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A CONSTRUCTION OF PEAK FUNCTIONS ON LOCALLY CONVEX DOMAINS IN $\mathbf{C}^n$

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1. Introduction

Let Ω be a smoothly bounded pseudoconvex domain in $\mathbf{C}^n$ and let $A(\Omega)$ denote the functions holomorphic on Ω and continuous on $\bar{\Omega}$. A point $p \in b\Omega$ is a peak point if there is a function $f \in A(\Omega)$ such that $f(p) = 1$, and $|f(z)| < 1$ for $z \in \bar{\Omega} - \{p\}$. The existence of peaking functions is a qualitative converse of the maximum principle: if $f \in A(\Omega)$, then

$$|f(z)| \leq \sup_{b\Omega} |f|, \quad z \in \Omega.$$

When Ω is strictly pseudoconvex, the situation with regard to peak functions is fairly well understood, but in the weakly pseudoconvex case we know very little. If $\Omega \subset\subset \mathbf{C}^2$ is pseudoconvex and $b\Omega$ is of finite type, Bedford and Fornaess [1], showed that there is a peak function in $A(\Omega)$. This method also works for finite type domains in $\mathbf{C}^n$ where the Levi-form of $b\Omega$ has $(n-2)$-positive eigenvalues. We also mention the work of Bloom [2], Hakim and Sibony [10], and Range [16] on the existence of peak functions with additional smoothness up to the boundary of Ω, i.e., in the various subclass of $A(\Omega)$.

Recently Fornaess and McNeal [9] proposed a new method to construct peak functions on finite type domains in $\mathbf{C}^2$ and on decoupled domains in $\mathbf{C}^n$. Their method depends on the solvability of $\bar{\partial}$-equation with L^∞ or Hölder estimates of the domain, and on the estimates of the Bergman kernel on and off the diagonal near $p \in b\Omega$. Here we propose a method different from Fornaess and McNeal's. Namely, we construct a regular bumping family of pseudoconvex domains outside V, and use Bishop's $\frac{1}{4} - \frac{3}{4}$ method directly on bumped domains. This method can be applied for the domains where the precise estimates of the Bergman kernel function on and off the diagonal are known but the L^∞ or Hölder estimates of the

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$\bar{\partial}$-equation is not known.

The precise estimates of the Bergman kernel functions near $p \in b\Omega$ are known for instance, finite type domains in $\mathbf{C}^2$ [4,11], decoupled domains in $\mathbf{C}^n$ [12], finite type domains in $\mathbf{C}^n$ with the Levi-form of $b\Omega$ $(n-2)$-positive eigenvalues [6,7]. Also the existence of peaking functions are known for these domains. In this paper we will construct a peak function for locally convex finite type domains in $\mathbf{C}^n$. We will use McNeal's [15] estimates of the Bergman kernel functions for locally convex domains. We state our main theorem as follows:

THEOREM 1.1. *Let $\Omega \subset\subset \mathbf{C}^n$ be a smoothly bounded pseudoconvex domain and let $b\Omega$ be of finite type. Suppose $p \in b\Omega$ and Ω is convex near p. For each small neighborhood V of p, there is a peak function that peaks at p and extends holomorphically up to $b\Omega \setminus V$.*

We will prove Theorem 1.1 in Section 4.

2. Smooth bumping families

Let Ω be a smoothly bounded pseudoconvex domain in $\mathbf{C}^n$ with smooth defining function r and let $0 \in b\Omega$. If $g : \mathbf{C} \to \mathbf{C}$ is any smooth function with $g(0) = 0$, let $\nu(g)$ denote the order of vanishing of g at 0. For a vector valued $G = (g_1, \ldots, g_n)$, let $\nu(G)$ denote the minimum order of vanishing of the g_i at 0.

DEFINITION 2.1 (D'Angelo). 0 is a point of finite 1-type if

$$\sup_G \frac{\nu(r \circ G)}{\nu(G)} = \Delta(0) < \infty,$$

where $G : \mathbf{C} \to \mathbf{C}^n$ is a complex analytic map with $G(0) = 0$; $\Delta(0)$ is called the type of 0.

