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DERIVATIONS ON WHITE NOISE FUNCTIONALS

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Introduction

The Gaussian space (E^*, μ) is a natural infinite dimensional analogue of Euclidean space with Lebesgue measure and a special choice of a Gelfand triple $(E) \subset L^2(E^*, \mu) \subset (E)^*$ gives a fundamental framework of white noise calculus [2] as distribution theory on Gaussian space. It is proved in Kubo-Takenaka [7] that (E) is a topological algebra under pointwise multiplication. The main purpose of this paper is to answer the fundamental question: what are the derivations on the algebra (E)?

Since (E) is a topological algebra, each $\Phi \in (E)^*$ gives rise to a multiplication operator $\phi \mapsto \Phi\phi = \phi\Phi \in (E)^*$, $\phi \in (E)$. In fact, this is a continuous operator, namely, $\Phi \in \mathscr{L}((E), (E)^*)$. We then adopt a slightly general definition: a linear operator Ξ from (E) into $(E)^*$ is called a *derivation* if

$$\Xi(\phi\psi) = \Xi\phi\cdot\psi + \phi\cdot\Xi\psi, \quad \phi, \psi \in (E). \tag{0-1}$$

In this paper we determine all continuous derivations on (E); more precisely, the derivations which belong to $\mathscr{L}((E), (E)^*)$ or $\mathscr{L}((E), (E))$. The main result is stated in Theorem 5.1.

As a result, we shall see that a continuous derivation is nothing but a *first order differential operator* with distribution coefficients. Its formal expression is given as

$$\Xi = \int_T \Phi_t \partial_t dt, \tag{0-2}$$

where ∂_t is Hida's differential operator, i.e., an annihilation operator at a point $t \in T$, and $\Phi_t \in (E)^*$ is (identified with) a multiplication operator with parameter $t \in T$. In fact, $t \mapsto \Phi_t$ is an $(E)^*$-valued distribution on T, namely, $\tilde{\Phi}(t, x) = \Phi_t(x)$ is an element in $E^*_{\mathbf{C}} \otimes (E)^* \cong (E_{\mathbf{C}} \otimes (E))^*$, for the rigorous definition see Section 3. Moreover, the operator Ξ defined as in (0-2) belongs to $\mathscr{L}((E), (E))$ if

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and only if $\tilde{\Phi} \in E_{\mathbf{C}}^{*} \otimes (E)$, namely, if and only if Ξ is a first order differential operator with smooth coefficients.

The discussion is based on the theory of Fock expansion of operators on white noise functionals established in a series of works [12], [13], [14]. The essence of this theory lies in the fact that *every* operator $\Xi \in \mathscr{L}((E), (E)^{*})$ admits an infinite series expansion in terms of *integral kernel operators,* see also Section 4. For another application of this effective theory, see e.g., [11].

There has been observed formal analogy between white noise calculus and the calculus on Euclidean space based on the Gelfand triple $\mathscr{S}(\mathbf{R}^n) \subset L^2(\mathbf{R}^n) \subset \mathscr{S}^{*}(\mathbf{R}^n)$, e.g., rotation groups, Laplacians, Fourier transform, see [3], [4], [9], [11]. A more informative expression of (0-2) would be

$$\Xi\phi(x) = \int_T \Phi_t(x)\partial_t\phi(x)\,dt, \quad \phi \in (E),\, x \in E^{*}.$$

We then easily understand that the operator Ξ is a white noise analogue of a usual first order differential operator on Euclidean space given as

$$D\phi(x) = \sum_{j=1}^{n} A_j(x)\frac{\partial\phi}{\partial x_j}(x), \quad \phi \in \mathscr{S}(\mathbf{R}^n),\, x = (x_1, \cdots, x_n) \in \mathbf{R}^n.$$

Thus the formal analogy is again reinforced with our result in this paper.

Obviously, a first order differential operator on E^{*} gives rise to a vector field on it and vice versa. It is then interesting to investigate a (local) one-parameter group of tranformations on E^{*} which generates the vector field, for a particular case see [4]. As a next step, it will be interesting to discuss an operator of the form:

$$\int_{T\times T} \partial_s^{*}\Phi_{s,t}(x)\partial_t\,ds\,dt, \tag{0-3}$$

which gives rise to a quadratic form on (E) in a natural way. In [5] a simple case $\Phi_{s,t}(x) = \tau(s, t)\Phi(x)$ is discussed in relation with Dirichlet forms on white noise functionals. Furthermore, there are similarities between the above mentioned operators and quantum stochastic integrals, see e.g., [10], [15]. In fact, (0-2) and (0-3) are considered as direct generalizations of quantum stochastic integrals against the annihilation process and the number process, respectively. Systematic approaches to those topics will be carried out elsewhere.

1. White noise functionals

To avoid tedious introduction of notation we use the same framework as settled in [4] under the name of standard setup of white noise calculus. Nevertheless we recapitulate minimal notation for readers' convenience, for details see also [12], [13], [14], etc.

