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LOCAL SOLVABILITY AND HYPOELLIPTICITY FOR PSEUDODIFFERENTIAL OPERATORS OF EGOROV TYPE WITH INFINITE DEGENERACY

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Dedicated to Professor Yoshio Kato on his 60th birthday

Introduction and results

Let P be a pseudodifferential operator of the form

$$P = D_t + it^s(D_{x_1} + f(t)x_1^b \mid D_x \mid) \quad \text{in } \boldsymbol{R}_t \times \boldsymbol{R}_x^n, \tag{1}$$

where s, $b \geq 0$ are even integers and $f(t) \in C^\infty$ is odd function with $f'(t) > 0$ $(t \neq 0)$. Here $|D_x|^2 = \sum_{j=1}^n D_{x_j}^2$. We shall call P an operator of Egorov type because P with $f(t) = t^k$, (k odd) is an important model of subelliptic operators studied by Egorov [1] and Hörmander [3], [4, Chapter 27]. Roughly speaking, any subelliptic operator can be reduced to this operator or Mizohata one after several steps of microlocalization arguments. In this paper we shall study the hypoellipticity of P and the local solvability of adjoint operator P^* in the case where $f(t)$ vanishes infinitely at the origin and moreover consider the case where t^s and x_1^b are replaced by functions with zero of infinite order. Our result is not general theory (see Theorems below), but there seems few literature that treats operators of Egorov type with infinite degeneracy. In the preceding paper [11], the hypoellipticity of P in case of either $s = 0$ or $b = 0$ was studied in a little more general situation. It seems hard to consider the general situation corresponding to the case of both s, $b > 0$. Lerner [7] recently has proved that L^2-*a priori* estimate can not hold for some infinitely degenerate version of operator of Egorov type though it satisfies $(\bar{\Psi})$ condition given by Nirenberg-Treves [13]. We remark that P satisfies $(\bar{\Psi})$ from the assumptions of s, b and $f(t)$. It is known by Moyer [12] (in two

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dimension case) and Hörmander [4, Theorem 26.4.7] (in general case) that Nirenberg-Treves (Ψ) condition is necessary for pseudodifferential operators of principal type to be locally solvable. On the other hand, Lerner's example says (Ψ) is not sufficient for L^2 local solvability. We hope the research in the present paper might contribute to understanding the local solvability for operators of principal type. About further historical remarks of operators of principal type and subelliptic operators we refer to Hörmander [4, Chapter 26, 27].

Let us state results, precisely.

THEOREM 1. *Let P be the above operator of the form* (1) *with*

$$f(t) = \int_0^t \exp(-|\sigma|^{-\kappa})d\sigma, \quad \kappa > 0. \tag{2}$$

Then for any $\varepsilon > 0$ and any compact set $K \subset \boldsymbol{R}^{n+1}_{t,x}$ there exists a constant $C = C_{\varepsilon,K}$ such that

$$\begin{aligned}\|(\log \Lambda)^{1/\kappa}u\| + \|(\log \Lambda)^{1+1/\kappa-\varepsilon}\chi(t(\log \Lambda)^{1/\kappa})u\| + \|D_t u\| \\ \leq C(\|Pu\| + \|u\|) \quad \text{for} \quad u \in C_0^\infty(K),\end{aligned} \tag{3}$$

where $\Lambda^2 = 2 + |D_x|^2$ and $\chi(t) \in C^\infty$ satisfies $\chi(t) = 0$ for $t \leq 2$ and $\chi(t) = 1$ for $t \geq 3$. Therefore, adjoint operator P^* is locally solvable. Furthermore, if $\kappa < s + 1$ then P is hypoelliptic.

A simple modification of the proof of Theorem 1 gives a slight generalization of Theorem 1 as follows:

THEOREM 2. *Let $g_\nu(t) = |t|^{|\log|t||^\nu}$ for $\nu > 0$. Let P be the operator of the form* (1) *with t^s and x_1^b replaced by $g_{\nu'}(t)$ $(0 < \nu' < 1)$ and $g_{\nu''}(x_1)$ $(\nu'' > 0)$, that is,*

$$P = D_t + ig_{\nu'}(t)(D_{x_1} + f(t)g_{\nu''}(x_1)|D_x|) \text{ in } \boldsymbol{R}_t \times \boldsymbol{R}^n_x. \tag{4}$$

Let $f(t)$ be C^∞ — odd function and satisfy for some $\nu_0 > 0$

$$|t|^{|\log|t||^{\nu_0}}\exp(-|t|^{-\kappa}) \leq f'(t) \leq |t|^{-|\log|t||^{\nu_0}}\exp(-|t|^{-\kappa}),\ \kappa > 0 \tag{5}$$

in a neighborhood of the origin. Assume that $f'(t)$ is monotone in each half axis. Then we have the estimate (3) *whth $\|D_t u\|$ in the left hand side replaced by $\|D_t\chi(|D_t|/\Lambda)u\|$, that is,*

$$\begin{aligned}\|(\log \Lambda)^{1/\kappa}u\| + \|(\log \Lambda)^{1+1/\kappa-\varepsilon}\chi(t(\log \Lambda)^{1/\kappa})u\| + \|D_t\chi(|D_t|/\Lambda)u\| \\ \leq C(\|Pu\| + \|u\|) \quad \text{for} \quad u \in C_0^\infty(K).\end{aligned} \tag{3$'$}$$

Hence adjoint operator P^* is locally solvable and moreover P is hypoelliptic.

The proof of Theorem 2 admits to treat the operator of principal type whose imaginary part vanishes on the interval.

COROLLARY. *Let P be the same operator as in Theorem 2 with $g_{\nu'}(t)$ replaced by $H(t)g_{\nu'}(t)$, where $H(t)$ is Heaviside function. Then P^* is locally solvable.*

By arranging proofs of Theorems 1 and 2 we obtain a further generalization concerning the local solvability: Let $\tilde{P}$ be a pseudodifferential operator of the form

$$\tilde{P} = D_t + i\alpha(t)(D_{x_1} + f(t)g(x_1)\,|\,D_x\,|) \quad \text{in} \quad \boldsymbol{R}_t \times \boldsymbol{R}^n_x, \tag{6}$$

where $f(0) = 0$, and $\alpha(t)$, $g(t)$ and $f'(t)$ are even C^∞-functions, monotone in each half axis, and they satisfy $\alpha(0) = g(0) = f'(0) = 0$ and $\alpha(t), g(t), f'(t) > 0$ for $t \neq 0$. Setting $\Phi(t) = \log f'(t)$ we assume that if $t > 0$ small

$$C_1 t\Phi'(t) \geq |\,\Phi(t)\,|^{\theta_0} \quad \text{for} \quad \exists\,\theta_0 > 0,\ \exists\,C_1 > 0, \tag{7}$$

$$\Phi'''(t) \geq 0, \quad |\,\Phi'(t)\,|^2 > 2\,|\,\Phi''(t)\,|\,|\,\Phi(t)\,|^{\theta_1} \text{ for } 0 < \exists\,\theta_1 < \theta_0, \tag{8}$$

and moreover

$$\frac{\alpha'(t)}{\alpha(t)} \leq C_2 \frac{(\log|\,\Phi(2t)\,|)^{\theta_2}}{t} \quad \text{for} \quad 0 < \exists\,\theta_2 < 1 \quad \text{and} \quad \exists\,C_2 > 0, \tag{9}$$

$$C_3\alpha(t)g(t\alpha(t)\,|\,\Phi(t)\,|^{-\theta_3})\exp(|\,\Phi(t)\,|^{\theta_1}) \geq t^{-2}\,|\,\Phi(t)\,|^{2\theta_3} \quad \text{for} \quad 0 < \exists\,\theta_3 \leq \theta_0 - \theta_1 \quad \text{and} \quad \exists\,C_3 > 0. \tag{10}$$

THEOREM 3. *Let $\tilde{P}$ be the above operator of the form* (6) *satisfying conditions* (7)-(10). *Then adjoint operator $\tilde{P}^*$ is locally solvable.*

It should be noted that the condition (7) forces $f'(t)$ vanish infinitely at the origin. If $f(t)$ is of Theorem 1 and $\alpha(t) = g_{\nu'}(t)$, $g(x_1) = g_{\nu''}(x_1)$ for those of Theorem 2, then all assumptions (7)-(10) of Theorem 3 are satisfied. Though assumptions (7)-(10) are rather complicated, they are fulfilled by other many functions with zero of infinite order at the origin, for example,

$$\begin{aligned} f'(t) &= \exp(-(\exp|\,t\,|^{-\kappa_1})), \quad \alpha(t) = \exp(-|\,t\,|^{-\kappa_2}), \\ g(x_1) &= \exp(-|\,x_1\,|^{-\kappa_3}) \text{ if } \kappa_1 > \kappa_2 > 0 \quad \text{and} \quad \kappa_3 > 0, \end{aligned} \tag{11}$$

where we can take $\theta_0 = 1$, θ_1 arbitrarily close to 1, $\theta_2 > \kappa_2/\kappa_1$ and $\theta_3 < \theta_1/\kappa_3$.

