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MICROSCOPIC ASYMPTOTICS FOR SOLUTIONS OF SOME SEMILINEAR ELLIPTIC EQUATIONS

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Dedicated to the memory of Professor Jongsik Kim

1. Introduction

In our previous work [8], we picked up the elliptic equation

$$\begin{cases} -\Delta u = \lambda f(u) e^{u^\alpha} & \text{in } B \equiv \{|x| < 1\} \subset \mathbf{R}^2, \\ u \quad\ = 0 & \text{on } \partial B \end{cases} \tag{1}$$

with the nonlinearity $f(u) \geq 0$ in C^1. We studied the asymptotics of the family $\{(\lambda, u(x))\}$ of classical solutions satisfying

$$\lambda \downarrow 0 \quad \text{and} \quad \|u\|_{L^\infty} \to +\infty. \tag{2}$$

Taking the result by Gidas-Ni-Nirenberg [5] into account, we may assume that the solution is radially symmetric and decreasing in $r = |x|$, i.e.,

$$u = u(|x|) \geq 0,\ u_r < 0\ (0 < r = |x| \leq 1).$$

Furthermore, the coefficient nonlinear term $f(u)$ is supposed to have the polynomial growth. More precisely,

$$f'(u) \geq 0\ (u \gg 1),$$
$$\lim_{u \to +\infty} (\log f)'(u) = 0,$$

and for some $k \in \mathbf{R}$,

$$0 < \liminf_{u \to +\infty} f(u) u^{-k+\alpha-1} \leq \limsup_{u \to +\infty} f(u) u^{-k+\alpha-1} < +\infty. \tag{3}$$

First, the *global asymptotics* is stated as follows.

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PROPOSITION 1 ([8]). *Let* (u, λ) *be a family of solutions of* (1) *with* (2).
1. *If* $0 < \alpha < 1$, *then for any* $x \in B$, $u(x) \to +\infty$ *as* $\lambda \to 0$.
2. *If* $\alpha > 1$, *then* $u(x) \to 0$ *for any* $x \in B \setminus \{0\}$ *as* $\lambda \to 0$.

It is well-known that the solutions are expressed explicitly if $f(u) \equiv 1$ and $\alpha = 1$. In this case the singular limit is explicitly determined as

$$u(x) \to 4\log \frac{1}{|x|} \text{ as } \lambda \downarrow 0.$$

Thus the exponent $\alpha = 1$ is the borderline of the global asymptotics.

Incidentally, the number of solutions for $f(u) = 1$ and $\alpha = 1$ is 0, 1, and 2 according to $\lambda > 2$, $\lambda = 2$, and $0 < \lambda < 2$, respectively. The unique solution for

$$\begin{cases} -\Delta u = 2e^u & \text{in } B, \\ u \quad\ \ = 0 & \text{on } \partial B \end{cases}$$

is given as

$$u(x) = 2\log \frac{2}{1+|x|^2}.$$

This function plays an important role in *microscopic asymptotics* in the following. Henceforth, we suppose that $\alpha > 1$.

PROPOSITION 2 ([8]). *Passing to a subsequence, it holds that*

$$u^\alpha(e^{-\tau/2}y) = u^\alpha(e^{-\tau/2}) + 2\log \frac{2}{1+|y|^2} + o(1) \tag{4}$$

locally uniformly in $y \in \mathbf{R}^2 \setminus \{0\}$, *where* $\tau \to +\infty$ *is taken appropriately.*

The purpose of the present paper is to study the uniformity of (4). When the exponent is in $1 < \alpha < 2$, the following fact has proven in [8] with the aid of o.d.e. approach by Atkinson-Peletier [2].

PROPOSITION 3 ([8]). *In the case of* $f(u) \equiv 1$ *and* $1 < \alpha < 2$, *the uniform convergence in* (4);

$$\sup_{|y| \le e^{\tau/2}} \left| u^\alpha(e^{-\tau/2}y) - u^\alpha(e^{-\tau/2}) - 2\log \frac{2}{1+|y|^2} \right| \to 0$$

never holds for any $\{\tau\}$.