DEFINITION 2.2. Let $p \in b\Omega$ be an arbitrary point and let V be a neighborhood of p. By a smooth bumping family for Ω outside V we mean a family $\{\Omega_t\}_{0 \leq t \leq 1}$ of pseudoconvex domains with C^∞ defining functions $\{r_t\}$ with the following properties:

(a) $\Omega = \Omega_0$,

(b) $\Omega_{t_1} \subset \Omega_{t_2}$ if $t_1 < t_2$, and $r_t(z)$ is smooth in z and t,

(c) for any neighborhood U of $\partial\Omega \setminus V$ there is a $t_0 > 0$ such that $\Omega_t \setminus U = \Omega \setminus U$ for all $t \in [0, t_0]$.

The following theorem can be found in [5].

THEOREM 2.3. *Let p be a point of finite 1-type in the boundary of a pseudoconvex domain Ω in $\mathbf{C}^n$ with smooth defining function $r(z)$. Then for each neighborhood V of p, there exists a smooth 1-parameter family of pseudoconvex domains $\{\Omega_t\}_{0\le t<t_0}$, each defined by $\Omega_t = \{z\,;\, r(z, t) < 0\}$, where $r(z, t)$ has the following properties:*

(a) *$r(z, t)$ is smooth in z near $b\Omega$, and in t for $0 \le t < t_0$,*

(b) *$r(z, t) = r(z)$ for $z \notin V$,*

(c) $\dfrac{\partial r}{\partial t}(z, t) \le 0$,

(d) $r(z, 0) = r(z)$,

(e) *for z in V,* $\dfrac{\partial r}{\partial t} < 0$.

DEFINITION 2.4. Suppose Ω, $p \in b\Omega$, V be as Theorem 2.3. Then we say $\{\Omega_t\}_{0\le t<t_0}$ a bumping family of Ω with front V.

THEOREM 2.5. *Let $\Omega \subset\subset \mathbf{C}^n$ be a smoothly bounded pseudoconvex domain and let $b\Omega$ be of finite 1-type. Assume $p \in b\Omega$ and V is a small neighborhood of p. Then there is a 1-parameter family of a smooth bumping family $\{\Omega_t\}$ outside V.*

Proof. Choose a neighborhood U of p such that $V \subset\subset U$. Since $b\Omega$ is compact, we can choose points $z_1, \ldots, z_N \in b\Omega$ and $\varepsilon_1, \ldots, \varepsilon_N > 0$ such that

(1) Ω is pseudoconvex and $b\Omega$ is of finite type,

(2) $\cup_{i=1}^N B(z_i, \varepsilon_i/2) \supset b\Omega \setminus U$,

(3) $V \cap B(z_i, \varepsilon_t) = \emptyset$, $i = 1,2,\ldots, N$,

(4) $\overline{B(z_i, \varepsilon_i)}$, is contained in a neighborhood V_i where Theorem 2.3 can be applicable, $i = 1,2,\ldots, N$.

Set $V_i = B(z_i, \varepsilon_i)$, $i = 1,2,\ldots, N$, for the convenience. Consider a bumping family of Ω with front V_1. Since the type condition is stable under small C^∞-perturbations of $b\Omega$, we will get a family $\{\Omega_{t_1}\}_{0\le t_1<\alpha_1}$ of smooth pseudoconvex domains satisfying (1)-(4) for the domains Ω_{t_1} (instead of Ω) provided α_1 is sufficiently small. For each Ω_{t_1}, $0 \le t_1 < \alpha_1$, we consider a bumping family of Ω_{t_1} with front V_2 and call it $\{\Omega_{t_1 t_2}\}_{0\le t_2<\alpha_2}$. Again $\{\Omega_{t_1 t_2}\}_{0\le t_2<\alpha_2}$ will satisfy (1)-(4) provided α_2 is sufficiently small. Continuing in this manner, we will get a bumping family of pseudoconvex domains $\{\Omega_{t_1 t_2 \cdots t_N}\}$ outside V. Obviously we can regard this family as a 1-parameter family of pseudoconvex domains. □

3. Estimates on the Bergman kernel

In this section, we estimate the Bergman kernel function on a locally convex domains in $\mathbf{C}^n$. For the estimates of the kernel on the other domains, one can refer [4,6,7,11,12].