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1. Proof of Theorem 1

Let $p(t, x, \tau, \xi)$ denote the symbol of P and let $\mathrm{Char}\, P = \{(t, x, \tau, \xi) \in T^*(\boldsymbol{R}^{n+1})\backslash 0\,;\, p = 0\}$. The local solvability of P^* and the hypoellipticity of P are obvious in the region $\{t \neq 0\}$. In fact, the Poisson bracket $\{\mathrm{Re}\, p, \mathrm{Im}\, p\}$ does not vanish on $\mathrm{Char}\, P$ if $x_1 \neq 0$ and $t \neq 0$. When $x_1 = 0$, $H^{b+1}_{\mathrm{Im}\, p}\, \mathrm{Re}\, p \neq 0$ on $\mathrm{Char}\, P \cap \{t \neq 0\}$, where H_q denotes the Hamilton vector field of q. By means of Egorov-Hörmander theorem we have subelliptic estimate there. Furthermore, subelliptic estimate holds microlocally at the point $((0, x_0), (\tau_0, \xi_{01}, \xi_0')) \in T^*(\boldsymbol{R}^{n+1})\backslash 0$ with $(\tau_0, \xi_{01}) \neq (0, 0)$ since $H^s_{\mathrm{Re}\, p}\, \mathrm{Im}\, p \neq 0$ if $\xi_{01} \neq 0$ and $t = 0$. Consequently, it suffices to prove the theorem microlocally at $\rho_0 = ((0, x_0), (0, \xi_0)) \in T^*(\boldsymbol{R}^{n+1})\backslash 0$ with $\xi_0 = (0, \xi_0')$, $|\xi_0'| = 1$. In order to get a microlocal version of the estimate (3) at ρ_0, we define a microlocalized operator of P at ρ_0 as follows: Let $h(x)$ be a $C_0^\infty(\boldsymbol{R}^n)$ function such that $0 \leq h \leq 1$, $h(x) = 1$ for $|x| \leq 1/5$ and $h(x) = 0$ for $|x| \geq 7/24$. For a $\delta > 0$ we set $h_\delta(x) = h(x/\delta)$ and $H_\delta(x, \xi\,;\, \lambda) = h_\delta(x - x_0) h_\delta(\lambda\xi - \xi_0)$, where $0 < \lambda \leq 1$ is a parameter. For a sufficiently small $\delta_1 > 0$ and a parameter $0 < \lambda \leq 1$ we set

$$
\begin{aligned}
P_\lambda &= D_t + i h_{\delta_1}(x - x_0) t^s (D_{x_1} + f(t) x_1^b |D_x|) h_{\delta_1}(\lambda D_x - \xi_0) \\
&\equiv D_t + i t^s B_\lambda(t, x, D_x).
\end{aligned}
\tag{1.1}
$$

Proposition 1.1. *Let P_λ be the above operator with $f(t)$ given in* (2). *If $\delta_1 > 0$ is small enough then for any $0 < \delta < \delta_1/100$ and any $\varepsilon > 0$ there exists a constant $C = C_{\varepsilon,\delta}$ such that for any $0 < \lambda \leq 1$*

$$
\begin{aligned}
&\| |\log \lambda|^{1/\kappa} u \|^2 + \| |\log \lambda|^{1+1/\kappa-\varepsilon} \chi(2t |\log \lambda|^{1/\kappa}) u \|^2 + \| D_t u \|^2 \\
&\qquad \leq C(\| P_\lambda u \|^2 + \lambda^{-2} \| (1 - H_{20\delta}) u \|^2 + \| u \|^2) \\
&\qquad\qquad \textit{for} \quad u \in C_0^\infty([-\delta_1, \delta_1]\,;\, \mathcal{S}(\boldsymbol{R}^n_x)).
\end{aligned}
\tag{1.2}
$$

Admitting this proposition for a while, we shall prove Theorem 1. Let $u \in C_0^\infty$ with $\mathrm{supp}\, u \subset \{|t| \leq \delta_1\}$ and substitute $h_\delta(\lambda D_x - \xi_0) h_\delta(x - x_0) u$ into (1.2). Since λ^{-1} is equivalent to $|\xi|$ on $\mathrm{supp}\, h(\lambda\xi - \xi_0)$ we have

$$
\begin{aligned}
&\| h_\delta(\lambda D_x - \xi_0) (\log \Lambda)^{1/\kappa} h_\delta(x - x_0) u \|^2 \\
&\quad + \| h_\delta(\lambda D_x - \xi_0) (\log \Lambda)^{1+1/\kappa-\varepsilon} \chi(t (\log \Lambda)^{1/\kappa}) h_\delta(x - x_0) u \|^2
\end{aligned}
$$

$$+ \| h_\delta(\lambda D_x - \xi_0) D_t h_\delta(x - x_0) u \|^2$$
$$\leq C(\| h_\delta(\lambda D_x - \xi_0) P u \|^2 + \| h_{2\delta}(\lambda D_x - \xi_0) u \|^2 + \lambda \| u \|^2).$$

Integrate λ from 0 to 1 after dividing both sides by λ. Then by means of [9, Proposition 1.7] we have

$$\| \psi_\delta(D_x) (\log \Lambda)^{1/\kappa} h_\delta(x - x_0) u \|^2 +$$
$$\| \psi_\delta(D_x) (\log \Lambda)^{1+1/\kappa-\varepsilon} \chi(t(\log \Lambda)^{1/\kappa}) h_\delta(x - x_0) u \|^2 + \tag{1.3}$$
$$\| \psi_\delta(D_x) D_t h_\delta(x - x_0) u \|^2 \leq C(\| Pu \|^2 + \| u \|^2),$$

where $\psi_\delta(\xi)$ is a suitable symbol in $S^0_{1,0}$ such that $\psi_\delta = 1$ in a small conic neighborhood of ξ_0 (see [9; Definition 1.6]). By using the usual partition of unity in the cotangent space, we obtain the estimate (3) because microlocal subelliptic estimates hold in the region except for $\rho_0 = ((0, x_0), (0,0, \xi_0')) \in T^*(\boldsymbol{R}^{n+1})\setminus 0$. The local solvability of P^* is a direct consequence of (3). In fact, noting the Poincaré inequality

$$\| u \| \leq \delta \| D_t u \| \quad \text{if} \quad \operatorname{diam}(\operatorname{supp} u) \leq \delta, \tag{1.4}$$

we have $\| u \| \leq C \| Pu \|$ if $\operatorname{supp} u$ is sufficiently small. For the proof of hypoellipticity of P in the case of $0 < \kappa < s + 1$, it suffices to show the microhypoellipticity of P at ρ_0, that is,

$$\rho_0 \notin \mathrm{WF}\,(Pu) \text{ implies } \rho_0 \notin \mathrm{WF}\,(u) \quad \text{for} \quad \forall u \in \mathscr{D}'(\boldsymbol{R}^{n+1}). \tag{1.5}$$

Since $(1 + 2\varepsilon)\kappa \leq s + 1$ for sufficiently small $\varepsilon > 0$ it follows from (1.2) that if $0 < \lambda \leq 1$

$$| \log \lambda |^{2/\kappa} \| u \|^2 + | \log \lambda |^{2+2\varepsilon} \| t^s u \|^2$$
$$\leq C(\| P_\lambda u \|^2 + \lambda^{-2} \| (1 - H_{20\delta}) u \|^2 + \lambda \| u \|^2), \tag{1.6}$$
$$\text{for} \quad u \in C_0^\infty([-\delta_1, \delta_1]\,; \mathscr{S}(\boldsymbol{R}^n_x)).$$

By means of this estimate we can easily see (1.5) if we employ [11, Theorem 1] as in the proof of [11, Theorem 2]. Indeed, we may assume $x_0 = 0$ by the translation after the estimate (1.6) was obtained. If we set $\Gamma = \operatorname{Char} P \cap \{t = 0\}$ then the hypotheses of [11, Theorem 1] are fulfilled for $\varphi(x, \xi) = (1 - h_{5\delta}(x)) + (1 - h_{5\delta}(\lambda\xi - \xi_0))$ and $\alpha(t, x, \xi) = t^s$. The detail is left to the reader.

We shall prove Proposition 1.1. It suffices to show (1.2) for a sufficiently small $\lambda > 0$. If $t_1 = | \log \lambda |^{-1/\kappa}$ then $f'(t_1)\lambda^{-1} = 1$. For a fixed x_0 there exists a constant $C > 0$ such that

$$h_{\delta_1}(x-x_0)t^s x_1^b\,|f(t)|\,\lambda^{-1} < C \quad \text{if} \quad |t| \le t_1 \tag{1.7}$$

If supp $u \subset \{|t| \le t_1\}$ then we have

$$C(\|P_\lambda u\|^2 + R_\lambda) \ge \|(D_t + it^s D_{x_1})u\|^2,$$

where $R_\lambda = \lambda^{-2}\|(1-H_{20\delta})u\|^2 + \|u\|^2$. Here and in what follows we denote different positive constants independent of λ by the same notation C. Since we have the maximal hypoelliptic estimate

$$C\|(D_t + it^s D_{x_1})u\|^2 \ge \|D_t u\|^2 + \|t^s D_{x_1} u\|^2, \tag{1.8}$$

it follows from the Poincaré inequality that

$$C(\|P_\lambda u\|^2 + R_\lambda) \ge |\log\lambda|^{2/\kappa}\|u\|^2 + \|D_t u\|^2 \quad \text{if} \quad \text{supp}\, u \subset \{|t| \le t_1\}. \tag{1.9}$$

Note that

$$\begin{aligned} \|P_\lambda u\|^2 &= \|D_t u\|^2 + \|t^s B_\lambda u\|^2 \\ &\quad + 2\mathrm{Re}(st^{s-1}B_\lambda u, u) + 2\mathrm{Re}(V_\lambda H_{\delta_1}\lambda|D_x|u, u), \end{aligned} \tag{1.10}$$

where

$$V_\lambda = V_\lambda(t, x_1) = t^s x_1^b f'(t)\lambda^{-1}. \tag{1.11}$$

Since $\lambda|\xi| \ge 1/2$ on supp $H_{\delta_1}(x, \xi; \lambda)$ it follows from the Gårding inequality that

$$2\mathrm{Re}(V_\lambda \lambda|D_x|H_{\delta_1}(x, D_x; \lambda)u, u) \ge (V_\lambda u, u) - CR_\lambda. \tag{1.12}$$

