\right].$$

Since $\Delta^2 h(z)$ is $(c-1)$-symmetric, we immediately get

$$q_t \leq 0 \quad \text{if} \quad c-1-\left[\frac{c-1}{2}\right] \leq t \leq c-1-a.$$

But

$$\left[\frac{c-1}{2}\right] \geq \frac{c-2}{2},$$

hence

$$c-1-\left[\frac{c-1}{2}\right] \le \left[\frac{c-1}{2}\right]+1$$

so that we finally get

(b′) $$q_t \le 0, \quad \text{if} \quad a \le t \le c-1-a.$$

Now let

$$\sum_{i=0}^{c} k_t z^i := \Delta^3 h(z) = \Delta\Big(\sum_{t=0}^{c-1} q_t z^t\Big) = h(z)(1-z)^3.$$

We have already remarked that this polynomial is c-antisymmetric. We have some strong informations on its coefficients.

1. $k_0 = 1,\ k_c = -1$.

2. $k_j = 0$ if $j \in [1, a-1] \cup [c-a+1, c-1]$.

This is easy to see since, by the definition of a, for every $i = 0, \ldots, a-1$ we have $h_i = \binom{i+2}{2}$, hence

$$h_i - h_{i-1} = \binom{i+2}{2} - \binom{i+1}{2} = i+1$$
$$q_i = h_i - h_{i-1} - (h_{i-1} - h_{i-2}) = i+1-i = 1$$

and finally

$$k_0 = 1, \quad k_i = q_i - q_{i-1} = 1-1 = 0, \text{ for every } i = 1, \ldots, a-1.$$

The c-antisymmetry of $\Delta^3 h(z)$ gives the conclusion.

3. $k_a < 0,\ k_{c-a} > 0$.

This is also clear since $k_a = q_a - q_{a-1} = q_a - 1$. It follows that $k_a < 0$ because $q_a \le 0$ by assumption.

Now the crucial remark is that by using the c-antisymmetry of $\sum_{i=0}^{c} k_i z^i$, we can write in a unique way

$$\sum_{i=0}^{c} k_i z^i = 1 - \sum_{i=1}^{p} z^{a_i} + \sum_{i=1}^{p} z^{b_i} - z^c$$

where p, a_i and b_i are positive integers such that

$$b_i = c - a_i \text{ for every } i \quad \text{and} \quad a_i \ne b_j, \text{ for every } i, j.$$

We may assume that

$$a_1 \leq a_2 \leq \cdots \leq a_p$$

so that

$$b_p \leq b_{p-1} \leq \cdots \leq b_1.$$

Since

$$\Delta^3 h(z) = 1 - \sum_{i=1}^{p} z^{a_i} + \sum_{i=1}^{p} z^{b_i} - z^c$$

is a multiple of $(1 - z)^3$, its derivative vanishes at 1, so that

$$- \sum_{i=1}^{p} a_i + \sum_{i=1}^{p} b_i - c = 0.$$

By using the fact that $b_i = c - a_i$, we get

4. $(p - 1)c = 2 \sum_{i=1}^{p} a_i$.

By 1, 2 and 3 above we also get

5. $a_1 = a$, $b_1 = c - a$, and a_i, $b_i \in [a, c - a]$ for every $i = 1, \ldots, p$.

Since

$$\sum_{i=0}^{s+2} q_i z^i = \frac{1 - \sum_{i=1}^{p} z^{a_i} + \sum_{i=1}^{p} z^{b_i} - z^c}{1 - z} = \Sigma\Big(1 - \sum_{i=1}^{p} z^{a_i} + \sum_{i=1}^{p} z^{b_i} - z^c\Big),$$

we also have

6. $q_t = 1 + \#\{m \mid b_m \leq t\} - \#\{m \mid a_m \leq t\}, \quad t = 0 \ldots, c - 1.$

We collect some other properties of the integers involved in our computation.

7. $p \geq 2$.

This follows immediately from 4, since $a_i \geq 2$ for every i.

8. $a_2 < b_p$, hence $a_p < b_2$.