Let $\Omega \subset\subset \mathbf{C}^n$ be smoothly bounded and convex in some neighborhood U of $p \in b\Omega$. Suppose that p is a point of finite type T in the sense of D'Angelo [8]. In [13], McNeal showed that T is actually the maximum order of contact of $b\Omega$ with complex lines at p. The convexity of $b\Omega$ in U is independent of the choice of the local defining function. Furthermore we may assume that the defining function r for $U \cap b\Omega$ has the property that all the sets $\{z\,; r(z) < \eta\} \cap U$ are convex for η in some range $-\eta_0 < \eta < \eta_0$, $\eta_0 > 0$. Let $S^n = \{\zeta \in \mathbf{C}^n\,; |\zeta| = 1\}$. Then each element of S^n, together with the point q near p determines a complex line in $\mathbf{C}^n$. If $\zeta \in S^n$ and $\eta_0 < \eta < \eta_0$, we denote the distance from q to the level set $\{z\,; r(z) = \eta\}$ along the complex line determined by ζ by $\delta_\eta(q, \zeta)$. Assume $p = 0 \in b\Omega$.

PROPOSITION 3.1 (15, Proposition 2.1). *After perhaps shrinking U, for every $q \in \Omega \cap U$ and every $\varepsilon > 0$ sufficiently close to 0, there exist coordinates $(z_1, \ldots, z_n)$ centered at q, positive numbers $\tau_1(q, \varepsilon), \ldots, \tau_n(q, \varepsilon)$, and points $p_1, \ldots, p_n \in \{z\,; r(z) = \varepsilon + r(q)\}$ such that in the coordinates $(z_1, \ldots, z_n)$, the defining function r satisfies*

(i) *for* $1 \le i \le n$,

$$\frac{\tau_1(q, \varepsilon)}{\tau_i(q, \varepsilon)} \lesssim \frac{\partial r}{\partial z_i}(p_i) \,| \lesssim \frac{\tau_1(q, \varepsilon)}{\tau_i(q, \varepsilon)},$$

(ii) *if* $i < j$,

$$\left|\frac{\partial r}{\partial z_i}(p_j)\right| \lesssim \frac{\tau_1(q, \varepsilon)}{\tau_i(q, \varepsilon)},$$

(iii) *if* $i > j$, $\left|\frac{\partial r}{\partial z_i}(p_j)\right| = 0.$

Also if we define the polydisc

$$P_\varepsilon(q) = \{z \in U\,; |z_1| < \tau_1(q, \varepsilon), \ldots, |z_n| < \tau_n(q, \varepsilon)\},$$

then there exists a constant $c > 0$, independent of $q \in \Omega \cap U$, such that $cP_\varepsilon(q) \subset \{z \in U\,; r(z) < \varepsilon + r(q)\}$.

Let us introduce a quantitative estimate on the weights $\tau_i(q, \varepsilon)$. Let $z_j = x_j + ix_{j+n}$ for $1 \le j \le n$ denote the underlying real coordinates. For each $2 \le i \le n$, an application of Taylor's theorem gives

$$\begin{aligned} r\circ z(0, \ldots, x_i, 0 \ldots, 0) &= r(q) + \sum_{k=2}^{T} a_k^i(q) x_i^k + \mathcal{O}(|x_i|^{T+1}) \\ &= r(q) + f_i(x_i). \end{aligned}$$

Since each $\tau_i(q, \varepsilon)$ is the maximum distance in $\mathrm{Re}z_j$ direction to the level surface $\{z; r(z) = r(q) + \varepsilon\}$, we have that

$$\varepsilon + r(q) = r(q) + \sum_{k=2}^{T} a_k^i(q)\tau_i(q, \varepsilon)^k + \mathcal{O}(|\tau_i(q, \varepsilon)|^{T+1}).$$

For $2 \leq k \leq T$, define

$$A_k^i(q) = |a_k^i(q)|$$

and for $\varepsilon > 0$, set

$$\sigma_i(q, \varepsilon) = \min\{(\varepsilon / A_k^i(q))^{\frac{1}{q}}; 2 \leq k \leq T\}. \tag{3.1}$$

Then it was shown in [15] that, for each $2 \leq i \leq n$,

$$\sigma_i(q, \varepsilon) \lesssim \tau_i(q, \varepsilon) \lesssim \sigma_i(q, \varepsilon). \tag{3.2}$$

From the construction in Proposition 3.1, $\tau_1(q, \varepsilon)$ is the distance from q to $bD_{q,\varepsilon}$ where $bD_{q,\varepsilon} = \{z \in U; r(z) = \varepsilon + r(q)\}$. Hence $\mathrm{Re}z_1$ is the normal direction and $\tau_1(q, \varepsilon) \approx \varepsilon$ independent of $q \in U$.

In these setting, McNeal got the following estimates for the Bergman kernel function.