If supp $u \subset \{|t| \ge t_1/2\}$ then we have

$$|(st^{s-1}B_\lambda u, u)| \le \frac{1}{4}\|t^s B_\lambda u\|^2 + C|\log\lambda|^{2/\kappa}\|u\|^2. \tag{1.13}$$

It follows from (1.10), (1.12) and (1.13) that

$$\begin{aligned} \|P_\lambda u\|^2 \ge \frac{1}{2}(\|D_t u\|^2 + \|t^s B_\lambda u\|^2 + (V_\lambda u, u)) - C(|\log\lambda|^{2/\kappa}\|u\|^2 + R_\lambda) \\ \text{if} \quad \text{supp}\, u \subset \{|t| \ge t_1/2\}. \end{aligned} \tag{1.14}$$

If $|t| \le \delta_1$ for δ_1 small enough then it follows from the implicit function theorem that there exist real symbols $0 \ne e(t, x, \xi) \in S^0_{1,0}$ and $r(t, x, \xi') \in S^1_{1,0}$ in a small conic neighborhood Γ_0 of (x_0, ξ_0) such that

$$\xi_1 + f(t)x_1^b|\xi| = e(t, x, \xi)(\xi_1 + r(t, x, \xi')) \quad \text{in} \quad \{|t| \le \delta_1\} \times \Gamma_0.$$

Choosing a sufficiently small $\delta_0 > 0$ again, we have for constants $c_0 > 0$ and

$C > 0$ independent of λ

$$\| B_\lambda u \|^2 \geq c_0 \| (D_{x_1} + \tilde{r}(t, x, D_{x'}))u \|^2 - CR_\lambda \quad \text{for} \quad u \in C_0^\infty([-\delta_1, \delta_1]; \mathcal{S}(\boldsymbol{R}_x^n)), \tag{1.15}$$

where $\tilde{r} = rh_{\delta_1}(x - x_0)\tilde{h}_{\delta_1}(\lambda\xi' - \xi_0')$ and $\tilde{h}(x') \in C_0^\infty(\boldsymbol{R}_{x'}^{n-1})$ is a similar function as $h(x)$.

$$\begin{aligned} \| P_\lambda u \|^2 \geq c_0(\| D_t u \|^2 &+ \| t^s(D_{x_1} + \tilde{r}(t, x, D_{x'}))u \|^2 + (V_\lambda u, u)) \\ &- C(\| \log \lambda \|^{2/\kappa} \| u \|^2 + R_\lambda) \\ &\text{if} \quad \operatorname{supp} u \subset \{\delta_1 \geq |t| \geq t_1/2\}. \end{aligned} \tag{1.16}$$

Here and in what follows we denote different small positive constants independent of λ by the same notation c_0.

In order to get (1.2) we shall see together (1.9) and (1.16) as gaining the positivity from the first three terms of the right hand side of (1.16). To this end we have to consider the behavior of the potential V_λ near $t = t_1$ and a special decomposition of the neighborhood of $t = t_1$. For an $l > 0$ we set

$$\mu = \mu(\lambda) = \frac{(\log|\log\lambda|)^{l+1}}{|\log\lambda|} = \frac{\log|\log\lambda|^{a(\lambda)}}{|\log\lambda|}, \tag{1.17}$$

where we have set $a(\lambda) = (\log|\log\lambda|)^l$. If $\lambda > 0$ is small then $\mu(\lambda) < 1/2$. Setting $t_2 = ((1 + \mu)/|\log\lambda|)^{1/\kappa}$ we have

$$\begin{aligned} f'(t_2)\lambda^{-1} &= \exp\left\{-\frac{|\log\lambda|}{1+\mu} + |\log\lambda|\right\} \\ &= \exp\left\{\frac{\log(|\log\lambda|)^{a(\lambda)}}{1+\mu}\right\}. \end{aligned}$$

Consequently we have

$$f'(t)\lambda^{-1} \geq |\log\lambda|^{a(\lambda)(1-\mu(\lambda))} \quad \text{if} \quad |t| \geq t_2. \tag{1.18}$$

Since $(1 + \mu)^{1/\kappa} = 1 + \mu/\kappa + O(\mu^2)$ we have

$$t_2 - t_1 = O(|\log\lambda|^{-1-1/\kappa}(\log|\log\lambda|)^{l+1}), \tag{1.19}$$

where $R = O(J)$ means $C^{-1} < R/J < C$ for some constant $C > 0$ independent of λ. Set $\gamma_0 = t_1/2$ and $\gamma_j = \gamma_{j-1}/2$ for $j = 1,2,3, \ldots$ inductively. Let N be the smallest integer such that $\gamma_N < 2(t_2 - t_1)$. It follows from (1.19) that

$$N = O(\log|\log\lambda|). \tag{1.20}$$

We take a convention that $\gamma_j = \gamma_N$ for $j \geq N+1$. Setting $t_0 = t_1 - \gamma_0$ we consider the following divisions of intervals $[t_0, t_1]$ and $[t_1, \infty)$: Put $I_1 = [t_0, t_0 + \gamma_1]$ and put $I_{j+1} = [\bar{\gamma}_j, \bar{\gamma}_{j+1}]$ with $\bar{\gamma}_j = t_0 + \sum_{k=1}^{j} \gamma_k$ for $j = 1,2,3,\ldots$. We have

$$[t_0, t_1] = I_1 \cup \cdots \cup I_N \cup I_{N+1} \quad \text{and} \quad [t_1, \infty) = \bigcup_{j=N+2}^{\infty} I_j.$$

Set

$$\mu_j = t_1^s \gamma_j = |\log \lambda|^{-s/\kappa} \gamma_j \tag{1.21}$$

for $j = 1, \ldots, N$ and let $\mu_j = \mu_N$ for $j \geq N+1$. If l is large enough then it follows from (1.11) and (1.18) that

$$V_\lambda(t, x_1) \geq \gamma_j^{-2} \quad \text{on} \quad \{|t| \geq t_2\} \times \{|x_1| \geq \mu_j\}. \tag{1.22}$$

Let Q_j denote a rectangle $I_j \times J_j$ in $\boldsymbol{R}^2_{t,x_1}$, where $J_j = [-\mu_j, \mu_j]$. Let I_j^* denote 4 times dilation of I_j, to the right direction, that is, $I_j^* = [\bar{\gamma}_{j-1}, \bar{\gamma}_j + 3\gamma_j]$ in case of $j = 1, \ldots, N+2$. We take a convention that $I_j^* = I_j$ if $j \geq N+3$. We denote by Q_j^* a rectangle $I_j^* \times [-2\mu_j, 2\mu_j]$ for $j = 1,2,3,\ldots$.

LEMMA 1.2. *If $\mathscr{H}'$ and $\|\cdot\|_{\mathscr{H}'}$ denote a Hilbert space $L^2(\boldsymbol{R}^{n-1}_{x'})$ and its norm, respectively, then there exist positive constants c_0 and C such that*

$$\begin{aligned} &\int_{Q_j^*} \|D_t u\|^2_{\mathscr{H}'} + t^{2s} \|(D_{x_1} + \tilde{r})u\|^2_{\mathscr{H}'} + V_\lambda(t, x_1) \|u\|^2_{\mathscr{H}'} dt dx_1 \\ &\geq c_0 \gamma_j^{-2} \int_{Q_j} \|u\|^2_{\mathscr{H}'} dt dx_1 - C \int_{Q_j^*} (\lambda^{-2} \|(1 - \tilde{H}_{20\delta})u\|^2_{\mathscr{H}'} + \|u\|^2_{\mathscr{H}'}) dt dx_1 \end{aligned} \tag{1.23}$$

$$\text{for} \quad u \in C^1(\boldsymbol{R}_t \times \boldsymbol{R}_{x_1}; \mathscr{H}'),$$

where $\tilde{H}_\delta = \tilde{H}_\delta(x', D_{x'}; \lambda) = \tilde{h}((x' - x_0')/\delta)\tilde{h}((\lambda D_{x'} - \xi_0')/\delta)$.

Proof. Regarding t as a parameter, we take a canonical transformation Φ_t keeping x_1 variable such that

$$\xi_1 + \tilde{r}(t, x, \xi') \to \xi_1.$$

Let U be an elliptic Fourier integral operator in $\boldsymbol{R}^{n-1}_{x'}$ realizing the canonical transformation Φ_t such that $(U(D_{x_1} + \tilde{r}) - D_{x_1}U)\tilde{H}_{40\delta}(x', D_{x'}; \lambda)$ belongs to $Op(S^0_{1,0})$ uniformly with respect to $\lambda > 0$. Note that $\tilde{H}_{40\delta}(\tilde{\Phi}_t(x, \xi); \lambda) = 1$ on supp $\tilde{H}_{20\delta}(x', \xi'; \lambda)$ if $|t|, |x_1| \leq \delta_1$, where $\tilde{\Phi}_t$ denotes the projection of Φ_t into $T^*(\boldsymbol{R}^{n-1}_{x'})$. For the sake of simplicity we denote x_1 variable by y until the end of the proof of this lemma. If $\tilde{u} = U\tilde{H}_{40\delta}u$ and if $|t|, |y| \leq \delta_1$ then

$$\begin{aligned}(1.24)\qquad &\| (D_y + \tilde{r}(t, y, x', D_{x'}))u \|^2_{\mathscr{H}'} \\ &\geq c_0 \| D_y \tilde{u} \|^2_{\mathscr{H}'} - C(\lambda^{-2} \| (1 - \tilde{H}_{20\delta})u \|^2_{\mathscr{H}'} + \| u \|^2_{\mathscr{H}'}) \quad \text{for} \quad u \in \mathscr{H}'.\end{aligned}$$

