H. Kang and J. K. Seo
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CAUCHY TRANSFORMS ON POLYNOMIAL CURVES AND RELATED OPERATORS

HYEONBAE KANG AND JIN KEUN SEO

1. Introduction and statement of results

Let Γ be a curve in $\mathbf{R}^2$ defined by $y = A(x)$. The Cauchy transform $\mathscr{C}_A$ on Γ is defined by the kernel

$$K(x, y) = \frac{1 + iA(y)}{(x - y) + i(A(x) - A(y))}.$$

When A is a Lipschitz function, the L^2 boundedness of $\mathscr{C}_A$ is well understood and several proofs of it have been produced (cf. [C, CJS, CMM, DJ, M]). If A is a C^1-smooth function, then the local L^2 boundedness of $\mathscr{C}_A$ is also well understood (cf. [FJR]). However, if A is a smooth, not necessarily Lipschitz function, the question of global L^2 boundedness of $\mathscr{C}_A$ has not been settled. In [KS], we observe that $\mathscr{C}_A$ is not, in general, bounded on L^2 if A is a smooth non-Lipschitz function, and prove that $\mathscr{C}_A$ is bounded on L^2 if A is either a polynomial of odd degree or an even polynomial. The purpose of this paper is to give a new proof of it and to extend the result to arbitrary polynomials.

THEOREM. *If A is a polynomial, then the Cauchy transform on the curve $y = A(x)$ is bounded on $L^p(\mathbf{R})$, $1 < p < \infty$.*

In [KS], we used a continuously varying cut-off function to separate the singularities of $K(x, y)$ at $x = y$ and at $x = -y$ and the T1-theorem of David and Journé. In this paper, instead of using a cut-off function, we use a direct decomposition of the Cauchy kernel. If A is a polynomial, then the Cauchy kernel $K(x, y)$ can be decomposed as

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$$K(x, y) = \frac{1 + iA(y)}{(x - y) + i(A(x) - A(y))} = \frac{1}{x - y} + i\frac{P(x, y)}{1 + iQ(x, y)} \tag{1.1}$$

where

$$Q(x, y) = \frac{A(x) - A(y)}{x - y} \quad \text{and } P(x, y) = \frac{A'(y) - Q(x, y)}{x - y}. \tag{1.2}$$

Moreover, if A is an even polynomial, one can easily see that $Q(x, y) = (x + y)R(x, y)$ for some polynomial R. Define an operator T_A by

$$T_A f(x) = \int_{-\infty}^{\infty} \frac{P(x, y)}{1 + iQ(x, y)} f(y)\,dy. \tag{1.3}$$

Then, $\mathscr{C}_A = \mathscr{H} + iT_A$ where $\mathscr{H}$ is the Hilbert transform. We prove that T_A is bounded on $L^p(\mathbf{R})$, $1 < p < \infty$. If A is a polynomial of odd degree, it is easy to prove the L^2 boundedness of T_A since $Q(x, y)$ does not have any zero when $x^2 + y^2$ is large. If A is an even polynomial, then we compare T_A with a linear combination of the Hilbert transform and various operators defined in Section 3. We show that these operators are bounded on $L^p(\mathbf{R})$ and that the difference of T_A and a linear combination of the Hilbert transform and these operators can be estimated by the Hardy Littlewood maximal operator which is well known to be bounded on L^p (cf. [S]). If A is a polynomial of even degree, then there exists a change of variables $\alpha(x)$ defined for large x such that $A(\alpha(x))$ becomes an even polynomial. By carefully studying the behavior of $\alpha(x)$ for large x, we are able to reduce matters to the case of even polynomials.

We organize this paper as follows; in Section 2, we prove some properties of $Q(x, y)$ which will be used in later sections. In Section 3, we introduce some related operators and prove that they are bounded on $L^p(\mathbf{R})$. In Section 4, we prove that T_A is bounded on $L^p(\mathbf{R})$ if A is a polynomial of odd degree. In the final section, we prove that T_A is bounded on $L^p(\mathbf{R})$ if A is a polynomial of even degree.

We use a standard notation of $A \lesssim B$ to imply that $A \leq CB$ for some constant C. $A \approx B$ means that both $A \lesssim B$ and $A \gtrsim B$ hold.

2. Preliminary on polynomials

Let A be a polynomial and let

$$Q(x, y) = \frac{A(x) - A(y)}{x - y}$$

In this section, we collect some properties of Q which will be used in later sections.