Let T be a topological space equipped with a Borel measure $d\nu(t) = dt$ and introduce a real Hilbert space $H = L^2(T, \nu; \mathbf{R})$ with norm $|\cdot|_0$ and inner product $\langle\cdot, \cdot\rangle$. Let A be a positive selfadjoint operator on H with Hilbert-Schmidt inverse and assume that inf Spec $(A) > 1$. Then there exist an increasing sequence of positive numbers $1 < \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots$ and a complete orthonormal basis $\{e_j\}_{j=0}^{\infty}$ for H such that $Ae_j = \lambda_j e_j$. We use the following constant numbers:

$$\delta \equiv \| A^{-1} \|_{\mathrm{HS}} = \left(\sum_{j=0}^{\infty} \lambda_j^{-2}\right)^{1/2} < \infty, \quad 0 < \rho \equiv \| A^{-1} \|_{\mathrm{OP}} = \lambda_0^{-1} < 1.$$

Then a Gelfand triple $E \subset H \subset E^*$ is constructed in the standard manner, where the nuclear Fréchet space E is equipped with Hilbertian norms:

$$|\xi|_p = |A^p \xi|_0 = \left(\sum_{j=0}^{\infty} \lambda_j^{2p} \langle\xi, e_j\rangle^2\right)^{1/2}, \quad \xi \in E, \quad p \in \mathbf{R}.$$

The canonical bilinear form on $E^* \times E$ is also denoted by $\langle\cdot, \cdot\rangle$. Hereafter we assume the usual conditions (H1)-(H3) to keep a delta function δ_t in E^*, see [4], [14].

The *Gaussian space* is by definition the probability space (E^*, μ), where μ is the Gaussian measure defined by

$$\exp\left(-\frac{1}{2}|\xi|_0^2\right) = \int_{E^*} e^{i\langle x, \xi\rangle} \mu(dx), \quad \xi \in E.$$

The Wiener-Itô-Segal isomorphism between $(L^2) \equiv L^2(E^*, \mu; \mathbf{C})$ and the Boson Fock space over $H_{\mathbf{C}}$ is given by means of the Wick ordered product as

$$\phi(x) = \sum_{n=0}^{\infty} \langle :x^{\otimes n}:, f_n\rangle, \quad x \in E^*, \tag{1-1}$$

where $\phi \in (L^2)$ and $f_n \in H_{\mathbf{C}}^{\hat{\otimes} n}$. Note also that each $\langle :x^{\otimes n}:, f_n\rangle$ is defined as an L^2-function and that the series is an orthogonal direct sum.

For $\phi \in (L^2)$ given as in (1-1) the second quantized operator $\Gamma(A)$ acts as

$$(\Gamma(A)\phi)(x) = \sum_{n=0}^{\infty} \langle :x^{\otimes n}:, A^{\otimes n} f_n\rangle.$$

With the maximal domain $\Gamma(A)$ becomes a positive selfadjoint operator with Hilbert-Schmidt inverse and thereby we obtain a complex Gelfand triple:

$$(E) \subset (L^2) = L^2(E^*, \mu ; \mathbf{C}) \subset (E)^*.$$

Here (E) is again a nuclear Fréchet space equipped with Hilbertian norms:

$$\| \phi \|_p^2 = \| \Gamma(A)^p \phi \|_0^2 = \sum_{n=0}^{\infty} n! \, | (A^{\otimes n})^p f_n |_0^2 = \sum_{n=0}^{\infty} n! \, | f_n |_p^2, \quad p \in \mathbf{R},$$

where $\phi \in (E)$ and $f_n \in H_{\mathbf{C}}^{\widehat{\otimes} n}$ are related as in (1-1) and $\| \cdot \|_0$ denotes the norm of (L^2). In particular, if $\phi \in (E)$ then $f_n \in H_{\mathbf{C}}^{\widehat{\otimes} n}$ for all $n = 0,1,2, \cdots$. Elements in (E) and $(E)^*$ are called a *test* (*white noise*) *functional* and a *generalized* (*white noise*) *functional*, respectively. We denote by $\langle\langle \cdot, \cdot \rangle\rangle$ the canonical bilinear form on $(E)^* \times (E)$.

By construction each $\phi \in (E)$ is a function on E^* determined only up to μ-null functions. Kubo-Yokoi's continuous version theorem [8] asserts that for $\phi \in (E)$ the series (1-1) converges absolutely at each $x \in E^*$ and becomes a unique continuous function on E^* which coincides with ϕ up to μ-null functions Hereafter we always assume that (E) consists of such continuous functions on E^*.

As for generalized white noise functionals similar but formal expression as in (1-1) is also possible and useful in many applications.

2. Gradient operator

We first recall basic differential operators on white noise functionals. Let $\phi \in (E)$ be given as

$$\phi(x) = \sum_{n=0}^{\infty} \langle : x^{\otimes n} :, f_n \rangle, \quad x \in E^*, \quad f_n \in E_{\mathbf{C}}^{\widehat{\otimes} n}. \tag{2-1}$$

Then for any $y \in E^*$ we put

$$\begin{aligned} D_y \phi(x) &= \lim_{\theta \to 0} \frac{\phi(x + \theta y) - \phi(x)}{\theta} \\ &= \sum_{n=1}^{\infty} n \langle : x^{\otimes (n-1)} :, y \otimes_1 f_n \rangle, \quad x \in E^*, \quad \phi \in (E), \end{aligned} \tag{2-2}$$

where $\otimes_1$ stands for the contraction of tensor products. The limit always exists and the series converges absolutely as numerical series. Moreover, it is known [4] that for any $p \geq 0$ and $q > 0$,

$$\| D_y\phi \|_p \le \left(\frac{\rho^{-2q}}{-2eq\log\rho}\right)^{1/2} | y |_{-(p+q)} \| \phi \|_{p+q}, \quad \phi \in (E). \tag{2-3}$$

In particular, $D_y \in \mathscr{L}((E), (E))$. *Hida's differential operator* is defined as $\partial_t = D_{\delta_t}$, where $\delta_t \in E^*$ is the delta function at $t \in T$. It is nothing but an annihilation operator at a point $t \in T$ in usual Fock space language.