Furthermore, if $|t|, |y| \leq \delta_1$ then there exists a $C_1 > 0$ such that

$$\begin{aligned}(1.25)\qquad C_1^{-1} \| u \|^2_{\mathscr{H}'} &\leq \| \tilde{u} \|^2_{\mathscr{H}'} + \| (1 - \tilde{H}_{20\delta})u \|^2_{\mathscr{H}'} + \lambda \| u \|^2_{\mathscr{H}'} \\ &\leq C_1 \| u \|^2_{\mathscr{H}'} \quad \text{for} \quad u \in \mathscr{H}'.\end{aligned}$$

We use the following elementary inequality given in [2; p. 148] (see also [10; Lemma 1.1]): If I is an interval in $\boldsymbol{R}_t$ then

$$\int_I | D_t v |^2 dt \geq c_0 \frac{(\operatorname{diam} I)^{-2}}{|I|} \int_{I \times I} | v(t) - v(t') |^2 dt dt' \quad \text{for} \quad v \in C^1(\boldsymbol{R}),$$

where $|I| = \operatorname{diam} I$ denotes the length of I. The absolute value $|\cdot|$ can be replaced by the norm $\|\cdot\|_{\mathscr{H}'}$. Substitute $v(t) = v(t, y) \in C^1(R_t \times R_y; \mathscr{H}')$ and integrate with respect to y on $[-2\gamma_j, 2\gamma_j]$. Since the volume $|Q_j^*|$ of Q_j^* is equal to $8|I_j||J_j| = 8|Q_j|$, we have

$$\int_{Q_j^*} \| D_t u(t, y) \|^2_{\mathscr{H}'} dt dy \geq c_0 \frac{\gamma_j^{-2}}{|Q_j|} \int_{Q_j^* \times Q_j^*} \| u(t, y) - u(t', y) \|^2_{\mathscr{H}'} dt dy dt' dy'.$$

Similarly, we have

$$\int_{Q_j^*} t^{2s} \| D_y \tilde{u}(t, y) \|^2_{\mathscr{H}'} dt dy \geq c_0 \frac{\gamma_j^{-2}}{|Q_j|} \int_{Q_j^* \times Q_j^*} (t')^{2s} \| \tilde{u}(t', y) - \tilde{u}(t', y') \|^2_{\mathscr{H}'} dt dy dt' dy'.$$

Obviously we have

$$\int_{Q_j^*} V_\lambda(t, y) \| u(t, y) \|^2_{\mathscr{H}'} dt dy = \frac{1}{|Q_j|} \int_{Q_j^* \times Q_j^*} V_\lambda(t', y') \| u(t', y') \|^2_{\mathscr{H}'} dt dy dt' dy'.$$

Note that the right hand sides of those formula are positive. If $Q_j^0 = Q_j^* \cap \{|t| \geq t_2\} \times \{|x_1| \geq \mu_j\}$ it follows from (1.21) and (1.22) that

$$\begin{aligned}&\int_{Q_j^*} \| D_t u \|^2_{\mathscr{H}'} + t^{2s} \| D_y \tilde{u} \|^2_{\mathscr{H}'} + V_\lambda(t, y) \| u \|^2_{\mathscr{H}'} dt dy \\ &\geq c_0 \frac{\gamma_j^{-2}}{|Q_j|} \int_{Q_j \times Q_j^0} \{\| u(t, y) - u(t', y) \|^2_{\mathscr{H}'} \\ &\quad + 2C_1 \| \tilde{u}(t', y) - \tilde{u}(t', y') \|^2_{\mathscr{H}'} + 2C_1^2 \| u(t', y') \|^2_{\mathscr{H}'}\} dt dy dt' dy' \\ &\geq c_0 \frac{\gamma_j^{-2}}{|Q_j|} \int_{Q_j \times Q_j^0} \{\| u(t, y) \|^2_{\mathscr{H}'}/2 - \| u(t', y) \|^2_{\mathscr{H}'} + C_1 \| \tilde{u}(t', y) \|^2_{\mathscr{H}'} \\ &\quad - 2C_1 \| \tilde{u}(t', y') \|^2_{\mathscr{H}'} + 2C_1^2 \| u(t', y') \|^2_{\mathscr{H}'}\} dt dy dt' dy',\end{aligned}$$

where C_1 is the constant in (1.25). Since $|Q_j^0|/|Q_j| \geq 1$, we have by (1.25)

$$\int_{Q_j^*} \| D_t u \|_{\mathscr{H}'}^2 + t^{2s} \| D_y \tilde{u} \|_{\mathscr{H}'}^2 + V_\lambda(t, y) \| u \|_{\mathscr{H}'}^2 dtdy$$

$$\geq c_0 \gamma_j^{-2} \int_{Q_j} \| u(t, y) \|_{\mathscr{H}'}^2 dtdy - C\gamma_j^{-2} \Big(\int_{Q_j^*} \| (1 - \tilde{H}_{20\delta}) u(t, y) \|_{\mathscr{H}'}^2 + \lambda \| u(t, y) \|_{\mathscr{H}'}^2 \Big) dtdy.$$

From this and (1.24) we obtain the desired estimate (1.23). The proof of Lemma 1.2 is completed.

Let $\mathscr{H}$ and $\| \cdot \|_{\mathscr{H}}$ denote a Hilbert space $L^2(R_x^n)$ and its norm, respectively. By the similar way as in the proof of Lemma 1.2, it follows from (1.22) that

$$\text{(1.26)} \qquad \begin{aligned} &\int_{I_j^*} \| D_t u \|_{\mathscr{H}'}^2 + (V_\lambda u, u)_{\mathscr{H}} dt \geq c_0 \gamma_j^{-2} \int_{I_j} \| u \|_{\mathscr{H}}^2 dt \\ &\text{for} \quad u \in C^1(\boldsymbol{R}_t; \mathscr{H}) \text{ with } \operatorname{supp} u \cap \{ |x_1| \leq \mu_j \} = \emptyset . \end{aligned}$$

By (1.23) and (1.26) we have

$$\text{(1.27)} \qquad \begin{aligned} &\int_{I_j^*} \| D_t u \|_{\mathscr{H}}^2 + t^{2s} \| (D_{x_1} + \tilde{r}(t, x, D_{x'})) u \|_{\mathscr{H}}^2 + (V_\lambda u, u)_{\mathscr{H}} dt \\ &\geq c_0 \gamma_j^{-2} \int_{I_j} \| u \|_{\mathscr{H}}^2 dt - C \int_{I_j^*} (\lambda^{-2} \| (1 - H_{20\delta}) u \|_{\mathscr{H}}^2 + \| u \|_{\mathscr{H}}^2) dt \\ &\qquad \text{for} \quad u \in C^1(\boldsymbol{R}_t; \mathscr{H}). \end{aligned}$$

Let $\psi(t) \in C^\infty$ be a function such that $0 \leq \psi \leq 1$, $\psi(t) = 0$ for $t \leq 0$, $\psi(t) = 1$ for $t \geq 1$ and $|\psi'(t)| \leq 2$. Setting

$$\text{(1.28)} \qquad \varphi_\lambda(t) = \begin{cases} \dfrac{1}{N} (j - 1 + \psi((t - \bar{\gamma}_{j-1})/\gamma_j)^2) & \text{if } t \in I_j \ (j = 1, \ldots, N) \\ 0 & \text{if } t < t_0 \\ 1 & \text{if } t > \bar{\gamma}_N, \end{cases}$$

then we have for $j = 1, \ldots, N$

$$\text{(1.29)} \qquad \frac{j-1}{N} \leq \varphi_\lambda(t) \leq \frac{j}{N} \quad \text{if} \quad t \in I_j,$$

$$\text{(1.30)} \qquad |\varphi_\lambda'(t)| \leq 4(\gamma_j N)^{-1} \quad \text{if} \quad t \in I_j,$$

$$\text{(1.31)} \qquad |(\sqrt{\varphi_\lambda(t)})'| \leq 4(\gamma_j \sqrt{Nj})^{-1} \quad \text{if} \quad t \in I_j.$$

Assume $u \in C_0^\infty([-\delta_1, \delta_1]; \mathscr{S}(\boldsymbol{R}_x^n))$ and substitute $\sqrt{N\varphi_\lambda(t)}\, u$ into (1.16). Then we have with suitable positive constants C, C_1, C_2 and c_0

$$\begin{aligned} &CN(\| P_\lambda u\|^2 + R_\lambda) \\ &\geq c_0 N\{\| \sqrt{\varphi_\lambda} D_t u\|^2 + \| t^s \sqrt{\varphi_\lambda}(D_{x_1} + \tilde{r})u\|^2 + (\varphi_\lambda V_\lambda u, u)\} \\ &\quad - C_1 N\| [D_t, \sqrt{\varphi_\lambda}]u\|^2 - C_2 N|\log\lambda|^{2/\kappa}\|\sqrt{\varphi_\lambda}u\|^2 \\ &\equiv \Omega_1 - \Omega_2 - \Omega_3. \end{aligned} \tag{1.32}$$

Since $I_j^* = I_j$ for $j \geq N+2$ it follows from (1.29) that

$$\begin{aligned} C\Omega_1 &\geq \sum_{j=2}^{N+2}\int_{I_j^*}\| D_t u\|_{\mathscr{H}}^2 + t^{2s}\|(D_{x_1} + \tilde{r})u\|_{\mathscr{H}}^2 + (V_\lambda u, u)_{\mathscr{H}} dt \\ &\quad + \sum_{j=N+3}^{\infty}\int_{I_j}\| D_t u\|_{\mathscr{H}}^2 + t^{2s}\|(D_{x_1} + \tilde{r})u\|_{\mathscr{H}}^2 + (V_\lambda u, u)_{\mathscr{H}} dt \\ &\geq \sum_{j=2}^{N+1}\gamma_j^{-2}\int_{I_j}\| u\|_{\mathscr{H}}^2 dt + \gamma_N^{-2}\int_{t_1}^{\infty}\| u\|_{\mathscr{H}}^2\, dt - CNR_\lambda. \end{aligned} \tag{1.33}$$