In case of $\alpha > 2$, it is not known whether classical solutions for (1) with (2) exist or not ([1], [2]). The exponent $\alpha = 2$ is considered to be a borderline for the existence. What we want to claim here is that it is also the borderline from the microscopic asymptotic point of view. We shall give a uniform convergence result for this borderline case. The following theorem is the main result of the present paper.

THEOREM 4. *If* $\alpha = 2$ *and*

$$E_0 \equiv \limsup_{\lambda \to 0} \int_B |\nabla u|^2 dx < 6\pi, \tag{5}$$

then the convergence (4) *is locally uniform in* $y \in \mathbf{R}^2$. *In other words, the uniform asymptotics near* $y = 0$ *is exactly expressed as in* (4).

Concerning the existence of such a family, we have the following theorem.

THEOREM 5. *If* $k > 2$ *in* (3), *there exists a family* $\{(\lambda, u(x))\}$ *of classical solutions of* (1) *with* $\alpha = 2$, *satisfying* (2) *and*

$$E \equiv \int_B |\nabla u|^2 dx \to 4\pi. \tag{6}$$

In case of $\alpha < 2$, $E_0 < \infty$ implies that

$$\lambda f(u) e^{u^\alpha} \in L^{1+\varepsilon}(\Omega)$$

for some $\varepsilon > 0$ because of the Trudinger-Moser inequality ([11], [7]) i.e.

$$\sup_v \left\{ \int_\Omega e^{v^2} dx \mid \|\nabla v\|_2^2 \leq 4\pi \right\} \leq C|\Omega|. \tag{7}$$

Consequently the blow-up (2) does not occur by the standard elliptic estimates. In this sense, Theorems 4 and 5 are peculiar to the case $\alpha = 2$.

A similar observation also yields for the case $\alpha = 2$, that

$$\liminf_{\lambda \to 0} \int_B |\nabla u|^2 dx \geq 4\pi,$$

for the solution of (1) with (2). We shall give a more specified estimate (Lemma 8) for the Dirichlet integral for the solution by using the scaling parameter which will be a key estimate to show Theorem 4.

The special case

$$- \Delta u = \lambda u e^{u^2}, \; u > 0 \text{ in } B = \{|x| < 1\} \subset \mathbf{R}^2$$

with

$$u = 0 \text{ on } \partial B$$

is closely related to the Trudinger-Moser inequality and also Carleson-Chang's theorem ([4], see also [6], [10]). However, this case of $k = 2$ is not treated in Theorem 5. We shall pick up such a kind of equations in a forthcoming paper.

For the proof of Theorem 4, we invoke the following uniform estimate by Brezis-Merle [3]. Assume that $\Omega \subset \mathbf{R}^2$ is a bounded domain. Consider a family of solutions to

$$\begin{cases} - \Delta u_n = V_n e^{u_n} & \text{in } \Omega, \\ u_n \quad = 0 & \text{on } \partial\Omega, \end{cases} \tag{8}$$

where $\{V_n\}$ is a given family of functions on Ω.

LEMMA 6 ([3]). *Let* $\{V_n\}$ *be given functions with*

$$\| V_n \|_{L^p(\Omega)} \leq \beta$$

for some $1 < p \leq \infty$ *and* u_n *be a solution of* (8) *in the sense of distribution. Suppose that*

$$\int_\Omega | V_n | e^{v_n} dx \leq \gamma < \frac{4\pi}{p'}, \; p' = p/p - 1 \tag{9}$$

then the solution u_n *is bounded independent of* n*, i.e.,*

$$\| v_n \|_{L^\infty} \leq C(\beta, \gamma, \Omega, p).$$

The smallness assumption (5) in Theorem 4 comes from the assumption (9).

The proof of Theorem 4 is based on that of Proposition 2. In §2 we shall review the latter to perform the former in §3. The proof of Theorem 5 is independent and shall be given in §4.

