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BOUNDARY BEHAVIOUR OF EXTREMAL PLURISUBHARMONIC FUNCTIONS

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1. Introduction

In [Mo.1], S. Momm studied the boundary behaviour of extremal plurisubharmonic functions by using the pluricomplex Green function g_Ω of a bounded convex domain Ω in $\mathbf{C}^n$ to exhaust the domain by a family of sublevel sets. Let Ω be a bounded convex domain in $\mathbf{C}^n$ containing the origin $0 \in \mathbf{C}^n$ in its interior. The pluricomplex green function of Ω with a pole $0 \in \Omega$ is defined by

$$(1.1) \qquad g_\Omega(z, 0) := \sup_u u(z), \; z \in \Omega,$$

where the supremum is taken over all plurisubharmonic functions $u : \to [-\infty, 0[$, $u \leq 0$ on Ω, with $u(z) \leq \log \| z \| + 0(1)$ as $z \to 0$. This function is plurisubharmonic and continuous on $\bar{\Omega} \setminus \{0\}$ if its restriction to the boundary $\partial\Omega$ of Ω is identically zero. It is clear that in this definition the point $0 \in \Omega$ can be replaced by any fixed point $w \in \Omega$.

The sublevel sets

$$(1.2) \qquad \Omega_x := \{z \in \Omega \, ; g_\Omega(z, 0) < x\}, \; x < 0, \; x \in \mathbf{R}$$

are convex sets by results of Lempert [Lem.1]. Next consider the supporting functions

$$(1.3) \qquad H_x(z) = H_{\Omega_x}(z) := \sup\{\mathrm{Re}\langle z, w\rangle \, ; g_\Omega(w, 0) < x\}, \; x < 0, \; x \in \mathbf{C}^n,$$

of $\{\Omega_x\}_{x<0}$, where $\langle z, w\rangle = z \cdot w := \sum_{j=1}^n z_j \bar{w}_j$ and $\| z \| = \langle z, \bar{z}\rangle^{1/2}$ is a norm on $\mathbf{C}^n$. Consider a type of directional Lelong number defined by

$$(1.4) \qquad \Delta_\Omega(\zeta) := \lim_{x \uparrow 0} \frac{H_0(\zeta) - H_x(\zeta)}{-x} \in \,]0, +\infty],$$

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where $\zeta \in \mathbf{S} := \{z \in \mathbf{C}^n ; \|z\| = 1\}$. This Lelong number is used to measure the rate of approximation of the boundary $\partial\Omega$ of Ω by the boundaries $\{\partial\Omega_x\}_{x<0}$ of the family of sublevel sets $\{\Omega_x\}_{x<0}$, in the direction of the vector $\zeta \in \mathbf{S}$. Momm showed that this lower semi-continuous function is connected with the boundary behaviour of the Siciak extremal plurisubharmonic functions [Si.1]. Set $H := H_0$ and consider

$$V_H := \sup_u u(z),\ z \in \mathbf{C}^n, \tag{1.5}$$

where the supremum is taken over all plurisubharmonic functions $u \leq H$ on $\mathbf{C}^n$ with

$$u(z) \leq \log\|z\| + O(1) \text{ as } z \to \infty.$$

This is a continuous plurisubharmonic function and attains the value H on a compact star-shaped set

$$P_H := \left\{\lambda\zeta ; 0 \leq \lambda \leq \frac{1}{\mathscr{C}_H(\zeta)},\ \zeta \in \mathbf{S}\right\}, \tag{1.6}$$

where the numbers $\mathscr{C}_H(\zeta) \in [0, +\infty]$. He then proved the following results.

THEOREM 1.1 [Mo.1]. *Let Ω be a bounded convex domain in $\mathbf{C}^n$ containing the origin and with supporting function H defined in* (1.4). *Then there is a constant $C \geq 1$ such that $\mathscr{C}_H \leq \Delta_\Omega \leq C\mathscr{C}_H$.*

THEOREM 1.2 [Mo.1]. *For Ω a bounded convex domain in $\mathbf{C}^n$ containing the origin and with supporting function H, the following statements are equivalent*:
(i) *Δ_Ω (or $\mathscr{C}_H$) is bounded.*
(ii) *There is a constant $C > 0$ with $\Omega \subset \Omega_x + C(-x)\mathbf{B}$, $x < 0$ where $\mathbf{B} := \{z \in \mathbf{C}^n ; \|z\| < 1\}$ is the unit ball in $\mathbf{C}^n$.*
(iii) *There is a plurisubharmonic function $v \leq H$ with $v(z) \leq \log\|z\| + O(1)$ as $z \to \infty$ which coincides with H on a neighbourhood of zero.*

In this article we generalize S. Momm's results to the larger class of bounded linearly convex domains $\Omega \subset \mathbf{C}^n$ (see Section 2 for definitions) and use pluricomplex Green functions with a single pole $w \in \Omega$ to exhaust the domains. We also consider bounded linearly convex domains having pluricomplex Green functions with finite singularities. Here one should recall that linear convexity is a notion of convexity which is intermediate between classical convexity and pseudoconvexity for bounded domains in $\mathbf{C}^n$. These domains are defined using the linear structure of $\mathbf{C}^n$ and as such are not holomorphically invariant. Nevertheless, they have certain remarkable analytic and geometric properties (see [Lem. 2,4,5,6,7]). Their complements can be represented as the union of complex hyperplanes.

First we give some relevant definitions and notations and then state the main results of this paper. Let $\Omega \subset \mathbf{C}^n$ be a bounded linearly convex domain with boundary $\partial\Omega$. Consider for Ω a pluricomplex Green function $g_\Omega(\cdot, w)$ with a pole $w \in \Omega$, $w \neq 0$ given by

(1.7) $$g_\Omega(z, w) := \sup_u u(z), \; z \in \Omega,$$

where the supremum is taken over all $u \in \mathrm{PSH}(\Omega)$, $u \leq 0$ on Ω with $u(z) \leq \log \| z - w \| + O(1)$, as $z \to w$ and $\mathrm{PSH}(\Omega)$ is the cone of plurisubharmonic functions on Ω. This is a plurisubharmonic and continuous function when $g_\Omega(z, w)$ restricted to $\partial\Omega$ is identically zero. Lempert's results [Lem.2] imply that in this case the sublevel sets $\Omega_x := \{z \in \Omega\,; g_\Omega(z, w) < x\}$, for $x < 0$, $x \in \mathbf{R}$ are linearly convex sets. Clearly $\Omega_x \subset \Omega$ for all $x < 0$ and each contains the origin $0 \in \mathbf{C}^n$ in its interior. We define the supporting functions of the family $\{\Omega_x\}_{x<0}$ by

(1.8) $$h(\zeta, x) = h_{\Omega_x}(\zeta) := \sup_{z \in \Omega_x} (-\log|\zeta_0 + z_1\zeta_1 + \cdots + z_n\zeta_n|), \; x < 0, \; \zeta \in \mathbf{C}^{1+n}.$$

This supporting function will be shown later to be a plurisubharmonic function for all $(\zeta, x) \in \mathbf{C}^n \times \mathbf{C}$. Here we take $\zeta = (\zeta_0, \zeta') \in \mathbf{C}^{1+n}$ with $\zeta_0 \in \mathbf{C} \setminus \{0\}$ and $\zeta' \in \mathbf{C}^n$. Next we generalize to the case of finite singularities. Let $W = \{w_j\,; w_j \in \Omega\}$, $1 \leq j \leq m < \infty$ be a finite set of singularities with weights the numbers $\nu := \{\nu(w_j) := \nu_j > 0\}$, $1 \leq j \leq m < \infty$. We define a pluricomplex multipole Green function $g_\Omega(z, W, \nu)$ by

(1.9) $$g_\Omega(z, W, \nu) := \sup_u u(z), \; z \in \Omega,$$

where the supremum is taken over all $u \in \mathrm{PSH}(\Omega)$, $u \leq 0$ on Ω with $u(z) \leq \sum_{j=1}^m \nu_j \log \| z - w_j \| + O(1)$, as $z \to w_j$. The sublevel sets of $g_\Omega(z, W, \nu)$ are given by $\Omega_x := \{z \in \Omega\,; g_\Omega(z, W, \nu) < x\}$, $x < 0$. It is clear that for all $x < 0$, $\Omega_x \subset \Omega$ and $\Omega_0 = \Omega$ for $x = 0$. However, for very small values of $x < 0$, the members of the family $\{\Omega_x\}_{x<0}$ may be disconnected. But since we are only interested in those sublevel sets for which x is near 0 rather than near $-\infty$ we can overcome this difficulty by considering

(1.10) $$-\infty < \inf(x < 0\,; \Omega_x \text{ is connected}) \neq 0.$$

We call this infimum x_*. For the purpose of this study, we restrict our attention to the sublevel sets $\Omega_x := \{z \in \Omega\,;\, g_\Omega(z, W, \nu) < x\}$, for which the values of x satisfy $x_* < x < 0$, such that $\{\Omega_x\}_{x_*<x<0}$ is a family of connected sets. For this family of sets we define the supporting functions as in (1.8) i.e.,

$$h(\zeta, x) = h_{\Omega_x}(\zeta) := \sup(-\log|\zeta_0 + z_1\zeta_1 + \cdots + z_n\zeta_n|\,;\, g_\Omega(z, W, \nu) < x), \tag{1.11}$$

for all $x_* < x < 0$, $\zeta \in \mathbf{C}^{1+n}$ with $\zeta = (\zeta_0, \zeta')$ where $\zeta' \in \mathbf{C}^n$, $\zeta_0 \in C \setminus \{0\}$ and each member of the family $\{\Omega_x\}_{x_*<x<0}$ contains the origin $0 \in \mathbf{C}^n$ in its interior. $h(\zeta, x)$ is again shown to be a plurisubharmonic function in all (ζ, x). To each of the supporting functions there is associated a directional Lelong number given by

$$\mathscr{D}_\Omega(\zeta) := \lim_{x \uparrow 0} \frac{h(\zeta, 0) - h(\zeta, x)}{-x} \in \,]0, \infty], \tag{1.12}$$

where $\zeta \in \Omega^* := \{z \in \mathbf{C}^n\,;\, h_\Omega(z) < \infty\}$.

Finally, for Ω a bounded linearly convex domain in $\mathbf{C}^n$ with supporting function h_Ω we define the function $V : \mathbf{C}^n \times\,]0, +\infty] \to \mathbf{R}_+$, $\alpha > 0$ by

$$V_\alpha(\zeta) := V(\zeta, \alpha) := \sup(\varphi(\zeta)\,;\, \varphi \in \alpha\mathscr{L},\ \varphi \le h_\Omega),$$

(for definition of the class $\mathscr{L}$ see (5.0)).

With the notations above we can now state our main results

THEOREM 1.3. *Let Ω be a bounded linearly convex domain in $\mathbf{C}^n$ which contains the origin with boundary $\partial\Omega$. Let $h_{\Omega_x} := h_x : \mathbf{C}^n \to \mathbf{R} \cup \{\infty\}$, $x \le 0$, be the supporting functions of the linearly convex sublevel sets $\Omega_x := \{z \in \Omega\,;\, g_\Omega(z) < x\}$ of the pluricomplex Green function g_Ω of Ω with a pole at the origin. If $V_\alpha : \mathbf{C}^n \to \mathbf{R}_+$, $\alpha > 0$, are Siciak functions for $h_\Omega := h_0$ with respect to the class $\alpha\mathscr{L}$, define for all $\zeta \in \Omega^* := \{z \in \mathbf{C}^n\,;\, h_\Omega(z) < \infty\}$, the functions*

$$\alpha_{h_\Omega} : \mathbf{C}^n \times\,]0, \infty] \to \mathbf{R}_+$$

by

$$\alpha_{h_\Omega}(\zeta) := \inf(\alpha\,;\, V_\alpha(\zeta) = h_\Omega(\zeta)),\ \alpha \in\,]0, \infty],$$

and

$$\mathscr{D}_\Omega(\zeta) := \lim_{x \nearrow 0} \frac{h_\Omega(\zeta) - h_x(\zeta)}{-x} \in [0, \infty].$$

Then

$$\alpha_{h_\Omega}(\zeta) = \|\zeta\| \mathscr{D}_\Omega^{-1}(\zeta) e^{h_\Omega(\zeta)}, \ \zeta \in \Omega^*$$

where $\mathscr{D}_\Omega$ *denotes the directional derivative in the case of the supporting functions for the pluricomplex Green function* $g_\Omega(z, 0)$ *at the point* $z = z(\zeta)$ *which minimizes* $|1 + \zeta \cdot z|$ *for a given* $\zeta \in \Omega^*$. *This is unique if the boundary* $\partial\Omega$ *is of class* C^1 *and the hyperplane* $\zeta \cdot z = 1$ *is the tangent hyperplane to* $\partial\Omega$ *at the point* z.

The following theorems are corollaries of Theorem 1.3.

THEOREM 1.4. *Let* Ω *be a bounded linearly convex domain in* $\mathbf{C}^n$ *containing the origin with supporting function* h *defined in* (1.8). *Then there exists a constant* $C \geq 1$ *such that* $\alpha_{h_\Omega}(\zeta) \leq \mathscr{D}_\Omega e^{h_\Omega(\zeta)} \leq C\alpha_{h_\Omega}(\zeta)$, $\zeta \in \Omega^* \subset \mathbf{C}^{n*} \cong \mathbf{C}^n$.

THEOREM 1.5. *For a bounded linearly convex domain* Ω *in* $\mathbf{C}^n$ *containing the origin and with supporting function* h *defined in* (1.8), *the following statements are equivalent*:

(i) $\mathscr{D}_\Omega$ *(or* α_{h_Ω}*) is bounded,*

(ii) *There is a constant* $C > 0$ *with* $\Omega \subset \Omega_x + C(-x)\mathbf{B}$, *where* $\mathbf{B} = \{z \in \mathbf{C}^n; \|z\| < 1\}$,

(iii) *There is a plurisubharmonic function* v *on* $\mathbf{C}^{n*} \setminus \Omega^*$ *with* $v(\zeta) \leq \log\|\zeta\| + O(1)$ *as* $\zeta \to \infty$, $v \leq h_\Omega$ *on a neighbourhood of the boundary* $\partial\Omega^*$ *of* Ω^* *and coincides with* h_Ω *on* $\Omega^* \subset \mathbf{P}_n^*(\mathbf{C}) \setminus \mathscr{H}_w$, *where* $\mathscr{H}_w$ *is the complex hyperplane dual to* $w \in \Omega$.

The organization of the paper is as follows. Section 2 gives a quick survey of the necessary preliminary material on linear convexity often without proofs. Section 3 is devoted to consideration of pluricomplex Green functions having several singularities with Lelong numbers as weights. Section 4 gives background material on complex Monge-Ampère operators and their relationship to pluricomplex Green functions. Section 5 introduces the Siciak and Lempert extremal functions and discusses their properties. Sections 6 and 7 are the core of the paper devoted to constructions leading to the proofs of our main results.

2. Linear convexity

In this section we give a brief résumé of some of the important properties of linearly convex sets which are a subclass of pseudoconvex sets, first introduced in [Be-P.1]. Their definitions are modelled on the definition of convex sets by supporting planes. Linear convexity is a notion of convexity which is stronger than pseudoconvexity and yet weaker than the traditional concept of convexity. Since the concept of linear convexity is more natural in projective spaces than in $\mathbf{C}^n$, we begin by reviewing some properties of the complex projective space $\mathbf{P}_n(\mathbf{C})$ of complex dimension n and its dual space $\mathbf{P}_n^*(\mathbf{C})$. We shall first consider a general situation. Let V be a vector space of finite dimension over $\mathbf{C}$ and let $\mathbf{P}(V)$ denote the projective space of V, defined as the set of all equivalence classes $[z]$ of $z \in V \setminus \{0\}$ with the equivalence relation $z \sim w$ if and only if $z = \lambda w$ for some $\lambda \in \mathbf{C} \setminus \{0\}$. The space $\mathbf{P}(V)$ has a natural quotient topology induced by the projection $\Pi : V \setminus \{0\} \to \mathbf{P}(V)$, $z \mapsto [z]$. We call a subset W of $\mathbf{P}(V)$ of the form $W := \Pi(U \setminus \{0\})$, where U is a subspace of V of dimension $k+1$, a projective subspace of dimension k. W is a hyperplane if $k = n-1$ and a projective line if $k = 1$. The natural pairing $\langle \cdot , \cdot \rangle : V \times V^* \to \mathbf{C} : (z, \zeta) \to \langle z, \zeta \rangle$ between V and its dual space V^* establishes a bijection between the hyperplanes in V and the points of V^*. Thus every hyperplane U in V containing the origin has the form $U := \{z \in V ; \langle z, \zeta \rangle = 0\}$ for some $\zeta \in V^* \setminus \{0\}\}$ where ζ is determined up to a non-zero multiple of a complex number. As a consequence we can identify the class $[\zeta] \in \mathbf{P}(V^*)$ with the projection of $U \setminus \{0\}$ and write $[\zeta] = \{[z] \in \mathbf{P}(V) ; \langle z, \zeta \rangle = 0\}$. By duality there is a one to one correspondence between the points of $\mathbf{P}(V^*)$ and the projective planes in $\mathbf{P}(V)$. For every $E \subset \mathbf{P}(V)$ we define the dual complement $E^* \subset \mathbf{P}(V^*)$ of E by

$$E^* := \{[\zeta] \in \mathbf{P}(V^*) ; \langle z, \zeta \rangle \neq 0, \text{ for all } z \in E\}, \tag{2.1}$$

in other words, E^* is the set of all hyperplanes in $\mathbf{P}(V)$ which do not intersect E. It can be easily shown (see [A-P-S.1]), that if $E \subset \mathbf{P}(V)$ is an open set, then the dual complement $E^* \subset \mathbf{P}(V^*)$ is compact, and if it is compact, then E^* is open.

Definition 2.1. A subset E of $\mathbf{P}(V)$ is said to be linearly convex if its complement $\mathbf{P}(V) \setminus E$ can be represented as a union of projective hyperplanes, or equivalently $E^{**} = E$.

Therefore, a linearly convex set is determined by its dual complement. Precisely, it is the dual complement of its dual complement. Here it is clear that we should identify the hyperplanes in the space of all hyperplanes with the points of the original space. This establishes the duality mentioned above. The dual complement is often called the projective complement. Indeed in [Ma.1], it was called *le complêmentaire projectif*. The term *dual complement* was first used in this connection in [A-P-S.1].

