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ON COMPACTNESS OF ISOSPECTRAL CONFORMAL METRICS OF 4-MANIFOLDS

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§1. Introduction

In this paper, we are interested in the compactness of isospectral conformal metrics in dimension 4.

Let us recall the definition of the isospectral metrics. Two Riemannian metrics g, g' on a compact manifold are said to be isospectral if their associated Laplace operators on functions have identical spectrum. There are now numeruos examples of compact Riemannian manifolds which admit more than two metrics such that they are isospectral but not isometric. That is to say that the eigenvalues of the Laplace operator Δ on the functions do not necessarily determine the isometry class of (M, g). If we further require the metrics stay in the same conformal class, the spectrum of Laplace operator still does not determine the metric uniquely ([BG], [BPY]).

Thus the problem is to know how much we can say about the metric from its spectrum.

In dimension 2, Osgood, Phillips and Sarnak [OPS] proved that the set of isospectral metrics on a compact surface form a compact family in the C^∞ topology. Remember that in dimension 2, all metrics on a compact Riemann surface are conformally equivalent.

In dimension 3, Brooks, Perry and Yang [BPY] had showed that if (M, g_0) has negative constant scalar curvature, an isospectral set of metrics $g = u^4 g_0$ which are pointwise conformal to g_0 is compact in the C^∞ topology. Later Chang and Yang [CY1, CY2] had showed that this is true for general compact 3-manifold without boundary. Recently, M. Anderson [An] and Brooks, Perry and Petersen [BPP], independently, show that this is still true for general metrics which are not necessary conformal to g_0 provided the Sobolev constants have uniform lower bound. Therefore for dimension 3, the compactness of isospectral metrics is reduced to how to estimate the Sobolev constant from the spectrum.

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When the dimension gets higher (i.e., $n \geq 4$), the problem becomes much more difficult. The reason is that at this moment we do not know how to compute all spectral invariants. For the first few computable heat invariants, when dimension is getting bigger, they tells us less information and so do Sobolev inequalities. Hence we can not expect get any cheap results without further assumption. What we can do in this paper is to show the following:

THEOREM 1. *Let M be a compact 4-manifold and $\{g_0\}$ a negative conformal class on M. Then for $g_1 \in \{g_0\}$, the set of the metrics g in $\{g_0\}$ which are isospectral to g_1 is compact in the C^∞ topology if and only if $\int_M s_g^4 dv_g \leq C_0$ for some constant C_0 and where s_g is the scalar curvature of the metric g.*

For standard 4-sphere, the conformal group G complicates the analysis. Since G is not compact, we can not expect to have same conclusion as Theorem 1 above.

DEFINITION. *For a positive function u on S^n and φ a conformal transformation, define*

$$u_\varphi = u \circ \varphi \,|\, d\varphi \,|^{1/n}$$

where $|\, d\varphi \,|$ is the linear stretch of $d\varphi$ measured with respect to the standard metric g_0.

Thus $u_\varphi^{4/(n-2)} g_0 = \varphi^*(u^{4/(n-2)} g_0)$. We set

$$[u] = \{u_\varphi \mid \varphi \in G, \text{ the conformal group of } S^4\}.$$

We also will show the following

THEOREM 2. *For (S^4, g_0), if $\{g_i = u_i^2 g_0\}$ is a sequence of isospectral metrics, then there exist conformal transformations φ_i for g_i such that $\{\varphi^* g_i\}$ is compact in the C^∞ topology provided there exists a constant $C_1 > 0$ such that $\int_{S^4} s_{g_i}^4 dv_{g_i} \leq C_1$.*

The condition we provided here is motivated by Chang and Yang's paper [CY2]. At the end of their paper, it was pointed out that if the condition $\int_{S^3} s_g^2 dv_g \leq C_0$ is replaced by $\int_{S^n} s_g^{n/2+\delta} dv_g \leq C_0$ for some $\delta > 0$ and $n \geq 4$ in their Theorem 1′, then their Theorem 1′ still holds. We follow from their argument to get pointwise estimates on conformal factors. Therefore our argument gives slightly more general results than what we have stated in above. It was informed by Paul Yang that Gursky [Gu] has gotten some estimates for higher dimensional manifolds in terms of L^p integrals of the whole curvature tensor for $p > n/2$.

The proof given here is similar to the proof given by Chang and Yang for three dimensional case ([CY1]). Since we restrict ourselves to the same conformal class, in order to get control on curvature tensors, we have to get estimates on conformal factors. Our starting point is to get pointwise lower bounds of conformal factors which will be given in next section. Once we have lower bounds of conformal factors at hand, we can employ them to get upper bounds too. Detail of this argument will provide in our section three. Section 4 will simply contribute to L^4 integral bound of full curvature tensors. With all those estimates done, we can apply Cheeger-Gromov's Compactness theorem ([C], [G]), reduce our compactness to get pointwise control on our curvature tensors and their higher derivatives. That will follow from higher order heat invariants whose leading coefficients have been computed by P. Gilkey [G2]. This is our main content in section five.