It is convenient to use a white noise analogy of the gradient. We put

$$\nabla\phi(t, x) = \partial_t\phi(x), \quad t \in T, \quad x \in E^*. \tag{2-4}$$

This operator is well known in various contexts, see e.g., [1], [5], [6]. For further discussion we need $E_{\mathbf{C}} \otimes (E)$, i.e., the space of $E_{\mathbf{C}}$-valued white noise test functionals. As usual the symbol $\otimes$ stands for the completed π-tensor product following our convention [4], [14]. It is known (see e.g., [13]) that the topology of $E_{\mathbf{C}} \otimes (E)$ is given by the norms

$$\| \omega \|_p = \| (A \otimes \Gamma(A))^p \omega \|_0, \quad \omega \in E_{\mathbf{C}} \otimes (E), \quad p \in \mathbf{R}. \tag{2-5}$$

With these notation we prove the following

PROPOSITION 2.1. *It holds that*

$$\nabla\phi = \sum_{j=0}^{\infty} e_j \otimes D_{e_j}\phi, \quad \phi \in (E), \tag{2-6}$$

where the series converges in $E_{\mathbf{C}} \otimes (E)$ as well as pointwisely. Moreover, for any $p \ge 0$

$$\| \nabla\phi \|_p^2 = \sum_{j=0}^{\infty} \| e_j \otimes D_{e_j}\phi \|_p^2 \le \left(\frac{\rho^{-2}\delta^2}{-2e\log\rho}\right) \| \phi \|_{p+1}^2, \quad \phi \in (E). \tag{2-7}$$

In particular, $\nabla \in \mathscr{L}((E), E_{\mathbf{C}} \otimes (E))$.

Proof. Suppose that $\phi \in (E)$ is given as in (2-1). Then, by definition

$$\nabla\phi(t, x) = \partial_t\phi(x) = \sum_{n=1}^{\infty} n\langle :x^{\otimes(n-1)}:, \delta_t \otimes_1 f_n\rangle. \tag{2-8}$$

Using the Fourier expansion of f_n in terms of $\{e_j\}_{j=0}^{\infty}$, we obtain

$$\nabla\phi(t, x) = \sum_{j=0}^{\infty}\sum_{n=1}^{\infty} n\langle :x^{\otimes(n-1)}:, e_j \otimes_1 f_n\rangle e_j(t). \tag{2-9}$$

For simplicity we put $\phi_j = D_{e_j}\phi$. Then, by (2-2)

$$\nabla\phi(t, x) = \sum_{j=0}^{\infty} e_j(t)\phi_j(x). \tag{2-10}$$

As is easily verified, the above infinite series (2-8), (2-9) and (2-10) converge absolutely at each $t \in T$ and $x \in E^*$.

We next investigate a norm estimate. In view of (2-3) we have

$$\|\phi_j\|_p \le \left(\frac{\rho^{-2q}}{-2eq\log\rho}\right)^{1/2} |e_j|_{-(p+q)} \|\phi\|_{p+q}, \quad p \ge 0, \quad q > 0. \tag{2-11}$$

On the other hand, since $\{e_j\}_{j=0}^{\infty}$ is an orthogonal set with respect to every norm $|\cdot|_p$, we see from (2-5) and (2-10) that

$$\|\nabla\phi\|_p^2 = \|(A \otimes \Gamma(A))^p \nabla\phi\|_0^2 = \sum_{j=0}^{\infty} |e_j|_p^2 \|\phi_j\|_p^2 = \sum_{j=0}^{\infty} \|e_j \otimes \phi_j\|_p^2, \tag{2-12}$$

which proves the first half of (2-7). Inserting (2-11) into (2-12) we obtain

$$\|\nabla\phi\|_p^2 = \sum_{j=0}^{\infty} |e_j|_p^2 \left(\frac{\rho^{-2q}}{-2eq\log\rho}\right) |e_j|_{-(p+q)}^2 \|\phi\|_{p+q}^2 = \left(\frac{\rho^{-2q}}{-2eq\log\rho}\right) \|\phi\|_{p+q}^2 \sum_{j=0}^{\infty} \lambda_j^{-2q}.$$

Thus the second half of (2-7) follows by taking $q = 1$. The rest of the assertion is now immediate. Q,E.D.

COROLLARY 2.2. *For $y \in E^*$ and $\Phi \in (E)^*$ it holds that*

$$\langle\langle y \otimes \Phi, \nabla\phi \rangle\rangle = \langle\langle \Phi, D_y\phi \rangle\rangle, \quad \phi \in (E). \tag{2-13}$$

Here the canonical bilinear form on $(E_{\mathbf{C}} \otimes (E))^* \times (E_{\mathbf{C}} \otimes (E))$ is also denoted by $\langle\langle \cdot, \cdot \rangle\rangle$. It is also possible to adopt (2-13) as the definition of $\nabla\phi$.

3. First order differential operators

Before going into the definition of an operator of the form (0-2) we recall the following

LEMMA 3.1 ([7]). *For each $p \ge 0$ there exist $q > 0$ and $C \ge 0$ such that*

$$\|\phi\psi\|_p \le C \|\phi\|_{p+q} \|\psi\|_{p+q}, \quad \phi, \psi \in (E). \tag{3-1}$$

In this paper we do not need a precise norm estimate though it is very interesting in itself, see e.g., [14]. We next prove the following

LEMMA 3.2. *For* ϕ, $\psi \in (E)$ *put*

$$\omega_{\phi,\psi}(t, x) = (\partial_t \phi)(x) \cdot \psi(x), \quad t \in T, \quad x \in E^*. \tag{3-2}$$

Then, $\omega_{\phi,\psi} \in E \otimes (E)$. *Moreover,* $(\phi, \psi) \mapsto \omega_{\phi,\psi}$ *is a continuous bilinear map from* $(E) \times (E)$ *into* $E_{\mathbf{C}} \otimes (E)$.