THEOREM 3.2. *Suppose $\Omega \subset\subset \mathbf{C}^n$ is smoothly bounded and pseudoconvex. Let $p \in b\Omega$ be a point of finite type T and assume there is some neighborhood U of p so that Ω is convex in U. There exists a neighborhood $V \subset\subset U$ so that if $q \in V \cap \Omega$,*

$$K_\Omega(q, q) \approx \prod_{i=1}^{n} \tau_i(q, \delta)^{-2}, \tag{3.3}$$

where $\delta = |r(q)|$.

Suppose that $q^1, q^2 \in U \cap \Omega$. Define

$$M(q^1, q^2) = \inf\{\varepsilon > 0; q^2 \in P_\varepsilon(q^1)\},$$

where $P_\varepsilon(q^1)$ is constructed from the coordinates about q^1 as in Proposition 3.1. Set $\delta = M(q^1, q^2)$. Then

$$M(q^1, q^2) \approx |q_1^1 - q_1^2| + \sum_{i=2}^{n}\sum_{l=2}^{T} A_l^i(q^1)|q_i^1 - q_i^2|^l,$$

where the coordinates of q^1 and q^2 are measured in the coordinates associated to q^1 and δ by Proposition 3.1. In the following, we let D_i denote the differential operator $\dfrac{\partial}{\partial z_i}$ where z_i is one of the coordinates constructed in Proposition 3.1

associated to q^1. For multi-indices α and β, $D^\alpha \bar{D}^\beta = D_1^{\alpha_1} \ldots D_n^{\alpha_n} \bar{D}_1^{\beta_1} \ldots \bar{D}_n^{\beta_n}$ with the convention that the holomorphic derivatives act on the first n variables of the kernel function and the anti-holomorphic derivatives act on the last n-variables.

THEOREM 3.3. *Let $\Omega \subset\subset \mathbf{C}^n$ be a smoothly bounded, pseudoconvex domain. Suppose that near $p \in b\Omega$, Ω is convex and that p is a point of finite type T. There exists a neighborhood U of p so that, for all multi-indices α, β, there exists a constant $C_{\alpha\beta}$ such that for all q^1, $q^2 \in U \cap \Omega$,*

$$(3.4) \qquad | D^\alpha \bar{D}^\beta K_\Omega(q^1, q^2) | \leq C_{\alpha\beta} \prod_{i=1}^{n} \tau_i(q^1, \delta)^{-2-\alpha_i-\beta_i},$$

where $\delta = (| r(q^1) | + | r(q^2) | + M(q^1, q^2))$.

LEMMA 3.4. *Let Ω and $p \in b\Omega$ be as in Theorem 3.3. Then there exist a neighborhood U of p and a constant C so that*

$$| K(z, q)/K(q, q) | \leq C$$

for all z, $q \in U$.

Proof. Take $\alpha = \beta = 0$ in (3.4). Then we have

$$| K_\Omega(z, q) | \leq C' \prod_{i=1}^{n} \tau_i(q, \delta)^{-2},$$

where $\delta_1 = (| r(z) | + | r(q) | + M(q^1, q^2))$. Also from (3.3) we have $K_\Omega(q, q) \approx \Pi_{i=1}^n \tau_i(q, \delta_2)^{-2}$, for $\delta_2 = | r(q) |$. From (3.1), (3.2) and from the definition of τ_i, one can see that $\tau_i(q, \delta)$ is an increasing function for δ. So $| K_\Omega(z, q)/K_\Omega(q, q) | \leq C$ for some $C > 0$. $\square$

4. Construction of peak functions

Suppose that $\Omega \subset\subset \mathbf{C}^n$ is a smoothly bounded pseudoconvex domain and $p \in b\Omega$ is a point of finite 1-type. Then Catlin's theorem [3] says that there are $\varepsilon > 0$ and a neighborhood U in which $\bar{\partial}$-Neumann problem satisfies a subelliptic estimate of order $\varepsilon > 0$ on (0,1)-forms. For each $w \in U \subset \Omega$, define the function

$$h_w(z) = \frac{K_\Omega(z, w)}{K_\Omega(w, w)}.$$

We will estimate $K_\Omega(z, w)$ for z outside a certain neighborhood of w and will show that $|h_w(z)|$ is quite small outside that neighborhood. For the convenience in notation, we denote by C the various constants that follows.