§2. Lower bounds of conformal factors

Notation first. Let M be a compact manifold of dimension n with metric g_0. dv_0 will be used to denote the volume element of g_0 and s_0 the scalar curvature of (M, g_0). Let $p = \dfrac{n+2}{n-2}$ and $g = u^{4/(n-2)} g_0$ for some positive function u. Then it is standard calculation that the volume element and the scalar curvature of g are given by

$$dv = u^{p+1} dv_0 \tag{1}$$

and

$$s_g = u^{-p}(s_0 u - c_n \Delta u) \tag{2}$$

where $c_n = 4(n-1)/(n-2)$ and Δ is understood to take with respect to the metric g_0.

First of all, we consider the negative conformal class. Without loss of generality, we can assume that g_0 is a negative constant.

Lemma 1. *If $s_0 < 0$ and $\int_M s_g^4 dv_g \leq C_0$, then there exists a constant $C_\alpha > 0$ such that $u \geq C_\alpha$ where $g = u^2 g_0$.*

Proof. Using (2), we can get

$$
\begin{aligned}
C_0 &\geq \int_M s_g^4 dv_g \\
&= \int_M [u^{-3}(s_0 u - c_4 \Delta u)]^4 u^4 dv_0 \\
&= \int_M s_0^4 u^{-4} dv_0 - 24 s_0^3 \int_M \Delta u u^{-5} dv_0 + 216 s_0^2 \int_M (\Delta u)^2 u^{-6} dv_0 \\
&\quad - 864 s_0 \int_M (\Delta u)^3 u^{-7} dv_0 + 1296 \int_M (\Delta u)^4 u^{-8} dv_0 \\
&= \int_M u^{-4} s_0^4 dv_0 - 24 s_0^3 \int_M u^{-5} (\Delta u) dv_0 \\
&\quad - 216 \int_M (\Delta u)^2 u^{-8} [s_0 - 2\Delta u]^2 dv_0 + 432 \int_M u^{-8} (\Delta u)^4 dv_0.
\end{aligned} \tag{3}
$$

Since $s_0 < 0$, $s_0^3 < 0$. Also

$$\int_M u^{-5} \Delta u = 5 \int_M u^{-6} |\nabla u|^2 > 0.$$

Thus in (3), each term on the right hand side is positive. In particular, we get

$$\int_M u^{-4} dv_0 \leq C_0 s_0^{-4} \tag{4}$$

and

$$\int_M u^{-8} (\Delta u)^4 dv_0 \leq C_0. \tag{5}$$

Now from the identity,

$$\Delta(u^{-1}) = -u^{-2} \Delta u + 2u^{-3} |\nabla u|^2,$$

it is easy to see that for any point $p \in M$,

$$u^{-1}(p) - \left(\int_M dv_0\right)^{-1} \int_M u^{-1} dv_0 = -\int_M G(p, q)[-u^{-2} \Delta u + 2u^{-3} |\nabla u|^2] dv_0, \tag{6}$$

where $G(p, q)$ is the Green's function on M which can be assumed to be positive everywhere on M. The well known fact is that the Green's function on a 4-dimensional manifold is L^p integrable for $p < 2$. Thus combining (4), (5) and Hölder inequality, the equation (6) simply implies that there is a constant $C_\alpha > 0$ which does not depend on u such that $u(p) \geq C_\alpha > 0$. We have thus finished the proof of Lemma 1.

Now we consider the sphere (S^4, g_0) case.

LEMMA 2. *On (S^4, g_0), if $g = u^2 g_0$ is a conformal metric satisfying*

$$\alpha_0 = \int_{S^4} u^4 dv_0, \quad \int_{S^4} |s_g|^{2+\delta_0} u^4 dv_0 \leq \alpha_2, \quad \lambda_1(g) > \Lambda > 0$$

where $0 < \delta_0 \leq 2$ and $\Lambda > 0$ are two constants, then there are a constant $C_{12} = C_{12}(\alpha_0, \alpha_2, \delta_0, \Lambda) > 0$ and a conformal transformation φ such that $v = u_\varphi$ satisfies $v \geq C_{12}$.

Proof. Applying Lemma 1 of [CY2], we have a $v \in [u]$ such that

$$\int_{S^4} v^4 x_j dv_0 = 0$$

for $j = 1,2,\ldots,5$ where x_j are the ambient coordinates of S^4. The key point is that $u^2 g_0$ is isometric to $v^2 g_0$. Thus they have same geometry, for example, the same volume and the same first eigenvalues. Thus if we denote $v^2 g_0$ by g again, we have

$$\begin{aligned} \int_{S^4} v^4 x_j^2 dv_0 &\leq [\lambda_1(g)]^{-1} \int_{S^4} |\nabla x_j|_g^2 v^4 dv_0 \\ &\leq [\lambda_1(g)]^{-1} \int_{S^4} v^2 |\nabla x_j|_{g_0}^2 dv_0. \end{aligned} \tag{7}$$

Remember $x_1^2 + x_2^2 + \cdots + x_5^2 = 1$ and $|\nabla x_1|^2 + \cdots + |\nabla x_5|^2 = 4 (= \lambda_1(S^4, g_0))$. Summary the inequality (7) from $j = 1$ to $j = 5$ to get

$$\alpha_0 = \int_{S^4} u^4 dv_0 = \int_{S^4} v^4 dv_0 \leq (\Lambda)^{-1} \int_{S^4} 4v^2 dv_0.$$