If not we have, by 5, $a_1 < b_p < a_2$, hence

$$\{m \mid a_m \leq b_p\} = \{1\},$$

and

$$\{m \mid b_m \leq b_p\} \supseteq \{p\},$$

so that, by 6,

$$q_{b_p} \geq 1 + 1 - 1 = 1.$$

Since

$$b_2 = c - a_2 < c - a_1 = b_1,$$

we have

$$a_1 < b_p \leq b_2 \leq b_1 - 1 = c - a - 1.$$

Hence we get

$$a < b_p \leq c - a - 1, \quad \text{and } q_{b_p} \geq 1,$$

a contradiction to b′).

9. $q_{a_2} < 0$.

By 8 we have

$$a_1 \leq a_2 < b_p,$$

hence

$$\{m \mid a_m \leq a_2\} \supseteq \{1,2\}$$

and

$$\{m \mid b_m \leq a_2\} = \emptyset .$$

By 6, this means

$$q_{a_2} \leq 1 - 2 = -1,$$

as wanted.

10. If p is even, then c is even.

This follows immediately from 4.

11. $a_2 \leq \left[\frac{s}{2}\right] + 1$.

Otherwise $c - b_2 > \left[\frac{s}{2}\right] + 1$, hence

$$b_2 < c - \left[\frac{s}{2}\right] - 1 = s + 2 - \left[\frac{s}{2}\right].$$

But, by 8, $a_2 < b_2$, hence

$$\left[\frac{s}{2}\right] + 1 < a_2 \leq b_2 - 1 \leq s - \left[\frac{s}{2}\right],$$

a contradiction.

12. Let us assume that there exists an integer t such that $a_2 < t < b_2$, and $q_t = 0$.

We claim that this has the following consequences:

12a. $q_t = 0$ for some integer t such that $a_2 < t \leq \left[\frac{s}{2}\right] + 1$.

If we have

$$\left[\frac{s}{2}\right] + 2 \leq t \leq b_2 - 1 = c - a_2 - 1$$

then we get

$$a_2 \leq c - 1 - t \leq c - 1 - \left[\frac{s}{2}\right] - 2 = s - \left[\frac{s}{2}\right] \leq \left[\frac{s}{2}\right] + 1.$$

Since $0 = q_t = q_{c-1-t}$ and by 9 $q_{a_2} < 0$, we must have

$$a_2 < c - 1 - t \leq \left[\frac{s}{2}\right] + 1$$

and the conclusion follows.

Thank to this last property we may then define the following integer:

$$n := \min\left\{t \mid a_2 < t \leq \left[\frac{s}{2}\right] + 1,\ q_t = 0\right\}.$$

12b. $n < c - n < b_2$.

Since $n \leq \left[\frac{s}{2}\right] + 1$ we have

$$2n \leq 2\left[\frac{s}{2}\right] + 2 \leq s + 2 = c - 1 < c.$$

On the other hand

$$c - b_2 = a_2 < n,$$

as desired.

12c. $q_d < 0$ if $d \in [a_2, n-1] \cup [c-n, b_2-1]$, $q_d = 0$ if $d \in [n, c-n-1]$.
Since

$$a \leq a_2 < n \leq \left[\frac{s}{2}\right] + 1$$

and $q_{a_2} < 0$ by 9, $q_n = 0$ by assumption, condition c) implies that

$$q_d = 0, \quad \text{if} \quad n \leq d \leq \left[\frac{s}{2}\right] + 1.$$

From the $(c-1)$-symmetry of $\Delta^2 h(z)$, we get

$$q_d = 0, \quad \text{if} \quad c - 2 - \left[\frac{s}{2}\right] \leq d \leq c - 1 - n.$$

From this we get that $q_d = 0$ for $n \leq d \leq c - n - 1$ since

$$c - 2 - \left[\frac{s}{2}\right] = s + 1 - \left[\frac{s}{2}\right] \leq \left[\frac{s}{2}\right] + 2.$$

Moreover, by the true definition of n and the condition b), it is clear that $q_d <$ if $a_2 \leq d \leq n-1$ and we get the conclusion by the $(c-1)$-symmetry of $\Delta^2 h(z)$.