If we choose $[\eta^*] \in \mathbf{P}(V^*)$ we can in a natural way (see [A-P-S.1]), define a unique affine structure on $\mathbf{P}(V) \setminus [\eta]$ which is independent of the choice of $[\eta^*]$. We call this $[\eta^*]$ the hyperplane at infinity in $\mathbf{P}(V)$. If we now fix a point $[\eta] \in \mathbf{P}(V) \setminus [\eta^*]$, we obtain a linear structure on $\mathbf{P}(V) \setminus [\eta^*]$ with $[\eta]$ as the origin, by the rule $[z] + [w] = \left[\frac{z}{\langle z, \eta^*\rangle} + \frac{w}{\langle w, \eta^*\rangle} - \frac{\eta}{\langle \eta, \eta^*\rangle}\right]$. In a similar way by a fixed choice of $[\eta] \in \mathbf{P}(V)$ we can define a unique affine structure on $\mathbf{P}(V^*) \setminus [\eta]$ and a choice of $[\eta^*]$ gives $\mathbf{P}(V^*) \setminus [\eta]$ a linear structure with $[\eta^*]$ as the origin. Next we set $\mathbf{P}_n(\mathbf{C}) := \mathbf{P}(\mathbf{C}^{n+1})$ and its dual $\mathbf{P}_n^*(\mathbf{C}) := \mathbf{P}(\mathbf{C}^{n+1*})$. We identify $\mathbf{C}^n$ with an open set in $\mathbf{P}_n(\mathbf{C}) := \mathbf{P}(\mathbf{C}^n \oplus \mathbf{C})$ by mapping $z \in \mathbf{C}^n$ to the class $[(z,1)]$. If Ω is an open set in $\mathbf{P}_n(\mathbf{C})$, we define Ω^* the compact dual complement set in $\mathbf{P}_n^*(\mathbf{C})$ as the set of all points, which viewed as hyperplanes in $\mathbf{P}_n(\mathbf{C})$, do not intersect Ω. When $0 \in \Omega$ then every hyperplane $\mathscr{H}$ with $\mathscr{H} \cap \Omega = \emptyset$ has a representation of the form $\mathscr{H} := \{z \in \mathbf{C}^n ; \langle z, \zeta\rangle - 1 = 0\}$ so that we can identify Ω^* with $\{\zeta \in \mathbf{C}^{n*} ; \langle z, \zeta\rangle \neq 1, \ \forall\, z \in \Omega\}$. It can be seen as demonstrated in [Ki.3], that the concept of dual complement defined in $\mathbf{P}_n(\mathbf{C})$ can be realized from a similar notion for certain subsets of $\mathbf{C}^{1+n} \setminus \{0\}$ or of subsets of $\mathbf{C}^n$ by simply adding the hyperplane at infinity and considering $\mathbf{C}^n$ as an open subset of $\mathbf{P}_n(\mathbf{C})$. In the former case, let Ω be a subset of $\mathbf{C}^{1+n} \setminus \{0\}$. We say that Ω is a homogeneous subset of $\mathbf{C}^{1+n} \setminus \{0\}$ if $\lambda z \in \Omega$ whenever $z \in \Omega$ and $\lambda \in \mathbf{C} \setminus \{0\}$. To any homogeneous subset Ω of $\mathbf{C}^{1+n} \setminus \{0\}$, we define its dual complement Ω^* to be the set of all hyperplanes $\mathscr{H}$ passing through the origin which do not intersect Ω. Since any such hyperplane has the form $\mathscr{H} := \langle z, \zeta\rangle = \zeta_0 z_0 + \zeta_1 z_1 + \cdots + \zeta_n z_n = 0$ for some $\zeta \in \mathbf{C}^{1+n} \setminus \{0\}$ we can define

$$\Omega^* := \{\zeta \in \mathbf{C}^{1+n} \setminus \{0\} ; \langle \zeta, z\rangle \neq 0 \text{ for every } z \in \Omega\rangle. \tag{2.2}$$

A homogeneous set Ω in $\mathbf{C}^{1+n} \setminus \{0\}$ is called linearly convex if $\mathbf{C}^{1+n} \setminus \Omega$ is a union of complex hyperplanes $\mathscr{H}$ passing through the origin. Observe that a dual complement Ω^* is always linearly convex, and we always have $\Omega^{**} \supset \Omega$. If a set Ω is linearly convex, then $\Omega \supset \Omega^{**}$. If $\Omega \supset \Omega^{**}$, then Ω is linearly convex. Thus linearly convex sets are characterized by $\Omega \supset \Omega^{**}$, as well as by $\Omega = \Omega^{**}$.

We shall write $z = (z_0, z') = (z_0, z_1, \ldots, z_n)$ for points in $\mathbf{C}^{1+n} \setminus \{0\}$ with $z_0 \in \mathbf{C}$ and $z' = \{z_1, \ldots, z_n) \in \mathbf{C}^n$. The homogeneous sets in $\mathbf{C}^{1+n} \setminus \{0\}$ correspond to subsets of the projective n-space $\mathbf{P}_n(\mathbf{C})$ and so we can transfer the concepts of dual complement and linear convexity to $\mathbf{P}_n(\mathbf{C})$. In the open set where $z_0 \neq 0$ we can use z' as the coordinates in $\mathbf{P}_n(\mathbf{C})$.

Since every real hyperplane contains a complex hyperplane, it is clear that every convex open set $\Omega \subset \mathbf{C}^n$ is linearly convex. Recall that the complement of a complex hyperplane is connected. This makes linear convexity a much weaker condition on open subsets Ω of $\mathbf{C}^n$ than the usual notion of convexity.

THEOREM 2.2 [A-P-S.1]. *Let $\Omega \subset \mathbf{C}^n$ be a linearly convex set and assume that Ω^* is connected.*
(i) *If Ω is compact, then it is polynomially convex.*
(ii) *If Ω is open, then it is a Runge domain.*

DEFINITION 2.3. An open subset Ω of $\mathbf{P}_n(\mathbf{C})$ is said to be weakly linearly convex if for every $z \in \partial\Omega$ there exists a complex hyperplane $\mathscr{H}_z$ through z not intersecting Ω. A compact subset K of $\mathbf{P}_n(\mathbf{C})$ is said to be weakly linearly convex if it can be represented as $K = \cap_{j=1}^{\infty} \Omega_j$ for some decreasing sequence of open weakly linearly convex sets $\{\Omega_j\}_{j=1}^{\infty}$.

DEFINITION 2.4. An open subset Ω of $\mathbf{P}_n(\mathbf{C})$ is said to be locally weakly linearly convex if for every $z \in \partial\Omega$ there exists a complex hyperplane through z that does not intersect $U_z \cap \Omega$ for some neighbourhood U_z of the point z.

THEOREM 2.5 [Hö.1]. *If Ω is an open set in $\mathbf{C}^n$, then the union Γ of all the complex hyperplanes $\mathscr{H} \subset \mathbf{C}^n \setminus \Omega$ is a closed set and $\mathbf{C}^n \setminus \Gamma$ is linearly convex. It is the smallest linearly convex open set containing Ω. The components of $\mathbf{C}^n \setminus \Gamma$ are weakly linearly convex, and if Ω is weakly linearly convex, then each component of Ω is a component of $\mathbf{C}^n \setminus \Gamma$.*

PROPOSITION 2.6 [Hö.1]. *Every locally weakly linearly convex open set $\Omega \subset \mathbf{C}^n$ is pseudoconvex.*

Pseudoconvexity is a local property. In general, weak linear or linear convexity is not a local property. However, for sets with C^1 boundary weak linear or linear convexity is a local property. Recall that at any boundary point the tangent plane is then defined and it contains a unique affine complex hyperplane which is the only possible candidate for the plane $\mathscr{H}$ in the definition of weak linear convexity. We call this plane $\mathscr{H}$ the complex tangent plane.

PROPOSITION 2.7 [Hö.1]. *Let $\Omega \subset \mathbf{C}^n$, $n > 1$, be a bounded connected open set with a C^1 boundary, and assume that Ω is locally weakly linearly convex in the sense that for every $z \in \partial\Omega$ there is a neighbourhood ω_z such that $\omega_z \cap \mathscr{H}_z \cap \Omega = \emptyset$, if $\mathscr{H}_z$ is the complex tangent plane of $\partial\Omega$ at the point z; then Ω is weakly linearly convex.*

Moreover, if $\mathcal{L}$ is any affine complex line contained in $\mathbf{C}^n$, then $\mathcal{L} \cap \Omega$ is connected and simply connected, and $\mathcal{L}$ intersects $\partial\Omega$ transversally.

Let $\Omega \subset \mathbf{C}^n$ be a bounded domain with C^2 boundary $\partial\Omega$ and assume that Ω is given as $\Omega := \{z \in \mathbf{C}^n ; \rho(z) < 0\}$ where ρ is a C^2 function defined in a neighbourhood Ω' of $\bar{\Omega}$; $\rho(z) = 0$ and $d\rho = \left(\frac{\partial \rho}{\partial z_1}, \dots, \frac{\partial \rho}{\partial z_n}\right) \neq 0$ on $\partial\Omega$. Then the complex tangent plane at $z \in \partial\Omega$ has the form

$$T_z^{\mathbf{C}}(\partial\Omega) := \left\{ w \in \mathbf{C}^n ; \textstyle\sum_{j=1}^n \frac{\partial \rho(z)}{\partial z_j} w_j = 0 \right\}.$$

We let $\mathrm{Hess}(\rho, z)$ denote the Hessian form of ρ,

$$\mathrm{Hess}(\rho, z) := 2 \sum_{j,k=1}^n \frac{\partial^2 \rho(z)}{\partial z_j \partial \bar{z}_k} w_j \bar{w}_k + 2 \,\mathrm{Re} \left(\sum_{j,k=1}^n \frac{\partial^2 \rho(z)}{\partial z_j \partial z_k} w_j w_k \right). \tag{2.4}$$

COROLLARY 2.8. *Let Ω be a weakly linearly convex open subset of $\mathbf{C}^n$ with C^2 boundary $\partial\Omega$. Suppose $\rho \in C^2(\mathbf{C}^n)$ is a defining function such that $\Omega = \{\rho < 0\}$, $\rho = 0$ and $d\rho \neq 0$ on $\partial\Omega$. Then it follows that the second differential $d^2\rho(\cdot) :=$ $\mathrm{Hess}(\rho, \cdot)$ of ρ is a positive semi-definite quadratic form in the complex tangent plane $T_z^{\mathbf{C}}(\partial\Omega) := \mathcal{H}_z$ at $z \in \partial\Omega$. Conversely, if Ω is open, bounded and connected set and $\mathrm{Hess}(\rho, z)$ is positive definite in $T_z^{\mathbf{C}}(\partial\Omega)$ for every $z \in \partial\Omega$, then Ω is weakly linearly convex.*

Proof. If Ω is weakly linearly convex then $\rho \geq 0$ in a neighbourhood of $z \in \partial\Omega$ in $T_z^{\mathbf{C}}(\partial\Omega)$ for every $z \in \partial\Omega$, and since $\rho(z) = 0$ on $\partial\Omega$, it follows that $d^2\rho$ is positive semi-definite. Conversely, if $d^2\rho$ is positive definite in $T_z^{\mathbf{C}}(\partial\Omega)$ at ζ then $\rho(\zeta) > 0$ if $z \neq \zeta \in T_z^{\mathbf{C}}(\partial\Omega)$ and $|\zeta - z|$ is sufficiently small, so that the statement follows from the preceding proposition. □

Remark 2.9. Observe that the condition in the corollary involves the full second differential $d^2\rho = \mathrm{Hess}(\rho, \cdot)$ of ρ. Compare this with the following condition for pseudoconvexity: $\Omega \subset \mathbf{C}^n$ is an open set with a C^2 boundary $\partial\Omega$. Let $\rho \in C^2(\mathbf{C}^n)$ be a defining function for Ω, where $\Omega := \{z \in \mathbf{C}^n ; \rho(z) < 0\}$, $\rho = 0$ and $d\rho \neq 0$ on $\partial\Omega$. Then Ω is pseudoconvex if and only if

$$\sum_{j,k=1}^n \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k} w_j \bar{w}_k \geq 0, \tag{2.5}$$

when $z \in \partial\Omega$ and $\sum_{j=1}^{n} \frac{\partial\rho}{\partial z_j} w_j = 0$, $w = (w_j) \in \mathbf{C}^n$. Recall that this condition for pseudoconvexity is holomorphically invariant. This stands in sharp contrast to the non holomorphically invariant condition (2.4).

Note that every open set in $\mathbf{C}$ is linearly convex. If $\Omega_1 \subset \mathbf{C}^{n_1}$, and $\Omega_2 \subset \mathbf{C}^{n_2}$ are open linearly convex sets, then $\Omega_1 \times \Omega_2 \subset \mathbf{C}^{n_1+n_2}$ is a linearly convex set.

DEFINITION 2.10. A bounded domain Ω in $\mathbf{C}^n$ with C^2 boundary $\partial\Omega$ is said to be strictly linearly convex if its small C^2 perturbations are linearly convex and for each boundary point $z \in \partial\Omega$ the holomorphic tangent space $T_z^{\mathbf{C}}(\partial\Omega)$ to $\partial\Omega$ through z is disjoint from $\bar{\Omega} \setminus \{z\}$ and has precisely first-order contact with $\partial\Omega$ at z, in the sense that with some constant $c > 0$, $\beta(w, T_z^{\mathbf{C}}(\partial\Omega)) \geq c\beta^2(w, z)$, $w \in \Omega$, where $\beta(\cdot\,,\cdot) := \mathrm{dist}(\cdot\,,\cdot)$. In addition, there is a C^2 defining function $\rho \in C^2(\mathbf{C}^n)$ with $\Omega = \{\rho(z) < 0\}$ such that for all $z \in \partial\Omega$ the inequality

$$\sum_{j,k=1}^{n} \frac{\partial^2\rho(z)}{\partial z_j \partial \bar{z}_k} w_j \bar{w}_k > \left| \sum_{j,k=1}^{n} \frac{\partial^2\rho(z)}{\partial z_j \partial z_k} w_j w_k \right|, \tag{2.6}$$

holds for any nonzero vector $w = (w_j)$ in the holomorphic tangent space $T_z^{\mathbf{C}}(\partial\Omega)$ to $\partial\Omega$ at z.

In particular, strictly convex domains are strictly linearly convex, and any strictly linearly convex domain is strictly pseudoconvex.

The following theorems are $\mathbf{C}^n$ versions of L. Lempert's $\mathbf{C}^2$ theorems in [Lem. 7].

THEOREM 2.11. *If $\Omega \subset \mathbf{C}^n$ is a strictly linearly convex domain with a C^2 boundary $\partial\Omega$ and $w \in \mathbf{C}^n \setminus \bar{\Omega}$, then there is a complex affine hyperplane $\mathscr{H}_w$ through w which is disjoint from $\bar{\Omega}$.*

Proof. For the proof in the case when $n = 2$ see Lempert [Lem.7]. The general case for $n > 2$ follows from some modification of the arguments presented there. □

THEOREM 2.12, *If $\Omega \subset \mathbf{C}^n$ is a strictly linearly convex domain with a C^2 boundary $\partial\Omega$, then $\Omega^* \subset \mathbf{P}_n^*(\mathbf{C})$ is also a strictly linealy convex domain with C^2 boundary $\partial\Omega^*$.*

Proof. The C^2 mapping $\varphi : \partial\Omega \rightarrow \partial\Omega^*$ which sends the point $w \in \partial\Omega$ to the complex hyperplane $w^* := \mathscr{H}_w$, tangent to $\partial\Omega$ at w is clearly a one to one mapping and onto $\partial\Omega^*$. To see that $\partial\Omega^*$ is a C^2 hypersurface we shall construct a C^2 left inverse of φ as follows: Let $\rho \in C^2(\mathbf{C}^n)$ be a defining function of Ω. Then for $w \in \partial\Omega$, $\varphi(w) = w^*$ implies that the restriction of ρ to the complex hyperplane $w^* := \mathscr{H}_w$ assumes its minimum in w. Strict linear convexity then implies that this minimum is nondegenerate. Thus if $\mathscr{H}_w$ varies in a small neighbourhood of the boundary $\partial\Omega^*$, there will be a unique point $\psi(w^*) \in \mathscr{H}_w \subset \mathbf{P}_n(\mathbf{C})$ where the restriction $\rho\lceil_{\mathscr{H}_w}$ assumes its minimum. Furthermore, this minimum point depends continuously on $\mathscr{H}_w$. Since $\psi \circ \varphi = \mathrm{id}_{\partial\Omega}$, $\partial\Omega^*$ is indeed C^2. The strict linear convexity of $\partial\Omega^*$ follows from Lemma 5.2 in [Lem.2] see also [E-M.1]. □

3. Pluricomplex multipole Green function

Let Ω be a bounded linearly convex domain in $\mathbf{C}^n$, $n > 1$, with boundary $\partial\Omega$. In [Lem.1], and later in a more general setting in [Kl.2], [Po-Sh.1], and [Za.1], the concept of pluricomplex Green function $g_\Omega(z, w)$ was introduced for every $z \in \Omega$, with a logarithmic pole at a point $w \in \Omega$. A point $w \in \Omega$ is said to be a logarithmic pole of a plurisubharmonic function u defined in a neighbourhood ω_w of w if $u(z) - \log\|z - w\| \leq O(1)$ as $z \to w$, or equivalently if there exists a positive number $C \in \mathbf{R}$ with the property that

$$\log\|z - w\| - C \leq u(z, w) \leq \log\|z - w\| + C, \quad \forall z \in \Omega. \tag{3.1}$$

Let us assume that Ω contains the origin $0 \in \mathbf{C}^n$ in its interior. This means on the one hand that the complex hyperplane at infinity is in the interior of the projective complement $\Omega^* \subset \mathbf{P}_n^*(\mathbf{C})$, and on the other hand that all the complex hyperplanes in $\mathbf{P}_n(\mathbf{C})$ that do not intersect Ω can be written as $\{z \in \mathbf{C}^n ; \zeta_1 z_1 + \cdots + \zeta_n z_n = 1\}$ with a unique $(\zeta_1, \ldots, \zeta_n) \in \mathbf{C}^n$.

The extremal function

$$g_\Omega(z, w) := \sup_u u(z), \ z \in \Omega, \tag{3.2}$$

where the supremum is taken over all $u \in \mathrm{PSH}(\Omega)$, $u \leq 0$, in Ω, with $u(z) \leq \log\|z - w\| + O(1)$, as $z \to w$ is called the pluricomplex Green function of Ω with a singularity concentrated on a single point $w \in \Omega$.

Definition 3.1. A set $E \subset \mathbf{C}^n$ is pluripolar if for every point $w \in E$ there is an open set U containing w and a $u \in \mathrm{PSH}(U)$, u is not identically equal to minus infinity such that $E \cap U \subset \{u = -\infty\}$.

Observe that the definition of pluripolarity is local i.e. a set is pluripolar if it is locally pluripolar. However, it can in fact be shown that a pluripolar set is globally pluripolar in the sense that if E is pluripolar, then there exists a plurisubharmonic function u on $\mathbf{C}^n$ such that $E \subset \{u = -\infty\}$.

Let Ω be an open subset of $\mathbf{C}^n$, and $u : \Omega \rightarrow \mathbf{R}$ a plurisubharmonic function. Following Sadullaev, [Sd.1], we say that u is maximal (or extremal) if for every relatively compact open subset $\mathcal{W}$ of Ω, and for each upper semi-continuous function v on $\bar{\mathcal{W}}$ such that $v \in \mathrm{PSH}(\mathcal{W})$ and $v \leq u$ on $\partial\mathcal{W}$, we have $v \leq u$ in $\mathcal{W}$. We denote the family of all maximal plurisubharmonic functions on Ω by $\mathcal{M}\mathrm{PSH}(\Omega)$.