That is,

$$\int_{S^4} v^2 dv_0 \geq 4^{-1} \alpha_0 \Lambda = C_3 > 0. \tag{8}$$

Let $\eta = (C_3/(2\mathrm{vol}(g_0)))^{1/2} > 0$ and $\Omega = \{x \in S^4 \mid v(x) \geq \eta\}$. Then we have estimate

$$0 < C_3 \leq \int_{S^4} v^4 dv_0 = \int_{\Omega} v^2 dv_0 + \int_{S^4 \setminus \Omega} v^2 dv_0$$

$$\leq \left(\int_\Omega v^4 dv_0\right)^{1/2} (\mathrm{vol}(\Omega))^{1/2} + \eta^2 \mathrm{vol}(S^4 \setminus \Omega)$$
$$\leq \alpha_0^{1/2} \mathrm{vol}(\Omega)^{1/2} + \frac{C_3}{2}.$$

Therefore we have

$$\mathrm{vol}(\Omega) \geq \frac{C_3^2}{(4\alpha_0)} > 0. \tag{9}$$

Now let $0 < \delta < \dfrac{\delta_0}{(2+\delta_0)}$ be chosen later, multiply the equation (2) with $n = 4$ by $v^{-1-2\delta}$ and apply integration by parts to get

$$6\int_{S^4} |\nabla v^{-\delta}|^2 dv_0 = -\frac{\delta_2}{(1+2\delta)} \int_{S^4} s_g v^{2-2\delta} dv_0 + \frac{(\delta^2 s_0)}{(1+2\delta)} \int_{S^4} v^{-2} dv_0.$$

Now let $\lambda_1(g_0) = \lambda_1$ denote the first eigenvalue of Δ acting on functions, i.e., $\lambda_1 = 4$ then by the Rayleigh-Ritz characterization for λ_1, we get

$$\begin{aligned}\int_{S^4} v^{-2\delta} dv_0 &\leq (\mathrm{vol}(S^4))^{-1}\left(\int_{S^4} v^{-\delta} dv_0\right)^2 + (1/\lambda_1)\int_{S^4} |\nabla v^{-\delta}|^2 dv_0.\\ &\leq \mathrm{vol}(S^4)^{-1}\left(\int_{S^4} v^{-\delta} dv_0\right)^2 + \frac{\delta^2 s_0}{(6\lambda_1(1+2\delta))}\int_{S^4} v^{-2\delta}\, dv_0\\ &\quad - \frac{\delta^2}{6\lambda_1(1+2\delta)}\int_{S^4} s_g v^{2-2\delta}\, dv_0.\end{aligned} \tag{10}$$

From the equation (9), we should have

$$\begin{aligned}\int_{S^4} v^{-\delta} dv_0 &= \int_\Omega v^{-\delta} dv_0 + \int_{S^4\setminus\Omega} v^{-\delta} dv_0\\ &\leq \eta^{-\delta}\mathrm{vol}(\Omega) + \left(\int_{S^4\setminus\Omega} v^{-2\delta} dv_0\right)^{1/2} (\mathrm{vol}(S^4\setminus\Omega))^{1/2}.\end{aligned} \tag{11}$$

Hence, by taking the square in equation (11) and being divided by the volume and using Hölder, we have

$$\mathrm{vol}(S^4)^{-1}\left(\int_{S^4} v^{-\delta} dv_0\right)^2 \leq (1 + 1/\gamma)[\eta^{-2\delta}\mathrm{vol}(\Omega)^2] + (1+\gamma)\frac{\mathrm{vol}(S^4\setminus\Omega)}{\mathrm{vol}(S^4)}\int_{S^4} v^{-2\delta} dv_0.$$

for all positive γ. Now as $\mathrm{vol}(\Omega) \geq C_3 > 0$, $\mathrm{vol}(S^4 \setminus \Omega) = (1-2\theta)\mathrm{vol}(S^4)$ for some $\theta = \theta(\mathrm{vol}(\Omega)) > 0$. Therefore we can take γ small enough such that $(1+\gamma)(1-2\theta) \leq (1-\theta)$. Now combining the equations (10) and (11), we obtain

$$
\begin{aligned}
\int_{S^4} v^{-2\delta} dv_0 &\leq C_4(1-\theta)\int_{S^4} v^{-2\delta} dv_0 + \frac{\delta^2 s_0}{(6\lambda_1(1+2\delta))}\int_{S^4} v^{-2\delta}\, dv_0 \\
&\quad - \frac{\delta^2}{6\lambda_1(1+2\delta)}\int_{S^4} s_g v^{-2\delta}\, dv_0 \\
&\leq C_4 + (1-\theta)\int_{S^4} v^{-2\delta} + \frac{\delta^2}{2(1+2\delta)}\int_{S^4} v^{-2\delta}\, dv_0 \\
&\leq \frac{\delta^2}{6\lambda_1(1+2\delta)}\left[\int_{S^4} s_g v^{\frac{4}{(2+\delta_0)}} v^{2-2\delta-\frac{4}{(2+\delta_0)}}\, dv_0\right].
\end{aligned}
\tag{12}
$$