Proof. For simplicity we write $\omega = \omega_{\phi,\psi}$. It then follows from Proposition 2.1 that

$$\omega(t, x) = \nabla\phi(t, x) \cdot \psi(x) = \sum_{j=0}^{\infty} e_j(t) \phi_j(x) \psi(x), \quad t \in T, \quad x \in E^*,$$

where $\phi_j = D_{e_j}\phi$. Suppose $p \geq 0$ is given. Then, in view of Lemma 3.1,

$$\| \omega \|_p^2 = \sum_{j=0}^{\infty} | e_j |_p^2 \| \phi_j \psi \|_p^2 \leq C^2 \| \psi \|_{p+q}^2 \sum_{j=0}^{\infty} | e_j |_p^2 \| \phi \|_{p+q}^2, \tag{3-3}$$

for some $C \geq 0$ and $q > 0$. Using $| e_j |_p = \rho^q | e_j |_{p+q}$, we obtain

$$\| \omega \|_p^2 \leq C^2 \rho^{2q} \| \psi \|_{p+q}^2 \sum_{j=0}^{\infty} | e_j |_{p+q}^2 \| \phi_j \|_{p+q}^2,$$

and therefore, by (2-7) we come to

$$\| \omega_{\phi,\psi} \|_p \leq M \| \phi \|_{p+q+1} \| \psi \|_{p+q}, \quad \phi, \psi \in (E), \tag{3-4}$$

where $M = C\rho^{q-1}\delta(-2e \log \rho)^{-1/2}$. This completes the proof. Q.E.D.

THEOREM 3.3. *For* $\tilde{\Phi} \in (E_{\mathbf{C}} \otimes (E))^*$ *there exists a unique operator* $\Xi \in \mathcal{L}((E), (E)^*)$ *such that*

$$\langle\langle \Xi\phi, \psi \rangle\rangle = \langle\langle \tilde{\Phi}, \omega_{\phi,\psi} \rangle\rangle, \quad \phi, \psi \in (E), \tag{3-5}$$

where $\omega_{\phi,\psi}$ *is defined as in* (3-2).

Proof. Choose $p \geq 0$ such as $\| \tilde{\Phi} \|_{-p} < \infty$. Then, by (3-4) we have

$$| \langle\langle \tilde{\Phi}, \omega_{\phi,\psi} \rangle\rangle | \leq \| \tilde{\Phi} \|_{-p} \| \omega_{\phi,\psi} \|_p \leq M \| \tilde{\Phi} \|_{-p} \| \phi \|_{p+q+1} \| \psi \|_{p+q}$$

for some $q > 0$ and $M \geq 0$. This means that $(\phi, \psi) \mapsto \langle\langle \tilde{\Phi}, \omega_{\phi,\psi} \rangle\rangle$ is a continuous bilinear form on $(E) \times (E)$, and therefore there exists a unique operator $\Xi \in \mathcal{L}((E), (E)^*)$ satisfying (3-5). Q.E.D.

The above constructed operator Ξ is called a *first order differential operator* with coefficient $\tilde{\Phi} \in (E_{\mathbf{C}} \otimes (E))^*$ and is denoted (somehow formally) by

$$\Xi = \int_T \Phi_t \partial_t dt. \tag{3-6}$$

Here we write $\Phi_t(x) = \tilde{\Phi}(t, x)$. In fact $t \mapsto \Phi_t$ is an $(E)^*$-valued distribution on T, namely, an element in $E_{\mathbf{C}}^* \otimes (E)^* \cong (E_{\mathbf{C}} \otimes (E))^*$.

We are now interested in first order differential operators acting on (E) into itself.

THEOREM 3.4. *Let Ξ be a first order differential operator with coefficient $\tilde{\Phi} \in (E_{\mathbf{C}} \otimes (E))^*$. Then $\Xi \in \mathscr{L}((E), (E))$ if and only if $\tilde{\Phi} \in E_{\mathbf{C}}^* \otimes (E)$.*

Proof. There is a canonical isomorphism $(E_{\mathbf{C}} \otimes (E))^* \cong \mathscr{L}(E_{\mathbf{C}}, (E)^*)$: the correspondence between $\tilde{\Phi} \in (E_{\mathbf{C}} \otimes (E))^*$ and $K \in \mathscr{L}(E_{\mathbf{C}}, (E)^*)$ is given by

$$\langle\langle \tilde{\Phi}, \xi \otimes \phi \rangle\rangle = \langle\langle K\xi, \phi \rangle\rangle, \quad \xi \in E_{\mathbf{C}}, \quad \phi \in (E). \tag{3-7}$$

Under this isomorphism, $\tilde{\Phi} \in E_{\mathbf{C}}^* \otimes (E)$ if and only if $K \in \mathscr{L}(E_{\mathbf{C}}, (E))$.