2. Summary of the Proof of Proposition 2

We take the case $\alpha = 2$ for simplicity. Namely, we consider the smooth solution u of

$$\begin{cases} - \Delta u = \lambda f(u) e^{u^2} & \text{in } B, \\ u \quad = 0 & \text{on } \partial B, \end{cases} \tag{10}$$

where $f(u) \geq 0$ is a C^1 function satisfying (3).

The solution u becomes radially symmetric and has the property that

$$u_r < 0 \ (0 < r = |x| \leq 1).$$

We put the scaling solution $v(r)$ as

$$v(r) = u^2(\gamma r) - u^2(\gamma) \tag{11}$$

for some scaling constant $\gamma \to 0$. This function v is subject to

$$\begin{cases} -\Delta \ v = k(r)e^v - \rho(r) & \text{in } B, \\ v > 0 & \text{in } B, \\ v = 0 & \text{on } \partial B, \\ v < 0 & \text{on } B_{\gamma^{-1}} \setminus B, \end{cases} \tag{12}$$

where we set

$$\begin{cases} k(r) = 2\lambda u(\gamma r) f(u(\gamma r)) e^{u(\gamma)} \gamma^2, \\ \rho(r) = 2\gamma^2 |\nabla u(\gamma r)|^2 \end{cases} \tag{13}$$

and $B_{\gamma^{-1}} = \{x \in \mathbf{R}^2, |x| < \gamma^{-1}\}$.

Writing both equations (10) and (12) into the ODE form, we introduce the transformation $r = e^{-t/2}$, $U(t) = u(r)$ and $V(t) = v(r)$ to get

$$\begin{cases} \ddot{U} + \dfrac{\lambda}{4} f(U) e^{U^2 - t} = 0, \\ U > 0 \ (t > 0), \\ \dot{U} > 0 \ (t > 0), \\ \dot{U} e^{t/2} \to 0 \ (t \to +\infty) \end{cases} \tag{14}$$

and

$$\begin{cases} \ddot{V} + \dfrac{1}{4} K(t) e^{V(t) - t} - 2\dot{U}_\tau^2(t) = 0, \\ V > 0 \ (t > 0), \\ \dot{V} > 0 \ (t > 0), \\ \dot{V} e^{t/2} \to 0 \ (t \to +\infty), \end{cases} \tag{15}$$

where $\tau = -2 \log \gamma$, $U_\tau(t) = U(t + \tau)$ and $K(t) \equiv 2\lambda U_\tau f(U_\tau) e^{U(\tau)^2 - \tau} = k(r)$.

The equation (15) has a representation of the integral equation as

$$V(t) - V(+\infty) + \int_t^\infty (s - t) \frac{K(s)}{4} e^{V(s) - s} ds = 2 \int_t^\infty (s - t) \dot{U}_\tau^2(s) ds. \tag{16}$$

In the proof of Proposition 1 (cf. [8]), the asymptotics

$$\eta \equiv \max_{0\le r\le 1} |ru_r| \to 0 \tag{17}$$

is proven. Since

$$\dot{U}_\tau(t) = -\frac{1}{2} ru_r |_{r=\exp(-\frac{t+\tau}{2})},$$

it holds that

$$\| \dot{U}_\tau \|_{L^\infty(-\tau,\infty)} \to 0. \tag{18}$$

This relation deduces that

$$K(t) \to \text{constant, locally uniformly in } t \in \mathbf{R}. \tag{19}$$

Two cases should be distinguished for the parameter $\tau \to +\infty$ to be specified. Let

$$\begin{aligned} m &= \max_{0\le r\le 1} 2\lambda u(r) f(u(r)) e^{u^2(r)} r^2 \\ &= \sup_{t\in\mathbf{R}} 2\lambda U(t) f(U(t)) e^{U^2(t)-t}. \end{aligned} \tag{20}$$

Case 1: $m \to +\infty$

In this case we can take $\tau \to +\infty$ as

$$K(0) = 2. \tag{21}$$

The asymptotics (18) and (19) imply that

$$\rho \to 0,\ k \to 2 \text{ locally uniformly in } \mathbf{R}^2 \setminus \{0\}. \tag{22}$$

The relations (12) is reduced to

$$\| -\Delta v \|_{L^\infty(1<|y|<R)} = O(1)$$

with

$$v = 0 \text{ on } | y = 1$$

for any $R > 1$. Hence $\{v\}$ never blows-up on $\{|y| \ge 1\}$.