The pluricomplex Green function has important properties which are formulated in the following well-known result, (see [Lem.1], [Kl.1], [De.1]),

Proposition 3.2. *If Ω and Ω' are linearly convex domains in $\mathbf{C}^n$ and $w \in \Omega$, then the following statements hold.*

(i) *If $z \in \Omega$ and $\Omega \subset \Omega'$, then $g_\Omega(z, w) \geq g_{\Omega'}(z, w)$.*

(ii) *If $z \in \Omega$, $\Omega \subset \Omega'$ and $\Omega' \setminus \Omega$ is pluripolar, then $g_\Omega(z, w) = g_{\Omega'}(z, w)$,*

(iii) *If $R > r > 0$ and $\bar{\mathbf{B}}(w, r) \subset \Omega \subset \bar{\mathbf{B}}(w, R)$, then $\log(\| z - w \|/R) \leq g_\Omega(z, w) \leq \log(\| z - w \|/r)$,*

(iv) *If Ω is bounded, then $z \mapsto g_\Omega(z, w)$ is a negative plurisubharmonic function with a logarithmic pole at $w \in \Omega$.*

(v) *If $F : \Omega \rightarrow \Omega'$ is a holomorphic mapping, then $g_{\Omega'}(F(z), F(w)) \leq g_\Omega(z, w)$, $z \in \Omega$,*

(vi) *If Ω is bounded, then $z \mapsto g_\Omega(z, w) \in \mathcal{M}\mathrm{PSH}(\Omega \setminus \{0\})$ and is the generalized solution of the homogeneous complex Monge-Ampère equation $(dd^c u)^n = 0$ in $\Omega \setminus \{0\}$.*

(vii) *If $\{\Omega\}_{j\in\mathrm{N}}$ is an increasing sequence of linearly convex domains in $\mathbf{C}^n$ and $\Omega = \bigcup_{j=1} \Omega_j$, then $g_\Omega(z, w) = \lim_{j\rightarrow\infty} g_{\Omega_j}(z, w)$, $(z, w \in \Omega_1)$.*

(viii) *If Ω is a bounded linearly convex domain in $\mathbf{C}^n$, then for each $w \in \Omega$ and $x \in \partial\Omega$, $\lim_{z\rightarrow x} g_\Omega(z, w) = 0$ for $z \in \Omega$.*

(ix) *If Ω is a bounded linearly convex domain in $\mathbf{C}^n$, then $z \mapsto g_\Omega(z, w)$ is lower semi-continuous in $\Omega \setminus \{w\}$ and hence continuous.*

(x) *Let $\Omega \subset \mathbf{C}^n$ be a bounded linearly convex domain and let $w \in \Omega$. Then for each $\varepsilon > 0$ and for each neighbourhood $U_w \subset \Omega$ of w, there exists a neighbourhood V_w of w such that V_w is a relatively compact subset of U_w and*

$$(1 + \varepsilon)^{-1} \leq \frac{g_\Omega(z, x)}{g_\Omega(z, y)} \leq (1 + \varepsilon), \tag{3.3}$$

for all (z, x), $(z, y) \in (\Omega \setminus U_w) \times V_w$.

(xi) *If $\Omega \subset \mathbf{C}^n$ is a bounded linearly convex domain, then the pluricomplex Green func-*

tion $\bar{\Omega} \times \Omega \to [-\infty, 0[$ *is continuous where* $g_\Omega \lceil \partial\Omega \times \Omega \equiv 0$.

In [Le.1], P. Lelong, generalized the construction of pluricomplex Green function in $\mathbf{C}^n$ given in [Kl.2], and [De.1], to bounded hyperconvex domains Ω in infinite dimensional complex Banach space E. Recall that an open bounded set $\Omega \subset E$ is hyperconvex if it is connected and there is a continuous plurisubharmonic function $\rho : \Omega \to [-\infty, 0[$ such that the set $\{z \in \Omega ; \rho(z) < \alpha\}$ is a relatively compact subset of Ω, for each $\alpha \in]-\infty, 0[$. Lelong did this by considering arbitrary singularities $\{w_j \in \Omega \subset E\}$, $1 \leq j \leq m < \infty$ with specified weights or densities which are positive numbers $\{\nu(w_j) = \nu_j > 0\}$. The pluricomplex Green functions for bounded hyperconvex domains Ω in E were defined as the upper envelopes of a family of plurisubharmonic functions $u \in \mathrm{PSH}(\Omega)$, $u \leq 0$ in Ω with the poles $\{w_1, \dots, w_m\}$ satisfying $\liminf_{z \to w_j} \frac{u(z)}{\log \| z - w_j \|} = \nu(w_j) := \nu_j > 0$. In this section, we adapt his techniques to bounded linearly convex domains Ω in $\mathbf{C}^n$, $n > 1$. First we introduce a finite set of singularities $W = \{w_j \in \Omega\}$, for $1 \leq j \leq m < \infty$ with weights, the numbers $\nu = \{\nu(w_j) := \nu_j > 0\}$, $1 \leq j \leq m < \infty$. Then the extremal function

$$g_\Omega(z, W, \nu) := \sup_u u(z), \ z \in \Omega, \tag{3.4}$$

where the supremum is taken over all plurisubharmonic functions $u(z)$, $u \leq 0$ in Ω such that

$$u(z) - \sum_{j=1}^{m} \nu_j \log \| z - w_j \| \leq O(1) \text{ as } z \to w_j. \tag{3.5}$$

with the points w_j satisfying

$$\nu(u, w_j) := \nu(w_j) := \nu_j = \liminf_{z \to w_j} \frac{u(z)}{\log \| z - w_j \|} < +\infty \text{ and} \tag{3.6}$$
$$\nu_j > 0, \ 1 \leq j \leq m < +\infty,$$

is called the pluricomplex multipole Green function or Green function of order m if $\#\{W\} = m$, relative to the finite set of singularities $W = \{w_j \in \Omega\}$ with weights, the numbers $\nu = \{\nu(w_j) := \nu_j > 0\}$, $1 \leq j \leq m < \infty$. If $\varphi \in \mathrm{PSH}(\Omega)$, recall that the Lelong number of φ at a point $w \in \Omega$ is given by

$$\nu(\varphi, w) := \liminf_{z \to w} \frac{\varphi(z)}{\log \| z - w \|} < +\infty. \tag{3.7}$$

More generally if T is a closed positive current of bidegree (p, p) defined in the neighbourhood U_w of a point $w \in \Omega \subset \mathbf{C}^n$, $n > 1$ then the positive measure $\nu = \frac{1}{(2\pi)^{n-p}} T \wedge (dd^c \log \| z - w \|)^{n-p}$ having a finite mass and $\nu(w) \geq 0$ at w is the Lelong number of T at w. If $\varphi \in \mathrm{PSH}(\Omega)$, we set $T = (2\pi)^{-1} dd^c \varphi$. If $v = \log \| F \|$, for any holomorphic mapping F, then $\nu(w)$ is an integer which is the multiplicity of the zeros of F at the points w. From condition (3.6) and definition (3.7) we see that the weights in the definition of the pluricomplex multipole Green function are in fact Lelong numbers.

The Lelong number $\nu(\varphi, \cdot)$ is a functional which applied to plurisubharmonic functions φ measures the size of the singularities of these functions at certain points of their domains. It also measures the densities of closed positive currents. We say that w is a pole of φ if $\varphi(w) = -\infty$ and that it is a logarithmic pole of φ if in addition $\nu(\varphi, w) > 0$. The study of the singularities (logarithmic poles) of plurisubharmonic functions is of interest because the singularities represent complex varieties and these can often be defined as the superlevel sets of the Lelong numbers, see for instance [Siu.1], [Ki.2,4] and [De.2,3]. We shall see later that pluricomplex Green functions having several poles represent a remarkable connection with a highly non-linear differential operator—the Complex Monge-Ampère Operator.

THEOREM 3.3. *Let $\Omega \subset \mathbf{C}^n$ be a bounded linearly convex domain with C^2 boundary $\partial\Omega$. Let $W = \{w_j \in \Omega\}$ be a finite set of points with the w_j's endowed with the numbers* $\nu(u, w_j) := \nu_j := \liminf_{z \to w_j} \frac{u(z)}{\log \| z - w_j \|} < +\infty$, $\nu_j > 0$, $u \in \mathrm{PSH}(\Omega)$, $1 \leq j \leq m < +\infty$, *which are called weights. Then the family $\mathcal{N}(\Omega, W, \nu)$ of plurisubharmonic functions, $u \in \mathrm{PSH}(\Omega)$, $u \leq 0$ in Ω, such that*

$$u(z) \leq \sum_{j=1}^{m} \nu_j \log \| z - w \| + O(1) \text{ as } z \to w_j, \tag{3.8}$$

contains a maximal element $g_\Omega(z, W, \nu)$, so that the restriction $g_\Omega(z, W, \nu)\lceil_{\partial\Omega} \equiv 0$ and $\tilde{g}_\Omega(z, W, \nu) := \exp g_\Omega(z, W, \nu)$ is uniformly continuous on $\bar{\Omega}$. Moreover, if W consists of only a single point $w \in \Omega$ with $\nu(w) = 1$, then the pluricomplex Green function $g_\Omega(z, W, \nu)$ is the same as the pluricomplex Green function defined in [Lem. 1], *with $g_\Omega(\cdot, \cdot, \cdot)$ continuous on $\bar{\Omega} \setminus \{w\}$ when $g_\Omega \lceil_{\partial\Omega} \equiv 0$.*

To prove this existence theorem for pluricomplex multipole Green functions, we use the following

PROPOSITION 3.4 [Le.1]. *The family* $\mathcal{N}(\Omega, W, \nu)$ *of functions* $v : \Omega \to [-\infty, 0[$ *defined by the conditions*;

(i) $\nu \in \mathrm{PSH}(\Omega)$, $v \leq 0$ *in* Ω, $v(z) \leq \sum_{j=1}^{m} \nu_j \log \| z - w_j \| + O(1)$ *as* $z \to w_j$, $w_j \in \Omega$, $1 \leq j \leq m$ *and*

(ii) $\nu(w) := \nu_j := \liminf_{z \to w_j} \dfrac{v(z)}{\log \| z - w_j \|} < +\infty$, $\nu_j > 0$ *at the points of the singularities* $W = \{w_j \in \Omega\}$, $1 \leq j \leq m$ *or equivalently* $\nu(w_j) := \inf_\eta \nu(w_j, \eta) = \inf_\eta \lim_{r \to 0} \dfrac{\sup_\theta v(w_j + \eta r e^{i\theta})}{\log r}$, $\eta \in \mathbf{C}^n \setminus \{0\}$, $0 \leq \theta \leq 2\pi$, $0 < r < R(w_j)$ *where* $R(w_j)$ *is the radius for the finite majorization of* v, *has the same upper envelope* $g_\Omega(\cdot, W, \nu)$ *as the family* $\mathcal{M}(\Omega, W, \nu)$ *of functions* $v : \Omega \to [-\infty, 0[$ *satisfying in addition to the condition* (i) *above the condition*

(ii)′ $l(z) \leq v(z) \leq V(z)$, *where we set*

$$l(z) = \sum_{j=1}^{m} \nu_j \frac{\log \| z - w_j \|}{d[\Omega]}, \tag{3.9}$$

and

$$V(z) = \inf_j \left[\inf\left(0, \nu_j \frac{\log \| z - w_j \|}{\gamma_\Omega}\right) \right], \tag{3.10}$$

with $d[\Omega] = \text{diameter } (\Omega)$ *and* $\gamma_\Omega = \mathrm{dist}(W[\nu], \partial\Omega)$, *where* $W[\nu] := (W, \nu)$.

We now have the following,

PROPOSITION 3.5 [Le.1]. *The family* $\mathcal{N}(\Omega, W, \nu)$ *has a maximal element.*

Proof of Theorem 3.3. The theorem follows from Proposition 3.4, Proposition 3.5, Théorème 2 in [Le.1] and the proof of Théorème 4 on pages 461-465 in [Lem.1]. Also see Proposition 7 in [Le.1]. □

We now examine the sublevel sets for the pluricomplex Green function $g_\Omega(z, w)$ of a bounded linearly convex domain $\Omega \subset \mathbf{C}^n$ given by

$$\Omega_x := \{z \in \Omega \,;\, g_\Omega(z, w) < x\}, \ x < 0. \tag{3.11}$$

PROPOSITION 3.6. *Let* $\Omega \subset \mathbf{C}^n$ *be a bounded linearly convex* C^1 *set and let* Ω_x *be defined as in* (3.11). *Then* Ω_x *is bounded linearly convex* C^1 *set for each* $x < 0$.

Proof. The proof is given in Section 4. □

For the general case of the pluricomplex multipole Green function $g_{\Omega}(\cdot, W, \nu)$ we have the sublevel sets

(3.12) $$\Omega_x := \{z \in \Omega ; g_{\Omega}(z, W, \nu) < x, \text{ for all } x_* < x < 0\},$$

each containing the origin $0 \in \Omega$ in its interior.

PROPOSITION 3.7. *Suppose $W = \{w_j \in \Omega\}$ is a finite set of singularities with weights the numbers $\nu = \{\nu(w_j) = \nu_j > 0\}$, $1 \leq j \leq m < \infty$ of the pluricomplex multipole Green function $g_{\Omega}(\cdot, W, \nu)$. Let $\{\Omega_x\}_{x_* < x < 0}$ be the family of the connected sublevel sets of $g_{\Omega}(\cdot, W, \nu)$ containing the origin in their interior. Then each of the sets Ω_x for $x_* < x < 0$ is a bounded linearly convex set.*

Proof. Similar to that of the case of a single pole above. □

THEOREM 3.8. *If Ω is a bounded linearly convex C^1 domain in $\mathbf{C}^n$ and $g_{\Omega}(\cdot, 0)$ its pluricomplex Green function with a logarithmic pole at the origin, then for each $z \in \Omega$ we have*

$g_{\Omega}(z, 0) = \inf\{\log|\sigma| ; 0 \leq |\sigma| < 1,$ *there exists a holomorphic mapping*

(3.13) $$f : \mathbf{D} \to \Omega \text{ such that } f(0) = 0, f(|\sigma|) = z\},$$

where $\mathbf{D} = \{\zeta \in \mathbf{C} ; |\zeta| < 1\}$.

Proof. Lempert's results, [Lem.4], on the characterization of extremal maps on bounded linearly convex domains Ω in $\mathbf{C}^n$ imply the existence of extremal holomorphic mappings $f : \mathbf{D} \to \Omega$ with $f(\partial \mathbf{D}) \subset \partial\Omega$ and f transverse to $\partial\Omega$ such that $f(0) = 0, f(|\sigma|) = z$, solving the variational problem (3.13). Also Theorem 1 of, [Lem.4], shows that for linearly bounded convex domain Ω, the Kobayashi distance k_{Ω} on Ω is equal to the Carathéodory distance c_{Ω} on the same domain. Results of [Kl.1,2], then give

(3.14) $$g_{\Omega}(z, 0) = \log\tanh k_{\Omega}(z, 0), \text{ for all } z \in \Omega.$$

The theorem follows from the definition of the Kobayashi distance (cf. [Lem.1]). □

4. Complex Monge-Ampère operators

In this section, following [Sib.1], we extend the definition of the complex Monge-Ampère operators $(dd^c\cdot)^n$ slightly so as to include certain plurisubharmonic functions which are not necessarily bounded on bounded linearly convex domains Ω in $\mathbf{C}^n$. For functions $u \in \mathrm{PSH}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$ this extension has already been successfully realized in [B-T.2], with interesting consequences in pluripotential theory and its applications in complex analysis. We apply the theory developed here to give an alternative description of the pluricomplex Green functions having finite singularities with weights constructed in the previous section. Let $\Omega \subset \mathbf{C}^n$ be a bounded linearly convex domain with a boundary $\partial\Omega$. Recall that the operator $(dd^c\cdot)^n$ for a C^2 function $u \in \mathrm{PSH}(\Omega)$ is given in a local coordinate patch by

$$(4.1) \qquad (dd^c u)^n = \det\left[\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}\right] \cdot 4^n n!\, \Pi_{\nu=1}^n \, i dz_\nu \wedge d\bar{z}_\nu.$$

Here we set $d^c = i(\bar\partial - \partial)$, $d = \partial + \bar\partial$, with $dd^c = 2i\partial\bar\partial$, $i = \sqrt{-1}$. We let $C^\infty_{(p,q)}(\Omega)$, $p, q \in \mathrm{N}$ denote the space of smooth differential forms on Ω of bidegree (p, q) and $\mathscr{D}_{(p,q)}(\Omega)$ the subspace of smooth differential forms in $C^\infty_{(p,q)}(\Omega)$ which have compact supports in Ω. The space of currents of bidimension (p, q) or bidegree $(n-p, n-q)$ is the dual of the space $\mathscr{D}_{(p,q)}(\Omega)$ and will be denoted throughout by $\mathscr{D}'_{(p,q)}(\Omega)$.

Definition 4.1. Let T be a current of bidimension (p, p) in Ω. We say that T is a positive current if for all differential forms $\alpha_1, \ldots, \alpha_p$ in $\mathscr{D}_{(1,0)}(\Omega)$, the distribution

$$(4.2) \qquad T \wedge i\alpha_1 \wedge \bar\alpha_1 \wedge \ldots \wedge i\alpha_p \wedge \bar\alpha_p,$$

is a positive measure. The current T is said to be a closed current if $dT = 0$. It is easy to check that positive currents have order zero. If $\alpha \in \mathbf{C}^n \equiv \mathbf{R}^{2n}$, we denote by $\|\alpha\|$ the norm of α. If T is a current of dimension k, with measure coefficients, we define the Borel measure of T by

$$(4.3) \qquad \|T\|(U) := \sup\{|T(\varphi)|,\ \varphi \in \mathscr{D}_k(U),\ |\varphi(x)| < 1, \text{ for } x \in U\},$$

where U is an open set.

For any Borel set $E \subset \Omega$, we set $\|T\|(E)$ to be the mass of T concentrated on the set E. Let $\beta = (\sqrt{-1}/2)\partial\bar\partial\|z\|^2$ be the standard Kähler form on $\mathbf{C}^n$. If T is a positive current of bidimension (p, p) in Ω, define the trace measure of T by the equation

$$\sigma(T, \Omega) := \sigma_T := T \wedge \frac{\beta^p}{p!}. \tag{4.4}$$

It is a well-known result that there exists a positive constant C which does not depend on n and p such that for every open set $U \subset \Omega$ we have

$$C^{-1}\sigma_T(U) \leq \| T \|(U) \leq C\sigma_T(U). \tag{4.5}$$

If A is a closed subset of Ω, we say that a current T of bidimension (p, p) defined in $\Omega \setminus A$ has a locally finite mass in every neighbourhood of A, if for every compact set $K \subset \Omega$ we have $\| T \|(K \setminus A) < \infty$. Let $\mathrm{PSH}(\Omega) \cap C^\infty(\Omega)$ denote the cone of C^∞ plurisubharmonic functions on Ω and $\mathrm{PSH}_+(\Omega)$ the subcone of those positive plurisubharmonic functions in $\mathrm{PSH}(\Omega)$. If ψ is a bounded function in Ω and E is a subset of Ω we set

$$\| \psi \|_\infty(E) := \sup_{z \in E} | \psi(z) |, \text{ and } \| \psi \|_\infty := \sup_{z \in \Omega} | \psi(z) |. \tag{4.6}$$

LEMMA 4.2 [Sib.1]. *Let $\Omega \subset \mathbf{C}^n$ be an open set, $M \subset \Omega$ a closed set and T a closed positive current of bidimension (1,1) in $\Omega \setminus M$. Then for every compact set $K \subset \Omega$, there exists a constant $C(K, \Omega)$ such that for every $v \in \mathrm{PSH}_+(\Omega)$, $v = 0$ in a neighbourhood of M we have*

$$\int_{K \cap \{a < v < b\}} T \wedge dv \wedge d^c v \leq C(K, \Omega)(b - a) \| T \|(\Omega \setminus M) \| v \|_\infty, \tag{4.7}$$

where a, b are arbitrary constants such that $a < b$.