Here we have used (1.27) in deriving the last inequality. By means of (1.31) we obtain

$$\Omega_2 \leq C\sum_{j=1}^{N}\frac{\gamma_j^{-2}}{j}\int_{I_j}\| u\|_{\mathscr{H}}^2 dt \tag{1.34}$$

It follows from (1.29) that

$$\Omega_3 \leq C|\log\lambda|^{2/\kappa}\Big(\sum_{j=1}^{N+1} j\int_{t_1}^{\infty}\| u\|_{\mathscr{H}}^2\, dt + N\int_{t_1}^{\infty}\| u\|_{\mathscr{H}}^2\, dt\Big). \tag{1.35}$$

There exists an integer j_0 independent of λ such that

$$2C^2\Big(\frac{\gamma_j^{-2}}{j} + (\min\{j, N\})\,|\log\lambda|^{2/\kappa}\Big) < \gamma_j^{-2} \quad \text{if} \quad j > j_0. \tag{1.36}$$

In view of (1.36), it follows from (1.32)-(1.35) that

$$\begin{aligned} CN(\| P_\lambda u\|^2 + R_\lambda) &\geq c_0\Big\{N\|\sqrt{\varphi_\lambda(t)}D_t u\|^2 \\ &\quad + \sum_{j=2}^{N+1}\gamma_j^{-2}\int_{I_j}\| u\|_{\mathscr{H}}^2 dt + \gamma_N^{-2}\int_{t_1}^{\infty}\| u\|_{\mathscr{H}}^2 dt\Big\} \\ &\quad - C\sum_{j=1}^{j_0}|\log\lambda|^{2/\kappa}\int_{I_j}\| u\|_{\mathscr{H}}^2 dt. \end{aligned} \tag{1.37}$$

Substitute $\sqrt{N\varphi_\lambda(-t)}\,u$ into (1.16) similarly. If $-I_j$ denotes the symmetric interval of I_j with respect to the origin, then we have

$$(1.38)\quad \begin{aligned} CN(\|P_\lambda u\|^2 + R_\lambda) \geq c_0\Big\{N\|\sqrt{\varphi_\lambda(|t|)}D_t u\|^2 \\ + \sum_{j=2}^{N+1} \gamma_j^{-2}\int_{I_j\cup(-I_j)} \|u\|^2_{\mathscr{H}}dt + \gamma_N^{-2}\int_{|t|\geq t_1}\|u\|^2_{\mathscr{H}}dt\Big\} \\ - C\sum_{j=1}^{j_0} |\log\lambda|^{2/\kappa}\int_{I_j\cup(-I_j)}\|u\|^2_{\mathscr{H}}dt. \end{aligned}$$

Substitute $(1-\varphi_\lambda(|t|))u$ into (1.9). Then

$$(1.39)\quad \begin{aligned} C(\|P_\lambda u\|^2 + R_\lambda) \geq \|(1-\varphi_\lambda(|t|))D_t u\|^2 \\ + |\log\lambda|^{2/\kappa}\|(1-\varphi_\lambda(|t|))u\|^2 \\ - \frac{C}{N^2}\sum_{j=2}^{N}\gamma_j^{-2}\int_{I_j\cup(-I_j)}\|u\|^2_{\mathscr{H}}dt \end{aligned}$$

because $|\log\lambda|^{2/\kappa}$ is much bigger than γ_1^{-2}/N^2 and it follows from (1.30) that

$$\|[D_t,\ \varphi_\lambda(|t|)]u\|^2 \leq \frac{16}{N^2}\sum_{j=1}^{N}\gamma_j^{-2}\int_{I_j\cup(-I_j)}\|u\|^2_{\mathscr{H}}dt.$$

Note that

$$(1.40)\quad \begin{aligned} 2|\log\lambda|^{2/\kappa}\|(1-\varphi_\lambda(|t|))u\|^2 \geq \sum_{j=1}^{j_0}|\log\lambda|^{2/\kappa}\int_{I_j\cup(-I_j)}\|u\|^2_{\mathscr{H}}dt \\ + \int_{-t_0}^{t_0}|\log\lambda|^{2/\kappa}\|u\|^2_{\mathscr{H}}dt. \end{aligned}$$

In view of (1.40), it follows from (1.38) and (1.39) multiplied by N that

$$(1.41)\quad \begin{aligned} CN(\|P_\lambda u\|^2 + R_\lambda) \geq N\|D_t u\|^2 + N\int_{-\bar{\gamma}_{[N/2]}}^{\bar{\gamma}_{[N/2]}}|\log\lambda|^{2/\kappa}\|u\|^2_{\mathscr{H}}dt \\ + \sum_{j=1}^{N+1}\gamma_j^{-2}\int_{I_j\cup(-I_j)}\|u\|^2_{\mathscr{H}}dt + \gamma_N^{-2}\int_{|t|\geq t_1}\|u\|^2_{\mathscr{H}}dt \end{aligned}$$

because $(1-\varphi_\lambda(|t|))^2 + \varphi_t(|t|) \geq 1/2$. Here $[N/2]$ is the largest integer smaller than $N/2$.

Dividing both sides by N we obtain (1.2) in view of (1.20) and (1.19) because $\gamma_j^{-2} > N|\log\lambda|^{2/\kappa}$ if $j \geq [N/2]+1$.

2. Proofs of Theorem 2 and Corollary

Let $\rho_0 = ((0, x_0), (0, \xi_0)) \in T^*(\boldsymbol{R}^{n+1})\backslash 0$ with $\xi_0 = (0, \xi_0'), |\xi_0'| = 1$ and let P_λ be the same microlocalized operator of (1.1) with t^s and x_1^b replaced by $g_{\nu'}(t)$ and $g_{\nu''}(x_1)$, respectively. By the almost same way as in the preceding section, we shall prove (1.2) with the third term on the left hand side dropped, that is,

$$\begin{aligned} &\| |\log\lambda|^{1/\kappa} u\|^2 + \| |\log\lambda|^{1+1/\kappa-\varepsilon}\chi(2t|\log\lambda|^{1/\kappa})u\|^2 \\ &\le C(\|P_\lambda u\|^2 + \lambda^{-2}\|(1-H_{20\delta})u\|^2 + \|u\|^2) \\ &\quad \text{for} \quad u \in C_0^\infty([-\delta_1, \delta_1]\,;\mathcal{S}(\boldsymbol{R}_x^n)). \end{aligned} \tag{2.1}$$

If $t_3 = ((1-\mu)/|\log\lambda|)^{1/\kappa}$ with $\mu = \mu(\lambda)$ in (1.17) and if l is larger than ν_0 then it follows from (5) and the monotoneness of f' that

$$\begin{aligned} f'(t)\lambda^{-1} &\le \exp(-|t|^{-\kappa} + |\log|t||^{\nu_0+1} + |\log\lambda|) \\ &\le \exp\Big\{-\frac{a(\lambda)}{1-\mu(\lambda)}\log|\log\lambda| + \Big(\frac{\log|\log\lambda| - \log(1-\mu(\lambda))}{\kappa}\Big)^{\nu_0+1}\Big\} \\ &\le (\log|\lambda|)^{-a(\lambda)/2} \quad \text{if} \quad |t| \le t_3. \end{aligned}$$

Instead of (1.7) we have

$$h_{\delta_1}(x-x_0)g_{\nu'}(t)g_{\nu''}(x_1)f(t)\lambda^{-1} \le C \quad \text{if} \quad |t| \le t_3. \tag{2.2}$$

If $\operatorname{supp} u \subset \{|t| \le t_3\}$ then we have by (2.2)

$$C(\|P_\lambda u\|^2 + R_\lambda) \ge \|(D_t + ig_{\nu'}(t)\tilde{B}_\lambda)u\|^2, \tag{2.3}$$

where $\tilde{B}_\lambda = D_{x_1}h_{\delta_1}(\lambda D_x - \xi_0)$. By using the Nirenberg-Treves estimate (see [4; Section 26.8]) we have the following lemma given by Lerner [6] (see also [5; Section 2]):

LEMMA 2.1. *There exists a $\delta' > 0$ independent of $\lambda > 0$ such that for any $u(t) \in C_0^1(\boldsymbol{R}_t\,;\mathcal{H})$ we have*

$$2\int \|(D_t + ig_{\nu'}(t)\tilde{B}_\lambda)u(t)\|_{\mathcal{H}}dt \ge \sup \|u(t)\|_{\mathcal{H}} \quad \text{if} \quad \operatorname{supp} u \subset \{|t| \le \delta'\}. \tag{2.4}$$

Since $\tilde{B}_\lambda$ is independent of t the condition (3.1) of [11] is fulfilled trivially. Hence the proof of this lemma is the same as that of the lemma in [11]. By means of the Schwartz inequality and the Poincaré one it follows from (2.3) and (2.4) that

$$C(\| P_\lambda u \|^2 + R_\lambda) \geq |\log \lambda|^{2/\kappa} \| u \|^2 \quad \text{if} \quad \operatorname{supp} u \subset \{|t| \leq t_3\}. \tag{2.5}$$