LEMMA 2.1.
(1) *If* $\deg A = 2n + 1$, *then there exists a positive constant* r *such that*

$$|Q(x, y)| \gtrsim x^{2n} + y^{2n}$$

if $x^2 + y^2 \geq r$.

(2) *If* A *is an even polynomial and* $\deg A = 2n + 2$, *then there exist a positive constant* r *and a polynomial* $R(x, y)$ *such that*

$$Q(x, y) = (x + y)R(x, y)$$

and

$$|R(x, y)| \gtrsim x^{2n} + y^{2n}$$

if $x^2 + y^2 \geq r$.

Proof. For (1), note that

$$\begin{aligned}
\frac{x^{2n+1} - y^{2n+1}}{x - y} &= \sum_{j=0}^{2n} x^{2n-j} y^j \\
&= \frac{1}{2}\left(x^{2n} + y^{2n}\right) + \frac{1}{2}(x + y)^2 \sum_{j=1}^{n} x^{2(n-j)} y^{2(j-1)} \\
&\geq \frac{1}{2}(x^{2n} + y^{2n}).
\end{aligned}$$

Let $A(x) = \sum_{j=0}^{2n+1} a_j x^j$ $(a_{2n+1} \neq 0)$. Then

$$\begin{aligned}
\left|\frac{A(x) - A(y)}{x - y}\right| &\geq \frac{|a_{2n+1}|}{2}(x^{2n} + y^{2n}) - \sum_{j=1}^{2n-1} j\,|a_j|\,(|x|^j + |y|^j) \\
&\gtrsim x^{2n} + y^{2n}
\end{aligned}$$

if $x^2 + y^2$ is large.

For (2), we note that

$$\begin{aligned}
|x^{2j} - y^{2j}| &= |x - y|\,|x + y|\,|x^{2j-2} + x^{2j-4}y^2 + \cdots + x^2 y^{2j-4} + y^{2j-2}| \\
&\approx |x - y|\,|x + y|\,(|x|^{2j-2} + |y|^{2j-2}).
\end{aligned}$$

Let $A(x) = \sum_{j=0}^{2n+2} a_j x^{2j}$ $(a_{2n+1} \neq 0)$. Then

$$
\begin{aligned}
|A(x)-A(y)| &\geq |a_{2n+1}|\,|x^{2n+2}+y^{2n+2}| - \sum_{j=1}^{2n} |a_j|\,|x^{2j}-y^{2j}| \\
&\geq |x-y|\,|x+y|\,[|a_{2n+2}|\,(|x|^{2n}+|y|^{2n}) - C(|x|^{2n-2}+|y|^{2n-2}+1)] \\
&\gtrsim |x-y|\,|x+y|\,(|x|^{2n}+|y|^{2n})
\end{aligned}
$$

for some constants C as long as x^2+y^2 is large. This completes the proof.

The next lemma and corollary show that a polynomial of even degree is essentially the same as an even polynomial for our purpose.

LEMMA 2.2. *Let A be a polynomial of even degree. Then, there exist $r>0$ and a smooth change of variable $\alpha(x)$ on $|x|>r$ such that*

(1) *$A(\alpha(x))$ is an even polynomial,*

(2) *$\alpha(x)=x+\beta(x)$ where $\beta(x)=O(1)$ and $\beta'(x)=O(1/x)$ as $x\to\infty$.*

Proof. Let $A(x)=\sum_{j=0}^{2n} a_j x^j$ and assume that $a_{2n}=1$ without loss of generality. Choose $r>0$ so that A is monotone if $|x|>r$ and define α on $|x|>r$ by

$$
A(\alpha(x)) = \frac{A(x)+A(-x)}{2}. \tag{2.1}
$$

Since $A(x)\approx x^{2n}$ and $A'(x)\approx x^{2n-1}$ if $|x|>r$ by increasing r if necessary, one can easily see that $\alpha(x)\approx x$ and $\alpha'(x)\approx 1$. From (2.1), we have

$$
\alpha(x)^{2n} + \sum_{j=0}^{2n-1} a_j\alpha(x)^j = x^{2n} + \sum_{j=0}^{n-1} a_{2j}x^{2j}.
$$

It then follows that

$$
\alpha(x)^{2n} = x^{2n} + O(x^{2n-1})
$$

and

$$
2n\alpha(x)^{2n-1}\alpha'(x) = 2nx^{2n-1} + O(x^{2n-2}).
$$

It follows immediately from these relations that

$$
\alpha(x) = x + O(1) \quad \text{and} \quad \alpha'(x) = 1 + O\Big(\frac{1}{x}\Big) \quad \text{as} \quad x\to\infty.
$$

This completes the proof.