Now make the choice of arbitrary constant δ. Choose $\delta < \dfrac{\delta_0}{(2+\delta_0)}$ such that

$$
\frac{\delta^2}{2(1+2\delta)} < \frac{1}{2}\theta. \tag{13}
$$

And also by Hölder inequality and the assumption we know that

$$
\begin{aligned}
&\left|\int_{S^4} s_g v^{\frac{4}{(2+\delta_0)}} v^{2-2\delta-\frac{4}{(2+\delta_0)}}\, dv_0\right| \\
&\qquad \leq \left\{\int_{S^4} |s_g|^{2+\delta_0} v^4 dv_0\right\}^{\frac{1}{(2+\delta_0)}} \left\{\int_{S^4} v^{\frac{2(\delta_0-\delta(2+\delta_0))}{1+\delta_0}}\, dv_0\right\}^{\frac{1+\delta_0}{2+\delta_0}}
\end{aligned}
$$

can be bounded by some constant $C_5 > 0$. Thus the equation (12) tells us that

$$
\begin{aligned}
\int_{S^4} v^{-2\delta} dv_0 &\leq \frac{2C_4}{\theta} + \frac{2}{\theta}\frac{\delta^2}{6\lambda_1(1+2\delta)} C_5 \\
&\leq \frac{2C_4}{\theta} + \frac{1}{12} C_5 \\
&\equiv C_6.
\end{aligned}
$$

Let $G(p, q)$ denote the Green's function for Δ with singularity at p. We may add a constant and assume $G(p, q)$ is positive. Then we have

$$
\begin{aligned}
v^{-\alpha}(p) &= \frac{1}{\mathrm{vol}(S^4)}\int_{S^4} v^{-\alpha} dv_0 - \int_{S^4} G(p, q)\Delta v^{-\alpha} dv_0(q) \\
&= \frac{1}{\mathrm{vol}(S^4)}\int_{S^4} v^{-\alpha} dv_0 - \frac{1}{6}\int_{S^4} G(p, q)\,[\alpha s_g v^{2-\alpha} - s_0 \alpha v^{-\alpha} \\
&\quad + 6\alpha(1+\alpha) v^{-\alpha-2} |\nabla v|^2]\, dv_0(q) \\
&\leq \frac{1}{\mathrm{vol}(S^4)}\int_{S^4} v^{-\alpha} dv_0 - \frac{(\alpha s_0)}{6}\int_{S^4} v^{-\alpha} G(p, q)\, dv_0
\end{aligned}
\tag{14}
$$

$$-\frac{\alpha}{6}\int_{S^4} G(p, q) s_g v^{2-\alpha} dv_0.$$

Now by Hölder inequality, we have

$$\int_{S^4} v^{-\alpha} G(p, q) dv_0 \leq \left[\int_{S^4} v^{-\alpha r/(r-1)} dv_0\right]^{(r-1)/r} \left[\int_{S^4} G^r(p, q) dv_0\right]^{1/r}. \tag{15}$$

If we choose $1 < r < 2$ and $\alpha < \frac{(2\delta)(r-1)}{r}$ where $\delta > 0$ is determined in the equation (13), then equation (15) says that the second term in the right hand side of the equation (14) is bounded. Also, using Hölder inequality, we get

$$\begin{aligned} &\left|\int_{S^4} G(p, q) s_g v^{2-\alpha} dv_0\right| \\ &= \left|\int_{S^4} G(p, q) s_g v^{4/(2+\delta_0)} v^{2-\alpha-4/(2+\delta_0)} dv_0\right| \\ &\leq \left[\int_{S^4} G^r(p, q) dv_0\right]^{1/r} \left[\int_{S^4} | s_g |^{2+\delta_0} v^4 dv_0\right]^{1/(2+\delta)} \\ &\quad \cdot \left[\int_{S^4} v^{\frac{(r(2-\alpha)(2+\delta_0)-4)}{(r(2+\delta_0)-(2+\delta_0+r))}} dv_0\right]^{\frac{(r(2+\delta_0)-(2+\delta_0+r)}{r(2+\delta_0)}}. \end{aligned} \tag{16}$$

Now we first choose r such that

$$2 > r > \frac{(2+\delta_0)}{(1+\delta_0)}.$$

We then choose α sufficiently small for a fixed r so that

$$0 < \alpha < \min\left\{2\delta, \frac{[2\delta(r(1+\delta_0) - (2+\delta_0)) + 2\delta_0 r]}{(r(2+\delta_0))}, \frac{[2\delta(r-1)]}{r}\right\}. \tag{17}$$

Finally by Hölder inequality, the first term and the third term in the right hand side of the equation (14) are bounded in terms of something which does not depend on the point p. Thus there is a constant $C_{12} > 0$ such that $u \geq C_{12}$. Thus this proves Lemma 2.

§3. Upper bounds of conformal factors

The main purpose of this section is to show the following

LEMMA 3. *Suppose that* $\lambda_1(\Delta_g) \geq \Lambda > 0$, $\int_M s_g^{2+\delta_0} dv_0 \leq C_0$ *and* $\int_M u^4 dv_0 = \alpha_0$.