Let Ω be a bounded strictly linearly convex domain with C^2 boundary $\partial\Omega$. Suppose ρ is a strictly plurisubharmonic function of class C^2 in a neighbourhood Ω' of $\bar{\Omega}$ satisfying $\Omega = \{z \in \Omega' ; \rho(z) < 0\}$. Let K be a compact subset of Ω. Set

$$u_K := \sup\{u ; u \text{ psh and continuous, } u \leq 1 \text{ in } \Omega,\ u \leq 0, \text{ on } K\}. \tag{4.8}$$

Then we have the following,

PROPOSITION 4.3 [Sib.1]. *Let K be a compact subset of an open strictly linearly convex set $\Omega \subset \mathbf{C}^n$. Let T be a closed positive current on $\Omega \setminus K$. Then the current $u_K T$ has a locally bounded mass in every neighbourhood of K.*

Let Ω be an open set in $\mathbf{C}^n$ and M be a closed subset of Ω. We impose the following convexity condition (C) on M:

(C) For every $x \in M$ there exists a strictly pseudoconvex neighbourhood ω of x, $\omega \subset\subset \Omega$, such that x is not contained in the holomorphically convex envelope of $\partial\omega \cap M$. That is to say, for any $x \in M$ there exists a strictly pseudoconvex neighbourhood ω of x, a function $\varphi \in \mathrm{PSH}(\Omega) \cap C^\infty(\Omega)$ such that,

(i) $\varphi(x) > 1$

(ii) $\varphi \leq 0$ on a neighbourhood V of $\partial\omega \cap M$.

We will now see that given a closed subset M of $\Omega \subset \mathbf{C}^n$ satisfying the condition (C) and a plurisubharmonic function u in Ω of class C^2 in $\Omega \setminus M$, for every compact subset $K \subset \Omega$ we have $\int_{K\setminus M} (dd^c u)^n < \infty$.

THEOREM 4.4 [Sib.1]. *Let $\Omega \subset \mathbf{C}^n$ be an open set and M a closed subset of Ω satisfying the convexity condition* (C). *Let T be a closed positive current of bidegree $(n-1, n-1)$ in Ω and u a plurisubharmonic function, negative in Ω, locally bounded in $\Omega \setminus M$. Then for all compact sets $K \subset \Omega$, we have*

$$\int_K |u|\, T \wedge \beta < \infty. \tag{4.14}$$

From Theorem 4.4 we obtain the following

COROLLARY 4.5. *Let K be a compact subset of $\Omega \subset \mathbf{C}^n$. Then there exists a compact set $X \subset \Omega \setminus M$ and a constant $C > 0$ such that for all $u \in \mathrm{PSH}(\Omega) \cap L^\infty_{\log}(\Omega \setminus M)$ we have*

$$\int_{K\setminus M} (dd^c u)^n \leq C\, [\| u \|_\infty (X)]^n. \tag{4.15}$$

It is clear from the above corollary that if T is a closed positive current of bidimension (p, p) in a bounded linearly convex domain $\Omega \subset \mathbf{C}^n$ and $u \in \mathrm{PSH}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega \setminus M)$ for $M \subset \Omega$ a closed set, then for any compact set $K \subset \Omega$ and a positive constant C we obtain the inequality

$$\| T \wedge (dd^c u)^q \| (K \setminus M) \leq C\, [\| u \|_\infty (X)]^q \, \| T \| (X),$$

where X is as in the corollary and $1 \leq q \leq n$.

The Theorem 4.4 enables us to define the operator $(dd^c \cdot)^n$ for the subcone $\mathrm{PSH}(\Omega, M) := \mathrm{PSH}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega \setminus M)$ of plurisubharmonic functions in the cone $\mathrm{PSH}(\Omega)$ which are locally uniformly bounded outside every neighbourhood of the closed set M.

Let T be a closed positive current of bidimension (p, p) in Ω, and let $u_1, \dots, u_q \in \mathrm{PSH}(\Omega) \cap L^{\infty}_{\mathrm{loc}}(\Omega \setminus M)$. If ψ is a test form of bidegree $(n-q, n-q)$ we define the closed positive current $dd^c u_1 \wedge \cdots \wedge dd^c u_q \wedge T$ by the formula

$$\int dd^c u_1 \wedge \cdots \wedge dd^c u_q \wedge T \wedge \psi = \int u_1 dd^c u_2 \wedge \cdots \wedge dd^c u_q \wedge T \wedge dd^c \psi.$$

The definiton is by induction on q. This definition is similar to the one given in [B-T.2], the main difference being that in [B-T.2] the functions u_j are assumed to be locally uniformly bounded, whereas here the Theorem 4.4 gives sense to the left hand side of the formula above.

Proposition 4.6 [Sib.1]. *Let $(u_1^j), \dots, (u_q^j)$ be a decreasing sequence of plurisubharmonic functions $\subset \mathrm{PSH}(\Omega) \cap L^{\infty}_{\mathrm{loc}}(\Omega \setminus M)$. Suppose that for every k, $1 \leq k \leq q$*

$$\lim_{j \to \infty} u_k^j = u_k \in \mathrm{PSH}(\Omega) \cap L^{\infty}_{\mathrm{loc}}(\Omega \setminus M).$$

Let T be a closed positive current of bidimension (p, p) on Ω. Then

$$\lim_{j \to \infty} dd^c u_1^j \wedge \cdots \wedge dd^c u_q^j \wedge T = dd^c u_1 \wedge \cdots \wedge dd^c u_q \wedge T. \tag{4.16}$$

That is, the convergence is in the sense of the weak convergence of currents.

Now we let Ω be a bounded linearly convex domain in $\mathbf{C}^n$ with a given fixed point $w \in \Omega$. In the theory developed above we let the closed set $M = \{w\}$ and define $\mathrm{PSH}(\Omega, w) := \mathrm{PSH}(\Omega) \cap L^{\infty}_{\mathrm{loc}}(\Omega \setminus \{w\})$ and

$$C_c^{\infty}(\Omega, w) = \{\varphi \in C_c^{\infty}(\Omega)\,;\, \mathrm{supp}\,(d\varphi) \subset \Omega \setminus \{w\}\}.$$

Then it is clear that $C_c^{\infty}(\Omega, w)$ if and only if φ is a test function in $C_c^{\infty}(\Omega)$ which is constant in a neighbourhood U_w of w.

Theorem 4.7 [Kl.1]. *The space $C_c^{\infty}(\Omega, w)$ is dense in $C_c^{\circ}(\Omega)$, the space of continuous functions with compact support in Ω.*

Proposition 4.8. *Let $\Omega \in \mathbf{C}^n$ be a linearly convex domain and let $u \in \mathrm{PSH}(\Omega, w)$. Then there exists a positive Borel measure μ on Ω such that, for any decreasing sequence $\{u_j\}_{j \in \mathrm{N}} \subset \mathrm{PSH}(\Omega) \cap L^{\infty}_{\mathrm{loc}}(\Omega)$ convergent to u at each point of Ω, the sequence of Radon measures $\{(dd^c u_j)^n\}_{j \in \mathrm{N}}$ is weak*-convergent to μ.*

Proof. Follows from Proposition 4.6 and the Chern-Levine-Nirenberg Inequality [C-L-N.1]. Also See [Kl.1]. □

Recall that if $u \in \mathrm{PSH}(\Omega) \cap L^{\infty}_{\mathrm{loc}}(\Omega)$, then the measure μ from Proposition 4.8 coincides with $(dd^c u)^n$; this is why for $u \in \mathrm{PSH}(\Omega, w)$ we define $(dd^c u)^n = \mu$.

COROLLARY 4.9. *Let $w \in \mathbf{C}^n$ and $R \in \mathbf{R}$, $R > 0$. If $u = \log(\| z - w \|/R)$ for all $z \in \mathbf{C}^n$, then*

$$(dd^c u)^n = (2\pi)^n \delta_w,$$

where δ_w is the Dirac delta function at w.

Proof. See [Kl.1], [De.1]. □

The following property is similar to the comparison theorem in [B-T.2].

LEMMA 4.10. *Let $\Omega \subset \mathbf{C}^n$ be a bounded hyperconvex domain. Let $w \in \Omega$ and $u, v \in \mathrm{PSH}(\Omega) \cap C^{\circ}(\bar{\Omega} \setminus \{w\})$. Suppose that*

$$P_u := \{u = -\infty\} = P_v := \{v = -\infty\} = \{w\}$$

are the pluripolar sets of u and v and $\lim_{z \to \partial\Omega}(u(z) - v(z)) = 0$ and $u < v$ in $\Omega \setminus \{w\}$. Then

$$\int_{\Omega} (dd^c v)^n \leq \int_{\Omega} (dd^c u)^n.$$

Proof. See [Kl.1]. □

THEOREM 4.11. *Let Ω be a bounded hyperconvex domain in $\mathbf{C}^n$, $w \in \Omega$, and let $u, v \in \mathrm{PSH}(\Omega) \cap C^{\circ}(\Omega, [-\infty, +\infty[)$ be such that*

$$P_u := \{u = -\infty\} = P_v := \{v = -\infty\} = \{w\},$$

$u < v$ in $\Omega \setminus \{w\}$, and

$$\limsup_{\Omega \setminus \{w\} \ni z \to w} \frac{u(z)}{v(z)} = 1. \tag{4.17}$$

Then $(dd^c u)^n(\{w\}) \leq (dd^c v)^n(\{w\})$.

Proof. (cf. [Kl.1], [De.2]). □

Let $\Omega \subset \mathbf{C}^n$ be a bounded hyperconvex domain and $w \in \Omega$. Consider the problem of finding a function $u : \bar{\Omega} \setminus \{w\} \to [-\infty, \infty[$ which satisfies the following conditions:

$$(4.18) \qquad \begin{cases} u \in \mathrm{PSH}(\Omega) \cap C^{\circ}(\bar{\Omega} \setminus \{w\}) \\ (dd^c u)^n = 0, \text{ in } \Omega \setminus \{w\} \\ (dd^c u)^n = (2\pi)^n \delta_w, \text{ in } \Omega \\ u(z) - \log \| z - w \| = O(1), \text{ as } z \to w \\ u(z) \to 0 \text{ as } z \to \partial\Omega. \end{cases}$$

THEOREM 4.12. *Let Ω be a bounded hyperconvex domain in $\mathbf{C}^n$. Then the function $u(z) = g_\Omega(z, w)$ is a unique solution to the problem* (4.18).

Proof. See [De.1], [Kl.1], [Lem.1], [E-M.1]. □

Just as in the case of the pluricomplex Green function with a single pole $w \in \Omega$, we extend the definition of the complex Monge-Ampère operator $(dd^c \cdot)^n$ slightly. We take for the closed set $M \subset \Omega$ above, the set

$$W[\nu] = \{w_j \in \Omega\, ;\, \nu(w_j) = \nu_j > 0\},\ 1 \leq j \leq m < \infty$$

of finite singularities of the pluricomplex multipole Green function $g_\Omega(z, W, \nu)$ for any $z \in \Omega$, where Ω is a bounded linearly convex domain and the points w_j have weights, the numbers $\nu = \{\nu(w_j) = \nu_j > 0\}$, $1 \leq j \leq m$. We set $M = W[\nu]$ and define as above $\mathrm{PSH}(\Omega, W, \nu) := \mathrm{PSH}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega \setminus W[\nu])$ together with

$$C^\infty_c(\Omega, W, \nu) := \{\varphi \in C^\infty_c(\Omega)\, ;\, \mathrm{supp}(d\varphi) \subset \Omega \setminus W[\nu]\}.$$

Then as before $\varphi \in C^\infty_c(\Omega \setminus W[\nu])$ if and only if φ is a test function in $C^\infty_c(\Omega)$, which is constant in the neighbourhood of $W[\nu] \subset \Omega$.

THEOREM 4.13. *The space $C^\infty_c(\Omega, W, \nu)$ is dense in $C^\circ_c(\Omega)$.*

Proof. The proof follows from a modification of the arguments for the case of a single pole (cf. [Kl.1]). Consider a family $\{V_\eta\}_{\eta>0}$ of neighbourhoods of $W[\nu]$ decreasing to $W[\nu]$ as $\eta \searrow 0$. Take $\varphi \in C^\circ_c(\Omega)$ and $\varepsilon > 0$. Since φ is uniformly continuous in Ω, we can find $\eta > 0$ such that $|\varphi(z) - \varphi(w)| < \varepsilon$ if $z, w \in \Omega$ and $\| z - w \| < \eta$. If $\partial\Omega \neq 0$, then by taking η sufficiently small, if necessary, we can suppose that $0 < \eta < \mathrm{dist}(\mathrm{supp}\ \varphi, \partial\Omega)$. Now define

$$\varphi_\eta(z) = \begin{cases} \varphi(v), & z \in V_\eta,\ v \in W[\nu] \\ \varphi((1 - \eta \| z - v \|^{-1})z), & z \in \Omega \setminus V_\eta. \end{cases}$$

Clearly $\varphi \in C_c^\circ(\Omega)$ and if $z \in \mathrm{supp}\ \varphi_\eta$, then $\mathrm{dist}(z, \mathrm{supp}\ \varphi) \leq \eta$.

Moreover, $\| \varphi - \varphi_\eta \|_\Omega \leq \varepsilon$. Consequently,

$$\mathscr{F} = \{\varphi \in C_c^\circ(\Omega)\ ;\ \varphi \text{ is constant in neighbourhood } V_\eta \text{ of } W[\nu]\}$$

is dense in $C_c^\circ(\Omega)$. The theorem then follows since for any compact set $K \subset \Omega$, if $u \in C^\circ(\Omega)$ then using a family of smooth C^∞ regularizing kernels $(\rho_j)_{j>0}$ and setting $u_j = u * \rho_j \in C^\infty(\Omega)$ we see that $u * \rho_j \to u$ uniformly as $j \searrow 0$. This is easily shown by taking $K \subset \Omega$ compact. Fix $j_0 > 0$ such that $K_{j_0} \subset \Omega$ where

$$K_j = \{z \in \mathbf{C}^n\ ;\ \mathrm{dist}(z, K) \leq j\},\ j > 0.$$

Let $0 < j < j_0$. Then we have

$$(u * \rho_j - u)(z) = (\rho_j * u - u)(z) = \int \rho_j(z - w)(u(w) - u(z))(z)\,dV(w),$$

where dV is the Lebesgue measure on $\mathbf{C}^n$. Therefore,

$$\| u * \rho_j - u \|_K \leq \sup_{z \in K} \sup_{w \in \overline{V}_\eta} | u(w) - u(z) |.$$

The right-hand side tends to zero as $j \searrow 0$, because u is continuous on K_{j_0}. □

Important properties of the extremal function $g_\Omega(\cdot, W, \nu)$ are stated in the following

THEOREM 4.14. *Let $\Omega \subset \mathbf{C}^n$ be a bounded hyperconvex domain. Let* $W[\nu] = \{w_j \in \Omega\ ;\ \nu = \nu(w_j) := \nu_j > 0\}$, $1 \leq j \leq m < 0 < +\infty$, *be a finite set of singularities with weights which are positive numbers. If the pluricomplex multipole Green function is given by*:

$$g_\Omega(z, W, \nu) = \sup_u u(z)\ ;\ z \in \Omega, \tag{4.19}$$

where the supremum is taken over all $u \in \mathrm{PSH}(\Omega)$; $u \leq 0$, *in* Ω, *with* $u(z) \leq \sum_{j=1}^n \nu_j \log \| z - w_j \| + O(1)$, *as* $z \to w_j$; *then*

(i) $g_\Omega \in \mathrm{PSH}_-(\Omega, W, \nu)$, *where* $\mathrm{PSH}_-(\Omega, W, \nu)$ *is the subcone of* $\mathrm{PSH}(\Omega, W, \nu)$ *of functions which are negative in* Ω.

(ii) $(dd^c g_\Omega(z, W, \nu))^n = 0$, *in* $\Omega \setminus W[\nu]$.

(iii) $(dd^c g_\Omega(z, W, \nu))^n = (2\pi)^n \sum_{j=1}^m \nu_j^n \delta_{w_j}$ *in* Ω *where* δ_w *is the Dirac delta function at* w_j.
(iv) $g_\Omega(\cdot, W, \nu) \in C^\circ(\bar{\Omega}, W, \nu)$ *if* $g_\Omega(\cdot, W, \nu)\lceil\partial\Omega \equiv 0$
(v) $\tilde{g}_\Omega(\cdot, W, \nu) = \exp g_\Omega(\cdot, W, \nu)$ *is uniformly continuous on* $\bar{\Omega}$.

Proof. (i) follows trivially from the definition of the pluricomplex multipole Green function. (ii) and (iii) follow from Proposition 8 in [Le.1]. (iv) and (v) follow from Proposition 5 and Théorème 2 in [Le.1]. □

Similar to Lemma 4.10 and Theorem 4.11 we have the following results for the case of pluricomplex multipole Green functions.