COROLLARY 2.3. *Let β and r be as above. Then,*

$$\left| \frac{\beta(x) - \beta(y)}{x - y} \right| \lesssim \frac{1}{|x| + |y|}$$

if $x^2 + y^2 \geq r$.

Proof. If $xy < 0$, then there is nothing to prove. Suppose that $xy > 0$ and that $y > x > 0$ without loss of generality. If $y > 2x$, then since $\beta(x) = O(1)$ as $x \to \infty$, we have

$$\left| \frac{\beta(x) - \beta(y)}{x - y} \right| \lesssim \frac{1}{y} \leq \frac{1}{|x| + |y|}.$$

If $x < y < 2x$, then since $\beta'(x) = O(1/x)$ as $x \to \infty$, we have

$$\left| \frac{\beta(x) - \beta(y)}{x - y} \right| = |\beta'(\xi)| \lesssim \frac{1}{\xi} \leq \frac{1}{x} \lesssim \frac{1}{|x| + |y|}$$

for some $x < \xi < y$. This completes the proof.

3. Related operators

In this section we introduce some related operators and show that they are bounded on $L^p(\mathbf{R})$ by comparing them with the Hardy Littlewood maximal operator and the Hilbert transform. Throughout this paper M denotes the Hardy Littlewood maximal operator.

PROPOSITION 3.1. *Let $P(x, y)$ and $R(x, y)$ be smooth functions such that there exists a positive constant r so that*

$$|P(x, y)| \lesssim |x|^n + |y|^n \text{ and } |R(x, y)| \gtrsim |x|^n + |y|^n \quad \text{if} \quad x^2 + y^2 \geq r.$$

Suppose that $0 \leq \alpha$ and $0 < \beta - \alpha \leq \gamma$. For $f \in C_0^\infty(\mathbf{R})$, define

$$Uf(x) = \int_{-\infty}^{\infty} \frac{|x - y|^\alpha |P(x, y)|}{1 + |x - y|^\beta |R(x, y)|^\gamma} |f(y)| \, dy. \tag{3.1}$$

If $\gamma \geq 1 + 1/n$, then

$$|Uf(x)| \gtrsim Mf(x) \tag{3.2}$$

for every x.

Proof. For simplicity, we assume that $r = 1$. Write

$$Uf(x) = \int_{|x-y|\leq 1} + \int_{|x-y|>1} := \mathrm{I} + \mathrm{II}.$$

Since $0 < \beta - \alpha$ and $\gamma \geq 1 + 1/n$, we have

$$\begin{aligned}
|\mathrm{II}| &\lesssim \int_{|x-y|>1} \frac{|x-y|^{\alpha}(|x|+|y|)^{n}}{1+|x-y|^{\beta}(|x|+|y|)^{\gamma n}} |f(y)|\,dy \\
&\lesssim \int_{1<|x-y|\leq 1+|x|} \frac{(1+|x|)^{n}}{1+|x|^{\gamma n}} |f(y)|\,dy \\
&\quad + \sum_{j=1}^{\infty} \int_{2^{j-1}(1+|x|)<|x-y|\leq 2^{j}(1+|x|)} \frac{2^{nj}(1+|x|)^{n}}{2^{(\beta-\alpha+\gamma n)j}(1+|x|)^{\beta-\alpha+\gamma n}} |f(y)|\,dy \\
&\lesssim \left(1+\sum_{j=1}^{\infty} 2^{(1+n-\beta+\alpha-\gamma n)j}(1+|x|)^{1+n-\beta+\alpha-\gamma n}\right) Mf(x) \\
&\lesssim Mf(x).
\end{aligned}$$

We now deal with I. If $|x| \leq 1$, then it is obvious that $|\mathrm{I}| \lesssim Mf(x)$. Suppose that $|x| \geq 1$. Then

$$\begin{aligned}
|\mathrm{I}| &\lesssim \sum_{j=1}^{\infty} \int_{2^{-j}<|x-y|<2^{-j+1}} \frac{2^{-\alpha j}(|x|+|y|^{n})}{1+2^{-\beta j}(|x|^{\gamma n}+|y|^{\gamma n})} |f(y)|\,dy \\
&\lesssim \sum_{j=1}^{\infty} \frac{|x|^{n}2^{-(\alpha+1)j}}{1+2^{-\beta j}|x|^{\gamma n}} Mf(x).
\end{aligned}$$