Let π be a projection onto $b\Omega$ and set $p = \pi(w)$. For $\eta > 0$, let $B(p, \eta)$ denote the euclidean ball centered at p of radius η. Let $\zeta \in C^\infty(\mathbf{C}^n)$ be a function with the property that $\zeta \equiv 1$ on $\Omega \setminus B(p, \eta)$ and $\zeta \equiv 0$ on $B\left(p, \frac{\eta}{2}\right)$, and let N denote the $\bar\partial$-Neumann operator on (0,1)-forms. The following theorem was proved in [14].

Proposition 4.1. *Let Ω, U, and $\varepsilon > 0$ be as above. Let $s, t \in \mathbf{R}^+$. If α is a smooth* (0,1)*-form in the domain of the Kohn Laplacian and* $\operatorname{supp}\alpha \subset B\left(p, \frac{\eta}{8}\right)$ *then there is a constant $C_{st} > 0$ so that*

$$\| \zeta N\alpha \|_s^2 \le C_{st}\eta^{-2\left(\frac{s+t}{\varepsilon}+4\right)} \| \alpha \|_{-t}^2.$$

Let $\phi \in C_0^\infty(0,1)$ be a non-negative radial function with $\int \phi = 1$. For $w \in U$, set

$$\phi_w(z) = \left(\frac{\delta(w)}{2}\right)^{-2n} \phi\left(\frac{z-w}{\delta(w)/2}\right).$$

Then from the Kohn's formula,

$$K_\Omega(z, w) = \phi_w(z) - \bar\partial^* N \bar\partial \phi_w(z).$$

Assume $\operatorname{supp}\phi_w \subset B\left(p, \frac{\eta}{8}\right)$. Then from the Proposition 4.1 with $s = r + 1$, we have

$$\begin{aligned} \| \zeta(\cdot) K_\Omega(\cdot, w) \|_r^2 &\le C \| \zeta N \bar\partial \phi_w(\cdot) \|_{r+1}^2 \\ &\le C\eta^{-2\left(\frac{r+1+t}{\varepsilon}+4\right)} \| \bar\partial \phi_w(\cdot) \|_{-t}^2. \end{aligned}$$

If $t > n + 1$, Sobolev's lemma gives

$$\begin{aligned} \| \phi_w \|_{-(t-1)}^2 &= \sup\{ |(\phi_w, f)| \,;\, f \in C_0^\infty, \ \| f \|_{t-1} \le 1 \} \\ &\le C \left| \int \phi_w \right| \le C \end{aligned}$$

for some $C > 0$. If we choose $r > n + 1$, another application of Sobolev's lemma shows

$$\sup_z | \zeta(z) K_\Omega(z, w) | \le C \| \zeta(\cdot) K_\Omega(\cdot, w) \|_r.$$

Hence

$$\sup_z |\zeta(z)K_\Omega(z, w)| \leq C\eta^{-2\left(\frac{2n+4}{\varepsilon}+4\right)}.$$

Now set $\eta = \delta(w)^{\left(\frac{\varepsilon}{2(2n+4)}\right)}$. If we combine Theorem 3.2 and Theorem 3.3, we have proved the following:

PROPOSITION 4.2. *Let Ω and η be as above. There exists a constant $C > 0$, independent of $w \in U$, so that*

$$|h_w(z)| \leq C\delta(w)$$

for $w \in U$ and $z \in \bar{\Omega} \setminus B(\pi(w), \eta)$.

Now we denote by N the interior normal to the boundary of Ω at p. In [15], McNeal showed that the sharp subelliptic estimates of $\bar{\partial}$-equation holds near $p \in b\Omega$. So we may take $\varepsilon = \frac{1}{T}$. Set $\eta = \frac{1}{2(2n+4)T}$.

LEMMA 4.3. *For every q on N, let $|q - p| = d$. There exists a constant $C > 0$ such that for every point q on N sufficiently close to p, there exist a neighborhood U_p of p and a holomorphic function $h = h_q$ on Ω such that*

(1) $|h| < C$ *on* Ω,

(2) $h(q) = 1$,

(3) $|h(z)| < Cd$ *for* $z \in \Omega \setminus U_p$,

(4) $|Dh| < \frac{C}{d}$.

Proof. Define $h_q(z) = K(z, q)/K(q, q)$. Property (2) is clear. From Proposition 4.2, we have $|h_q(z)| \leq Cd$ for $z \in \bar{\Omega} \setminus B(\pi(q), d^\eta)$. This proves (3) with $U_p = B(p, d^\eta)$. Lemma 3.4 gives $|h| \leq C$ for $z \in U_p = B(\pi(q), d^\eta)$. This fact together (3) gives (1). Also from the estimated of Theorem 3.2 and Theorem 3.3, we have $|Dh| < \frac{C}{d}$. □

We now ready to prove Theorem 1.1.