Noting that $|g'_{\nu'}(t)| \leq \frac{\nu'+1}{|t|} |\log|t||^{\nu'} |g_{\nu'}(t)|$ we have instead of (1.14)

$$\begin{aligned} \| P_\lambda u \|^2 \geq \frac{1}{2} (\| D_t u \|^2 + \| g_{\nu'}(t) B_\lambda u \|^2 + (V_\lambda u, u)) \\ - C(\zeta(\lambda)^2 \| u \|^2 + R_\lambda) \quad \text{if} \quad \operatorname{supp} u \subset \{|t| \geq t_1/2\}, \end{aligned} \tag{2.6}$$

where $\zeta(\lambda) = |\log \lambda|^{1/\kappa} (\log|\log \lambda|)^{\nu'}$ and the formula (1.11) is replaced by

$$V_\lambda = V_\lambda(t, x_1) = g_{\nu'}(t) g_{\nu''}(x_1) f'(t) \lambda^{-1}. \tag{2.7}$$

Since t^s is replaced by $g_{\nu'}(t)$ we have in place of (1.16)

$$\begin{aligned} \| P_\lambda u \|^2 \geq c_0(\| D_t u \|^2 + \| g_{\nu'}(t)(D_{x_1} + \tilde{r}(t, x, D_{x'})) u \|^2 + (V_\lambda u, u)) \\ - C(\zeta(\lambda)^2 \| u \|^2 + R_\lambda) \\ \text{if} \quad \operatorname{supp} u \subset \{\delta_1 \geq |t| \geq t_1/2\}. \end{aligned} \tag{2.8}$$

Choosing a positive θ satisfying $\nu' < \theta < 1$ we set $\rho(\lambda) = (\log|\log \lambda|)^\theta$. We need replace the definition of γ_0 in Section 1 by $\gamma_0 = t_1/\rho(\lambda)$. We set γ_j for $j = 1,2,3,\ldots$ similarly as in Section 1. Set $\mu_j = g_{\nu'}(t_1)\gamma_j = |\log \lambda|^{-(\log|\log\lambda|)^{\nu'}/\kappa^{\nu'+1}} \gamma_j$ for $j = 1,\ldots, N$ instead of (1.21) and $\mu_j = \mu_N$ for $j \geq N+1$. If l is larger than $(\nu'+1)(\nu''+1)$ then we have (1.22) in view of (2.7) because it follows from (5) and the monotoneness of f' that

$$f'(t)\lambda^{-1} \geq |\log \lambda|^{a(\lambda)/2} \quad \text{if} \quad |t| \geq t_2.$$

On account of (1.22), Lemma 1.2 still holds with the factor t^{2s} of (1.23) replaced by $g_{\nu'}(t)^2$. From (1.26) we have (1.27) with the same modification. Substituting $\sqrt{N\varphi_\lambda(t)}\, u$ into (2.8) we have (1.32) with factors t^{2s} and $|\log \lambda|^{2/\kappa}$ replaced by $g_{\nu'}(t)^2$ and $\zeta(\lambda)^2$, respectively. By the same exchange, all formulae (1.33)-(1.37) still hold. In particular, (1.36) is valid on account of the factor $\rho(\lambda)^2$ included in γ_j^{-2} since $\nu' < \theta$. Hence we have in place of (1.38)

$$\begin{aligned} CN(\| P_\lambda u \|^2 + R_\lambda) \geq c_0 \Big\{ N \| \sqrt{\varphi_\lambda(|t|)}\, D_t u \|^2 \\ + \sum_{j=2}^{N+1} \gamma_j^{-2} \int_{I_j \cup (-I_j)} \| u \|_{\mathscr{H}}^2 dt + \gamma_N^{-2} \int_{|t| \geq t_1} \| u \|_{\mathscr{H}}^2 dt \Big\} \\ - C \sum_{j=1}^{j_0} \zeta(\lambda)^2 \int_{I_j \cup (-I_j)} \| u \|_{\mathscr{H}}^2 dt. \end{aligned} \tag{2.9}$$

Because of $t_1 - \gamma_N < t_3$ we can substitute $(1 - \varphi_\lambda(|t|))u$ into (2.5). Hence we have instead of (1.39)

$$C(\|P_\lambda u\|^2 + R_\lambda) \geq |\log\lambda|^{2/\kappa}\|(1-\varphi_\lambda(|t|))u\|^2 - \frac{C}{N^2}\sum_{j=2}^{N}\gamma_j^{-2}\int_{I_j\cup(-I_j)}\|u\|_{\mathscr{H}}^2 dt \tag{2.10}$$

because $|\log\lambda|^{2/\kappa}$ is much bigger than γ_1^{-2}/N^2 in spite of the factor $\rho(\lambda)^2$ included in γ_1^{-2}. If we choose a positive $\omega \geq 1$ such that $2 > \omega > 2\theta\ (> 2\nu')$ we have instead of (1.40)

$$N^\omega|\log\lambda|^{2/\kappa}\|(1-\varphi_\lambda(|t|))u\|^2 \geq \sum_{j=1}^{j_0}\zeta(\lambda)^2\int_{I_j\cup(-I_j)}\|u\|_{\mathscr{H}}^2 dt + \int_{-t_0}^{t_0}\zeta(\lambda)^2\|u\|_{\mathscr{H}}^2 dt.$$

In view of this, it follows from (2.9) and (2.10) multiplied by N^ω that

$$\begin{aligned} &C(N+N^\omega)(\|P_\lambda u\|^2 + R_\lambda) \\ &\geq N^\omega\int_{-\bar\gamma_{[N/2]}}^{\bar\gamma_{[N/2]}}|\log\lambda|^{2/\kappa}\|u\|_{\mathscr{H}}^2 dt + \sum_{j=1}^{N+1}\gamma_j^{-2}\int_{I_j\cup(-I_j)}\|u\|_{\mathscr{H}}^2 dt + \gamma_N^{-2}\int_{|t|\geq t_1}\|u\|_{\mathscr{H}}^2 dt \end{aligned} \tag{2.11}$$

because N^ω/N^2 is much smaller than 1. Dividing both sides by N^ω we obtain (2.1).

Let $\rho_1 = ((0, x_0), (0, \tilde\xi_0)) \in T^*(\boldsymbol{R}^{n+1})\setminus 0$ with $\tilde\xi_0 = (\xi_{01}, \tilde\xi_0')$, $\xi_{01} \neq 0$ and let $P_\lambda = D_t + ig_{\nu'}(t)B_\lambda(t, x, D_x)$ also denote the microlocalized operator at ρ_1 in this paragraph. Since $B_\lambda(t, x, \xi)$ has a definite sign if δ_1 is small enough, similarly as in Lemma 2.1 we have for any $u(t) \in C_0^1(\boldsymbol{R}_t; \mathscr{H})$

$$2\int\|(D_t + ig_{\nu'}(t)B_\lambda)u(t)\|_{\mathscr{H}} dt \geq \sup\|u(t)\|_{\mathscr{H}} \quad \text{if} \quad \operatorname{supp} u \subset \{|t| \leq \delta'\}. \tag{2.12}$$

By means of Schwartz's inequality and Poincaré's one it follows from (2.12) that

$$C\|P_\lambda u\|^2 \geq |\log\lambda|^{2+2\varepsilon}\|u\|^2 \quad \text{if} \quad \operatorname{supp} u \subset \{|t| \leq 2|\log\lambda|^{-1-\varepsilon}\}. \tag{2.13}$$

Note that $g_{\nu'}(t)B_\lambda(t, x, \xi) \geq \lambda^{-3/4}H_{\delta_1}(x, \xi; \lambda)$ if $|t| \geq |\log\lambda|^{-1-\varepsilon}$. Using the similar method as in the proof of Lemma 1.2 we have by (2.6) with $\zeta(\lambda)$ and $t_1/2$ replaced by $|\log\lambda|^{1+\varepsilon}(\log|\log\lambda|)^{\nu'}$ and $|\log\lambda|^{-1-\varepsilon}$

$$\begin{aligned} 2C(\|P_\lambda u\|^2 + R_\lambda) &\geq C(\|D_t u\|^2 + \|g_{\nu'}(t)B_\lambda u\|^2) + R_\lambda) \\ &\geq \|D_t u\|^2 + \lambda^{-3/2}\|u\|^2 \text{ if } \operatorname{supp} u \subset \{|t| \geq |\log\lambda|^{-1-\varepsilon}\}. \end{aligned} \tag{2.14}$$

If $\chi_\lambda(t) = \chi(2|t||\log\lambda|^{1+\varepsilon})$ then $|\chi_\lambda(t)^{(j)}| \leq C|\log\lambda|^{j(1+\varepsilon)}$ and it follows from (2.14) that

$$\|[P_\lambda, \chi_\lambda]u\|^2 = \|\chi_\lambda' u\|^2$$

$$\leq \lambda^{3/2}C\{\|[P_\lambda, \chi'_\lambda]u\|^2 + |\log\lambda|^{2+2\varepsilon}(\|P_\lambda u\|^2 + R_\lambda)\}.$$

Since a similar estimate holds with χ_λ replaced by $\chi'_\lambda|\log\lambda|^{-1-\varepsilon}$, in view of $u = \chi_\lambda u + (1-\chi_\lambda)$, it follows from (2.14) and (2.13) we have

$$|\log\lambda|^{2+2\varepsilon}\|u\|^2 \leq C(\|P_\lambda u\|^2 + R_\lambda) \quad \text{for} \quad u \in C_0^\infty([-\delta_1, \delta_1]; \mathcal{S}(\boldsymbol{R}_x^n)). \tag{2.15}$$

On account of microlocal estimates (2.1) and (2.15) we have (3)′ for $u \in C_0^\infty$ whose support is contained in a sufficiently small neighborhood of $(0, x_0)$. From this estimate we see P^* is locally solvable at $(0, x_0)$ because for any $\varepsilon' > 0$ there exists a $\delta(\varepsilon') > 0$ such that

$$\|u\| \leq \varepsilon'\|(\log(|D_t| + \Lambda))^{1/\kappa}u\| \quad \text{for} \quad \text{diam}(\text{supp}\, u) \leq \delta(\varepsilon'). \tag{2.16}$$

The Poincaré type estimate (2.16) can be easily seen by the similar way as in the proof of [8; Lemma 3.3].