Pick N so that $1 \leq |x|^{n}2^{-N} \leq 10$. Then we obtain

$$\sum_{j=N}^{\infty} \frac{|x|^{n}2^{-(\alpha+1)j}}{1+2^{-\beta j}|x|^{\gamma n}} \leq |x|^{n}2^{-N+1} \leq 20.$$

We also obtain

$$\sum_{j=1}^{N} \frac{|x|^{n}2^{-(\alpha+1)j}}{1+2^{-\beta j}|x|^{\gamma n}} \leq \sum_{j=N}^{N} 2^{(\beta-\alpha-1)j}|x|^{(1-\gamma)n} \lesssim 2^{-(\gamma-1)N}\sum_{j=1}^{N} 2^{(\beta-\alpha-1)j} \leq C$$

because $|x|^{(1-\gamma)n} \leq 2^{-(\gamma-1)N}$ and $\beta - \alpha \leq \gamma$. This completes the proof.

Remark 3.2. The condition on γ in Proposition 3.1 are sharp in the sense that if $\gamma < 1 + 1/n$, then (3.2) does not hold.

PROPOSITION 3.3. *Let $P(x, y)$ be a homogeneous polynomial of degree $2n-1$. Let*

$$Vf(x) = \int_{-\infty}^{\infty} \frac{|P(x, y)|}{x^{2n} + y^{2n}} |f(y)|\, dy \tag{3.3}$$

for $f \in C_0^\infty(\mathbf{R})$. Then, V extends to be an operator bounded on $L^p(\mathbf{R})$, $1 < p < \infty$.

Proof. Let $f \in C_0^\infty(\mathbf{R})$. Let

$$V_1 f(x) = \int_{-\infty}^{\infty} \frac{|x|^{2n-1}}{x^{2n} + y^{2n}} |f(y)|\, dy.$$

We will show that

$$|V_1 f(x)| \lesssim Mf(x).$$

Assume $x \neq 0$.

$$\begin{aligned}
|V_1 f(x)| &\leq |x|^{2n-1}\Big(\int_{|y| \leq 2|x|} + \sum_{j=1}^{\infty} \int_{2^{j-1}|x| < |x-y| \leq 2^j|x|} \frac{1}{x^{2n} + y^{2n}} |f(y)|\, dy\Big) \\
&\lesssim |x|^{2n-1}\Big(\int_{|x-y| \leq 3|x|} x^{-2n} |f(y)|\, dy + \sum_{j=1}^{\infty} \int_{|x-y| \leq 2^j|x|} (2^j x)^{-2n} |f(y)|\, dy\Big) \\
&\lesssim \Big(1 + \sum_{j=1}^{\infty} 2^{j(1-2n)}\Big) Mf(x) \lesssim Mf(x).
\end{aligned}$$

Hence V_1 extends to be an operator bounded on $L^p(\mathbf{R})$, $1 < p < \infty$.

Since $|P(x, y)| \lesssim |x|^{2n-1} + |y|^{2n-1}$, we have

$$|Vf| \lesssim V_1 f + V_1^* f$$

where V_1^* is the adjoint of V_1. Hence Proposition 3.3 follows from L^p-boundedness of V_1 and V_1^*. This completes the proof.

COROLLARY 3.4. *Let $P(x, y)$ and $R(x, y)$ be homogeneous polynomials of degree $2n$. Assume that*

$$|R(x, y)| \gtrsim x^{2n} + y^{2n}.$$

For $f \in C_0^\infty(\mathbf{R})$, define

$$Wf(x) = \int_{-\infty}^{\infty} \frac{P(x, y)}{(x-y)R(x, y)} f(y)\, dy.$$

Then, W extends to be an operator bounded on $L^p(\mathbf{R})$, $1 < p < \infty$.

Proof. Let $a = P(x, x)/R(x, x)$. Then, we have

$$\frac{P(x, y)}{(x-y)R(x, y)} - \frac{a}{x-y} = \frac{P(x, y) - aR(x, y)}{x-y} \frac{1}{R(x, y)}.$$

Note that $(P(x, y) - aR(x, y))/(x-y)$ is a homogeneous polynomial of degree $2n-1$. Hence Corollary 3.4 follows from the L^p-boundedness of the Hilbert transform and Proposition 3.3. This completes the proof.