12d. For every $i = 2, \ldots, p$

$$a_i,\ b_i \in [a_2, n] \cup [c-n, b_2].$$

We know by 8 that $a_p < b_2$ hence, if $i \geq 2$, we have

$$a_2 \leq a_i \leq a_p < b_2.$$

But by 12c we have $q_n = \cdots = q_{c-1-n} = 0$ hence, by 6, we cannot have any a_i's or b_i's in the interval $[n+1, c-1-n]$. This gives the conclusion for the a_i's. On the other hand, if $i \geq 2$ we have by 8

$$a_2 \leq b_i \leq b_2$$

and we get the conclusion as before.

13. If $p \geq 4$ then for every $r = 3, \ldots, \left[\frac{p}{2}\right] + 1$ we have

$$b_r > a_{p+3-r}.$$

If not there exists $r \in \left[3, \left[\frac{p}{2}\right] + 1\right]$ such that $b_r < a_{p+3-r}$ and we have

$$\{m \mid b_m \leq b_r\} \supseteq \{r, r+1, \ldots, p\}$$

and

$$\{m \mid a_m \leq b_r\} \subseteq \{1,2, \ldots, p+2-r\}.$$

We get by 6

$$q_{b_r} \geq 1 + (p-r+1) - (p+2-r) = 0.$$

Since

$$a < b_r < b_1 = c - a,$$

by b′) we get $q_{b_r} = 0$ so that

$$\{m \mid a_m \leq b_r\} = \{1,2, \ldots, p+2-r\}.$$

This implies

$$a_{p+2-r} < b_r < a_{p+3-r}.$$

Since $q_{b_r} = 0$ and, by 8,

$$a_2 < b_r < b_2,$$

we have the assumption as in 12. Then by 12c we get

$$b_r \in [n, c-n-1],$$

while by 12d

$$b_r \in [a_2, n] \cup [c-n, b_2].$$

This implies

$$b_r = n.$$

Since by 12c

$$q_{c-n-1} = 0, \quad q_{c-n} < 0,$$

we must have $c - n = a_i$ for some i. But we have

$$a_{p+2-r} < n < a_{p+3-r},$$

hence by 12d we get

$$c - n = a_{p+3-r}.$$

It follows that $a_{p+3-r} = a_r$. Since $r \leq \left[\frac{p}{2}\right] + 1$ we get $r \leq \frac{p}{2} + 1$ which implies $r \leq p + 2 - r$. Finally we get

$$a_r \leq a_{p+2-r} < a_{p+3-r} = a_r,$$

a contradiction. This proves 13.

14. Conclusion. We have two possibilities: either p is odd, say $p = 2g + 1$, or p is even, say $p = 2g$.

$\mathbf{p} = 2g + 1$.

In this case we have $\left[\frac{p}{2}\right] + 1 = g + 1$. Hence, if $g \geq 2$, we may apply 13 to get

$$b_r > a_{p+3-r} = a_{2g+4-r}, \quad r = 3, \ldots, g + 1.$$

If $p = 3$, we certainly have by 8

$$b_2 > a_3.$$

In any case we have integers

$$2 \leq a_1 \leq \cdots \leq a_{2g+1}$$

such that by 4,

$$cg = \sum_{t=1}^{2g+1} a_t.$$

Now, if $g = 1$, we have $b_2 > a_3$, while, if $g \geq 2$, we have

$$b_r > a_{2g+4-r}, \quad r = 3, \ldots, g + 1.$$

As remarked after Lemma 2, this implies that, in any case, the matrix $(u_{tj} := b_i - a_j)$, verifies the conditions a) , b) and c) in Lemma 1. Hence we can find a codimension three Gorenstein ideal $I \subseteq R = k[X, Y, Z]$, such that R/I has minimal free resolution

$$0 \to R(-c) \to \bigoplus_{i=1}^{2g+1} R(-b_i) \to \bigoplus_{i=1}^{2g+1} R(-a_i) \to R \to R/I \to 0.$$

This means that

$$h(z) = \frac{1 - \sum_{i=1}^{2g+1} z^{a_i} + \sum_{i=1}^{2g+1} z^{b_i} - z^c}{(1-z)^3}$$

is the h-polynomial of R/I. By Lemma 3 we get the conclusion.