PROPOSITION 4.15. *Let* $\Omega \subset \mathbf{C}^n$ *be a bounded hyperconvex domain. Let* $W[\nu] = \{w_j \in \Omega\ ;\ \nu(w_j) = \nu_j > 0\}$ *and* $u, v \in \mathrm{PSH}(\Omega) \cap C^\circ(\bar{\Omega}\setminus W[\nu])$. *Suppose that*

$$P_u := \{u = -\infty\} = P_v := \{v = -\infty\} = W[\nu]$$

are the pluripolar sets of u *and* v, $\lim_{z\to\partial\Omega}(u(z) - v(z)) = 0$ *and* $u < v$ *in* $\Omega\setminus W[\nu]$. *Then*

$$\int_{\Omega\setminus W[\nu]} (dd^c v)^n \leq \int_{\Omega\setminus W[\nu]} (dd^c u)^n.$$

Proof. Given $\varepsilon > 0$, let $\{V_\varepsilon\}_{\varepsilon>0}$ be a family of neighbourhoods of $W[\nu]$ decreasing to $W[\nu]$ as $\varepsilon \searrow 0$. Choose K large such that $\{u < -K\} \subset\subset \Omega\setminus W[\nu]$. Let $u_1 = \max\{u, -K-1\}$ and $v_1 = \max\{v, -K\}$. Let $\varphi \in C_c^\infty(\Omega, [0, 1])$ be such that $\varphi = 1$ in the neighbourhood of $\overline{\{u < -K\}}$. $u = u_1$ and $v = v_1$ in the neighbourhood of $\Omega\setminus V_\varepsilon$. Then

$$\int_{\Omega\setminus W[\nu]} \varphi(dd^c u)^n = \int_{\Omega\setminus W[\nu]} \varphi(dd^c u_1)^n$$

and

$$\int_{\Omega\setminus W[\nu]} \varphi(dd^c v)^n = \int_{\Omega\setminus W[\nu]} \varphi(dd^c v_1)^n$$

by Proposition 4.8 or Proposition 4.10. Therefore,

$$\int_{\Omega\setminus W[\nu]} (dd^c v)^n = \int_{\Omega\setminus W[\nu]} (dd^c v_1)^n$$

and

$$\int_{\Omega\setminus W[\nu]} (dd^c u)^n = \int_{\Omega\setminus W[\nu]} (dd^c u_1)^n.$$

Also $u_1 < v_1$ in $\Omega\setminus W[\nu]$ and $u = u_1$, $v = v_1$ in the neighbourhood of $\partial\Omega$. So the result follows from an application of the comparison principle for bounded plurisubharmonic functions (see [Kl.1] Theorem 3.7.1). □

THEOREM 4.16. *Let Ω be a bounded hyperconvex domain in $\mathbf{C}^n$. Let $W[\nu] = \{w_j \in \Omega ; \nu(w_j) = \nu_j > 0\}$. Suppose u, $v \in \mathrm{PSH}(\Omega) \cap C^\circ(\bar{\Omega}\setminus W[\nu])$ so that*

$$P_u := \{u = -\infty\} = P_v := \{v = -\infty\} = W[\nu],$$

$u < v$ in $\Omega\setminus W[\nu]$, and

$$\limsup_{\Omega\setminus W[\nu]\ni z\to w\in W[\nu]} \frac{u(z)}{v(z)} = 1. \tag{4.20}$$

Then $(dd^c u)^n(W[\nu]) \le (dd^c v)^n(W[\nu])$.

Proof. Take a sufficiently small neighbourhood V of $W[\nu]$ such that $V \subset\subset \Omega$ and assume that u, v extend continuously to $\partial\Omega$ so that $u < v \le 0$ in $\bar{\Omega}\setminus W[\nu]$. Choose $\varepsilon > 0$ such that

$$\| u \|_{\partial\Omega}\left(1 - \frac{1}{1+\varepsilon}\right) < \inf_{z\in\partial\Omega} \{u(z) - v(z)\}.$$

Then for any $\eta \in]0, \varepsilon[$ we have $\dfrac{u}{1+\eta} < v$ on $\partial\Omega$. The condition (4.20) implies that for each $\eta \in]0, \varepsilon[$ we can find a $\delta > 0$ so that $\bar{\mathbf{B}}(w, \delta) \subset \Omega$, $\forall\, w \in W[\nu]$ and $u/v < 1 + \eta$ on $\bar{\mathbf{B}}(w, \delta)\setminus W[\nu]$. Define

$$\mathcal{W}_\eta = \left\{z \in \Omega ; \frac{u(z)}{1+\eta} > v(z)\right\} \cup W[\nu].$$

Then the set $\mathcal{W}_\eta$ is a relatively compact neighbourhood of $W(\nu]$ in Ω.

Moreover, $\bigcap_{\eta\in]0,\varepsilon[} \mathcal{W}_\eta = W[\nu]$. Therefore, if $\eta \in]0, \varepsilon[$, we have

$$\frac{1}{(1+\eta)^n} (dd^c u)^n(W[\nu]) \le \int_{\mathcal{W}_\eta} \left(dd^c \frac{u}{1+\eta}\right)^n \le \int_{\mathcal{W}_\eta} (dd^c v)^n = (dd^c v)^n(W[\nu])$$

by either Proposition 4.6 or Proposition 4.8. Now letting $\eta \searrow 0$ we obtain the desired result. □

Let $E \to \mathbf{P}_n(\mathbf{C})$ denote the tautological line bundle on the complex projective space $\mathbf{P}_n(\mathbf{C})$, which is the space of all complex lines $\mathscr{L} \subset \mathbf{C}^{n+1}$ through the origin. The total space $\mathbf{E}$ of the bundle E is the set of all pairs $(\mathscr{L}, z) \in \mathbf{P}_n(\mathbf{C}) \times \mathbf{C}^{n+1}$ with $z \in \mathscr{L}$. We identify E with the space obtained from $\mathbf{C}^{n+1}$ by blowing up the origin. The blow up map will be written as $\sigma : E \to \mathbf{C}^{n+1}$. In fact, let $\Xi \subset \mathbf{C}^{1+n} \times \mathbf{P}_n(\mathbf{C})$ be the set of points $((z_0, \ldots, z_n), (\zeta_0, \ldots, \zeta_n))$ satisfying the equations;

$$z_i\zeta_j = z_j\zeta_i, \quad 0 \leq i, j \leq n \quad n \geq 1.$$

Observe that $((z_0, \ldots, z_n), (\lambda z_0, \ldots, \lambda z_n)) \in \Xi$, for all $(z_0, \ldots, z_n) \neq 0$ and $\lambda \in C \setminus \{0\}$. That is to say, given $w \in \mathbf{C}^{1+n}$, $w \neq 0$, Ξ contains the point corresponding to w and the line through w. Since $\{0\} \times \mathbf{P}_n(\mathbf{C}) \subset \Xi$, we see that Ξ contains the points corresponding to zero and all the lines through zero. So that, in this way we obtain all the points contained in Ξ. Let $\sigma : \Xi \to \mathbf{C}^{1+n}$ denote the restriction of the projection of $\mathbf{C}^{1+n} \times \mathbf{P}_n(\mathbf{C})$ on $\mathbf{C}^{1+n}$ to Ξ. If we set $\mathscr{U}_0 = \sigma^{-1}(0)$, then $\mathscr{U}_0$ is biholomorphic to $\mathbf{P}_n(\mathbf{C})$ and σ maps $\Xi \setminus \mathscr{U}_0$ bijectively onto $\mathbf{C}^{1+n} \setminus \{0\}$. We call Ξ the blow up of $\mathbf{C}^{1+n}$ at the origin. So we can identify Ξ with E. Let ω be the hermitian metric on E and $h : E \to \mathbf{R}$ be given by $h := \| \omega \|^2$. Using the new metric h we define the unit disk bundle $\mathbf{B}_h \subset E$ and its boundary the unit circle bundle $\mathbf{S}_h \subset E$. Notice that $\mathbf{B}_h$ is just the blow up of the unit ball $\mathbf{B}_p$ or the indicatrix at the point $p \in \Omega$ with respect to the infinitesimal Kobayashi metric at $p \in \Omega$, (see [Lem.1]). In [Lem.5], Lempert showed that $\mathbf{B}_p$ is a strictly convex circular domain and the exponential map suitably normalized to be defined on $\mathbf{B}_p$ is a homeomorphism $\Psi_p : \mathbf{B}_p \to \Omega$ between the indicatrix and the domain, called the circular representation. He constructed the circular representation by first showing that given a point $p \in \Omega$ and a complex line $\mathscr{L} \subset T_p\Omega$, there is a unique Kobayashi extremal disk containing the point p and with $\mathscr{L}$ as its tangent space at p. The family of all extremal disks through p forms a foliation $\mathscr{F}_p$ of Ω which we call the Lempert foliation. The indicatrix $\mathbf{B}_p$ is also foliated by the family of disks obtained by intersecting it with the family of complex lines in $T_p\Omega$ containing the origin. Denote this foliation by $\mathscr{F}$. The circular representation is now constructed by sending the leaf $\mathscr{M}$ of $\mathscr{F}$ determined by the line $\mathscr{L} \subset T_p\Omega$ biholomorphically to the extremal disk determined by $\mathscr{L}$. Because the Kobayashi metric is a biholomorphic invariant, the circular representation is then also a biholomorphic invariant of the domain Ω with the given fixed point $p \in \Omega$. Denote the affine coordinates on $\mathbf{P}_n(\mathbf{C})$ by $w = (w_1, \ldots, w_n)$, and the fibre coordinate ζ on E is defined by $z = (\zeta, \zeta w_1, \ldots, \zeta w_n)$ where $z = (z_0, \ldots, z_n)$ are the linear coordinates on $\mathbf{C}^{n+1}$. Next let $\tilde{\Omega}_p$ denote the space obtained from Ω by blowing up the point $p \in \Omega$. Without loss of generality, we assume that p is the origin, and thus identify $\tilde{\Omega}_p$ with an open subset $\mathcal{U}$ of E and the blow up of p is identified with $\mathbf{P}_n(\mathbf{C})$. The Lempert foliation on Ω then lifts naturally to a nonsingular foliation $\tilde{\mathcal{F}}$ on $\tilde{\Omega}_p$ whose leaves are transverse to $\mathbf{P}_n(\mathbf{C})$. The circular representation also lifts to a map $\tilde{\Psi}_p : \mathbf{B}_p \to \tilde{\Omega}_p$ which we again refer to as the circular representation. Let $K_p : \Omega \to \mathbf{R}$ denote the Kobayashi distance from the point p and γ_p the real-valued function on Ω defined by

$$\gamma_p(s) =: \tanh^2(K_p(s)). \tag{4.21}$$

The function $\log(\gamma_p)$ is smooth away from p and can be extended to a smooth function on $\mathbf{C}^{n+1}$ so that $\Omega = \{z \in \mathbf{C}^{n+1} ; \gamma_p(z) < 1\}$.

THEOREM 4.17 ([Pa.1], [Lem.1]). *Let Ω be a strictly linearly convex domain in $\mathbf{C}^{n+1}$ with smooth boundary $\partial\Omega$. Let p be a given fixed point in Ω. Then*
(i) *The circular representation $\Psi_p : \mathbf{B}_p \to \Omega$ is a C^1 map on all of $\mathbf{B}_p$ and after identifying the tangent space to $T_p\Omega$ at 0 with $T_p\Omega$ itself, we have $\Psi_{p,*} = \mathrm{id}_{T_p\Omega}$. In particular, $\tilde{\Psi}_p$ is the identity on $\mathbf{P}_n(\mathbf{C}) \subset \mathbf{B}_h$.*
(ii) *The map $\tilde{\Psi}_p : \mathbf{B}_h \to \tilde{\Omega}_p$ is a C^∞ diffeomorphism which is holomorphic on the fibres of $\mathbf{B}_h \to \mathbf{P}_n(\mathbf{C})$ and maps the fibres to the leaves $\mathcal{M}$ of the Lempert foliation $\mathcal{F}$.*
(iii) *γ_p extends to a C^∞ function on $\tilde{\Omega}_p$ satisfying $h = \gamma_p \circ \tilde{\Psi}_p$. The function $u = \log(\gamma_p)$ is the solution of the degenerate complex Monge-Ampère equation*

$$\begin{cases} u \in \mathrm{PSH}(\Omega) \\ (dd^c u)^n = 0 \ \text{in}\ \Omega \setminus \{0\} \\ u(z) = \log\|z\| + O(1) \ \text{as}\ z \to 0 \\ u(z) = 0 \ \text{for}\ z \in \partial\Omega \end{cases}$$

and $\mathcal{F}_p$ is its Monge-Ampère foliation, i.e. the tangent bundle of $\mathcal{F}_p$ is given by

$$T\mathcal{F}_p = \{X \in T\Omega \,;\, X \rfloor \Lambda = 0\}$$

where $\Lambda = \partial\bar{\partial}\log(\gamma_p)$ and $\rfloor$ is its contraction with X. The function $u = \log(\gamma_p)$ is called the potential of the domain Ω with the given fixed point $p \in \Omega$.

Proof of Proposition 3.6. Without loss of generality we can assume that $w = 0$ is the origin. Theorem 4.17, then establishes the existence of a C^∞ diffeomorphism $\Psi_0 : \bar{\mathbf{B}}_0 \setminus \{0\} \to \bar{\Omega} \setminus \{0\}$ (see [Lem.1]), where $\bar{\mathbf{B}}_n \setminus \{0\} = \{z \in \mathbf{C}^n ; 0 < \|z\| < 1\}$ such that $u_0 = \log\|\Psi_0\|$ satisfies

$$\begin{cases} u_0 \in \mathrm{PSH}(\Omega) \\ (dd^c u_0)^n = 0 \text{ in } \Omega \setminus \{0\} \\ u_0(z) = 0 \text{ if } z \in \partial\Omega \\ u_0(z) = \log \| z \| + O(1) \text{ as } z \to 0. \end{cases}$$

The existence of the C^∞ function u_0 implies that the sublevel sets $\{\Omega_x\}_{x<0}$ are smooth. Next we show that these sublevel sets are linearly convex. Take a point $z \in \partial\Omega_x = \{u_0(z) = x\}$, $x \leq 0$, the boundary of the sublevel sets $\Omega_x = \{z \in \Omega\,;\, u_0(z) < x,\, x < 0\}$. Lempert [Lem.4], has shown that there exists a unique extremal mapping f from the closed unit disk $\bar{\mathbf{D}}$ in $\mathbf{C}$ into the set $\bar{\Omega}_x$ with $\xi = \exp x$ such that $f(0) = 0$, $f(\xi) = z$ and f is an embedding. In particular, f is transverse to $\partial\Omega_x$. There exists an inverse mapping $F : \bar{\Omega}_x \to \bar{\mathbf{D}}$ to f (cf. [Lem.2]), with the following properties:
(1) The fibres $F^{-1}(\xi)$, for all $\xi \in \mathbf{D}$ are hypersurface restricted to the neighbourhood of $\bar{\Omega}_x$.
(2) $|F(z)| < 1$ if $z \in \bar{\Omega}_x \setminus f(\partial\mathbf{D})$,
(3) $dF \neq 0$ on $\bar{\Omega}_x$. We claim that $F^{-1}(\xi)$ is a complex tangent hyperplane to $\partial\Omega_x$ at the point z. Since $\xi = F(z)$ the fact that $f(\xi) = z$ implies that z lies on the complex hyperplane $F^{-1}(\xi)$. To verify the claim it is therefore enough to show that no other point $w \in \bar{\Omega}_x$ can lie on $F^{-1}(\xi)$. Now for any $w \in \bar{\Omega}_x$ there exists a holomorphic mapping $g : \bar{\mathbf{D}} \to \bar{\Omega}_x$ such that $g(0) = 0$ and $g(\omega) = w$ with $\omega = \exp u_0(w) \leq \xi$. Now we apply Schwartz's lemma to $F \circ g : \bar{\mathbf{D}} \to \bar{\mathbf{D}}$ to give $\xi \geq \omega \geq |F(g(\omega))| = |F(\omega)|$ and $\xi = F(w)$, that is to say, $w \in F^{-1}(\xi)$. This can happen only if $F \circ g = \mathrm{id}_{\mathbf{D}}$. Hence by property (2) and the uniqueness of the extremal mapping f this holds only if $f = g$, i.e. $w = z$. Thus $F^{-1}(\xi)$ is indeed the complex tangent hyperplane to $\partial\Omega_x$, i.e. a complex hyperplane which does not intersect $\bar{\Omega}_x$.
□

5. Siciak and Lempert extremal functions

Let $\mathscr{L}$ represent the class of all functions u which are plurisubharmonic (psh) on $\mathbf{C}^n$ and satisfy the condition:

$$u(z) \leq \log(1 + \| z \|) + O(1), \text{ as } \| z \| \to +\infty. \tag{5.0}$$

This is the space of plurisubharmonic functions of logarithmic growth. Since $\mathscr{M}(r) := \sup_{\|z\|=R} u(z)$ is a convex, increasing function of R, we see easily that $\mathscr{L}$ consists of plurisubharmonic functions of minimal growth. Of particular interest is the subclass

(5.1) $$\mathscr{L}_+ = \{u \in \mathscr{L}\,;\, u(z) = \log^+ \|z\| + O(1)\}.$$

For $E \subset \mathbf{C}^n$ a bounded set, we define the $\mathscr{L}$-extremal psh function of E by setting

(5.2) $$V_E(z) := \sup\{v(z)\,;\, v \in \mathscr{L}, \text{ and } v(\zeta) \leq 0, \text{ for all } \zeta \in E\},$$

and letting

(5.3) $$V_E^*(z) := \limsup_{\zeta \to z} V_E(\zeta),$$

be the upper semi-continuous regularization of V_E. This function has been studied extensively in [Si.1,2], [Za.1] and [Sd.1]. The function V_E^* is in general not smooth on $\mathbf{C}^n \setminus E$ when $n > 1$. It is a theorem in [Si.1], that either $V_E^* \equiv +\infty$, in which case the set E is pluripolar or else V_E^* is psh and

(5.4) $$V_E^*(z) \leq \log(1 + \|z\|) + O(1), \text{ as } \|z\| \to +\infty.$$

If V_E is continuous on $\mathbf{C}^n$, then $V_E = V_E^* \in \mathscr{L}$. If $n = 1$, V_E^* is exactly the generalized Green function for $\mathbf{C} \setminus E$ with a pole at infinity and the definition is, essentially, given by the Perron method for its construction. In particular, V_E^* is harmonic in $\mathbf{C} \setminus E$ and smooth. However, when $n > 1$, there are some significant differences. In general the function V_E^* need not be continuous in the exterior of E. But, it is a result of [B-T.2], that the $\mathscr{L}_+$-extremal function V_E^* satisfies the homogeneous complex Monge-Ampère equation, $(dd^c V_E^*)^n = 0$, in a generalized sense on $\mathbf{C}^n \setminus E$ when E is a compact set. Thus for non pluripolar sets E,

(5.5) $$\lambda_E := (dd^c V_E^*)^n,$$

is a positive Borel measure supported on E.

The class $\mathscr{L}$ is closely related to the study of polynomials in n-complex variables. For a compact set $K \subset \mathbf{C}^n$, we have

(5.6) $$V_K(z) = \sup\left\{\frac{1}{d}\log|p(z)|\,;\, d = \deg(p),\ \|p\|_K \leq 1\right\}.$$

Now observe that if $E = \mathbf{B}(w, R) := \{z \in \mathbf{C}^n\,;\, \|z - w\| \leq R\}$ is a ball of centre w and radius R, where $\|\cdot\|$ is any norm on $\mathbf{C}^n$ then $V_E = \log^+(\|z - w\|/R)$ (see [Si.1]).

Let $\Psi : \mathbf{C}^n \to [-\infty, +\infty[$ be any function defined on $\mathbf{C}^n$ which may take the value $-\infty$ but not $+\infty$. Then for such functions we define the subclass $\mathscr{L}_\Psi$ of the class $\mathscr{L}$ by

(5.7) $$\mathscr{L}_\Psi := \{v(z)\,;\, v \in \mathscr{L},\ v \leq \Psi\},$$

and

$$\mathcal{L}_{\Psi}^{+} := \{v(z) \,;\, v \in \mathcal{L}_{+},\, v \leq \Psi\}. \tag{5.8}$$

Now set $V_{\Psi} = \sup\{v(z) \,;\, v \in \mathcal{L}_{\Psi}\}$ and $V_{\Psi}^{+} = \sup\{v(z) \,;\, v \in \mathcal{L}_{\Psi}^{+}\}$. The functions V_{Ψ} and V_{Ψ}^{+} will be called $\mathcal{L}_{\Psi}$-extremal and $\mathcal{L}_{\Psi}^{+}$-extremal functions associated to Ψ respectively.