Denote the Sobolev constant by C_1 *and the geometry by the constant* C_2. *If the function* $u \geq C > 0$ *for some constant* C, *then there are a constant* $\varepsilon_0 = \varepsilon_0(\Lambda, \alpha_0, C_0, C_1, C_2, C) > 0$ *and a constant* $C_7 = C_7(\Lambda, C_0, \alpha_0, C_1, C_2, C) > 0$ *such that*

$$\int_M u^{4+\delta_0} dv_0 \leq C_7. \tag{18}$$

Remark. It will be used to get the pointwise upper bound of function u.

Proof. Let $w = u^{1+\varepsilon}$. From the Sobolev inequality for w ([Au]), we have

$$\left(\int_M w^4 dv_0\right)^{1/2} \leq C_1 \int_M |\nabla w|^2 dv_0 + C_2 \int_M w^2 dv_0 \tag{19}$$

where C_1 and C_2 depend only on the geometry of M, C_1 is called the Sobolev constant.

On the other hand, multiply (2) by $u^{1+2\varepsilon}$ for $n = 4$, to obtain,

$$6(\Delta u) u^{1+2\varepsilon} + s_g u^{4+2\varepsilon} = s_0 u^{2+2\varepsilon}. \tag{20}$$

And now integrating (20) and using integration by parts, we get

$$6 \frac{(1+2\varepsilon)}{(1+\varepsilon)^2} \int_M |\nabla w|^2 dv_0 = \int_M s_g u^2 w^2 dv_0 - s_0 \int_M w^2 dv_0. \tag{21}$$

Notice that for $\varepsilon < 1$, $\int_M w^2 dv_0$ is bounded by some constant multiplying $\int_M u^4 dv_0$. We conclude that

$$\left(\int_M w^4 dv_0\right)^{1/2} \leq C_1 \frac{(1+\varepsilon)^2}{(6(1+2\varepsilon))} \int_M s_g u^2 w^2 dv_0 + C_2(\varepsilon). \tag{22}$$

For any $\eta > 0$, let $E = \{x \in M \mid |s_g| \geq (C_0 \eta^{-1})^{1/\delta_0}\}$. Then

$$\begin{aligned} C_0 &\geq \int_M |s_g|^{2+\delta_0} u^4 dv_0 \\ &\geq \int_E |s_g|^{2+\delta_0} u^4 dv_0 \\ &\geq C_0 \eta^{-1} \int_E s_g^2 u^4 dv_0. \end{aligned}$$

Therefore we have

$$\int_E s_g^2 u^4 dv_0 \leq \eta, \tag{23}$$

and

$$| s_g | \leq C_0^{1/g_0} \eta^{-1/\delta_0} \text{ on } M \backslash E. \tag{24}$$

This implies that

$$\begin{aligned} &\left| \int_M s_g u^2 w^2 dv_0 \right| \\ &\leq \int_M | s_g | w^2 u^2 dv_0 \\ &\geq C_0^{1/\delta_0} \eta^{-1/\delta_0} \int_M u^2 w^2 dv_0 + \eta^{1/2} \left[\int_M w^4 dv_0 \right]^{1/2}. \end{aligned} \tag{25}$$

To estimate $\int_M u^2 w^2 dv_0$, we apply the Rayleigh-Ritz characterization of $\lambda_1(\Delta_g)$

$$\lambda_1(\Delta_g) \leq \frac{\int_M | \nabla \phi |_g^2 u^4 dv_0}{\int_M \left[\phi - (\mathrm{vol}(M))^{-1} \int_M \phi u^4 dv_0 \right]^2 u^4 dv_0}, \tag{26}$$

or equivalently,

$$\int_M \phi^2 u^4 dv_0 \leq (\mathrm{vol}(M))^{-1} \left(\int_M \phi u^4 dv_0 \right)^2 + \Lambda^{-1} \int_M u^2 | \nabla \phi |^2 dv_0 \tag{26'}$$

to $\phi = u^\varepsilon$ to obtain

$$\begin{aligned} \int_M u^2 w^2 &\leq \left[\int_M u^4 dv_0 \right]^{-1} \left[\int_M u^{4+\varepsilon} dv_0 \right]^2 + \Lambda^{-1} \int_M u^2 | \nabla u^\varepsilon |^2 dv_0 \\ &= \left[\int_M u^4 dv_0 \right]^{-1} \left[\int_M u^{4+\varepsilon} dv_0 \right]^2 + \frac{\varepsilon^2}{(\Lambda(1+\varepsilon))} \int_M | \nabla w |^2 dv_0 \\ &= \left[\int_M u^4 dv_0 \right]^{-1} \left[\int_M u^{4+\varepsilon} dv_0 \right]^2 + \frac{\varepsilon^2}{(6\Lambda(1+2\varepsilon))} \left[\int_M s_g u^2 w^2 dv_0 \right. \\ &\qquad \left. - s_0 \int_M w^2 dv_0 \right], \end{aligned} \tag{27}$$

where we have used the equation (21) to obtain the second equality.