$\mathbf{p} = 2g$.

Under this assumption we have by 10 that c is even, say

$$c = 2f.$$

We also have

$$a_i, b_i \neq f, \quad i = 1, \ldots, p$$

otherwise, for example, $2a_i = 2f = c = a_i + b_i$, hence $a_i = b_i$.

Also it is clear that

$$a_2 < f$$

otherwise $f < a_2$ would imply

$$b_2 = 2f - a_2 < a_2,$$

a contradiction to 8.

Let

$$h := \max\{i \mid a_i < f\}.$$

Then $2 \leq h \leq 2g$. If $h < 2g$, then

$$a_h < f < a_{h+1},$$

so that

$$b_{h+1} = 2f - a_{h+1} < f < 2f - a_h = b_h.$$

If $h = 2g$, then $a_{2g} < f$, so that $c - b_{2g} < f$ which implies

$$a_{2g} < f < b_{2g}.$$

We let

$$a'_j = \begin{cases} a_j & 1 \leq j \leq h \\ f & j = h+1 \\ a_{j-1} & h+2 \leq j \leq 2g+1 \end{cases}$$

and

$$b'_j = \begin{cases} b_j & 1 \le j \le h \\ f & j = h+1 \\ b_{j-1} & h+2 \le j \le 2g+1. \end{cases}$$

Then it is clear that we have

$$a_1 = a'_1 \le a_2 = a'_2 \le \cdots \le a_h = a'_h < a'_{h+1} = f < a_{h+1} = a'_{h+2} \le \cdots \le a_{2g} = a'_{2g+1}$$

and

$$b_1 = b'_1 \ge b_2 = b'_2 \ge \cdots \ge b_h = b'_h > b'_{h+1} = f > b_{h+1} = b'_{h+2} \ge \cdots \ge b_{2g} = b'_{2g+1}.$$

By 4 with $p = 2g$ and $c = 2f$, we have

$$(2g-1)f = \sum_{i=1}^{2g} a_i,$$

hence

$$\sum_{i=1}^{2g+1} a'_i = \sum_{i=1}^{2g} a_i + f = (2g-1)f + f = 2fg = cg.$$

Further

$$b'_i = c - a'_i, \quad i = 1, \ldots, 2g+1.$$

Now let $g = 1$; then $p = 2$ and $h = 2$ hence

$$a_1 < a_2 < f < b_2 < b_1,$$

so that

$$b'_2 = b_2 > f = a'_3.$$

If $g \ge 2$ we claim that

$$b'_r > a'_{2g+4-r}, \quad r = 3, \ldots, g+1.$$

Let us assume by contradiction that $b'_r \le a'_{2g+4-r}$ for some r with $3 \le r \le g+1$. Then it is clear that

$$b'_r < a'_{2g+4-r}$$

since we can only have equality for $r = h+1 = 2g+4-r$. But this would mean $r = 2g+4-r$, so that $r = g+2$, which is absurd.

Since

$$g+1=\left[\frac{2g}{2}\right]+1=\left[\frac{p}{2}\right]+1,$$

we have

$$3\leq r\leq\left[\frac{p}{2}\right]+1.$$

We have three possibilities: either $b'_r>f$ or $b'_r=f$ or $b'_r<f$.

If $b'_r>f$ then $b'_r=b_r$ and $a'_{2g+4-r}>b_r>f$. This implies

$$a'_{2g+4-r}=a_{2g+3-r}.$$

Hence

$$b_r<a_{2g+3-r},$$

a contradiction to 13.

If $b'_r=f$, then

$$b'_{r+1}=b_r<f$$

and

$$f<a'_{2g+4-r},$$

hence

$$a'_{2g+4-r}=a_{2g+3-r}.$$

This implies

$$b_r<f<a_{2g+3-r}$$

which again contradicts 13.