Let the $\mathcal{L}_{\Psi}$-extremal function for a bounded linearly convex domain $\Omega \subset \mathbf{C}^n$ be given as $V_{\Omega,\Psi}(z) = \sup\{v(z) \,;\, v \in \mathcal{L}_{\Psi},\, v \leq 0 \text{ on } \Omega\}$ and its upper semi-continuous regularization $V_{\Omega,\Psi}^{*}(z) = \limsup_{\zeta \to z} V_{\Omega,\Psi}(\zeta)$. We also call the subclass

$$\mathcal{L}_{\Omega,\Psi} = \{v(z) \,;\, v \in \mathcal{L}_{\Psi},\, v \leq 0, \text{ on } \Omega\},$$

Siciak class of plurisubharmonic functions with respect to Ψ and the set Ω. $V_{\Omega,\Psi}$ and its upper semi-continuous regularization $V_{\Omega,\Psi}^{*}$ are the Siciak extremal functions of Ω with respect to Ψ. More generally, we fix $\Psi \in PSH(\mathbf{C}^n) \cap C^{\circ}(\mathbf{C}^n)$ and introduce the Siciak extremal functions $V : \mathbf{C}^n \times]0, \infty[\to \mathbf{R}_{+} := \{x \in \mathbf{R} \,;\, x > 0\}$ with respect to Ψ given by

$$V_{\alpha}(z) := V(z, \alpha) := \sup_{u} u(z),\ z \in \mathbf{C}^n, \quad \alpha \in]0, \infty[, \tag{5.9}$$

where the supremum is taken over all plurisubharmonic functions u with $u \in \alpha\mathcal{L}$ and $u \leq \Psi$.

THEOREM 5.1 [Mo.1], [Si.1]. *Let $u \in \mathcal{L}$, with $u \leq \Psi$. Then*

$$u(z) \leq \log^{+}(\| z \|/R) + \max_{\|\zeta\| \leq R} \Psi(\zeta), \quad z \in \mathbf{C}^n, \quad R \in \mathbf{R}_{+}.$$

Proof. Fix $R > 0$. By the hypothesis we have $u(z) \leq \max_{\|z\| \leq R} \Psi(z) =: M$ on $R\mathbf{B} = \{z \in \mathbf{C}^n \,;\, \| z \| \leq R\}$. Furthermore, $u - M \in \mathcal{L}$. So that $u - M$ is dominated by the pluricomplex Green function of $\mathbf{C}^n \setminus R\mathbf{B}$ with logarithmic pole at infinity, i.e. $u(z) - M \leq \log^{+}(\| z \|/R)$, $z \in \mathbf{C}^n$. □

The Siciak extremal psh function V_{Ψ} attains the value Ψ on the set $E_{\Psi} = \{z \in \mathbf{C}^n \,;\, V_{\Psi}(z) = \Psi(z)\}$. Next we give a description of what will become known as Lempert extremal psh function of bounded linearly convex domains Ω in $\mathbf{C}^n$. First let $u \in C^3(\Omega)$ be such that

$$(dd^c u)^{k-1} \neq 0, \text{ and } (dd^c u)^{k} = 0. \tag{5.10}$$

If we integrate the form $(dd^c u)^{k-1}$, then by Frobenius Theorem we obtain a foliation $\mathscr{F}_{n-k+1}$ of Ω, by complex manifolds of dimension $n-k+1$ which have the property that u is harmonic and $\dfrac{\partial u}{\partial z_j}$ is holomorphic on each leaf $\mathscr{M}$ of $\mathscr{F}$ (see [B-K. 1]). Conversely, if we are given a foliation $\mathscr{F}$ and if we can find a function u such that $\dfrac{\partial u}{\partial z_j}$ is holomorphic on each leaf of $\mathscr{F}$ for $1 \le j \le n$, then (5.10) holds. The mapping

$$z \mapsto \gamma_u(z) := \left(\frac{\dfrac{\partial u(z)}{\partial z_1}}{\sum_\nu z_\nu \dfrac{\partial u(z)}{\partial z_\nu}}, \dots, \frac{\dfrac{\partial u(z)}{\partial z_n}}{\sum_\nu z_\nu \dfrac{\partial u(z)}{\partial z_\nu}} \right), \tag{5.11}$$

which can be thought of as the complex Gauss map of the hypersurface $\{u = c\}$, is holomorphic on the leaves of $\mathscr{F}$. If γ_u is a local diffeomorphism, then γ_u pushes $\mathscr{F}$ forward to a new foliation $\mathscr{F}^\star$ and u to a new function $u^\star$. It is easily checked that the gradient of $u^\star$ is holomorphic on $\mathscr{F}^\star$ and that $u^\star$ satisfies (5.10), even when it is not necessarily plurisubharmonic. This duality map was introduced in [Lem.2] to transform an exterior Dirichlet problem to an interior one.

Let Ω be a bounded linearly convex domain with boundary $\partial\Omega$. Consider its pluricomplex Green function $g_\Omega(\cdot, w)$ having a logarithmic pole at $w \in \Omega$. This function is plurisubharmonic and continuous when its restriction to $\partial\Omega$ is identically zero. It follows that

$$\begin{cases} g_\Omega(z, w) = \log\|z - w\| + O(1), \text{ as } z \to w \\ (dd^c g_\Omega(z, w))^n = 0 \text{ on } \Omega \setminus \{w\}. \end{cases}$$

If there exists a bounded plurisubharmonic exhaustion function ψ of Ω i.e., $\psi \in C^\circ(\bar\Omega) \cap \mathrm{PSH}(\Omega, W, \nu)$ with $\psi < 0$ and $\{\psi < -c\} \subset\subset \Omega$ for all $c > 0$ then it follows that $g_\Omega(z, w)$ is continuous for all $(z, w) \in \bar\Omega \times \Omega$, $z \neq w$, and $g_\Omega(z, w) = 0$ for $z \in \partial\Omega$, (see [De.1]). In fact, $g_\Omega(z, w)$ satisfies $(dd^c g_\Omega(z, w))^n = (2\pi)^n \delta_w$ where δ_w is the Dirac function at w. Note that here, we cannot define $g_\Omega(\cdot, w)$ in terms of the Perron-Bremermann envelope $W(z) = \sup\{v(z)\, ;\, v \in \mathscr{B}(\varphi, \mu)\}$ of the Perron-Bremermann family

$$\mathscr{B}(\varphi, \mu) = \left\{ v \in \mathrm{PSH}(\Omega)\, ;\, (dd^c v)^n \ge \mu,\ \limsup_{\zeta \to z} v(\zeta) \le \varphi(z),\ z \in \partial\Omega \right\},$$

where $\varphi \in C^\circ(\partial\Omega, \mathbf{R}_+)$. To be precise, not as the Perron-Bremermann envelope of $\mathscr{B}(0, (2\pi)^n\delta_w)$, since $u(0, (2\pi)^n\delta_w) = 0$, (see [B-T.1]). In the case that Ω is a strictly convex and smoothly bounded domain, L. Lempert [Lem.1], has shown that $g_\Omega(\cdot, w) \in C^\infty(\bar\Omega \setminus \{w\})$. This was done using foliation $\mathscr{F}$ of the domain Ω whose leaves correspond to the extremal disks for the Kobayashi metric. In the strongly pseudoconvex case, $g_\Omega(z, w) \notin C^2(\bar{\Omega} \setminus \{w\})$, (see [B-De.1]). In [Lem.2], L. Lempert established that there is a close connection between the pluricomplex Green function $g_\Omega(\cdot, w)$ and the $\mathcal{L}$-extremal function V_Ω. If as before we identify $\mathbf{C}^n$ with $\mathbf{P}_n(\mathbf{C}) - \mathbf{P}_{n-1}(\mathbf{C})$, i.e. the n-dimensional complex projective space with a hyperplane at infinity removed, then given a strictly linearly convex domain $\Omega \subset \mathbf{C}^n$ containing the origin, we let Ω^* denote the dual complement domain inside the dual projective space $\mathbf{P}_n^*(\mathbf{C}) \setminus \mathcal{H}_w$ where $\mathcal{H}_w$ is the complex hyperplane dual to $w \in \Omega$.

The complex Gauss map $\gamma_{g_\Omega(\cdot,w)}$ of the pluricomplex Green function $g_\Omega(\cdot, w)$ extends the complex Gauss mapping of $\partial\Omega$ to $\partial\Omega^*$. This mapping gives a non holomorphic diffeomorphism between $\Omega \setminus \{w\}$ and $\mathbf{P}_n^*(\mathbf{C}) \setminus \{\mathcal{H}_w \cup \bar{\Omega}^*\}$, such that

$$g_\Omega(z, w) = - V_{\Omega^*}(\gamma_{g_\Omega(z,w)}(z)). \tag{5.12}$$

The $\mathcal{L}$-extremal function

$$V_{\Omega^*}(z) = \sup\{u(z) \,;\, u \in \mathcal{L},\ u \leq 0 \text{ on } \Omega^*\}, \tag{5.13}$$

is called the Lempert extremal function of Ω^*. To be precise, we define $\mathcal{L}_w$ to be the set of all psh functions u in a neighbourhood $\mathcal{W}$ of $\{z = w\}$ such that $u(z) = \log \| z - w \| + O(1)$, and $(dd^c u)^n = 0$ on $\{z \neq w\}$. Further we require that for some small $\delta > 0$ the set $\left\{u < -\dfrac{1}{\delta}\right\}$ is linearly convex. We also consider $\mathcal{L}_\infty$ the family of psh functions outside a compact set such that $U(z) = \log^+ \| z \| + O(1)$, and $(dd^c U)^n = 0$ with $\left\{U < \dfrac{1}{\delta}\right\}$ for small $\delta > 0$, linearly convex. Then Lempert's result above can be interpreted as saying that the complex Gauss map establishes a correspondence between $\mathcal{L}_w$ and $\mathcal{L}_\infty$.

We generalize this to the case of finite singularities. Let $W = \{w_j \in \Omega,\ 1 \leq j \leq m < \infty\}$ be a finite set of singularities with weights the numbers $\nu = \{\nu(w_j) := \nu_j > 0,\ 1 \leq j \leq m < +\infty\}$. Let $g_\Omega(\cdot, W, \nu)$ denote the pluricomplex multipole Green function of the bounded linearly convex domain $\Omega \subset \mathbf{C}^n$ containing the origin. We set $w_1 = 0$ the origin and let $\mathcal{H}_{w_1} = \mathcal{H}_0, \ldots, \mathcal{H}_{w_m}$ be the complex hyperplanes dual to the distinct points $w_1 = 0, w_2, \ldots, w_m \in W \subset \Omega$. Next let Ω^* denote the dual complement of Ω inside the dual projective space $\mathbf{P}_n^*(\mathbf{C}) \setminus \{\mathcal{H}_{w_1} \cup \ldots \cup \mathcal{H}_{w_m}\}$, then the complex Gauss map $\gamma_{g_\Omega(\cdot,W,\nu)}$ of the pluricomplex multipole Green function $g_\Omega(\cdot, W, \nu)$ extends as before the complex Gauss mapping of $\partial\Omega$ to $\partial\Omega^*$. This mapping gives a non holomorphic diffeomorphism between $\Omega \setminus W[\nu]$ and $\mathbf{P}_n * (\mathbf{C}) \setminus \{\mathcal{H}_{w_1} \cup \ldots \cup \overline{\Omega^*}\}$ with

$$g_\Omega(z, W, \nu) = -V_{\Omega^*}(\gamma_{g_\Omega(z,W,\nu)}(z)). \tag{5.14}$$

Analogous to the earlier situation, we let $\mathcal{L}_{W[\nu]}$ be the family of psh functions u in the neighbourhood of $W[\nu]$ such that $u(z) = \sum_{j=1}^m \nu_j \log \| z - w_j \| + O(1)$, and $(dd^c u)^n = 0$ for $z \notin W[\nu]$. Moreover, we can assume that for a small positive δ the set $\left\{u < \frac{1}{\delta}\right\}$ is linearly convex. Let $\mathcal{L}_\infty$ be the family of psh functions U outside $\{\cup_{j=1}^m \mathcal{H}_{w_j} \cup \overline{\Omega^*}\}$ with $U(z) = \log \| z \| + O(1)$, $(dd^c U)^n = 0$ also with $\left\{U < \frac{1}{\delta}\right\}$ linearly convex for $\delta < 0$ small. Then the complex Gauss map again establishes a correspondence between $\mathcal{L}_{W[\nu]}$ and $\mathcal{L}_\infty$.

We can consider the space $\mathcal{L}$ in greater generality (see [B-T.3]), where we let M be a complex manifold, and $D \subset M$ a subvariety of pure dimension 1 (a divisor). For $w \in D$ we let U_w be an open set containing w such that there exists a holomorphic function h on U_w with $U_w \cap D = \{h = 0\}$ and $\operatorname{grad}(h) \neq 0$ on the regular points of $U_w \cap D$, (see [B-T.3]). We can now define

$$\mathcal{L}(M, D) := \{u \in \mathrm{PSH}(M \setminus D) \,;\, u \leq \log |h| + O(1)\}. \tag{5.15}$$

Note that any $v \in \mathcal{L}(M, D)$ can in this situation be written on the open set U_w as $v = \log \left| \frac{1}{h} \right| + \tilde{v}$ for some $\tilde{v} \in \mathrm{PSH}(U_w) \cap L^\infty_{\mathrm{loc}}(U_w)$. We now let $M = \mathbf{P}_n(\mathbf{C})$ and the hyperplane at infinity chosen so that $\mathbf{C}^n = \mathbf{P}_n(\mathbf{C}) \setminus \mathbf{P}_{n-1}(\mathbf{C})$. We wish to consider the case of functions with logarithmic decrease at isolated singularities. Let $\Omega \subset \mathbf{C}^n$ be a bounded linearly convex domain containing a point $w \neq 0$, and let $\tilde{\Omega}_w$ be Ω with the point w blown up. Let $\mathbf{P}_{n-1}(\mathbf{C})$ be the fibre over w. Consider

$$\begin{aligned} \mathcal{L}(\Omega_w, \mathbf{P}_{n-1}(\mathbf{C})) = \{u \in \mathrm{PSH}(\tilde{\Omega}_w \setminus \mathbf{P}_{n-1}(\mathbf{C}))\,;\\ u(z) \leq \log \operatorname{dist}(w, \mathbf{P}_{n-1}(\mathbf{C})) + C\}, \end{aligned} \tag{5.16}$$

where C is a positive constant. If we choose $\mathcal{L}$ as above, we can study the pluricomplex Green function of a bounded domain with logarithmic singularity at the point w. For Ω a bounded convex domain in $\mathbf{C}^n$ this has been studied in depth in [Lem.1], and the case of hyperconvex domain Ω in [Kl.2], and [De.1]. It is known that in both cases there exists a unique psh function (the pluricomplex Green function), u_w on Ω that is continuous up to the boundary, vanishes there and satisfies

$$\begin{cases} u_w(z) = \log \| z - w \| + O(1), \text{ as } z - w \to 0 \\ (dd^c u_w)^n \equiv 0 \text{ on } \bar{\Omega}_w. \end{cases}$$

Recall that the first condition is just that u_w belongs to

$$\mathscr{L}_+(\tilde{\Omega}_w, \mathbf{P}_{n-1}(\mathbf{C})) := \{u \in \mathrm{PSH}(\tilde{\Omega}_w \setminus \mathbf{P}_{n-1}(\mathbf{C}))\,; \quad u(z) = \log \mathrm{dist}(w, \mathbf{P}_{n-1}(\mathbf{C})) + C\}. \tag{5.17}$$

In analogy with the previous situation we let $\tilde{\Omega}_{W[\nu]}$ be Ω with the singularities $w_1, \ldots, w_m$ blown up in succession and we let $\mathscr{H}_{w_1}, \ldots, \mathscr{H}_{w_m}$ be the fibres over the points $w_1, \ldots, w_m$, then we define

$$\mathscr{L}(\tilde{\Omega}_{W[\nu]}, \bigcup_{j=1}^{m} \mathscr{H}_{w_j}) = \{u \in \mathrm{PSH}(\tilde{\Omega}_{W[\nu]} \setminus \bigcup_{j=1}^{m} \mathscr{H}_{w_j})\,; \quad u(z) \leq \sum_{j=1}^{m} \nu_j \log \mathrm{dist}(w_j, \mathscr{H}_{w_j}) + C\}, \tag{5.18}$$

with a positive constant C. So as above we have that

$$\begin{cases} V_{W[\nu]}(z) = \sum_{j=1}^{m} \nu_j \log \| z - w_j \| + O(1), \text{ as } z \to w_j \\ (dd^c V_{W[\nu]})^n \equiv 0 \text{ on } \tilde{\Omega}_{W[\nu]}. \end{cases}$$

Here again, $V_{W[\nu]}$ belongs to the restricted class

$$\mathscr{L}_+(\tilde{\Omega}_{W[\nu]}, \bigcup_{j=1}^{m} \mathscr{H}_{w_j}) = \{u \in \mathrm{PSH}(\tilde{\Omega}_{W[\nu]} \setminus \bigcup_{j=1}^{m} \mathscr{H}_{w_j})\,; \quad u(z) = \sum_{j=1}^{m} \nu_j \log \mathrm{dist}(w_j, \mathscr{H}_{w_j}) + C\}, \tag{5.19}$$

with a positive constant C.

6. Duality of functions and supporting functions

A function $f : \mathbf{C}^{1+n} \setminus \{0\} \to [-\infty, +\infty]$ is called logarithmically homogeneous if

$$f(\lambda z) = -\log|\lambda| + f(z), \quad z \in \mathbf{C}^{1+n} \setminus \{0\}, \quad \lambda \in \mathbf{C} \setminus \{0\}. \tag{6.1}$$

A stronger version of this condition of logarithmic homogeneity is the following, $f(\lambda z) = -C \log|\lambda| + f(z)$, where C is a positive constant.

For such functions, following [Ki.3], we define the dual function $\tilde{f}$:

$$\tilde{f}(\zeta) = \sup_z(-\log|\zeta \cdot z| - f(z)\,; f(z) < +\infty), \ \zeta \in \mathbf{C}^{1+n} \setminus \{0\}. \tag{6.2}$$

Here $\zeta \cdot z = \zeta_0 z_0 + \cdots + \zeta_n z_n$ is the inner product, and we define $\log 0 = -\infty$ so that $f(\zeta) = +\infty$ if there is a z such that $\zeta \cdot z = 0$ and $f(z) < +\infty$. The difference $-\log|\zeta\cdot z|-f(z)$ is well-defined if $f(z)<+\infty$. It is clear that $\tilde{f}$ is also homogeneous.