Suppose that Ξ is given as in (3-6). Then, by definition (3-5) holds. On the other hand, it has been established during the proof of Lemma 3.2 that

$$\omega_{\phi,\psi} = \sum_{j=0}^{\infty} e_j \otimes (\phi_j \psi), \quad \phi_j = D_{e_j}\phi,$$

converges in $E_{\mathbf{C}} \otimes (E)$. Then (3-5) becomes

$$\langle\langle \Xi\phi, \psi \rangle\rangle = \sum_{j=0}^{\infty} \langle\langle \tilde{\Phi}, e_j \otimes (\phi_j \psi) \rangle\rangle = \sum_{j=0}^{\infty} \langle\langle Ke_j, \phi_j \psi \rangle\rangle = \sum_{j=0}^{\infty} \langle\langle \phi_j K e_j, \psi \rangle\rangle.$$

Hence for any $p \geq 0$,

$$|\langle\langle \Xi\phi, \psi \rangle\rangle| \leq \sum_{j=0}^{\infty} \| \phi_j K e_j \|_p \| \psi \|_{-p}, \tag{3-8}$$

though the sum is possibly infinite. We now suppose that $\tilde{\Phi} \in E_{\mathbf{C}}^* \otimes (E)$, or equivalently, $K \in \mathscr{L}(E_{\mathbf{C}}, (E))$. Then $Ke_j \in (E)$ and by Lemma 3.1 there exist $q > 0$ and $C_1 \geq 0$ such that

$$\| \phi_j K e_j \|_p \leq C_1 \| \phi_j \|_{p+q} \| K e_j \|_{p+q}.$$

Moreover, there exist $r \geq 0$ and $C_2 \geq 0$ such that

$$\| Ke_j \|_{p+q} \le C_2 | e_j |_{p+q+r}.$$

Thus (3-8) becomes

$$\begin{aligned} | \langle\langle \Xi\phi, \psi \rangle\rangle | &\le C_1 C_2 \| \psi \|_{-p} \sum_{j=0}^{\infty} \| \phi_j \|_{p+q} | e_j |_{p+q+r} \\ &= C_1 C_2 \| \psi \|_{-p} \sum_{j=0}^{\infty} \| \phi_j \|_{p+q} | e_j |_{p+q+r+1} \lambda_j^{-1} \\ &\le C_1 C_2 \| \psi \|_{-p} \left(\sum_{j=0}^{\infty} \| \phi_j \|_{p+q}^2 | e_j |_{p+q+r+1}^2 \right)^{1/2} \left(\sum_{j=0}^{\infty} \lambda_j^{-2} \right)^{1/2} \\ &\le C_1 C_2 \delta \| \psi \|_{-p} \left(\sum_{j=0}^{\infty} \| e_j \otimes \phi_j \|_{p+q+r+1}^2 \right)^{1/2}. \end{aligned}$$

It then follows from Proposition 2.1 that

$$| \langle\langle \Xi\phi, \psi \rangle\rangle | \le C_1 C_2 \delta \| \psi \|_{-p} \| \nabla\phi \|_{p+q+r+1},$$

and hence

$$\| \Xi\phi \|_p \le C_1 C_2 \delta \| \nabla\phi \|_{p+q+r+1} \le C_1 C_2 \delta \left(\frac{\rho^{-2}\delta^2}{-2e \log \rho} \right)^{1/2} \| \phi \|_{p+q+r+2}.$$

We have thus seen that $\Xi \in \mathscr{L}((E), (E))$.

Conversely, suppose that $\Xi \in \mathscr{L}((E), (E))$. Then for any $p \ge 0$ there exist $q \ge 0$ and $C \ge 0$ such that

$$\| \Xi\phi \|_p \le C \| \phi \|_{p+q}, \quad \phi \in (E). \tag{3-9}$$

Let $\xi \in E_{\mathbf{C}}$ be fixed and consider

$$\phi(x) = \langle x, \xi \rangle, \quad x \in E^*.$$

As is easily verified, $\omega_{\phi,\psi} = \xi \otimes \psi$ for any $\psi \in (E)$. Hence by (3-5) and (3-7) we obtain

$$\langle\langle \Xi\phi, \psi \rangle\rangle = \langle\langle \tilde{\Phi}, \omega_{\phi,\psi} \rangle\rangle = \langle\langle \tilde{\Phi}, \xi \otimes \psi \rangle\rangle = \langle\langle K\xi, \psi \rangle\rangle.$$

Then by (3-9) we obtain

$$| \langle\langle K\xi, \psi \rangle\rangle | = | \langle\langle \Xi\phi, \psi \rangle\rangle | \le \| \Xi\phi \|_p \| \psi \|_{-p} \le C \| \phi \|_{p+q} \| \psi \|_{-p}.$$

Therefore,

$$\| K\xi \|_p \le C \| \phi \|_{p+q} = C | \xi |_{p+q}, \quad \xi \in E_{\mathbf{C}}.$$

Consequently, $K \in \mathscr{L}(E_{\mathbf{C}}, (E))$ Q.E.D.

Such an operator Ξ as described in Theorem 3.4 is called a *first order differential operator with smooth coefficients.* This would be reasonable because in that case $t \mapsto \Phi_t$ is an (E)-valued distribution on T.

4. Integral kernel operators and Fock expansion

With each $\kappa \in (E_{\mathbf{C}}^{\otimes(l+m)})^*$ we may associate an *integral kernel operator* whose formal expression is given by

$$\Xi_{l,m}(\kappa) = \int_{T^{l+m}} \kappa(s_1, \cdots, s_l, t_1, \cdots, t_m) \partial_{s_1}^* \cdots \partial_{s_l}^* \partial_{t_1} \cdots \partial_{t_m} ds_1 \cdots ds_l dt_1 \cdots dt_m,$$

where κ is called the *kernel distribution.* More precisely, it is defined through two canonical bilinear forms:

$$\langle\langle \Xi_{l,m}(\kappa)\phi, \psi \rangle\rangle = \langle \kappa, \langle\langle \partial_{s_1}^* \cdots \partial_{s_l}^* \partial_{t_1} \cdots \partial_{t_m} \phi, \psi \rangle\rangle \rangle, \quad \phi, \psi \in (E).$$

It is proved that $\Xi_{l,m}(\kappa) \in \mathcal{L}((E), (E)^*)$, see [4] for further details. Without loss of generality we may assume that the kernel distribution κ is symmetric with respect to the first l and the last m variables independently. We denote by $(E_{\mathbf{C}}^{\otimes(l+m)})^*_{\mathrm{sym}(l,m)}$ the space of such $\kappa \in (E_{\mathbf{C}}^{\otimes(l+m)})^*$.