Consider the microlocalized operator P_λ at $\rho_2 = ((\tilde{t}_0, x_0), (0, \xi_0)) \in T^*(\boldsymbol{R}^{n+1})\backslash 0$ with $\tilde{t}_0 \neq 0$ and $\xi_0 = (0, \xi'_0)$, $|\xi'_0| = 1$. Then we have instead of (2.6)

$$\|P_\lambda u\|^2 \geq \frac{1}{2}(\|D_t u\|^2 + \|B_\lambda u\|^2 + (V_\lambda u, u)) - CR_\lambda \quad \text{if} \quad \text{supp}\, u \subset \{|t - \tilde{t}_0| \leq \delta_1\}. \tag{2.17}$$

Note that

$$V_\lambda \geq \lambda^{-1/2} \quad \text{if} \quad |t - \tilde{t}_0| > |\tilde{t}_0|/2 \quad \text{and} \quad |x_1| \geq |\log\lambda|^{-1-\varepsilon}.$$

If $J = [-|\log\lambda|^{-1-\varepsilon}, |\log\lambda|^{-1-\varepsilon}]$ and J^* denotes twice dilation of J then by the similar way as in the proof of Lemma 1.2 we have uniformly for $t \in \{|t - \tilde{t}_0| \leq \delta_1\}$

$$C\left(\int_{J^*}\|B_\lambda u\|^2_{\mathcal{H}'} + V_\lambda(t, x_1)\|u\|^2_{\mathcal{H}'}dx_1\right)$$
$$\geq c_0|\log\lambda|^{2+2\varepsilon}\int_J\|u\|^2_{\mathcal{H}'} - C\int_{J^*}(\lambda^{-2}\|(1 - \tilde{H}_{20\delta})u\|^2_{\mathcal{H}'} + \|u\|^2_{\mathcal{H}'})dx_1 \quad \text{for} \quad u \in C^1(\boldsymbol{R}_{x_1}; \mathcal{H}').$$

Hence we have

$$|\log\lambda|^{2+2\varepsilon}\|u\|^2 \leq C(\|B_\lambda u\|^2 + (V_\lambda u, u) + R_\lambda) \leq C(\|P_\lambda u\|^2 + R_\lambda) \quad \text{if} \quad \text{supp}\, u \subset \{|t - \tilde{t}_0| \leq \delta_1\}. \tag{2.18}$$

From this we have

$$\|(\log \Lambda)^{1+\varepsilon}u\|^2 + \|D_t\chi(|D_t|/\Lambda)u\|^2 \le C(\|Pu\|^2 + \|u\|^2) \tag{2.19}$$

for $u \in C_0^\infty$ with supp u contained in a sufficiently small neighborhood of $(\tilde{t}_0, x_0)$ with $\tilde{t}_0 \neq 0$ since (2.15) holds even if $\rho_1 = ((\tilde{t}_0, x_0), (0, \tilde{\xi}_0)) \in T^*(\boldsymbol{R}^{n+1})\backslash 0$ with $\tilde{t}_0 \neq 0$ and $\tilde{\xi}_0 = (\xi_{01}, \tilde{\xi}_0')$ for $\xi_{01} \neq 0$. The local solvability of P^* in the region $\{t \neq 0\}$ is a direct consequence of (2.19) and the similar formula as (2.16).

By means of [8; Theorem 1] it follows from (2.19) that P is microhypoelliptic outside of $\Gamma = \mathrm{Char}\, P \cap \{t = 0\}$. Using [11; Theorem 1] we can easily see the microhypoellipticity of P at ρ_0 because from (2.1) we obtain (1.6) with t^s replaced by $g_{\nu'}(t)$. The microhypoellipticity at ρ_1 also follows from (2.15). Now the proof of Theorem 2 is completed.

We shall prove Corollary. Let $\rho_0 = ((0, x_0), (0, \xi_0)) \in T^*(\boldsymbol{R}^{n+1})\backslash 0$ with $\xi_0 = (0, \xi_0'), |\xi_0'| = 1$ and let P_λ be the microlocalized operator at ρ_0 in the beginning of this section if $t > 0$ and $P_\lambda = D_t$ if $t \ge 0$. If $u(t) \in C_0^1(\boldsymbol{R}_t; \mathscr{H})$, in view of (2.2) we have for $t \le t_3$

$$C(\|P_\lambda u(t)\|_{\mathscr{H}} + \tilde{R}_\lambda(t)) \ge \|(D_t + ig_{\nu'}(t)H(t)\tilde{B}_\lambda)u(t)\|_{\mathscr{H}}, \tag{2.3$'$}$$

where $\tilde{R}_\lambda(t) = \lambda^{-1}\|(1 - H_{20\delta}(x, D_x; \lambda))u(t)\|_{\mathscr{H}} + \|u(t)\|_{\mathscr{H}}$. Since (2.4) still holds with $g_{\nu'}(t)$ replaced by $g_{\nu'}(t)H(t)$ we have

$$C\int_{-\delta'}^{t_3} (\|P_\lambda u(t)\|_{\mathscr{H}} + \tilde{R}_\lambda(t))dt \ge \sup \|u(t)\|_{\mathscr{H}} \text{ for } u \in C_0^1([-\delta', t_3]; \mathscr{H}). \tag{2.20}$$

If $\delta_1 < \delta'$ then by the Schwartz inequality we have

$$\begin{aligned} &C\left(\delta_1\int_{-\delta_1}^{0} \|P_\lambda u(t)\|_{\mathscr{H}}^2 dt + |\log \lambda|^{-1/\kappa}\int_0^{t_3} \|P_\lambda u(t)\|_{\mathscr{H}}^2 dt + \delta_1\int_{-\delta_1}^{t_3} \tilde{R}_\lambda(t)^2 dt\right) \\ &\quad \ge \delta_1^{-1}\int_{-\delta_1}^{0} \|u(t)\|_{\mathscr{H}}^2 dt + |\log \lambda|^{1/\kappa}\int_0^{t_3} \|u(t)\|_{\mathscr{H}}^2 dt \\ &\qquad \text{if} \quad \mathrm{supp}\, u \subset \{-\delta_1 \le t \le t_3\}. \end{aligned} \tag{2.21}$$

Assume $u \in C_0^\infty([-\delta_1, \delta_1]; \mathscr{S}(\boldsymbol{R}_x^n))$ and substitute $(1 - \varphi_\lambda(t))$ into (2.21). Then we have instead of (2.10)

$$\begin{aligned} C|\log \lambda|^{1/\kappa}(\|P_\lambda u\|^2 + R_\lambda) &\ge \delta_1^{-1}|\log \lambda|^{1/\kappa}\int_{-\delta_1}^{0} \|u\|_{\mathscr{H}}^2 dt \\ &\quad + |\log \lambda|^{2/\kappa}\int_0^{t_3} \|(1 - \varphi_\lambda(t))u\|_{\mathscr{H}}^2 dt \end{aligned} \tag{2.22}$$

$$-\frac{C}{N^2}\sum_{j=2}^{N}\gamma_j^{-2}\int_{I_j}\|u\|_{\mathscr{H}}^2dt.$$

Note that (2.9) of course holds with $\varphi_\lambda(|t|)$ and $I_j\cup(-I_j)$ replaced by $\varphi_\lambda(t)$ and I_j, respectively. From this and (2.22) multiplied by N^ω we obtain

$$C(\|P_\lambda u\|^2+R_\lambda)\geq\delta_1^{-1}\int_{-\delta_1}^{0}\|u\|_{\mathscr{H}}^2dt+|\log\lambda|^{1/\kappa}\int_0^{\delta_1}\|u\|_{\mathscr{H}}^2dt.$$

By taking a $\delta_1>0$ small enough, we have

$$C(\|P_\lambda u\|^2+\lambda^{-2}\|(1-H_{20\delta})u\|^2)\geq\delta_1^{-1}\|u\|^2. \tag{2.23}$$

Assume that $u\in C_0^\infty$ with $\operatorname{supp}u\subset\{|t|<\delta_1\}\times\{|x-x_0|<\delta/10\}$. Substituting $h_\delta(\lambda D_x-\xi_0)h_\delta(x-x_0)u$ into (2.23), we have

$$\delta_1^{-1}\|h_\delta(\lambda D_x-\xi_0)u\|^2\leq C(\|h_\delta(\lambda D_x-\xi_0)Pu\|^2+\|h_{2\delta}(\lambda D_x-\xi_0)u\|^2+\lambda\|u\|^2).$$

From this we have

$$\delta_1^{-1}\|\psi_\delta(D_x)u\|^2\leq C(\|Pu\|^2+\|u\|^2), \tag{2.24}$$

if $\psi_\delta(\xi)$ is the same symbol as in (1.3). If $P_\lambda=D_t+ig_{\nu'}(t)H(t)B_\lambda(t,x,D_x)$ also denote the microlocalized operator at $\rho_1=((0,x_0),(0,\tilde{\xi}_0))\in T^*(\boldsymbol{R}^{n+1})\backslash 0$ with $\tilde{\xi}_0=(\xi_{01},\tilde{\xi}_0')$, $\xi_{01}\neq 0$, then from estimates corresponding to (2.12) and (2.14) we obtain

$$C(\|P_\lambda u\|^2+R_\lambda)\geq\delta_1^{-1}\int_{-\delta_1}^{0}\|u\|_{\mathscr{H}}^2dt+|\log\lambda|^{1+\varepsilon}\int_0^{\delta_1}\|u\|_{\mathscr{H}}^2dt. \tag{2.25}$$