On the other hand, by (22), the boundedness of the equation (12) near ∂B follows and this implies

$$\| v_r \|_{L^\infty(\partial B)} \le C.$$

Therefore, for any $\varepsilon > 0$,

$$\int_{B/B_{\varepsilon/2}} \{k(r)e^{v} - \rho(r)\}dx$$
$$= \int_{B/B_{\varepsilon/2}} (-\Delta v)\,dx$$
$$\leq -\omega_2 v_r(1) \leq C.$$

While by (22),

$$\int_{B/B_{\varepsilon/2}} \{k(r)e^{v} - \rho(r)\}dx$$
$$\geq \int_{B/B_{\varepsilon/2}} \left\{\frac{1}{2}e^{v} - \frac{1}{4}\right\}dx$$
$$\geq \pi \int_{\varepsilon/2}^{\varepsilon} v(r)\,rdr$$
$$\leq C\varepsilon^2 v(\varepsilon).$$

Hence we obtain an apriori estimate for v on $\mathbf{R}^2 \setminus \{0\}$. Together with the equation (12), we may obtain the limit function v_0 as

$$v(r) \rightarrow v_0(r) \text{ locally uniformly on } \mathbf{R}^2 \setminus \{0\}$$

by the Ascori-Arzela theorem.

Finally, the singular limit $v_0(y) = 2\log\dfrac{2}{1+|y|^2}$ is specified through

$$-\Delta v_0 = 2e^{v_0} \text{ in } \mathbf{R}^2 \setminus \{0\}$$

and

$$v_0 \geq 0 \text{ on } |y| \leq 1.$$

Case 2: $m = O(1)$

In this case, we choose $r \rightarrow +\infty$ by

$$U^2(+\infty) = U^2(\tau) + 2\log 2. \tag{23}$$

The condition $m = O(1)$ implies that

$$\|K\|_{L^\infty(-\tau,\infty)} = O(1) \tag{24}$$

and hence by passing to a subsequence,

$$K(0) \rightarrow 2\mu$$

for some $\mu \geq 0$.

From (19), it follows the convergence

$$K(t) \to 2\mu \text{ locally uniformly in } t \in (-\infty, +\infty),$$

while

$$\| v \|_{L^\infty(|y|<e^{\tau/2})} = 2 \log 2$$

holds by (23). Utilizing the elliptic estimate, we see that a subsequence of $\{v\}$ converges locally uniformly in $\mathbf{R}^2 \setminus \{0\}$. The limiting function $v_0(y)$ satisfies

$$-\Delta v_0 = 2\mu e^{v_0} \text{ in } \mathbf{R}^2,$$
$$v_0 = 0 \text{ on } |y| = 1$$

and

$$\| v_0 \|_{L^\infty} = v_0(0) \leq 2 \log 2.$$

The conclusion $v_0(y) = 2 \log \dfrac{2}{1 + |y|^2}$ follows from $\mu = 1$ or equivalently $v_0(0) = 2 \log 2$. However, the right-hand side of (16) is non-negative and $V(+\infty) = 2 \log 2$. Therefore, the dominated convergence theorem implies that

$$0 \leq V_0(t) + \int_t^\infty (s - t) \frac{\mu}{2} e^{V_0(s)-s} ds - 2 \log 2$$
$$= V_0(+\infty) - 2 \log 2$$

for $V_0(t) = v_0(r)$, or equivalently,

$$v_0(0) \geq 2 \log 2.$$

This completes the proof. □

We note that the relation

$$\lambda u(e^{-\tau/2}) f(u(e^{-\tau/2})) e^{u^2(e^{-\tau/2})-\tau} = 1 + o(1)$$

follows from the proof.