To estimate $\int_M u^{4+\varepsilon} dv_0$, first of all, by assumption we have $C > 0$ with $u - C \geq 0$. Apply this to get

$$\begin{aligned}\int_M u^{4+\varepsilon} dv_0 &= \int_M (u^4 - C^4) u^{\varepsilon} dv_0 + C^4 \int_M u^{\varepsilon} dv_0 \\ &\leq \left[\int_M (u^4 - C^4) u^{2\varepsilon} dv_0\right]^{1/2} \left[\int_M (u^4 - C^4) dv_0\right]^{1/2} + C^4 \int_M u^{\varepsilon} dv_0,\end{aligned}$$

where inequality comes from Cauchy-Schwartz inequality and the positivity of $u^4 - C^4$. Thus we have

$$\begin{aligned}\left[\int_M u^{4+\varepsilon} dv_0\right]^2 &\leq (1+\gamma)\left[\int_M (u^4 - C^4) u^{2\varepsilon} dv_0\right] \left[\int_M (u^4 - C^4) dv_0\right] \\ &\quad + \left(1 + \frac{1}{\gamma}\right) C^8 \left[\int_M u^{\varepsilon} dv_0\right]^2,\end{aligned} \tag{28}$$

where γ will be chosen later. But

$$\int_M (u^4 - C^4) dv_0 = \alpha \int_M u^4 dv_0$$

where $\alpha = 1 - \frac{(C^4 \mathrm{vol}(M))}{\int_M u^4 dv_0}$ is a positive constant less than 1 and we conclude that

$$\begin{aligned}&\left[\int_M u^4 dv_0\right]^{-1} \left[\int_M u^{4+\varepsilon} dv_0\right]^2 \\ &\leq (1+\gamma)\alpha\left[\int_M (u^4 - C^4) u^{2\varepsilon} dv_0\right] + \left(1 + \frac{1}{\gamma}\right) C^8 \frac{\left[\int_M u^{\varepsilon} dv_0\right]^2}{\left[\int_M u^4 dv_0\right]} \\ &= (1+\gamma)\alpha \int_M u^{4+2\varepsilon} dv_0 + \left[\left(1 + \frac{1}{\gamma}\right) C^8 \frac{\left[\int_M u^{\varepsilon} dv_0\right]^2}{\left[\int_M u^4 dv_0\right]} - C^4 (1+\gamma)\alpha \int_M u^{2\varepsilon} dv_0\right].\end{aligned} \tag{29}$$

Since we assume $\varepsilon < 1$ and $\alpha_0 = \int_M u^4 dv_0 > 0$, we get the conclusion that the second term in the right hand side of the inequality (29) is bounded by some constant. Choosing γ so that $(1+\gamma)\alpha = (1-\beta) < 1$, from (27) we then have

$$\int_M u^2 w^2 dv_0 \leq (1-\beta) \int_M u^2 w^2 dv_0 + C_8 + \frac{\varepsilon^2}{(6\Lambda(1+2\varepsilon))} \int_M s_g u^2 w^2 dv_0.$$

It is equivalent to

$$\int_M u^2 w^2 dv_0 \leq \frac{1}{\beta} \frac{\varepsilon^2}{(6\Lambda(1+2\varepsilon))} \int_M s_g u^2 w^2 dv_0 + \frac{2}{\beta} C_8. \tag{30}$$

Combine (30) and (25) to obtain

$$\begin{aligned}
&\int_M s_g u^2 w^2 dv_0 \\
&\leq C_0^{\frac{1}{\delta_0}} \eta^{\frac{-1}{\delta_0}} \int_M u^2 w^2 dv_0 + \eta \left[\int_M w^4 dv_0\right]^{1/2} \\
&\leq \frac{(C_0^{1/\delta_0}\eta^{-1/\delta_0}\varepsilon^2)}{[6\beta\Lambda(1+2\varepsilon)]} \int_M s_g u^2 w^2 dv_0 + \frac{(C_0^{1/\delta_0}\eta^{-1/\delta_0}C_8)}{\beta} + \eta\left[\int_M w^4 dv_0\right]^{1/2}.
\end{aligned}$$

Therefore

$$\left(1 - \frac{(C_0^{1/\delta_0}\eta^{-1/\delta_0}\varepsilon^2)}{(6\beta\Lambda(1+2\varepsilon))}\right) \int_M s_g u^2 w^2 dv_0 \leq \eta \left[\int_M w^4 dv_0\right]^{1/2} + C_9$$

where $C_9 = \dfrac{(C_0^{1/\delta_0}\eta^{-1/\delta_0}C_8)}{\beta}$. From the equation (22), if we set $\mu = \dfrac{(C_0^{1/\delta_0}\eta^{-1/\delta_0}\varepsilon^2)}{(6\beta\Lambda(1+2\varepsilon))}$,

$$\begin{aligned}
&\frac{(6(1+2\varepsilon))}{(C_1(1+\varepsilon)^2)}(1-\mu)\left[\int_M w^4 dv_0\right]^{1/2} \\
&\leq \eta\left[\int_M w^4 dv_0\right]^{1/2} + (1-\mu)C_2(\varepsilon)\frac{6(1+2\varepsilon)}{C_1(1+\varepsilon)^2} + C_9.
\end{aligned} \tag{31}$$