Finally if $b'_r<f$ then $b'_r=b_{r-1}$ and $r\geq h+2\geq 4$. We have either

$$b'_r=b_{r-1}<a'_{2g+4-r}<f$$

or

$$b'_r=b_{r-1}<f<a'_{2g+4-r}.$$

In any case we get

$$b_{r-1}<a_{2g+4-r}=a_{2g+3-(r-1)}.$$

Since

$$3 \leq r - 1 \leq \left[\frac{p}{2}\right] + 1,$$

we have again a contradiction to 13.

The conclusion now follows as in the case $p = 2g + 1$ by considering the integers $a'_1, a'_2, \ldots, a'_{2g+1}$ instead of $a_1, a_2, \ldots, a_{2g+1}$.

Let us consider the 7-symmetric polynomial

$$h(z) = 1 + 3z + 4z^2 + 5z^3 + 5z^4 + 4z^5 + 3z^6 + z^7.$$

This is the h-polynomial of the codimension three Gorenstein G-algebra R/I where

$$I = (Z^2, YZ, Y^4 - X^3Z, X^4Y, X^7)$$

is the ideal of $R = k[X, Y, Z]$ generated by the pfaffians of the skew-symmetric matrix

$$\begin{pmatrix} 0 & 0 & 0 & X^3 & Y \\ 0 & 0 & X^4 & Y^3 & Z \\ 0 & -X^4 & 0 & Z & 0 \\ -X^3 & -Y^3 & -Z & 0 & 0 \\ -Y & -Z & 0 & 0 & 0 \end{pmatrix}.$$

But we have $a = 2$, $s = 7$ so that $\left[\frac{s}{2}\right] + 1 = 4$. Since clearly

$$\Delta^2 h(z) = h(z)(1 - z)^2 = 1 + z - z^2 - z^4 - z^5 - z^7 + z^8 + z^9,$$

the given polynomial cannot be the h-polynomial of a codimension three Gorenstein domain.

Given the polynomial

$$h(z) = 1 + 3z + 6z^2 + 10z^3 + 13z^4 + 14z^5 + 14z^6 + 13z^7 + 10z^8 + 6z^9 + 3z^{10} + z^{11}$$

we now explicitly construct a Gorenstein codimension three ideal whose h-polynomial is $h(z)$.

We have $a = 4$, and $s = 11$ so that $\left[\frac{s}{2}\right] + 1 = 6$. We get

$$\Delta^3 h(z) = 1 - 2z^4 - z^5 + z^6 - z^8 + z^9 + 2z^{10} - z^{14}.$$

Hence we let

$$a_1 = a_2 = 4,\ a_3 = 5,\ a_4 = 8$$

and

$$b_1 = b_2 = 10,\ b_3 = 9,\ b_4 = 6.$$

Since we have $p = 4$, we must consider

$$a_1' = a_2' = 4,\ a_3' = 5,\ a_4' = 7,\ a_5' = 8$$

and

$$b_1' = b_2' = 10,\ b_3' = 9,\ b_4' = 7,\ b_5' = 6.$$

If we let $u_{ij} := b_i - a_j$ we get the matrix

$$\begin{pmatrix} 6 & 6 & 5 & 3 & 2 \\ 6 & 6 & 5 & 3 & 2 \\ 5 & 5 & 4 & 2 & 1 \\ 3 & 3 & 2 & 0 & -1 \\ 2 & 2 & 1 & -1 & -2 \end{pmatrix}.$$

The ideal generated by the pfaffians of the skew-symmetric matrix

$$\begin{pmatrix} 0 & 0 & 0 & X^3 & Y^2 \\ 0 & 0 & X^5 & Y^3 & Z^2 \\ 0 & -X^5 & 0 & Z^2 & 0 \\ -X^3 & -Y^3 & -Z^2 & 0 & 0 \\ -Y^2 & -Z^2 & 0 & 0 & 0 \end{pmatrix}$$

is the ideal

$$I = (Z^4,\ Y^2Z^2,\ Y^5 - X^3Z^2,\ X^5Y^2,\ X^8).$$

It is clear that R/I has $h(z)$ as h-polynomial. Since $g = 2$ and $u_{35} = 1 > 0$, the degree matrix of I verifies the assumptions as in Lemma 3. Hence we can find a codimension three Gorenstein prime ideal whose h-polynomial is $h(z)$.