3. Proof of Theorem 4

We have to prepare a few lemmas.

LEMMA 7. *The function* $k(|y|)$ *defined by* (13) *satisfies that*

$$\|k\|_{L^p(|y|<1)} = O(1) \text{ for } 1 < p < \infty. \tag{25}$$

Proof. In the case of $m = O(1)$, the uniform estimate (24) holds. Therefore, we have only to consider the case $m \to +\infty$.

Then, $\tau \to +\infty$ is determined through (21), i.e.,

$$2 = K(0) = 2\lambda U(\tau) f(U(\tau)) e^{U^2(\tau)-\tau}.$$

Hence

$$\begin{aligned} K(t) &= 2\lambda U_\tau(t) f(U_\tau(t)) e^{U^2(\tau)-\tau} \\ &= 2\frac{U_\tau(t) f(U_\tau(t))}{U(\tau) f(U(\tau))} \approx \left(\frac{U_\tau(t)}{U(\tau)}\right)^k \end{aligned} \tag{26}$$

by (3).

Writing

$$\frac{U_\tau(t)}{U(\tau)} = 1 + \frac{1}{U(\tau)} \int_0^t \dot{U}_\tau(s)\, ds,$$

we reach

$$0 \leq \frac{U_\tau(t)}{U(\tau)} \leq C(1+t) \quad (t \geq 0) \tag{27}$$

by (18). The conclusion (25) follows from (26), (27), and

$$k(r) = K(t) \text{ for } r = e^{-t/2}.$$

□

LEMMA 8. *For any fixed* $R > 0$, *we have*

$$4\pi \leq \liminf_{\lambda \to 0} \int_{\{Re^{-\tau/2} < |x| < 1\}} |\nabla u|^2 dx. \tag{28}$$

Proof. As is described in the previous section,

$$V(t) \to V_0(t) \text{ locally uniformly in } t \in (-\infty, +\infty)$$

for

$$V_0(t) = 2 \log \frac{2}{1+e^{-t}}.$$

Making use of the elliptic estimate in (15), this implies that

$$\dot{V}(t) \to \dot{V}_0(t) \text{ locally uniformly in } t \in (-\infty, +\infty).$$

Here,

$$\dot{V}(t) = 2U(t+\tau)\dot{U}(t+\tau)$$

and

$$\dot{V}_0(t) = \frac{2e^{-t}}{1+e^{-t}}.$$

Therefore, writing $R = e^{-t/2}$, we obtain

$$-Re^{-\tau/2}u(Re^{-\tau/2})u_r(Re^{-\tau/2}) \to \frac{2R^2}{1+R^2}. \tag{29}$$

The equation (10) deduces that

$$\int_{\{Re^{-\tau/2}<|x|<1\}} |\nabla u|^2 dx = -2\pi Re^{-\tau/2}u(Re^{-\tau/2})u_r(Re^{-\tau/2}) + \int_{Re^{-\tau/2}<|x|<1} \lambda uf(u)e^2 dx. \tag{30}$$

Therefore, combining (29) and (30), we see for $\tilde{R} > R$

$$\begin{aligned} \frac{4\pi\tilde{R}^2}{1+\tilde{R}^2} &\le \liminf_{\lambda\to 0} \int_{\{\tilde{R}e^{-\tau/2}<|x|<1\}} (|\nabla u|^2 - \lambda uf(u)e^{u^2})dx \\ &\le \liminf_{\lambda\to 0} \int_{\{\tilde{R}e^{-\tau/2}<|x|<1\}} |\nabla u|^2 dx \\ &\le \liminf_{\lambda\to 0} \int_{\{Re^{-\tau/2}<|x|<1\}} |\nabla u|^2 dx. \end{aligned} \tag{31}$$

By taking $\tilde{R}$ arbitrarily large, we obtain (28). □

LEMMA 9. *Under the assumption of*

$$\limsup_{\lambda\to 0} \int_B |\nabla u|^2 dx < 6\pi, \tag{32}$$

we have

$$\| v \|_{L^\infty(|y|<R)} = O(1) \tag{33}$$

for any $R > 0$. *Here, the function* $v(|y|)$ *is defined by* (11).