Now we choose $\eta = \dfrac{1}{C_1}$ where C_1 is a Sobolev constant given in (19). Then choose $\varepsilon > 0$ small enough such that

$$\mu < \frac{1}{2}.$$

Finally we have reached the following

$$\begin{aligned}
&\frac{(6(1+2\varepsilon))}{(C_1(1+\varepsilon)^2)}(1-\mu) - \eta \\
&\geq \frac{(6(1+2\varepsilon))}{(C_1(1+\varepsilon)^2)}\frac{1}{2} - \frac{1}{C_1} = \frac{1}{C_1}\frac{[(2-\varepsilon^2)+4\varepsilon]}{(1+\varepsilon)^2} > 0
\end{aligned}$$

because $0 < \varepsilon < 1$. Hence from equation (31).

$$
\begin{aligned}
\left[\int_M w^4 dv_0\right]^{1/2} & \\
&\leq \frac{[((1-\mu)C_2(\varepsilon) + C_9)((2-\varepsilon^2)+4\varepsilon)]}{(C_1(1+\varepsilon)^2)} \\
&\equiv (C_7)^{\frac{1}{2}}.
\end{aligned}
$$

This completes the proof of Lemma 3.

Now we can state our main conclusion at this section as

PROPOSITION 1. *Let* $a_0 = \int_M u^4 dv_0$, $\alpha_0 = \int_M s_g^{2+\delta_0} u^4 dv_0$ *and* $\lambda_1(\Delta_g) \geq \Lambda > 0$, $u \geq C > 0$. *Then there exists a constant* $C_{10} = C_{10}(a_0, a_0, \Lambda, C) > 0$ *such that*

$$
u \leq C_{10}. \tag{32}
$$

Proof. Applying Green's function to equation (2), we have

$$
\begin{aligned}
u(p) - \mathrm{vol}(M)^{-1}\int_M u dv_0 &= \int_M (-\Delta u)(q) G(p,q) dv_0(q) \\
&= \frac{1}{6}\int_M (s_g u^3 - s_0 u) G dv_0.
\end{aligned} \tag{33}
$$

Since $\mathrm{vol}(M)^{-1}\int_M u dv_0$ and $\int_M uG dv_0$ are a priori bounded, to bound $u(p)$, it suffices to bound $\int_M s_g u^3 G dv_0$.

It is well known that $|G(p,q)| \leq \dfrac{K}{d^2(p,q)}$ for some constant K [Au, p.108]. Recall the following estimate [Au, p.37]: for $h(y) = \int_{R^4} \dfrac{f(x)}{\|x-y\|^2} dx$, we have

$$
\|h\|_r \leq C(r')\|f\|_{r'} \tag{34}
$$

where $\dfrac{1}{r} = \dfrac{1}{2} + \dfrac{1}{r'} - 1 = \dfrac{1}{r'} - \dfrac{1}{2}$ with $r > 1$.

We will iterate this estimate with a sequence of suitable choice of r_j and r_j'. Start with $r_0' = \dfrac{r_0(2+\delta_0)}{2+3\delta_0+r_0}$, $r_0 = 4 + 4\varepsilon$ for $4\varepsilon \leq 4\varepsilon_0$, we have

$$
\begin{aligned}
\int_M |s_g u^3|^{r_0'} dv_0 &\leq \left[\int_M |s_g|^{2+\delta_0} u^4 dv_0\right]^{\frac{r_0'}{2+\delta_0}} \left[\int_M u^{\frac{r_0'(2+3\delta_0)}{2+\delta_0-r_0'}}\right]^{\frac{r_0'(1+\delta_0)}{2+\delta_0}} \\
&= \left\{\int_M |s_g|^{2+\delta_0} u^4 dv_0\right\}^{\frac{r_0'}{2+\delta_0}} \left\{\int_{S^4} u^{r_0} dv_0\right\}^{\frac{1+\delta_0}{2+\delta_0} r_0'}.
\end{aligned} \tag{35}
$$

Thus applying (34), we get

$$\int_M u^{r_1'} dv_0^{\frac{1}{r_1}} \le C(r_0') \left\{ \int_M | s_g u^3 |^{r_0'} dv_0 \right\}^{\frac{1}{r_0'}} + C_{14}$$

where C_{14} is a constant and $\frac{1}{r_1} = \frac{1}{r_0'} - \frac{1}{2}$, i.e.,

$$\begin{aligned} r_1 & \frac{(2r_0')}{2 - r_0'} \\ &= \frac{2(2+\delta_0) r_0}{4 + 6\delta_0 + 2r_0 - (2+\delta_0) r_0} \\ &= \frac{2(2+\delta_0) r_0}{4 + 2\delta_0 - 4\varepsilon\delta_0} \\ &> r_0. \end{aligned}$$

Note that if we can choose ε such that $4 + 2\delta_0 - 4\varepsilon\delta_0 < 0$, then $4 + 4\varepsilon > 6 + \frac{4}{\delta_0}$. Thus $r_0' > 2$, we are done already from estimate (35) and Hölder inequality. If $4 + 2\delta_0 - 4\varepsilon\delta_0 = 0$, then we can replace ε by $\varepsilon' < \varepsilon$. So we have $4 + 2\delta_0 - 4\varepsilon'\delta_0 = 4\delta_0(\varepsilon - \varepsilon') > 0$. Thus we can assume that $4 + 2\delta_0 - 4\varepsilon\delta_0 > 0$.