If we are given a sequence

$$(1,\ 2,\ c_2, \ldots, c_t)$$

of non negative integers, we say that it is admissible if the corresponding numerical function is admissible in the sense we defined before.

By using the classical theorem of Macaulay as in [S], it is easy to see that $(1, 2, c_2, \ldots, c_t)$ is admissible if and only if for some integer $a \geq 2$ we have

$$c_i = i + 1, \quad 0 \leq i \leq a - 1,$$

and

$$c_{i+1} \leq c_i, \quad a - 1 \leq i \leq t - 1.$$

Let

$$h(z) = 1 + 3z + h_2 z^2 + \cdots + h_s z^s \in \mathbf{Z}[z]$$

be a s-symmetric polynomial. If, as before, a is the initial degree of $h(z)$ and we let

$$\sum_{i=0}^{t+2} q_i z^i = \Delta^2 h(z) = h(z)(1 - z)^2,$$

the following conditions are equivalent:

a) $q_i \leq 0$ for every $i \in \left[a, \left[\frac{s}{2}\right] + 1\right]$.

b) The sequence $(1,2, h_2 - 3, \ldots, h_{[\frac{s}{2}]} - h_{[\frac{s}{2}]-1})$ is admissible.

This can be easily proved in the following way.
Since a is the initial degree of $h(z)$, it is clear that

$$q_i = 1, \quad i \in [0, a - 1],$$

and

$$h_i - h_{i-1} = \sum_{j=0}^{i} q_j, \quad i \geq 1.$$

The result follows easily if we can prove that

$$h_{[\frac{s}{2}]} - h_{[\frac{s}{2}]-1} \geq 0 \Leftrightarrow q_{[\frac{s}{2}]+1} \leq 0.$$

But if $s = 2t + 1$, then $h_{[\frac{s}{2}]+1} = h_{[\frac{s}{2}]}$, hence

$$h_{[\frac{s}{2}]} - h_{[\frac{s}{2}]-1} = h_{[\frac{s}{2}]+1} - h_{[\frac{s}{2}]} - q_{[\frac{s}{2}]+1} = -q_{[\frac{s}{2}]+1}.$$

If $s = 2t$, then $h_{[\frac{s}{2}]-1} = h_{[\frac{s}{2}]+1}$, hence

$$h_{[\frac{s}{2}]} - h_{[\frac{s}{2}]-1} = h_{[\frac{s}{2}]+1} - h_{[\frac{s}{2}]} - q_{[\frac{s}{2}]+1} = -(h_{[\frac{s}{2}]} - h_{[\frac{s}{2}]-1}) - q_{[\frac{s}{2}]+1},$$

and

$$2(h_{[\frac{s}{2}]} - h_{[\frac{s}{2}]-1}) = -q_{[\frac{s}{2}]+1}.$$

In both cases the conclusion follows.

We will say that an admissible sequence $(1, 2, c_2, \ldots, c_t)$ is of decreasing type if for some integer $b \in [a, t+1]$, we have

$$c_i = \begin{cases} i+1, & i \in [0, a-1] \\ a & i \in [a-1, b-1] \end{cases}$$

and for $i \in [b-1, t-1]$ either $c_i = 0$ or $c_{i+1} < c_i$.

Proposition 6. *The polynomial*

$$h(z) = 1 + 3z + h_2 z^2 + \cdots + h_s z^s$$

verifies the conditions a), b) *and* c) *as in Theorem* 5, *if and only if* $h(z)$ *is* s*-symmetric and the sequence*

$$(1, 2, h_2 - 3, \ldots, h_{[\frac{s}{2}]} - h_{[\frac{s}{2}]-1})$$

is admissible of decreasing type.

Proof. As before we have

$$q_t = 1, \quad t \in [0, a-1],$$

and

$$h_i - h_{i-1} = \sum_{j=0}^{i} q_j, \quad i \geq 1.$$

Hence

$$h_i - h_{i-1} = i+1, \quad i \in [1, a-1].$$

Let us assume that the given polynomial verifies the conditions as in Theorem 5. Then the sequence $(1, 2, h_2 - 3, \ldots, h_{[\frac{s}{2}]} - h_{[\frac{s}{2}]-1})$ is admissible by the preceding remark and we need only to prove that it is of decreasing type.