Proof. We prove this lemma by the aid of Lemma 6 in Section 1.

The estimates (28) and (32) imply that

$$\limsup_{\lambda \to 0} \int_{\{|x| < Re^{-\tau/2}\}} |\nabla u|^2 dx < 2\pi \quad \text{for any } R > 0.$$

Hence the function $\rho(|y|)$ introduced in (13) satisfies that

$$\limsup_{\lambda \to 0} \|\rho\|_{L^1(|y|<1)} = 4\pi \int_0^{\gamma^{-1}} |\nabla u(r)|^2 dr < 4\pi. \tag{34}$$

We may suppose that $0 < R \ll 1$ in showing (33). Let us take the functions h_1 and h_2 as

$$-\Delta h_1 = 0 \quad \text{in } |y| < R, \quad h_1 = v \quad \text{on } |y| = R$$

and

$$-\Delta h_2 = -\rho \quad \text{in } |y| < R, \quad h_2 = 0 \quad \text{on } |y| = R.$$

We have already proven that

$$\|v\|_{L^\infty(|y|=R)} = O(1)$$

so that

$$\|h_1\|_{L^\infty(|y|<1)} = O(1) \tag{35}$$

holds by the maximum principle. On the other hand, $\rho \geq 0$ and hence

$$h_2 \leq 0 \text{ in } |y| < R. \tag{36}$$

This implies the estimate

$$\|e^h\|_{L^\infty(|y|<R)} = O(1) \tag{37}$$

for

$$h = h_1 + h_2.$$

Because of (12), the function $w = v - h$ solves that

$$\begin{aligned} -\Delta w &= -\Delta v + \Delta h = -\Delta v + \rho \\ &= ke^v = ke^h e^w \text{ in } |y| < R \end{aligned} \tag{38}$$

and

$$w = 0 \text{ on } |y| = R. \tag{39}$$

Here, Lemma 7 and (37) are utilized to deduce

$$\| ke^h \|_{L^p(|y|<R)} = O(1) \text{ for } 1 < p < \infty.$$

On the other hand we have

$$\begin{aligned} \| ke^h e^w \|_{L^1(|y|<R)} &= \| ke^v \|_{L^1(|y|<R)} \\ &= \| \rho \|_{L^1(|y|<R)} + \int_{\{|y|<R\}} (-\Delta v)\, dx \end{aligned} \tag{40}$$

by (12) and $k, \rho \geq 0$. By Proposition 2, we have

$$v_r(R) = v_{0r}(R) + o(1).$$

Therefore, from (40) and (29) we obtain

$$\begin{aligned} \| ke^h e^w \|_{L^1(|y|<R)} &= \| \rho \|_{L^1(|y|<R)} - 2\pi R v_r(R) \\ &\leq \| \rho \|_{L^1(|y|<1)} - 2\pi R v_{0r}(R) + o(1) \\ &\leq \| \rho \|_{L^1(|y|<1)} + \frac{4\pi R^2}{1 + R^2} + o(1) \end{aligned}$$

for $0 < R < 1$.

Here, we take $R > 0$ sufficiently small to deduce that

$$\limsup_{\lambda \to 0} \| ke^h e^w \|_{L^1(|y|<R)} < 4\pi$$

by (34).

Now, we can apply Lemma 6 for (38) with (29). Then it follows that

$$\| w \|_{L^\infty(|y|<R)} = O(1).$$

However, we have from (36),

$$0 \leq v = w + h_1 + h_2 \leq w + h_1 \text{ in } |y| < R. \tag{41}$$

Consequently (33) follows from (35) and (41). □

We are in position to complete the proof of Theorem 4.