Proof of Theorem 1.1. We may assume that $p = 0$. Let the type of p is equal to T and choose a neighborhood U of p such that the subelliptic estimates for $\bar{\partial}$-equation of order $\frac{1}{T}$ hold on U and Proposition 3.1, Theorem 3.2 and Theorem

3.3 hold on U. We denote by N the interior normal to the boundary of Ω at p. For each neighborhood $V \subset\subset U$ of p, choose a neighborhood V_1, $V_1 \subset\subset V \subset\subset U$, so that $V_1 \cap \Omega$ is convex. Next we consider a 1-parameter family of a smooth bumping family $\{\Omega_t\}_{0 \leq t < t_0}$ outside V_1. We may assume that $V_1 \cap \Omega = V_1 \cap \Omega_t$ for all $t > 0$, after perhaps shrinking V_1, and $\Omega \setminus V \subset\subset \Omega_t \setminus V$ for all $0 < t < t_0$. Now fix $0 < t_1 < t_0$ and consider the pseudoconvex domain Ω_{t_1}. Since the type condition is stable, we may assume that $b\Omega_{t_1}$ is of finite type. Also we have $d = \operatorname{dist}(\Omega \setminus V,\ b\Omega_{t_1} \setminus V) > 0$. Choose (by induction) a sequence q_n coverging to p along N and define $h_n(z) = h_{q_n}(z + q_n - p)$, where h_{q_n} is the function defined on Ω_{t_1} (instead of Ω) as in Lemma 4.3 associated with q_n. Notice that $h_n(z)$ is defined on $\bar{\Omega} \setminus V$, and on U intersect a translate of Ω which contains $\bar{\Omega} \cap \bar{V}$, provided q_0 is sufficiently close to p. Therefore h_n is well defined and holomorphic on $\bar{\Omega}$ for each $n \geq 0$. Let U_n denote the neighborhood corresponding to h_n as in Lemma 4.3. Without loss of generality, we may assume that $U_{n+1} \subset\subset U_n$. For a suitable constant $0 < c < 1$, to be determined later, let $r = 1 - c$ and define a peak function as $H = r \sum_{n=0}^{\infty} c^n h_n$. Let us estimate H on various sets. First outside U_0. Then $|h_n| < \frac{1}{2}$ for every n by the property (3) of Lemma 4.3 provided that q_0 is sufficiently close to p. So we get that $|H| < r \sum_{n=0}^{\infty} \frac{c^n}{2} = \frac{1}{2}$. From the continuity of h_k, $0 \leq k \leq n-1$, and from the fact that $h_k(p) = 1$, we may (inductively) choose q_n and hence a neighborhood U_n so that $|h_k| < r_n$ for $k < n$ on U_n, where $r_n > 1$ is arbitrary close to 1. Hence for $z \in U_n \setminus U_{n+1}$, we have that $|h_n| < C$ and $|h_k(z)| < \frac{1}{2}$ (by the property (3) of the Lemma 4.3) if $k > n$. Hence for $z \in U_n \setminus U_{n+1}$, we estimate H as follows

$$\begin{aligned}
|H| &< r\left[\sum_{k<n} r_n c^k + c^n C + \sum_{k>n} c^k 2\right] \\
&= r\frac{(1-c^n)}{1-c} r_n + rc^n C + \frac{r}{2}\frac{c^{n+1}}{2(1-c)} \\
&= (1-c^n) r_n + \frac{1}{2} c^{n+1} + rCc^n \\
&= 1 - c^n s_n + \frac{1}{2} c^{n+1} + rc^n C
\end{aligned}$$

where $s_n < 1$ can be chosen arbitrary close to 1. If $rC = \frac{1}{3}$, for instance, we have $|H| < 1$ and $r = \frac{1}{3C}$, $c = \frac{1}{3C}$. Notice that $H(p) = 1$ and H is continuous on $\bar{\Omega}$

and holomorphic on $\bar{\Omega}\setminus V$ and this proves Theorem 1.1. □

Remark 4.4. *In the proof of Theorem* 1.1, *we may take, for example,* $r_n = 1 + \frac{1}{10}c^n$ *and* $s_n = 1 - \frac{1}{10} + \frac{c^n}{10}$.

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