THEOREM 4.1 ([12]). *For any $\Xi \in \mathcal{L}((E), (E)^*)$ there exists a unique family of distributions $\kappa_{l,m} \in (E_{\mathbf{C}}^{\otimes(l+m)})^*_{\mathrm{sym}(l,m)}$ such that*

$$\Xi\phi = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m})\phi, \quad \phi \in (E), \tag{4-1}$$

where the right hand side converges in $(E)^*$.

More complete results are found in [14]. The unique expression of $\Xi \in \mathcal{L}((E), (E)^*)$ given in Theorem 4.1 is called the *Fock expansion* of Ξ and denoted simply by

$$\Xi = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m}). \tag{4-2}$$

It is also known that the series converges in $\mathcal{L}((E), (E)^*)$.

Given $\Xi \in \mathcal{L}((E), (E)^*)$, the kernel distributions $\kappa_{l,m}$ are easily found by using an operator symbol. For each $\xi \in E_{\mathbf{C}}$ the *exponential vector* $\phi_\xi \in (E)$ is defined by

$$\phi_\xi(x) = \exp\left(\langle x, \xi\rangle - \frac{1}{2}\langle \xi, \xi\rangle\right) = \sum_{n=0}^{\infty} \left\langle :x^{\otimes n}:, \frac{\xi^{\otimes n}}{n!}\right\rangle, \quad x \in E^*. \tag{4-3}$$

For $\Xi \in \mathscr{L}((E), (E)^*)$ a function on $E_{\mathbf{C}} \times E_{\mathbf{C}}$ defined by

$$\hat{\Xi}(\xi, \eta) = \langle\!\langle \Xi\phi_\xi, \phi_\eta \rangle\!\rangle, \quad \xi, \eta \in E_{\mathbf{C}}, \tag{4-4}$$

is called the *symbol* of Ξ. For example, for Ξ with Fock expansion (4-2) we have

$$e^{-\langle \xi,\eta\rangle}\hat{\Xi}(\xi, \eta) = \sum_{l,m=0}^{\infty} \langle \kappa_{l,m}, \eta^{\otimes l} \otimes \xi^{\otimes m}\rangle, \quad \xi, \eta \in E_{\mathbf{C}}. \tag{4-5}$$

Hence, in order to find kernel distributions $\kappa_{l,m}$ one needs only to consider the Taylor expansion of (4-5).

We also note the following

PROPOSITION 4.2. *For a first order differential operator Ξ with coefficient $\tilde{\Phi} \in (E_{\mathbf{C}} \otimes (E))^*$ we have*

$$e^{-\langle \xi,\eta\rangle}\hat{\Xi}(\xi, \eta) = \langle\!\langle \tilde{\Phi}, \xi \otimes \phi_{\xi+\eta}\rangle\!\rangle, \quad \xi, \eta \in E_{\mathbf{C}}. \tag{4-6}$$

Proof. By definition (3-2) we have

$$\omega_{\phi_\xi,\phi_\eta}(t, x) = \partial_t\phi_\xi(x)\phi_\eta(x) = \xi(t)\phi_\xi(x)\phi_\eta(x), \quad \xi, \eta \in E_{\mathbf{C}},$$

namely, $\omega_{\phi_\xi,\phi_\eta} = e^{\langle \xi,\eta\rangle}\xi \otimes \phi_{\xi+\eta}$. Then

$$\hat{\Xi}(\xi, \eta) = \langle\!\langle \Xi\phi_\xi, \phi_\eta\rangle\!\rangle = \langle\!\langle \tilde{\Phi}, \omega_{\phi_\xi,\phi_\eta}\rangle\!\rangle = e^{\langle \xi,\eta\rangle}\langle\!\langle \tilde{\Phi}, \xi \otimes \phi_{\xi+\eta}\rangle\!\rangle.$$

This shows (4-6). Q.E.D.

COROLLARY 4.3. *Let $\kappa \in E_{\mathbf{C}}^*$ and put $\tilde{\Phi} = \kappa \otimes 1 \in E_{\mathbf{C}}^* \otimes (E)$. Then the first order differential operator with coefficient $\tilde{\Phi}$ coincides with $\Xi_{0,1}(\kappa)$.*

Such an operator described as in Corollary 4.3 is called a *first order differential operator with constant coefficients.*

5. Main result

Recall that a linear operator $\Xi : (E) \to (E)^*$ is called a *derivation* if

$$\Xi(\phi\psi) = \Xi\phi\cdot\psi + \phi\cdot\Xi\psi, \quad \phi, \psi \in (E). \tag{5-1}$$

We then come to the main result.

THEOREM 5.1. *Any continuous derivation in $\mathscr{L}((E), (E)^*)$ is a first order differential operator and vice versa. Furthermore, any continuous derivation in $\mathscr{L}((E), (E))$ is a first order differential operator with smooth coefficients and vice versa.*

For the proof we prepare a few lemmas.