Proof of Theorem 4. As we have shown,

$$v \to v_0 \text{ locally uniformly in } \mathbf{R}^2 \setminus \{0\}$$

so that

$$V_0(+\infty) = 2\log 2 \leq \liminf_{\lambda\to 0} V(+\infty). \tag{42}$$

Furthermore,

$$K(t) \to 2,\ V(t) \to V_0(t) \text{ locally uniformly in } (-\infty, +\infty) \tag{43}$$

and also

$$\| V \|_{L^\infty(t_1,\infty)} = O(1) \text{ for any } t_1 \in \mathbf{R} \tag{44}$$

by Lemma 9. Finally, we have

$$| K(t) | \leq C(1+t)^m \text{ for } t \gg 1 \tag{45}$$

from (26) and (27).

Here, the dominated convergence theorem is utilized to take the limit in (16). We obtain

$$\begin{aligned} 0 &\leq \liminf_{\lambda\to 0} \int_t^\infty 2(s-t)\dot{U}_\tau^2(s)\,ds \leq \limsup_{\lambda\to 0} \int_t^\infty 2(s-t)\dot{U}_\tau^2(s)\,ds \\ &\leq V_0(t) + \int_t^\infty \frac{1}{2}(s-t)e^{V_0(s)-s}ds - \liminf_{\lambda\to 0} V(+\infty) \\ &= V_0(+\infty) - \liminf_{\lambda\to 0} V(+\infty) \leq 0 \end{aligned}$$

by (42). Therefore,

$$\int_t^\infty 2(s-t)\dot{U}_\tau^2(s)\,ds \to 0 \quad (t \in \mathcal{R}). \tag{46}$$

Furthermore,

$$\begin{aligned} 0 &= \lim_{\lambda\to 0} \int_t^\infty 2(s-t)\dot{U}_\tau^2(s)\,ds \\ &= V_0(t) + \int_t^\infty \frac{1}{2}(s-t)e^{V_0(s)-s}ds - \lim_{\lambda\to 0} V(+\infty) \\ &= V_0(+\infty) - \lim_{\lambda\to 0} V(+\infty). \end{aligned}$$

Hence

$$V(+\infty) \to V_0(+\infty). \tag{47}$$

Going back to (16), we have

$$| V(t) - V_0(t) | \leq \int_t^\infty (s-t)2\dot{U}_\tau^2(s)\,ds + | V(+\infty) - V_0(+\infty) |$$

$$+\int_t^\infty (s-t)\left|\frac{K(s)}{4}e^{V(s)}-\frac{1}{2}e^{V_0(s)}\right|e^{-s}ds$$

so that

$$\sup_{t\geq t_1}|V(t)-V_0(t)|\leq\int_{t_1}^\infty (s-t_1)2\dot{U}_\tau^2(s)ds+|V(+\infty)-V_0(+\infty)|$$
$$+\int_{t_1}^\infty (s-t_1)\left|\frac{K(s)}{4}e^{V(s)}-\frac{1}{2}e^{V_0(s)}\right|e^{-s}ds,$$

where $t_1\in\mathcal{R}$.

The first two terms converges to zero by (46) and (47). For the last term, we utilize (43)-(45) and the dominated convergence theorem. Thus we obtain

$$\|V-V_0\|_{L^\infty(t_1,\infty)}\to 0,$$

which means that

$$v\to v_0 \text{ locally uniformly in } \mathbf{R}^2,$$

the desired convergence. □

4. Proof of Theorem 5

The Trudinger-Moser inequality mentioned in Section 1 is expressed as

$$\sup_v\left\{\int_B e^{v^2}dx \mid \|\nabla v\|_2^2\leq 4\pi\right\}\leq C|\Omega|. \tag{48}$$

The constant 4π in (48) is shown to be best possibly by [7]. The following proposition is a slight refinement.

PROPOSITION 10. *For any continuous function* $k(u)\geq 0$ *with*

$$\lim_{u\to+\infty}k(u)=+\infty,$$

there exists a family $\{w\}\subset H_0^1(B)$ *satisfying*

$$w\geq 0,\ \int_B|\nabla w|^2dx<4\pi$$

and

$$\int_B k(w)e^{w^2}dx\to+\infty.$$

This fact is combined with the following lemma proven by Shaw via the Lagrange multiplier principle.