Given f defined in $\mathbf{C}^{1+n}\setminus\{0\}$ we can define a function F in $\mathbf{C}^n$ by setting $F(z')=f(z_1/z_0,\ldots,z_n/z_0)$, $z'\in\mathbf{C}^n$. Conversely, if F is defined in $\mathbf{C}^n$, we define a homogeneous function f in $\mathbf{C}^{1+n}\setminus\{0\}$ by

$$f(z)=\begin{cases}F(z_1/z_0,\ldots,z_n/z_0)-\log\|z_0\|, & z\in\mathbf{C}^{1+n}\setminus\{0\},\ z_0\neq 0;\\ +\infty & z\in\mathbf{C}^{1+n}\setminus\{0\},\ z_0=0.\end{cases}$$

The transform (6.2) then takes the form

$$\tilde{F}(\zeta')=\sup_{z'}(-\log|1+\zeta'\cdot z'|-F(z');F(z')<+\infty),\quad \zeta'\in\mathbf{C}^n.$$

In particular, if F is a function of $\|z'\|=r$, then the transform becomes

$$\tilde{F}(\rho)=\sup_{r}(-\log(1-\rho r)-F(r);F(r)<+\infty),\quad \rho=\|\zeta'\|\geq 0.$$

The radial function $F(r)=-\dfrac{1}{2}\log(1-r^2)$ is self-dual, i.e. $\tilde{F}(\rho)=-\dfrac{1}{2}(1-\rho^2)$. Returning to $\mathbf{C}^{1+n}\setminus\{0\}$, we see that the function

$$f(z)=\begin{cases}-\dfrac{1}{2}\log(\|z_0\|^2-\|z'\|^2), & z\in\mathbf{C}^{1+n}\setminus\{0\},\ \|z_0\|>\|z'\|;\\ +\infty & z\in\mathbf{C}^{1+n}\setminus\{0\},\ \|z_0\|\leq\|z'\|,\end{cases}$$

has this property.

Now let Ω be a homogeneous set in $\mathbf{C}^{1+n}\setminus\{0\}$. We define a function d, the distance to the complement of Ω, by

$$d(z)=d_\Omega(z)=\inf(\|z-w\|;w\notin\Omega),\quad z\in\mathbf{C}^{1+n}\setminus\{0\}. \tag{6.3}$$

The function $-\log d$ is homogeneous, and it is precisely less than $+\infty$ in the interior of Ω. Analogously we define a function δ by

$$\delta(\zeta)=d_{\Omega^*}(\zeta)=\inf(\|\zeta-\alpha\|;\alpha\notin\Omega^*),\quad \zeta\in\Omega^{*\circ},\quad \zeta\in\mathbf{C}^{1+n}\setminus\{0\}, \tag{6.4}$$

where Ω^* is the dual complement of Ω and $\Omega^{*\circ}$ is its interior.

THEOREM 6.1 [Ki.3]. *Let f be any function on $\mathbf{C}^{1+n}\setminus\{0\}$, let $\tilde{f}$ be the transform defined by* (6.2), *and denote by Ω^* the set of all points where $\tilde{f}<+\infty$, by δ the distance to the complement of Ω^*. Then $\tilde{f}$ is Lipschitz continuous in the interior of Ω^*; more precisely*

$$\limsup_{t\to 0+}\frac{\tilde{f}(\zeta+t\theta)-\tilde{f}(\zeta)}{t}\leq\frac{\|\theta\|}{\delta(\zeta)},\quad \zeta\in\Omega^{*\circ},\quad \theta\in\mathbf{C}^{1+n}.$$

THEOREM 6.2 [Ki.3]. *Let f be a homogeneous function on $\mathbf{C}^{1+n}\setminus\{0\}$ which is bounded from below on the unit sphere. Then $\tilde{f}$ is plurisubharmonic in the open set $\{\zeta\,;\ \delta(\zeta)>0\}$, where δ is the distance to the points where $\tilde{f}$ is $+\infty$.*

Examples of functions in duality are

$$f_c(z)=\begin{cases}-(1-c)\log\|z\|-c\log d(z), & z\in\Omega\,;\\ +\infty & z\notin\Omega,\end{cases}\tag{6.5}$$

and

$$\varphi_c(\zeta)=\begin{cases}-(1-c)\log\|\zeta\|-c\log\delta(\zeta), & \zeta\in\Omega^*\,;\\ +\infty & \zeta\notin\Omega^*,\end{cases}\tag{6.6}$$

where $0\leq c\leq 1$, Ω is any homogeneous subset of $\mathbf{C}^{1+n}\setminus\{0\}$, Ω^* its dual complement, and d and δ are defined by (6.3) and (6.4) respectively.

We shall call $f_0=\iota_\Omega$ the indicator function of the set Ω. Its restriction to the unit sphere is the indicator function in that usual sense. We call $\tilde{f}_0=\tilde{\iota}_\Omega$ the supporting function of Ω. The supporting function of Ω is given explicitly by

THEOREM 6.3 [Ki.3]. *Let Ω be a non-empty subset of $\mathbf{C}^{1+n}\setminus\{0\}$. Then $\tilde{f}_0=\varphi_1$, i.e. the supporting function of Ω is $\tilde{\iota}_\Omega=-\log d_{\Omega^*}$.*

If Ω is empty, its supporting function is identically $-\infty$, whereas $\varphi_1(\zeta)=-\log\|\zeta\|$.

Let Ω be a bounded linearly convex domain in $\mathbf{C}^n$. To Ω we associate a supporting function

$$h_\Omega(\zeta)=\sup_{z\in\Omega}(-\log|1+\zeta\cdot z|),\quad \zeta\in\mathbf{C}^n.\tag{6.7}$$

We know that for a homogeneous set Ω of all $z\in\mathbf{C}^{1+n}\setminus\{0\}$ such that $z_0\neq 0$ and $z'/z_0\in\Omega_0$ where Ω_0 is a subset of $\mathbf{C}^n$, the supporting function of Ω is given by

$$\tilde{\iota}_\Omega(\zeta)=-\log\delta(\zeta)=\sup_\Omega(-\log|\zeta\cdot z|-\log\|z\|),\quad \zeta\in\mathbf{C}^{1+n}\setminus\{0\}.$$

Now we modify ι_Ω a little and define

$$h_\Omega(\zeta)=\sup_\Omega(-\log|\zeta\cdot z|-\log\|z_0\|),\quad \zeta\in\mathbf{C}^{1+n}\setminus\{0\},$$

so that if Ω_0 is bounded, then h_Ω and $\tilde{\iota}_\Omega$ differ at most by an additive constant M.

EXAMPLE 6.4. If Ω is the ball with centre the origin and radius $r \in \mathbf{R}$, $r > 0$, i.e., $\Omega = r\mathbf{B} = \{z \in \mathbf{C}^n ; \|z\| < r\}$ then

$$h_\Omega(\zeta) = \begin{cases} -\log(1 - r\|\zeta\|), & \text{if } \|\zeta\| < \dfrac{1}{r}; \\ +\infty, & \text{if } \|\zeta\| \geq \dfrac{1}{r}. \end{cases}$$

Since the ball is bounded, $0 \in \Omega$, then there exists $R_1, R_2 \in \mathbf{R}_+$ with $0 < R_1 \leq R_2 < +\infty$ such that

$$-\log(1 - R_1\|\zeta\|) \leq h_\Omega(\zeta) \leq -\log(1 - R_2\|\zeta\|). \tag{6.8}$$

Here we adopt the convention that $-\log s = +\infty$, if $s \leq 0$.

THEOREM 6.5 [Ki.3]. *A closed linearly convex set can be recovered from its supporting function. Indeed, if Ω is a non-empty set with these properties, then $\iota_\Omega \geq \tilde{\iota} \geq \iota_\Omega - M$, with M a constant so that Ω is the set where $\tilde{\iota}_\Omega$ is finite.*

Let $g_\Omega(\cdot, 0)$ be the pluricomplex Green function in a bounded linearly convex domain Ω with a logarithmic pole at the origin $0 \in \mathbf{C}^n$, and $0 \in \Omega^\circ$. Since the pluricomplex Green function of the ball $R\mathbf{B} = \{z \in \mathbf{C}^n ; \|z\| < R \in \mathbf{R}_+\}$ is $g_{R\mathbf{B}}(z, 0) = \log(\|z\|/R)$ (cf. Proposition 3.2), we have

$$\log(\|z\|/R_2) \leq g_\Omega(z, 0) \leq \log(\|z\|/R_1), \text{ for } z \in \Omega. \tag{6.9}$$

Let

$$\Omega_{\operatorname{Re} x} = \{z \in \Omega ; g_\Omega(z, 0) < \operatorname{Re} x, \operatorname{Re} x < 0\}, \ x \in \mathbf{C},$$

be the sublevel sets of the pluricomplex Green function $g_\Omega(\cdot, 0)$. The supporting functions associated to the sublevel sets $\{\Omega_{\operatorname{Re} x}\}_{\operatorname{Re} x < 0}$ are given by

$$h_{\Omega_{\operatorname{Re} x}}(\zeta) = u(\zeta, x) = \sup_{\Omega_{\operatorname{Re} x}}(-\log|1 + \zeta \cdot z| ; g_\Omega(z, 0) < \operatorname{Re} x), \tag{6.10}$$

where $(\zeta, x) \in \mathbf{C}^n \times \mathbf{C}$, $\operatorname{Re} x < 0$.

THEOREM 6.6. *Let Ω be a bounded linearly convex domain in $\mathbf{C}^n$ and*

$$g_\Omega(z, 0) = \sup\{u \in \operatorname{PSH}(\Omega) ; u \leq 0,\ u(z) \leq \log^+\|z\| + O(1) \text{ as } z \to 0,\ \forall z \in \Omega\},$$

be its pluricomplex Green function with a logarithmic pole at the origin. Let

$$h_{\Omega_{\operatorname{Re} x}}(\zeta) = u(\zeta, x) = \sup_{z \in \Omega_{\operatorname{Re} x}} (-\log|1 + \zeta \cdot z| ; g_\Omega(z, 0) < \operatorname{Re} x),$$

where $(\zeta, x) \in \mathbf{C}^n \times \mathbf{C}$ *and* $\operatorname{Re} x < 0$, *be the associated supporting functions for* $\{\Omega_{\operatorname{Re} x}\}_{\operatorname{Re} x<0}$. *Then the functions* $u(\zeta, x)$ *are plurisubharmonic functions in all the variables* $(\zeta, x) \in \mathbf{C}^n \times \mathbf{C}$ *with* $\operatorname{Re} x < 0$. *Moreover,* $u(\zeta, x) = u(\zeta, \operatorname{Re} x)$ *and* $u(\zeta, 0) = \lim_{x \uparrow 0} u(\zeta, x) = h_\Omega(\zeta)$.

Proof. To prove the plurisubharmonicity of $u(\zeta, x)$ for all (ζ, x) with $\operatorname{Re} x < 0$, we first, use Theorem 3.8, to represent $u(\zeta, x)$ as follows,

$$u(\zeta, x) = \sup_f [\sup(-\log|1 + \zeta \cdot f(\sigma)|; \log|\sigma| < \operatorname{Re} x)], \tag{6.11}$$

with $(\zeta, x) \in \mathbf{C}^n \times \mathbf{C}$ and $\operatorname{Re} x < 0$, where the supremum is taken over all holomorphic mappings $f : \mathbf{D} \to \Omega_{\operatorname{Re} x}$ with $f(0) = 0$. Clearly, $u(\zeta, x)$ is a continuous function of $\zeta \in \mathbf{C}^n$. Now for f an extremal function solving the variational problem (6.11), we obtain

$$u(\zeta, x) = \sup(-\log|1 + \zeta \cdot f(\sigma)|; \log|\sigma| < \operatorname{Re} x), \tag{6.12}$$

where $\operatorname{Re} x < 0$ and $g_\Omega(f(\sigma), 0) = \log|\sigma|$. Thus $\sup(-\log|1 + \zeta \cdot f(\sigma)|; \log|\sigma| < \operatorname{Re} x < 0)$ is equal to $+\infty$ if ζ cuts $\bar{\Omega}_{\operatorname{Re} x}$ and is less than $+\infty$ if ζ does not cut $\bar{\Omega}_{\operatorname{Re} x}$ or $\zeta \in \bar{\Omega}^*_{\operatorname{Re} x}$, the dual complement of $\Omega_{\operatorname{Re} x}$. Since $g_\Omega(f(\sigma), 0)$, $\operatorname{Re} x < 0$, is an extremal psh function, it follows that $u(\zeta, x)$ is plurisubharmonic all variables where it is less than $+\infty$. In particular, it is clear that in this case $h_{\Omega_{\operatorname{Re} x}} \in \operatorname{PSH}(\Omega^*_{\operatorname{Re} x})$. Results in [Ki.3] then imply that $h_{\Omega_{\operatorname{Re} x}}$ is more or less $-\log d_{\Omega^*_{\operatorname{Re} x}}$ the interior distance function in $\Omega^*_{\operatorname{Re} x}$ and hence plurisubharmonic in all $(\zeta, x) \in \mathbf{C}^n \times \mathbf{C}$. The last statements are clear. □

In the general case of finite singularities with the pluricomplex multipole Green function $g_\Omega(z, W, \nu)$ we have

THEOREM 6.7. *Let* Ω *be a bounded linearly convex domain in* $\mathbf{C}^n$ *and* $g_\Omega(z, W, \nu)$ *be its pluricomplex multipole Green function defined in* (3.4), (3.5) and (3.6). *Let*

$$\Omega_{\operatorname{Re} x} = \{z \in \Omega ; g_\Omega(z, W, \nu) < \operatorname{Re} x, \quad \operatorname{Re} x < 0\}, \quad x \in \mathbf{C},$$

be the sublevel sets of the pluricomplex multipole Green function $g_\Omega(z, W, \nu)$. *Let*

$$h_{\Omega_{\operatorname{Re} x}}(\zeta) = u(\zeta, x) = \sup_{z \in \Omega_{\operatorname{Re} x}} (-\log|1 + \zeta \cdot z|; g_\Omega(z, W, \nu) < \operatorname{Re} x), \quad (\zeta, x) \in \mathbf{C}^n \times \mathbf{C},$$

with $\operatorname{Re} x < 0$, *be the associated supporting functions for* $\{\Omega_{\operatorname{Re} x}\}_{\operatorname{Re} x<0}$. *Then the functions* $u(\zeta, x)$ *are plurisubharmonic in all the variables* $(\zeta, x) \in \mathbf{C}^n \times \mathbf{C}$ *with* $\operatorname{Re} x < 0$. *Furthermore,* $u(\zeta, x) = u(\zeta, \operatorname{Re} x)$ *and* $u(\zeta, 0) = \lim_{x \uparrow 0} u(\zeta, x) = h_\Omega(\zeta)$.

Proof. We reduce the proof to the case of one pole in Theorem 6.6 by fixing $w_1 = 0$ and considering extremal mappings through $w_1 = 0$ and any one of the distinct poles w_j, $2 \leq j \leq m$. Then in each case Theorem 6.6 gives the required result. □

Now set $u(\zeta, x) = u(\zeta, \operatorname{Re} x)$ which is independent of $\operatorname{Im} x$. In we re-label $\operatorname{Re} x$ as x, then we have

$$R_1 e^x \mathbf{B} \subset \Omega_x \subset R_2 e^x \mathbf{B}, \quad x \leq 0,$$

which in turn implies that

$$-\log(1 - R_1 e^x \| \zeta \|) \leq u(\zeta, x) \leq -\log(1 - R_2 e^x \| \zeta \|). \tag{6.13}$$

The partial Fenchel (Legendre) transform of $u(\cdot, x)$ with respect to x is given by

$$\tilde{u}(\zeta, \alpha) = \sup_{x<0}(\alpha x - u(\zeta, x)), \tag{6.14}$$

where $\alpha \in]0, +\infty]$ and $\zeta \in \mathbf{C}^n$.

We know from Kiselman's minimum principle for psh functions [Ki.1], that

$$-\tilde{u}(\zeta, \alpha) = \inf_{x<0}(u(\zeta, x) - \alpha x)\,; \quad \zeta \in \mathbf{C}^n, \tag{6.15}$$

is plurisubharmonic in the open set where it is less than $+\infty$. But (6.13) gives

$$\begin{aligned} &\inf_{x<0}(-\log(1 - R_2 e^x \| \zeta \|) - \alpha x) \leq -\tilde{u}(\zeta, \alpha) \leq \\ &\leq \inf_{x<0}(-\log(1 - R_1 e^x \| \zeta \|) - \alpha x)\,; \alpha \in]0, +\infty], \ \zeta \in \mathbf{C}^n. \end{aligned} \tag{6.16}$$

We calculate

$$I(\alpha) := \inf_{x<0}(-\log(1 - Re^x \| \zeta \|) - \alpha x), \tag{6.17}$$

for all $\alpha \in]0, +\infty]$, $\zeta \in \mathbf{C}^n$ and $R \in \mathbf{R}$. Consider $\frac{d}{dx}(-\log(1 - \beta e^x) - \alpha x)$ with $\beta = R \| \zeta \|$. Setting $\frac{\beta e^x}{1 - \beta e^x} - \alpha = 0$ we obtain

$$-\alpha x = \log \frac{(1 + \alpha)^\alpha R^\alpha \| \zeta \|^\alpha}{\alpha^\alpha}.$$

The net result of this calculation is that

$$I(\alpha) = \begin{cases} +\infty & \text{if } \alpha < 0; \\ 0 & \text{if } \alpha = 0; \\ \log\dfrac{(1+\alpha)^{1+\alpha}(R\|\zeta\|)^{\alpha}}{\alpha^{\alpha}} = \alpha\log\|\zeta\| + K \text{ if } R\|\zeta\| < 1 \text{ and } 0 < \alpha < \dfrac{R\|\zeta\|}{1-R\|\zeta\|}, \\ \text{or if } R\|\zeta\| \geq 1; \\ -\log(1 - R\|\zeta\|) \text{ if } R\|\zeta\| < 1 \text{ and } \alpha \geq \dfrac{R\|\zeta\|}{1-R\|\zeta\|}. \end{cases}$$

Thus for $\|\zeta\| > \dfrac{1}{R}\dfrac{\alpha}{1+\alpha}$ or $\|\zeta\| > \dfrac{1}{R}$ we have $-\tilde{u}(\zeta, \alpha) \leq \alpha\log\|\zeta\| + K_1$ where $K_1 = \log\dfrac{(1+\alpha)^{1+\alpha}R_1^{\alpha}}{\alpha^{\alpha}}$. As a result we have that $-\tilde{u}(\zeta, \alpha) \in \alpha\mathscr{L}$ for all $\zeta \in \mathbf{C}^n$ and all $\alpha \in]0, +\infty]$. On the other hand from (6.15) and (6.16) we see that $-\tilde{u}(\zeta, \alpha) \leq h_{\Omega}(\zeta)$.

THEOREM 6.8. *Let Ω be a bounded linearly convex domain in $\mathbf{C}^n$. Let h_{Ω} be its supporting function. Consider the function $V : \mathbf{C}^n \times]0, +\infty] \to \mathbf{R}_+$ defined by*

$$V_{\alpha}(\zeta) := V(\zeta, \alpha) := \sup(\varphi(\zeta)\,;\, \varphi \in \alpha\mathscr{L},\ \varphi \leq h_{\Omega}).$$

Then $-\tilde{u}(\cdot, \alpha) \in \alpha\mathscr{L}$ and $V_{\alpha}(\zeta) = -\tilde{u}(\zeta, \alpha)$.