LEMMA 5.2. *Let $\Xi \in \mathscr{L}((E), (E)^*)$. Then, it is a derivation if and only if*

$$e^{\langle\xi,\eta\rangle}\hat{\Xi}(\xi+\eta,\zeta) = e^{\langle\eta,\zeta\rangle}\hat{\Xi}(\xi,\eta+\zeta) + e^{\langle\xi,\zeta\rangle}\hat{\Xi}(\eta,\xi+\zeta), \quad \xi,\eta,\zeta \in E_{\mathbf{C}}. \tag{5-2}$$

Proof. Since the exponential vectors (4-3) span a dense subspace of (E), Ξ is a derivation if and only if

$$\Xi(\phi_\xi\phi_\eta) = \Xi\phi_\xi\cdot\phi_\eta + \phi_\xi\cdot\Xi\phi_\eta, \quad \xi,\eta \in E_{\mathbf{C}}^*. \tag{5-3}$$

With the obvious relation $\phi_\xi\phi_\eta = e^{\langle\xi,\eta\rangle}\phi_{\xi+\eta}$, we see easily that (5-3) is equivalent to (5-2). Q.E.D.

LEMMA 5.3. *Any first order differential operator is a derivation in $\mathscr{L}((E), (E)^*)$.*

Proof. Immediate from Proposition 4.2 and Lemma 5.2. Q.E.D.

LEMMA 5.4. *Let $\Xi \in \mathscr{L}((E), (E)^*)$ be a derivation with Fock expansion:*

$$\Xi = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m}). \tag{5-4}$$

Then, $\kappa_{l,0} = 0$ for all $l \geq 0$ and

$$\langle\kappa_{l,m+1}, \eta^{\otimes l}\otimes\xi^{\otimes(m+1)}\rangle = \binom{l+m}{l}\langle\kappa_{l+m,1}, (\eta^{\otimes l}\otimes\xi^{\otimes m})\otimes\xi\rangle, \quad \xi,\eta \in E_{\mathbf{C}},$$

for all $l, m \geq 0$.

Proof. By assumption the symbol $\hat{\Xi}$ satisfies (5-2). Then, in view of expansion (4-5) we obtain

$$\begin{aligned}\binom{m+n}{n}&\langle\kappa_{l,m+n}, \zeta^{\otimes l}\otimes\xi^{\otimes m}\otimes\eta^{\otimes n}\rangle \\ &= \binom{l+m}{m}\langle\kappa_{l+m,n}, \zeta^{\otimes l}\otimes\xi^{\otimes m}\otimes\eta^{\otimes n}\rangle\end{aligned} \tag{5-5}$$

$$+\binom{l+n}{n}\langle\kappa_{l+n,m},\ \zeta^{\otimes l}\otimes\eta^{\otimes n}\otimes\xi^{\otimes m}\rangle,\quad \xi,\eta,\zeta\in E_{\mathbf{C}}.$$

Then, putting $m=n=0$ in (5-5), we see that $\kappa_{l,0}=0$ for all $l\geq 0$. We next put $n=1$ and $\eta=\xi$ in (5-5) to obtain

$$\begin{aligned} l!(m+1)!\langle\kappa_{l,m+1},\ \zeta^{\otimes l}\otimes\xi^{\otimes(m+1)}\rangle &= (l+m)!\langle\kappa_{l+m,1},\ (\zeta^{\otimes l}\otimes\xi^{\otimes m})\otimes\xi\rangle \\ &\quad + (l+1)!m!\langle\kappa_{l+1,m},\ (\zeta^{\otimes l}\otimes\xi)\otimes\xi^{\otimes m}\rangle. \end{aligned}$$

Applying this argument to the second term successively, we come to

$$l!(m+1)!\langle\kappa_{l,m+1},\ \zeta^{\otimes l}\otimes\xi^{\otimes(m+1)}\rangle = (m+1)(l+m)!\langle\kappa_{l+m,1},\ (\zeta^{\otimes l}\otimes\xi^{\otimes m})\otimes\xi\rangle,$$

which completes the proof. Q.E.D.

Proof of Theorem 5.1. Suppose that we are given a continuous derivation Ξ with Fock expansion as in (5-4). We first introduce a continuous bilinear form Ω on $E_{\mathbf{C}}\times(E)$ by

$$\Omega(\xi,\phi)=\sum_{n=0}^{\infty}n!\langle\kappa_{n,1},\ f_n\otimes\xi\rangle,\quad \xi\in E_{\mathbf{C}},\quad \phi\in(E), \tag{5-6}$$

where $\phi(x)=\sum_{n=0}^{\infty}\langle :x^{\otimes n}:,\ f_n\rangle$. We shall prove the convergence of (5-6). In fact, for any $p,\ q\geq 0$ we have

$$\begin{aligned} &\sum_{n=0}^{\infty}n!\,|\langle\kappa_{n1},\ f_n\otimes\xi\rangle| \\ &\leq \sum_{n=0}^{\infty}n!\,|\kappa_{n,1}|_{-(p+q+1)}\,|f_n\otimes\xi|_{p+q+1} \\ &\leq \left(\sum_{n=0}^{\infty}n!\,|\kappa_{n,1}|_{-(p+q+1)}^2\right)^{1/2}\left(\sum_{n=0}^{\infty}n!\,|f_n|_{p+q+1}^2\right)^{1/2}|\xi|_{p+q+1} \\ &= |\xi|_{p+q+1}\,\|\phi\|_{p+q+1}\left(\sum_{n=0}^{\infty}n!\,|\kappa_{n,1}|_{-(p+q+1)}^2\right)^{1/2}. \end{aligned} \tag{5-7}$$

Since $\Xi\in\mathcal{L}((E),\ (E)^*)$, there exist $C\geq 0$, $K\geq 0$ and $p\geq 0$ such that

$$|\hat{\Xi}(\xi,\ \eta)|\leq C\exp K\,(|\xi|_p^2+|\eta|_p^2),\quad \xi,\eta\in E_{\mathbf{C}}.$$