LEMMA 11 ([9]). *Suppose the existence of a non-negative function* $w \in H_0^1(B)$ *such that*

$$\int_B |\nabla w|^2 dx = \gamma < 4\pi.$$

Then, there exists a solution $(\lambda, u(x))$ *for* (10) *such that*

$$\int_B G(u)\, dx = \int_B G(w)\, dx$$

and

$$\int_B |\nabla u|^2 dx \leq \gamma,$$

where

$$G(u) = \int_0^u f(u) e^{u^2} du.$$

The condition (3) with $k > 2$ implies that

$$\lim_{u \to +\infty} f(u)/u = +\infty.$$

If

$$G(u) = k(u) e^{u^2},$$

this means that

$$\lim_{u \to +\infty} k(u) = +\infty.$$

Hence Proposition 10 and Lemma 11 are applicable.

We get a family $\{(\lambda, u(x))\}$ of solutions for (10) satisfying

$$\limsup_{\lambda \to 0} \int_B |\nabla u|^2 dx \leq 4\pi \tag{49}$$

and

$$\int_B G(u) dx \to +\infty. \tag{50}$$

The asymptotics (50) holds only when

$$\| u \|_{L^\infty} \to +\infty .$$

Furthermore, $f'(u) > 0$ for $u \gg 1$ so that there exists a constant $C > 0$ such that

$$G(u) = \int_0^u f(u) e^{u^2} du \leq C u f(u) e^{u^2} \quad (u \geq 0).$$

Therefore,

$$\begin{aligned} \lambda \int_B G(u)\, dx &\leq C \int_B \lambda u f(u) e^{u^2} dx \\ &= C \int_B |\nabla u|^2 dx = O(1) \end{aligned}$$

by (49) and hence

$$\lambda \downarrow 0$$

by (50).

In this way Theorem 5 has been reduced to Proposition 10. For the sake of completeness we show the proof, although it is quite similar to [7].

Proof of Proposition 10. The family is constructed from $W(t) = w(r)$ for $r = e^{-t/2}$. We have

$$\int_0^\infty \dot{W}^2 dt = \frac{1}{4\pi} \int_B |\nabla w|^2 dx$$

and

$$\int_0^\infty k(W) e^{W^2 - t} dt = \frac{1}{\pi} \int_B k(w) e^{w^2} dx .$$

Therefore, the desired relations are reduced to

$$\{W\} \subset AC[0, \infty),\ W(0) = 0,\ W \geq 0, \tag{51}$$

$$\int_0^\infty \dot{W}^2 dt < 1 \tag{52}$$

and

$$\int_0^\infty k(W) e^{W^2 - t} dt \to +\infty ,$$

where AC denotes the set of absolutely continuous functions.

Taking ε sufficiently small, we put

$$\eta_\varepsilon(s) = \min(s, 1 - \varepsilon)$$

and

$$W(t) = \varepsilon^{-1/2}\eta_\varepsilon(\varepsilon t).$$

For this function, the requirement (51) is obvious. The inequality (52) is examined as

$$\int_0^\infty \dot{W}^2 dt = \int_0^\infty \eta_\varepsilon'(s)^2 ds = (1 - \varepsilon)^2 < 1.$$

Finally, we conclude that

$$\int_0^\infty k(W)e^{W^2-t}dt = \int_0^\infty k(\varepsilon^{-1/2}\eta\varepsilon(s))e^{\varepsilon^{-1}\eta_\varepsilon-\eta^{-1}s}\eta^{-1}ds$$
$$\geq k(\eta^{-1/2}(1 - \varepsilon))\int_{1-\eta}^\infty e^{(1-\varepsilon-s)\varepsilon^{-1}}\varepsilon^{-1}ds$$
$$= k(\varepsilon^{-1/2}(1 - \varepsilon)) \to +\infty \text{ as } \varepsilon \downarrow 0.$$

Thus the proof has been completed. □

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