Proof. We have already noted before the statement of the theorem that $-\tilde{u}(\cdot\,, \alpha) \in \alpha\mathscr{L}$. Since $-\tilde{u}(\zeta, \alpha) \leq h_{\Omega}(\zeta)$, we deduce that $-\tilde{u}(\zeta, \alpha)$ is a candidate competing in the definition of $V_{\alpha}(\zeta)$ hence we get from this that $-\tilde{u}(\zeta, \alpha) \leq V_{\alpha}(\zeta)$. Applying Zaharyuta's two-constants theorem for analytic functions in [Za.1,2], it is proved as in Momm's [Mo.1] that $x \mapsto \sup_{\zeta\in\Omega^*}(V_{\alpha}(\zeta) - h_x(\zeta))$ is convex function of $x < 0$, from which we obtain the reverse inequality and together with the previous inequality we deduce that $-\tilde{u}(\zeta, \alpha) = V_{\alpha}(\zeta)$.) □

In the general case of finite singularities, we consider the balls $\{\mathbf{B}(w_j, R_j)\}_{j=1}^{m}$ in Ω with centres the singularities $w_1 = 0, w_2, \ldots, w_m$ and radii $R_1, \ldots, R_m$. Let $R_1 = \inf_j R_j$ and $R_2 = \sup_j \mathbf{R}_j$ work for all the points $w_1 = 0, w_2, \ldots, w_m$ so that as before we have

$$R_1 e^x \mathbf{B}(w_j) \subset \Omega_x \subset R_2 e^x \mathbf{B}(w_j), \quad x \leq 0, \quad 1 \leq j \leq m. \tag{6.18}$$

From this inclusion of sets we deduce that

$$-\log(1 - R_1 e^x \| \zeta - \omega_j \|) \leq u(\zeta, x) \leq -\log(1 - R_2 e^x \| \zeta - \omega_j \|), \tag{6.19}$$

where $\omega_j \in \mathcal{H}_{w_j}$ and the $\mathcal{H}_{w_j}$ are the complex hyperplanes dual to the singularities $w_1 = 0, w_2, \ldots, w_m$. The Fenchel (Legendre) transform of $u(\zeta, x)$ with respect to x is given as

$$\tilde{u}(\zeta, \alpha) = \sup_{x<0}(ax - u(\zeta, x)) \tag{6.20}$$

where $\alpha \in]0, +\infty]$ and $\zeta \in \mathbf{C}^n$.

Again as in the previous case, we know from Kiselman's minimum principle for psh functions that

$$-\tilde{u}(\zeta, \alpha) = \inf_{x<0}(u(\zeta, x) - ax) ; \quad \zeta \in \mathbf{C}^n, \tag{6.21}$$

is plurisubharmonic in the open set where it is less than $+\infty$. But (6.19) gives

$$\begin{aligned} &-\inf_{x<0}(-\log(1 - R_2 e^x \| \zeta - \omega_j \|) - \alpha x) \leq -\tilde{u}(\zeta, \alpha) \leq \\ &\leq \inf_{x<0}(-\log(1 - R_1 e^x \| \zeta - \omega_j \|) - \alpha x) ; \alpha \in]0, +\infty], \zeta \in \mathbf{C}^n, \end{aligned} \tag{6.22}$$

where the $\omega_j \in \mathcal{H}_{w_j}$ and $\mathcal{H}_{w_j}$ are the complex hyperplanes dual to the singularities $w_1 = 0, w_2, \ldots, w_m \in W \subset \Omega$. As before for $R \in \mathbf{R}$ we calculate

$$I(\alpha) = \inf_{x<0}(-\log(1 - Re^x \| \zeta - \omega_j \|) - \alpha x \|), \tag{6.23}$$

for all $\alpha \in]0, +\infty]$, $\zeta - \omega_j \in \mathbf{C}^n$. The result even in this case leads to the same conclusion as in Theorem 6.8.

7. Relation between D_Ω and α_{h_Ω}

In this final section we prove our main results by investigating the relationship between the function α_{h_Ω} and the directional derivative $\mathscr{D}_\Omega$ in the case of the pluricomplex Green function $g_\Omega(z, 0)$ with a pole at the origin.

THEOREM 7.1. *Let Ω be a bounded linearly convex domain in $\mathbf{C}^n$ which contains the origin and with boundary $\partial\Omega$. Let $h_{\Omega x} := h_x : \mathbf{C}^n \to \mathbf{R} \cup \{\infty\}$, $x \leq 0$, be the supporting functions of the linearly convex sublevel sets $\Omega_x := \{z \in \Omega ; g_\Omega(z) < x\}$ of the pluricomplex Green function g_Ω of Ω with a pole at the origin. If $V_\alpha : \mathbf{C}^n \to \mathbf{R}_+$, $\alpha > 0$, are the Siciak functions with respect to the class $\alpha\mathscr{L}$, define for all $\zeta \in \Omega^* := \{z \in \mathbf{C}^n ; h_\Omega(z) < \infty\}$ the functions $\alpha_{h_\Omega} : \mathbf{C}^n \times]0, +\infty] \to \mathbf{R}_+$ by*

$$\alpha_{h_\Omega}(\zeta) := \inf(\alpha\,;\, V_\alpha(\zeta) = h_\Omega(\zeta)),\ \alpha \in]0, +\infty],$$

and

$$\mathscr{D}_\Omega(\zeta) := \lim_{x \nearrow 0} \frac{h_\Omega(\zeta) - h_x(\zeta)}{-x} \in [0, \infty].$$

Then

$$\alpha_{h_\Omega}(\zeta) = \|\zeta\| \mathscr{D}_\Omega^{-1}(\zeta) e^{h_\Omega(\zeta)},\ \zeta \in \Omega^*,$$

where $\mathscr{D}_\Omega$ *denotes the directional derivative in the case of the supporting functions for the pluricomplex green function at the point* $z = z(\zeta)$ *which minimizes* $|1 + \zeta \cdot z|$ *for a given* $\zeta \in \Omega^*$. *This is unique if the boundary* $\partial\Omega$ *is of class* C^1 *and the hyperplane* $\zeta \cdot z = 1$ *is the tangent hyperplane to* $\partial\Omega$ *at the point* z.

Proof. Let Ω_x, $x < 0$ be the sublevel sets of a bounded linearly convex domain Ω. Let h_Ω and h_{Ω_x} be the supporting functions of Ω and Ω_x respectively. Consider the directional derivative:

$$\mathscr{D}_\Omega(b) = \lim_{x \uparrow 0} \frac{h_\Omega(b) - h_{\Omega_x}(b)}{-x} \in]0, \infty]$$

in the direction of the vector $b \in \Omega^* \subset \mathbf{C}^{n*} \cong \mathbf{C}^n$. Consider the drop in the level of the supporting function from $x = 0$ to $x < 0$. At the point $z = z(\zeta)$ which minimises $h_\Omega(\zeta) = \sup(-\log|1 + \zeta \cdot z|)$ we have $|1 + \zeta \cdot z| = e^{-h_\Omega(\zeta)}$. For this $z = z(\zeta)$ we have:
(i) $h_\Omega(\zeta) = -\log|1 + \zeta \cdot z|$, $z \in \mathbf{C}^n$,
(ii) $h_{\Omega_x}(\zeta) = -\log|1 + \zeta \cdot z_a|$, $z_a \in \partial\Omega_x$ is the point with $a = \|z - z_a\|$. For the given $\zeta \in \Omega^* \subset \mathbf{C}^{n*} \cong \mathbf{C}^n$ corresponding to the point $z = z(\zeta)$ we also have at the area of contact that $V_\alpha(\zeta) = h_\Omega(\zeta) = \varphi(\zeta)$ and that $\alpha_{h_\Omega}(\zeta) \le \alpha$. The reverse inequality is clear from Theorem 6.8 for the unique choice of $\zeta \in \Omega^*$. Hence $\alpha_{h_\Omega}(\zeta) = \dfrac{\|\zeta\| e^{h_\Omega(\zeta)}}{\mathscr{D}_\Omega(\zeta)}$. To complete the proof we let $\partial\Omega$ be of class C^1 and let Ω have an outward normal vector N at $z_0 \in \partial\Omega$. Then the real tangent plane at z_0 is given by $\mathrm{Re}\bar{N} \cdot (z - z_0) = 0$. The complex tangent plane is $1 + \zeta_0 \cdot z = 0$ equivalently $\bar{N} \cdot (z - z_0) = 0$. Hence $\zeta_0 = -\dfrac{1}{\bar{N} \cdot z_0}\bar{N}$. Next let Ω^* have an exterior normal vector ν at the point $\zeta_0 \in \partial\Omega^*$. The real tangent plane at this point is given by the equation $\mathrm{Re}\,\bar{\nu} \cdot (\zeta - \zeta_0) = 0$. The complex tangent plane has the equation $1 + z_0 \cdot \zeta = 0$ or equivalently $\bar{\nu} \cdot (\zeta - \zeta_0) = 0$. From which we

deduce that $z_0 = \frac{1}{\bar{\nu}\cdot\zeta_0}\cdot\bar{\nu}$. A complex hyperplane which is paralled to $1+\zeta_0\cdot z=0$ and passes through a point $z_1 = z_0 + \lambda N$ with $\mathrm{Re}\,\lambda \geq 0$ and $|\lambda|$ small is disjoint from Ω, (rather $\mathrm{Re}\,\lambda \geq A(\mathrm{Im}\,\lambda)^2$), where A is a positive constant. Such a hyperplane has the equation $1 + y\zeta_0\cdot z = 0$ where $y = \frac{1}{1-\lambda\zeta_0\cdot N} \approx 1 + \lambda\zeta_0\cdot N = 1 - \frac{\lambda}{\bar{N}\cdot z_0}$. So $y\cdot\zeta_0 \in \Omega^*$ for these y. Thus $\mathrm{Re}\,\bar{\nu}(y-1)\zeta_0 \leq 0$ for these y and $\bar{\nu}\cdot\zeta_0 = c(\bar{N}\cdot z_0)$, $c \geq 0$. Hence $N = \theta\bar{\zeta}_0$, $\nu = t\bar{z}_0$, with $t = c\theta$, $\theta = \frac{1}{\overline{N\cdot\zeta_0}}$, $t = \frac{1}{\overline{z_0\cdot\nu}}$. So that

$$\nu = \frac{\|\zeta_0\|}{N\cdot\zeta_0}\cdot\frac{\bar{z}_0}{\|\bar{z}_0\|} = \frac{-N\cdot\bar{z}_0}{\|N\cdot\bar{z}_0\|}\cdot\frac{\bar{z}_0}{\|\bar{z}_0\|}$$

and we see that ν is determined by z_0 and N. Similarly

$$N = \frac{-\nu\cdot\bar{\zeta}_0}{\|\nu\cdot\bar{\zeta}_0\|}\cdot\frac{\bar{\zeta}_0}{\|\bar{\zeta}_0\|}.$$

Since the boundary $\partial\Omega$ is C^1 it follows from Theorem 2.12 that the boundary $\partial\Omega^*$ is also C^1 and hence the points z_0 and ζ_0 correspond uniquely. Thus $\zeta \in \mathbf{C}^n$ for which $z = z(\zeta)$ minimizes is unique. Observe also that since $\zeta_0 = -\frac{1}{\bar{N}\cdot z_0}\cdot\bar{N}$ we can choose a point $w \in \mathbf{C}^n$ on the exterior normal N to get $\zeta = -\frac{1}{\bar{N}\cdot w}\cdot\bar{N}$. Let $\varepsilon > 0$ be given sufficiently small such that

$$\zeta = -\frac{1}{\bar{N}\cdot w}\cdot\bar{N} = \frac{\bar{N}}{-(z_0+\varepsilon\bar{N})\cdot\bar{N}} = \frac{\bar{N}}{-z_0\cdot\bar{N}-\varepsilon}.$$

Then

$$h_\Omega(\zeta) = h_\Omega\Big(\frac{\bar{N}}{-z_0\cdot\bar{N}-\varepsilon}\Big) \nearrow +\infty, \text{ as } \varepsilon \searrow 0,$$

and

$$\alpha_{h_\Omega}(\zeta) = \alpha_{h_\Omega}\Big(\frac{\bar{N}}{-z_0\cdot\bar{N}-\varepsilon}\Big) \nearrow +\infty, \text{ as } \varepsilon \searrow 0.$$

Also we have $\frac{\bar{N}}{-z_0\cdot\bar{N}-\varepsilon} \in \Omega^*$. However, when $\varepsilon = 0$ we see that $\frac{\bar{N}}{-z_0\cdot\bar{N}} \in \partial\Omega^*$. Now from C. O. Kiselman's Theorem 6.3 in [Ki.3] we have $h_\Omega(\zeta) =$

$-\log d_{\Omega^*}(\zeta)$ so we see that $-\log d_{\Omega^*}\left(\frac{\bar{N}}{-z_0 \cdot \bar{N} - \varepsilon}\right) = -\log \frac{\cos\gamma}{|z_0 \cdot \bar{N}|^2}\varepsilon$ where γ is the angle between the exterior normal vector ν at ζ_0 and the vector joining ζ_0 and ζ. One also sees that $d_{\Omega^*}\left(\frac{\bar{N}}{-z_0 \cdot \bar{N}}\right) = 0$. □

Remark 7.2. Let Ω_x, $x < 0$ be the sublevel sets of bounded convex domain Ω. Let H_Ω and H_{Ω_x} be the supporting functions of Ω and Ω_x respectively. Momm defined a type of directional Lelong number as follows:

$$\Delta_\Omega(b) = \lim_{x \uparrow 0} \frac{H_\Omega(b) - H_{\Omega_x}(b)}{-x} \in]0, +\infty], \ b \in \mathbf{S} = \{z \in \mathbf{C}^n ; \|z\| = 1\},$$

which measures the rate of approximation of $\partial\Omega$ by $\partial\Omega_x$, $x < 0$, in the direction of the vector $b \in \mathbf{S}$. We shall compare the drop in the level of the supporting functions from $x = 0$ to $x < 0$ in this case with the case of bounded linearly convex domains Ω with linearly convex sublevel sets Ω_x, $x < 0$. Now consider the case of linearly convex domains Ω with sublevel sets Ω_x, $x < 0$ and their supporting functions h_Ω and h_{Ω_x} respectively. The type of directional Lelong number $\mathscr{D}_\Omega(\zeta) = \lim_{x \nearrow 0} \frac{ae^{-h_\Omega(\zeta)}}{-x}$ in this case. So $\Delta_\Omega(\zeta) = \mathscr{D}_\Omega e^{h_\Omega(\zeta)}$, $\zeta \in \Omega^* \subset \mathbf{C}^{n*} \cong \mathbf{C}^n$, where $a = \|z - z_a\|$, $z_a \in \partial\Omega_x$.

The following two theorems are corollaries of Theorem 7.1. Using the fact from the Remark 7.2 that $\Delta_\Omega = \mathscr{D}_\Omega e^{h_\Omega}$ one can apply Momm's results to obtain

THEOREM 7.3. *Let Ω be a bounded linearly convex domain in $\mathbf{C}^n$ containing the origin with supporting function h defined in* (1.8). *Then there exists a constant $C \geq 1$ such that $\alpha_{h_\Omega}(\zeta) \leq \mathscr{D}_\Omega e^{h_\Omega}(\zeta) \leq C\alpha_{h_\Omega}(\zeta)$, $\zeta \in \Omega^*$.*

THEOREM 7.4. *For a bounded linearly convex domain Ω in $\mathbf{C}^n$ containing the origin and with supporting function h defined in* (1.8), *the following statements are equivalent*:

(i) *$\mathscr{D}_\Omega$ is bounded on the sphere $\mathbf{S}$ which is the boundary of the ball with centre the hypersurface at infinity dual to the origin containing Ω^*.*

(ii) *α_{h_Ω} is bounded on the sphere $\mathbf{S}$ which is the boundary of the ball with centre the hypersurface at infinity dual to the origin containing Ω^*.*

(iii) *There exists a constant $C > 0$ with*

$$\Omega \subset \Omega_x + C(-x)\mathbf{B}, \ x < 0$$

where $\mathbf{B} = \{z \in \mathbf{C}^n ; \|z\| < 1\}$.
(iv) *There is a plurisubharmonic function* v *on* $\mathbf{C}^{n*} \setminus \Omega^*$ *with* $v(\zeta) \le \log\|\zeta\| + O(1)$ *as* $\zeta \to \infty$, $v \le h_\Omega$ *on a neighbourhood of the boundary* $\partial\Omega^*$ *of* Ω^* *and coincides with* h_Ω *on* $\Omega^* \subset \mathbf{P}_n^*(\mathbf{C}) \setminus \mathscr{H}_0$, *where* $\mathscr{H}_0$ *is the complex hyperplane dual to the origin* $0 \in \Omega$.

Remark 7.5. Notice that as in Momm's paper on bounded convex domains, we get a relation between the derivative $\mathscr{D}_\Omega$ and the function α_{h_Ω} in the set where the Siciak extremal function agrees with the supporting function of the bounded linearly convex domain Ω. However, there is one thing which makes the linearly convex case less agreeable: in Momm's case it is enough to consider $\alpha = 1$, the other sets are simply homothetic images of this special case. This is because $\alpha\varphi(\zeta) \le H_\Omega(\zeta) \Leftrightarrow \varphi(\zeta) \le H_\Omega\left(\frac{\zeta}{\alpha}\right) \Leftrightarrow \varphi(\alpha\zeta) \le H_\Omega(\zeta)$, and $\zeta \mapsto \varphi(\alpha\zeta) \in \mathscr{L} \Leftrightarrow \varphi \in \mathscr{L}$. Hence we use the simple fact that H_Ω is positively homogeneous. But h_Ω does not admit such a simple trick. As a result we are forced to consider V_α for each α separately.

In any case, we have that

$$\mathscr{D}_\Omega^{-1}(\zeta) = \frac{\alpha_{h_\Omega}(\zeta)}{\|\zeta\|} e^{-h_\Omega(\zeta)}, \tag{7.1}$$

and

$$\mathscr{D}_\Omega^{-1}(z_0) = \lim_{\zeta \to \zeta_0} \frac{\alpha_{h_\Omega}(\zeta)}{\|\zeta\|} e^{-h_\Omega(\zeta)}, \tag{7.2}$$

where ζ_0 is the point on $\partial\Omega^*$ that corresponds to $z_0 \in \partial\Omega$ assuming that $\partial\Omega$ is C^1 smooth. Perhaps, it would be preferable to consider

$$W_\alpha = \sup(\varphi \in \mathscr{L} ; \varphi \le h_\Omega/\alpha) ;$$

and

$$\alpha_{h_\Omega}(\zeta) = \inf_{\alpha > 0} (\alpha ; W_\alpha(\zeta) = h_\Omega(\zeta)/\alpha).$$

In this case $h_\Omega/\alpha \searrow \iota_{\Omega^*}$ as $\alpha \searrow 0$.

In conclusion, we note that there is a clear analogy between the linearly convex case and Momm's results for bounded convex domains, but it is as yet not satisfactory, because it seems difficult to study the function α_{h_Ω} to get information about $\mathscr{D}_\Omega$, which was Momm's objective.

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