4. Cauchy transform I

Recall that T_A is the operator defined by the kernel

$$\frac{P(x, y)}{1 + iQ(x, y)}$$

where

$$Q(x, y) = \frac{A(x) - A(y)}{x-y} \quad \text{and} \quad P(x, y) = \frac{A'(y) - Q(x, y)}{x-y}. \tag{4.1}$$

and that

$$\mathscr{C}_A = \mathscr{H} + iT_A$$

where $\mathscr{H}$ is the Hilbert transform. We now prove that the operator T_A is bounded on $L^p(\mathbf{R})$. We first deal with the case when $\deg A$ is odd in this section.

THEOREM 4.1. *Let $P(x, y)$ and $Q(x, y)$ be polynomials of degree $2n-1$ and $2n$, respectively. Assume that there exists a constant r such that*

$$| Q(x, y) | \gtrsim x^{2n} + y^{2n}$$

if $x^2 + y^2 \geq r$. For $f \in C_0^\infty(\mathbf{R})$, define

$$Tf(x) = \int_{-\infty}^{\infty} \frac{P(x, y)}{1 + iQ(x, y)} f(y) dy.$$

Then, T extends to be an operator bounded on $L^p(\mathbf{R})$, $1 < p < \infty$.

Proof. Let q be the conjugate of p, and let

$$k(x, y) = \frac{P(x, y)}{1 + iQ(x, y)}.$$

Then

$$\int |Tf(x)|^p dx \lesssim \int_{|x| \leq r} |Tf(x)|^p dx + \int_{|x|>r} |Tf(x)|^p dx := \mathrm{I} + \mathrm{II}.$$

If $|x| \leq r$, then

$$\int_{-\infty}^{\infty} |k(x, y)|^q dy \lesssim 1 + \int_{|y|>r} \frac{y^{(2n-1)q}}{1 + y^{2qn}} dy \leq C.$$

It then follows from the Hölder inequality that

$$I \leq \int_{|x| \leq r} \left(\int |k(x, y)|^q dy \right)^{p/q} dx \, \|f\|_p^p \lesssim \|f\|_p^p.$$

For II, observe that if $|x| > r$, then

$$|Tf(x)| \lesssim \int_{-\infty}^{\infty} \frac{|x|^{2n-1} + |y|^{2n-1}}{x^{2n} + y^{2n}} |f(y)| \, dy.$$

Hence, it follows from the proof of Proposition 3.3 that

$$\int_{|x|>r} |Tf(x)|^p dx \lesssim \|f\|_p^p.$$

This completes the proof.

COROLLARY 4.2. *If A is a polynomial of odd degree, then the Cauchy transform $\mathscr{C}_A$ is bounded on $L^p(\mathbf{R})$, $1 < p < \infty$.*

Proof. It follows from Theorem 4.1 and Lemma 2.1.

5. Cauchy transform II

In this section we prove that if A is a polynomial of even degree, then the operator T_A is bounded on L^p. We first deal with the case when A is an even polynomial.

THEOREM 5.1. *Suppose that $P(x, y) = P_0(x, y) + P_1(x, y)$ and $R(x, y) = R_0(x, y) + R_1(x, y)$ satisfy the following*

(1) *$P_0(x, y)$ and $R_0(x, y)$ are homogeneous polynomial of degree $2n$,*
(2) *$|P_1(x, y)| \lesssim |x|^{2n-1} + |y|^{2n-1}$ and $|R_1(x, y)| \lesssim |x|^{2n-1} + |y|^{2n-1}$ if $x^2 + y^2 > r$ for some r,*
(3) *$|R(x, y)| \gtrsim |x|^{2n} + |y|^{2n}$ if $x^2 + y^2 > r$.*

For $f \in C_0^\infty(\mathbf{R})$, define

$$Tf(x) = \int_{-\infty}^{\infty} \frac{P(x, y)}{1 + i(x-y)R(x, y)} f(y)\,dy.$$

Then, T extends to be an operator bounded on $L^p(\mathbf{R})$, $1 < p < \infty$.