It is proved in [12] that the kernel distributions $\kappa_{l,m}$ of Ξ satisfies

$$|\kappa_{l,m}|_{-(p+1)}\leq C\,(l^l m^m)^{-1/2}(2e^3\delta^2)^{(l+m)/2}\left(\frac{\rho^{2p}}{2}+K\right)^{(l+m)/2}.$$

In particular,

$$
\begin{aligned}
|\kappa_{n,1}|_{-(p+q+1)} &\le \rho^{q(n+1)} |\kappa_{n,1}|_{-(p+1)} \\
&\le C\rho^{q(n+1)} n^{-n/2} (2e^3\delta^2)^{(n+1)/2} \Big(\frac{\rho^{2p}}{2} + K\Big)^{(n+1)/2}.
\end{aligned}
$$

Therefore,

$$
\sum_{n=0}^{\infty} n! \, |\kappa_{n,l}|^2_{-(p+q+1)} \le C^2 \sum_{n=0}^{\infty} \frac{n!}{n^n} \Big\{2e^3\delta^2\rho^{2q}\Big(\frac{\rho^{2p}}{2} + K\Big)\Big\}^{n+1} < \infty \tag{5-8}
$$

for a sufficiently large $q \ge 0$. In conclusion, we see from (5-7) and (5-8) that

$$
\Big| \sum_{n=0}^{\infty} n! \langle \kappa_{n,1}, f_n \otimes \xi \rangle \Big| \le C_1 |\xi|_{p+q+1} \|\phi\|_{p+q+1}, \quad \xi \in E_{\mathbf{C}}, \ \phi \in (E),
$$

for some $C_1 \ge 0$, $p \ge 0$ and $q \ge 0$. Therefore Ω in (5-6) is well defined on $E_{\mathbf{C}} \times (E)$ and becomes a continuous bilinear form.

Let $\tilde{\Phi} \in (E_{\mathbf{C}} \otimes (E))^*$ be the element corresponding to Ω, namely,

$$
\langle\langle \tilde{\Phi}, \xi \otimes \phi \rangle\rangle = \Omega(\xi, \phi), \quad \xi \in E_{\mathbf{C}}, \quad \phi \in (E).
$$

Let Ξ' be the first order differential operator with coefficient $\tilde{\Phi}$. It then follows from Proposition 4.2 that

$$
e^{-\langle \xi, \eta \rangle} \hat{\Xi}'(\xi, \eta) = \langle\langle \tilde{\Phi}, \xi \otimes \phi_{\xi+\eta} \rangle\rangle = \Omega(\xi, \phi_{\xi+\eta}). \tag{5-9}
$$

On the other hand, in view of (5-6) we have

$$
\begin{aligned}
\Omega(\xi, \phi_{\xi+\eta}) &= \sum_{n=0}^{\infty} n! \Big\langle \kappa_{n,1}, \frac{(\xi+\eta)^{\otimes n}}{n!} \otimes \xi \Big\rangle \\
&= \sum_{n=0}^{\infty} \sum_{l=0}^{n} \binom{n}{l} \langle \kappa_{n,1}, (\eta^{\otimes l} \otimes \xi^{\otimes (n-l)}) \otimes \xi \rangle \\
&= \sum_{l,m=0}^{\infty} \binom{l+m}{l} \langle \kappa_{l+m,1}, (\eta^{\otimes l} \otimes \xi^{\otimes m}) \otimes \xi \rangle.
\end{aligned}
$$

Now, applying the relations:

$$
\begin{cases}
\kappa_{l,0} = 0, \quad l \ge 0, \\
\langle \kappa_{l,m+1}, \eta^{\otimes l} \otimes \xi^{\otimes (m+1)} \rangle = \binom{l+m}{l} \langle \kappa_{l+m,1}, (\eta^{\otimes l} \otimes \xi^{\otimes m}) \otimes \xi \rangle, \quad l, m \ge 0,
\end{cases}
$$

which are obtained in Lemma 5.4, we come to

$$(5\text{-}10)\qquad \Omega(\xi, \phi_{\xi+\eta}) = \sum_{l,m=0}^{\infty} \langle \kappa_{l,m}, \eta^{\otimes l} \otimes \xi^{\otimes m} \rangle = e^{-\langle \xi, \eta \rangle} \hat{\Xi}(\xi, \eta).$$

It follows from (5-9) and (5-10) that

$$e^{-\langle \xi, \eta \rangle} \hat{\Xi}'(\xi, \eta) = e^{-\langle \xi, \eta \rangle} \hat{\Xi}(\xi, \eta), \quad \xi, \eta \in E_{\mathbf{C}},$$

so that $\Xi = \Xi'$. Consequently, Ξ is the first order differential operator with coefficient $\tilde{\Phi}$. The rest of the assertion is now immediate from Lemma 5.3 and Theorem 3.4. Q.E.D.

Note that any derivation maps constant functions into zero.

COROLLARY 5.5. *Any continuous derivation on* (E) *which maps linear functionals into constants is a first order differential operator with constant coefficients and vice versa.*

The *Gross Laplacian* and the *number operator* are defined respectively as integral kernel operators:

$$\Delta_G = \int_{T\times T} \tau(s, t) \partial_s \partial_t ds dt, \quad N = \int_{T\times T} \tau(s, t) \partial_s^* \partial_t ds dt,$$

where $\tau \in E^* \otimes E$ is given by $\langle \tau, \xi \otimes \eta \rangle = \langle \xi, \eta \rangle$, $\xi, \eta \in E$.

COROLLARY 5.6. $\Delta_G + N$ *is a derivation in* $\mathscr{L}((E), (E))$.

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