We have two possibilities.

Case 1. $q_i = 0$, for every $i \in \left[a, \left[\frac{s}{2}\right] + 1\right]$.

In this case $h_i - h_{i-1} = a$ for every $i \in \left[a-1, \left[\frac{s}{2}\right]\right]$ and the sequence $(1, 2, h_2 - 3, \dots, h_{[\frac{s}{2}]} - h_{[\frac{s}{2}]-1})$ is of decreasing type.

Case 2. $q_i < 0$ for some $i \in \left[a, \left[\frac{s}{2}\right] + 1\right]$.

In this case let b be the least integer with this property. Then we have

$$h_{a-1} - h_{a-2} = h_a - h_{a-1} = \cdots = h_{b-1} - h_{b-2} = a,$$

and

$$h_b - h_{b-1} = a + q_b < a.$$

Now, if for some $i \in \left[b, \left[\frac{s}{2}\right] - 1\right]$ we have $h_i - h_{i-1} > 0$ and $h_{i+1} - h_i = h_i - h_{i-1}$, then $q_{i+1} = 0$. By condition c) this implies $q_j = 0$ for every $j \in \left[i+1, \left[\frac{s}{2}\right] + 1\right]$. In turn, this implies

$$0 < h_i - h_{i-1} = h_{i+1} - h_i = \cdots = h_{[\frac{s}{2}]} - h_{[\frac{s}{2}]-1} = h_{[\frac{s}{2}]+1} - h_{[\frac{s}{2}]}.$$

Now if $s = 2t$, then $h_{[\frac{s}{2}]-1} = h_{[\frac{s}{2}]+1}$, hence

$$h_{[\frac{s}{2}]} - h_{[\frac{s}{2}]-1} = h_{[\frac{s}{2}]-1} - h_{[\frac{s}{2}]},$$

which implies

$$h_i - h_{i-1} = h_{[\frac{s}{2}]} - h_{[\frac{s}{2}]-1} = 0.$$

If $s = 2t + 1$, then $h_{[\frac{s}{2}]} = h_{[\frac{s}{2}]+1}$, which implies

$$h_i - h_{i-1} = h_{[\frac{s}{2}]+1} - h_{[\frac{s}{2}]} = 0.$$

In both cases we get the conclusion.

Conversely, let us assume that the sequence $(1, 2, h_2 - 3, \dots, h_{[\frac{s}{2}]} - h_{[\frac{s}{2}]-1})$ is admissible of decreasing type. If, by contradiction, we have

$$q_t < 0, \quad q_v = 0, \quad q_r < 0,$$

with $a \leq t < v < r \leq \left[\frac{s}{2}\right] + 1$, then

$$h_t - h_{t-1} < h_{t-1} - h_{t-2} \leq h_{a-1} - h_{a-2} = a.$$

Also

$$h_v - h_{v-1} = h_{v-1} - h_{v-2}.$$

Since

$$b \le t < v < r \le \left[\frac{s}{2}\right] + 1,$$

we get

$$b - 1 \le v - 1 \le \left[\frac{s}{2}\right] - 1.$$

Since the sequence is of decreasing type this means that

$$0 = h_{v-1} - h_{v-2}.$$

Hence we get

$$0 = h_{v-1} - h_{v-2} \ge \cdots \ge h_{r-1} - h_{r-2} = h_r - h_{r-1} - q_r > h_r - h_{r-1} \ge 0$$

a contradiction.

We remark that because of the above proposition, one can see a strong analogy of our result with the characterization of the h-polynomial of a perfect codimension two ideal as given by Grouson and Peskine in [GP].

They proved that

$$1 + 2z + h_2 z^2 + \cdots + h_s z^s$$

is the h-polynomial of a codimension two standard Cohen-Macaulay G-domain if and only if $h(z)$ is admissible of decreasing type.

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