Proof. By a straightforward computation, we have

$$\left| \frac{P(x, y)}{1 + i(x-y)R(x, y)} - \frac{P_0(x, y)}{1 + i(x-y)R_0(x, y)} \right|$$

$$\leq \frac{|P(x, y) - P_0(x, y)|}{1 + |x-y|^2 |R_0(x, y)R(x, y)|} + \frac{|R_0(x, y)P(x, y) - P_0(x, y)R(x, y)|}{1 + |x-y||R_0(x, y)R(x, y)|}.$$

Note that $|P(x, y) - P_0(x, y)| \lesssim |x|^{2n} + |y|^{2n}$, $|R_0(x, y)P(x, y) - P_0(x, y)R(x, y)| \lesssim |x|^{4n-1} + |y|^{4n-1}$, and $|R_0(x, y)R(x, y)| \gtrsim |x|^{4n} + |y|^{4n}$ if $|x| + |y|$ is large. It follows from Proposition 3.1 that

$$\int_{-\infty}^{\infty} \left| \frac{P(x, y)}{1 + i(x-y)R(x, y)} - \frac{P_0(x, y)}{1 + i(x-y)R_0(x, y)} \right| |f(y)|\,dy \lesssim Mf(x).$$

Hence we may assume $P(x, y)$ and $R(x, y)$ are homogeneous polynomials of degree $2n$. Note that

$$\frac{P(x, y)}{1 + i(x-y)R(x, y)} = \frac{P(x, y)}{1 + (x-y)^2 R(x, y)^2} + i\frac{(x-y)R(x, y)P(x, y)}{1 + (x-y)^2 R(x, y)^2}.$$

The operator defined by the first kernel on the right hand side is proved to be bounded on $L^p(\mathbf{R})$ in Proposition 3.1. Let

$$T_1 f(x) = \int_{-\infty}^{\infty} \frac{(x-y)R(x, y)P(x, y)}{1 + (x-y)^2 R(x, y)^2} f(y)\,dy.$$

We compare T_1 with the operator W defined in Corollary 3.4 with the same P and R. Let $T_2 f(x) = T_1 f(x) - Wf(x)$. Then

$$T_2 f(x) = \int_{-\infty}^{\infty} \frac{P(x, y)}{(x-y)R(x, y)(1 + (x-y)^2 R(x, y)^2)} f(y)\,dy.$$

With $a = P(x, x)/R(x, x)$, define

$$T_3 f(x) = T_2 f(x) - a\mathscr{H} f(x)$$

where $\mathscr{H}$ is the Hilbert transform. Then

$$\frac{1}{x-y}\left(\frac{P(x, y)}{R(x, y)(1 + (x-y)^2 R(x, y)^2)} - a\right)$$
$$= \frac{E(x, y)}{R(x, y)(1 + (x-y)^2 R(x, y)^2)} - a\frac{(x-y)R^2(x, y)}{1 + (x-y)^2 R^2(x, y)}$$

where $E(x, y) = (P(x, y) - aR(x, y))/(x-y)$ is a polynomial of degree $2n-1$. Hence, by Proposition 3.3, it suffices to show L^p-boundedness of the operator T_4 defined by

$$T_4 f(x) = \int_{-\infty}^{\infty} \frac{(x-y)R^2(x, y)}{1 + (x-y)^2 R^2(x, y)} f(y)\, dy.$$

Let $\varphi(x) = |x|^{-2n}$ if $|x| \geq 1$ and $\varphi(x) = 1$ if $|x| < 1$. By similar estimates as above and Proposition 3.1, we obtain

$$\left| T_4 f(x) - \int_{|x-y| \geq \varphi(x)} \frac{f(y)}{x-y}\, dy \right| \lesssim Mf(x).$$

Note that

$$\left| \int_{|x-y| \geq \varphi(x)} \frac{f(y)}{x-y}\, dy \right| \lesssim \sup_{\varepsilon > 0} \left| \int_{|x-y| \geq \varepsilon} \frac{f(y)}{x-y}\, dy \right|.$$

Since the right hand side is bounded on L^2 (cf. p. 42, [S]), the proof is completed.

COROLLARY 5.2. *If A is an even polynomial, then the Cauchy transform $\mathscr{C}_A$ is bounded on $L^p(\mathbf{R})$, $1 < p < \infty$.*

Proof. It follows from Theorem 5.1 and Lemma 2.1.

We now deal with the case when A is a polynomial of even degree.