From (2.25) we obtain (2.24) with $\psi_\delta(\xi)$ replaced by a suitable symbol $\tilde{\psi}_\delta(\xi)$ in $S_{1,0}^0$ such that $\tilde{\psi}_\delta=1$ in a small conic neighborhood of $\tilde{\xi}_0$. By means of the partition of unity on the unit sphere in $\boldsymbol{R}_\xi^n$ we have

$$\delta_1^{-1}\|u\|^2\leq C(\|Pu\|^2+\delta_1^{-1}\|\Lambda^{-1}u\|^2).$$

Since for any $\varepsilon'>0$ there exists a $\delta(\varepsilon')>0$ such that

$$\|\Lambda^{-1}w\|_{\mathscr{H}}\leq\varepsilon'\|w\|_{\mathscr{H}}\text{ for }w\in\mathscr{H}\text{ with }\operatorname{diam}(\operatorname{supp}w)\leq\delta(\varepsilon')$$

we have $\|u\|^2\leq C\|Pu\|^2$ if δ is sufficiently small. Hence P^* is locally solvable at $(0,x_0)$. The local solvability of P^* in the region $\{t\neq 0\}$ is obvious because by means of the similar way as in the derivation of (1.14) we have for any $\delta>0$ and some $C_\delta>0$

$$C_\delta(\|Pu\|+\|u\|)\geq\|D_tu\|\quad\text{if}\quad\operatorname{supp}u\subset\{|t|\geq\delta\}. \tag{2.26}$$

The proof of Corollary is completed.

3. Proof of Theorem 3

Let $t_1 = t_1(\lambda)$ be a positive such that $\Phi(t_1) = \log \lambda$ for a small $\lambda > 0$, which is uniquely determined since it follows from (7) that $\Phi(t)$ is monotone for $t > 0$. By the first inequality of (8) and Taylor's expansion we see that if $t > t_1$

$$(3.1) \qquad \Phi(t) - \log \lambda \geq \Phi'(t_1)(t - t_1) + \frac{\Phi''(t_1)}{2}(t - t_1)^2 \ (\equiv F_\lambda(t)).$$

If $\Phi''(t_1) < 0$ then the equation

$$(3.2) \qquad F_\lambda(t) = | \Phi(t_1) |^{\theta_1}$$

has two real roots on account of the second inequality of (8). If t_2 denotes a smaller root then we have instead of (1.15)

$$(3.3) \qquad t_2 - t_1 \leq 2 \frac{| \Phi(t_1) |^{\theta_1}}{\Phi'(t_1)} \leq 2C_1 t_1 | \log \lambda |^{\theta_1 - \theta_0}.$$

In fact, the first inequality follows from the fact that

$$\sqrt{1 - X} > 1 - X \quad \text{if} \quad X = 2 | \Phi''(t_1) | | \Phi(t_1) |^{\theta_1} / (\Phi'(t_1))^2 < 1.$$

The second inequality is a direct consequence of (7). Since $f'(t)$ is increasing in $\boldsymbol{R}_+$ we have from (3.1)

$$(3.4) \qquad f'(t)\lambda^{-1} \geq \exp | \log \lambda |^{\theta_1} \quad \text{if} \quad t \geq t_2 \quad \text{for} \quad t_2 \quad \text{satisfying (3.3).}$$

This is also true in the case where $\Phi''(t_1) \geq 0$, if t_2 is chosen as the root of (3.2) with $F_\lambda(t)$ replaced by $\Phi'(t_1)(t - t_1)$.

Let $\rho_0 = ((0, x_0), (0, \xi_0)) \in T^*(\boldsymbol{R}^{n+1}) \backslash 0$ with $\xi_0 = (0, \xi_0')$, $| \xi_0' | = 1$ and let P_λ denote the similar microlocalized operator at ρ_0 as (1.1) for the operator $\tilde{P}$ of the form (6). Since we have

$$h_{\delta_1}(x - x_0)\alpha(t)g(x_1)f(t)\lambda^{-1} \leq C \quad \text{if} \quad | t | \leq t_1$$

by means of Poincaré's inequality we get (instead of (2.5))

$$(3.5) \qquad C(\| P_\lambda u \|^2 + R_\lambda) \geq | t_1(\lambda) |^{-2} \| u \|^2 \quad \text{if} \quad \text{supp}\, u \subset \{| t | \leq t_1\}.$$

In view of (9) we have in place of (2.8) and (1.16)

(3.6) $$\|P_\lambda u\|^2 \geq \frac{1}{2}(\|D_t u\|^2 + \|\alpha(t)(D_{x_1} + \tilde{r})u\|^2 + (V_\lambda u, u)) - C(\zeta(\lambda)^2\|u\|^2 + R_\lambda)$$
$$\text{if} \quad \text{supp}\, u \subset \{|t| \geq t_1/2\},$$

where $\zeta(\lambda) = t_1^{-1}(\log|\Phi(t_1)|)^{\theta_2}$ and $V_\lambda = V_\lambda(t, x_1) = \alpha(t)g(x_1)f'(t)\lambda^{-1}$. Choosing $\theta > 0$ such that $\theta_2 < \theta < 1$ we set $\rho(\lambda)$ by the same way as in Section 2. Let N be the smallest integer such that $2^{-N} < 2|\log\lambda|^{-\theta_3}\rho(\lambda)$. Then we have still (1.20). We define γ_j and μ_j by the same way as in Section 2 with $g_{\nu'}(t_1)$ replaced by $\alpha(t_1)$. By means of (3.4) and (10) we have the similar formula as (1.22), that is,

(3.7) $$C_3 V_\lambda(t, x_1) \geq \alpha(t_1)g(\mu_N)\exp|\Phi(t_1)|^{\theta_1}$$
$$\geq \gamma_N^{-2} \quad \text{on} \quad \{|t| \geq t_2\} \times \{|x_1| \geq \mu_N\}.$$

It follows from (3.3) that $t_2 - t_1 \leq 2C_1\gamma_N$. By taking I_j^* equal to $4C_1$ times dilation of I_j to the right direction we obtain (2.9) by the same way as in Section 1. From (3.5) we get (2.10) with $|\log\lambda|^{2/\kappa}$ replaced by t_1^{-2}. From those two estimates we obtain (2.11) with $|\log\lambda|^{2/\kappa}$ replaced by t_1^{-2}. In view of (1.20), we have, if $\lambda > 0$ is sufficiently small,

(3.8) $$\|t_1^{-2}(\lambda)u\|^2 \leq C_\varepsilon(\|P_\lambda u\|^2 + \lambda^{-2}\|(1 - H_{2\delta})u\|^2)$$
$$\text{for} \quad u \in C_0^\infty([-\delta_1, \delta_1]; \mathcal{S}(\boldsymbol{R}_x^n)).$$

If $\rho_1 = ((0, x_0), (0, \tilde{\xi}_0)) \in T^*(\boldsymbol{R}^{n+1})\backslash 0$ with $\tilde{\xi}_0 = (\xi_{01}, \tilde{\xi}_0'), \xi_{01} \neq 0$ and if P_λ denotes the microlocalized operator of $\tilde{P}$ at ρ_1 we obtain

(3.9) $$C\|P_\lambda u\|^2 \geq \delta'^{-2}\|u\|^2 \text{ if supp}\, u \subset \{|t| \leq \delta'\}$$

by the similar way as in the derivation of (2.12) and (2.13). From (3.8) and (3.9) we see that $\tilde{P}^*$ is locally solvable at $(0, x_0)$, similarly as in the proof of Corollary of Theorem 2. The local solvablity of $\tilde{P}^*$ in the region $\{t \neq 0\}$ is obvious since we have the similar formula as (2.26).

REFERENCES

[1] Egorov Yu. V., Subelliptic operators, Russian Math. Surveys, **30:2** (1975), 59–118, **30:3** (1975), 55–105.

[2] Fefferman C., The uncertainty principle, Bull. Amer. Math. Soc., **9** (1983), 129–206.

[3] Hörmander, L., Subelliptic operators, Seminar on Singularities of Solution of Linear Partial Defferential Equations, Princeton University Press 1979, 3–49.

[4] ——, The analysis of Linear Partial Differential Operators IV, (1985), Springer-Verlag, Berlin, Heiderberg, New York, Tokyo.

[5] Lerner N., Sufficiency of condition (Ψ) for local solvability in two dimensions,

Ann. of Math., **128** (1988), 243–258.
[6] ——, An iff solvability condition for the oblique derivative problem, Séminar EDP 90–91, Ecole Polytechnique, exposé n° 18.
[7] ——, Nonsolvability in L^2 for a first-order operator satisfying condition (Ψ), Ann. of Math., **139** (1994), 363–393.
[8] Morimoto Y., Criteria for hypoellipticity of differential operators, Publ. RIMS Kyoto Univ., **22** (1986), 1129–1154.
[9] ——, A criterion for hypoellipticity of second order differential operators, Osaka J. Math., **24** (1987), 13–35.
[10] ——, Estimates for degenerate Schrödinger operators and hypoellipticity for infinitely degenerate elliptic operators, J. Math. Kyoto Univ., **32** (1992), 333–372.
[11] ——, Hypoelliptic operators of principal type with infinite degeneracy, Tsukuba J. Math., **19** (1995), 187–200.
[12] Moyer R. D., Local solvability in two dimensions: Necessary conditions for the principal-type case, Mimeographed manuscript, University of Kansas 1978.
[13] Nirenberg L. and Treves F., On local solvability of linear partial differential equations. I. Necessary conditions. II. Sufficient conditions, Correction. Comm. Pure Appl. Math., **23** (1970), 1–38 and 459–509; **24** (1971), 279–288.

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