THEOREM 5.3. *Let A be a polynomial of even degree. Then, the operator T_A is bounded on $L^p(\mathbf{R})$, $1 < p < \infty$.*

Proof. Let $A(x) = \sum_{j=0}^{2n+2} a_j x^j$ and let r be the number given in Lemma 2.2. It suffices to prove

$$\mathrm{I} := \int_{|x|>r} \left| \int_{|y|>r} \frac{P(x, y)}{1 + iQ(x, y)} f(y) dy \right|^p dx \lesssim \| f \|_p^p.$$

In fact, the rest cases can be treated by the Hölder inequality since $| Q(x, y) | \approx | x |^{2n+1} + | y |^{2n+1}$ if either $| x | < r$ and $| y | > 2r$, or $| x | > 2r$ and $| y | < r$. In order to estimate I, we make changes of variables $y = \alpha(s)$ and $x = \alpha(t)$ defined in Lemma 2.2. Then, since $\alpha'(s) \approx 1$, we have

$$\mathrm{I} \lesssim \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} \frac{P(\alpha t), \alpha(s))}{1 + iQ(\alpha(t), \alpha(s))} F(s) ds \right|^p dt,$$

where $F(s) = f(\alpha(s))\alpha'(s)\chi$ while χ is the characteristic function on $\{| \alpha(s) | > r\}$. Let $B(t) = A(\alpha(t))$. Then, $B(t)$ is an even polynomial and

$$Q(\alpha(t), \alpha(s)) = \frac{B(t) - B(s)}{\alpha(t) - \alpha(s)}.$$

Since $\alpha(t) = t + O(1)$, we have

$$P(\alpha(t), \alpha(s)) = P_0(t, s) + E(t, s)$$

where $P_0(t, s)$ is a homogeneous polynomial of degree $2n$ and $E(t, s) = O(| t |^{2n-1} + | s |^{2n-1})$ if $| t | + | s |$ is large. Since

$$| Q(\alpha(t), \alpha(s)) | = \left| \frac{B(t) - B(s)}{\alpha(t) - \alpha(s)} \right| \approx | t + s | (| t |^{2n} + | s ^{2n})$$

for $| t | + | s |$ large, it is already proved in Proposition 3.1 that the operator defined by the kernel $E(t, s)/[1 + iQ(\alpha(t), \alpha(s))]$ is bounded on L^p. For convenience, put

$$k_0(t, s) = \frac{P_0(t, s)}{1 + iQ(\alpha(t), \alpha(s))} \quad \text{and} \quad k(t, s) = \frac{P_0(t, s)}{1 + i\dfrac{B(t) - B(s)}{t - s}}.$$

Then, $k(t, s)$ defines an operator bounded on L^p by Theorem 4.1. A straightforward computation gives

$$k_0(t, s) - k(t, s) = \frac{iP_0(t, s)Q(\alpha(t), \alpha(s)) \dfrac{\beta(t) - \beta(s)}{t - s}}{[1 + iQ(\alpha(t), \alpha(s))]\left[1 + i\dfrac{B(t) - B(s)}{t - s}\right]}$$

where $\alpha(t) = t + \beta(t)$ as defined in Lemma 2.1. It then follows from Corollary 2.3 that

$$|k_0(t, s) - k(t, s)| \lesssim \frac{(|t|^{2n} + |s|^{2n})(|t|^{2n} + |s|^{2n})\,|t+s|\,\frac{1}{|t|+|s|}}{1 + |t+s|^2(|t|^{2n} + |s|^{2n})^2}$$

$$\lesssim \frac{|t+s|\,(|t|^{4n-1} + |s|^{4n-1})}{1 + |t+s|^2(|t|^{2n} + |s|^{2n})^2}.$$

Hence, $|k_0(t, s) - k(t, s)|$ defines an operator bounded on L^p by Proposition 3.1. This completes the proof.

Finally, we have the main theorem of this paper.

THEOREM. *If A is a polynomial, then the Cauchy transform on the curve $y = A(x)$ is bounded on $L^p(\mathbf{R})$, $1 < p < \infty$.*

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Hyeonbae Kang
Department of Mathematics
Korea University
Seoul 136-701, Korea
e-mail: kang@kuccnx.korea.ac.kr

Jin Keun Seo
Department of Mathematics
POSTECH
P. O. Box 125, Pohang 790-600, Korea
e-mail: *seoj@posmath.postech.ac.kr*

Current address of J. K. Seo
Department of Mathematics
Yonsei University
Seovl 120-749, Korea
e-mail: *seoj@bubble.yonsei.ac.kr*