Continue this process with

$$r^2 = \frac{2(2+\delta_0) r_0}{2(2+\delta_0) - 4\varepsilon\delta_0}; \quad r_1' = \frac{(2+\delta_0) r_1}{2 + 3\delta_0 + r_1};$$

$$\cdots\cdots$$

$$r_k = \frac{2r_{k-1}'}{2 - r_{k-1}'} = \frac{2(2+\delta_0) r_{k-1}}{4 + 2\delta_0 - 4\varepsilon\delta_0};$$

$$r_{k-1}' = \frac{(2+\delta_0) r_{k-1}}{2 + 3\delta_0 + r_{k-1}}.$$

Notice that

$$r_{k+1} - r_k = \frac{2\varepsilon\delta_0}{4 + 2\delta_0 - 4\varepsilon\delta_0} r_k > 0.$$

Thus there will be a k_0 with $r_{k_0} > 6 + \frac{4}{\delta_0}$ and $r_0 < r_1 < \cdots < r_{k_0-1} < 6 + \frac{4}{\delta_0} < r_{k_0}$ with

$$r_{k_0}' = \frac{(2+\delta_0) r_{k_0}}{2 + 3\delta_0 + r_{k_0}} > 2.$$

So at the end of the iteration, we can find a bound for $\| u \|_{r_{k_0}}$, $2 < r_{k_0}' < 2 + \delta_0$.

This, together with Hölder inequality, implies that $u \in L^{\infty}$,

$$\| u \|_{\infty} \leq \| u \|_{1} + \| s_g u^3 \|_{r'_{k_0}} \| G \|_{q'}$$

where $\frac{1}{r'_{k_0}} + \frac{1}{q'} = 1$ with $q' < 2$. This finishes the proof of Proposition 1.

§4. L^4 bounds of full curvature tensors

What we are going to prove in this section is the following

PROPOSITION 2. *Suppose $g = u^2 g_0$ on a 4 dimensional Riemannian manifold M without boundary. If $0 < C_\alpha \leq u \leq C_\beta$ and $\int_M s_g^4 u^4 dv_0 \leq C_0$, then $\int_M R^4 dv = \int_M | R |^4 u^4 dv_0 \leq C_{15}$ where R is the full curvature tensor of metric g and C_{15} is a constant depending only on C_α, C_β, C_0 and the geometry of metric g_0.*

Proof. It is well known that the curvature tensor R_{ijkl} can be decomposited to

$$R_{ijkl} = W_{ijkl} + \frac{1}{n-2} (g_{ik} B_{jl} - g_{il} B_{jk} + B_{ik} g_{il} - B_{il} g_{jk}) + \frac{s}{n(n-1)} [g_{jl} g_{ik} - g_{il} g_{jk}) \tag{36}$$

where W_{ijkl}, B_{il}, s, g_{il} are called the Weyl conformal curvature tensor, the traceless Ricci curvature scalar curvature and metric tensor respectively. On a four dimensional manifold, if $g = u^2 g_0$, then

$$W_{ijkl} = u^2 (W_0)_{ijkl}, \tag{37}$$

$$B_{ij} = B_{0ij} - 2\left[\frac{u_{ij}}{u} - 2\frac{u_i u_j}{u_2} - \frac{1}{4}\left(\frac{\Delta_0 u}{u} - 2\frac{|\nabla u|_0^2}{u^2}\right) g_{0ij}\right]. \tag{38}$$

See [B].

First of all, from (2)

$$\begin{aligned}\int_M [\Delta u]^4 dv_0 &= \frac{1}{36^2} \int_M (s_0 u - s_g u^3)^4 dv_0 \\ &\leq \frac{2^4}{36^2} \left[\int_M s_0^4 u^4 dv_0 + \int_M s_g^4 u^{12} dv_0\right]\end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{9^2} s_0^4 \int_M u^4 dv_0 + \frac{C_\beta^8}{9^2} \int_M s_g^4 u^4 dv_0 \\
&\le \alpha_0 s_0^4 + C_\beta^8 C_0 \\
&\equiv C_{17}.
\end{aligned}
$$

Then by elliptic theory, there exists a constant β_0 such that

$$
\int_M |\nabla^2 u|^4 dv_0 = \int_M (\sum u_{ij}^2)^2 dv_0 \le \beta_0 \int_M (\Delta u)^4 dv_0 \le C_{17}. \tag{39}
$$

From (38), Sobolev inequality and Hölder inequality, we have

$$
\begin{aligned}
\left[\int_M |\nabla u|^8 dv_0\right]^{\frac{1}{2}} &\le C_1 \int_M |\nabla |\nabla u|^2|^2 + C_2 \int_M |\nabla u|^4 dv_0 \\
&\le 4C_1 \int_M |\nabla u|^2 |\nabla^2 u|^2|^2 dv_0 + C_2 \int_M |\nabla u|^4 dv_0 \\
&\le 4C_1 \left[\int_M |\nabla u|^4 dv_0\right]^{\frac{1}{2}} \left[\int_M |\nabla u|^4 dv_0\right]^{\frac{1}{2}} \\
&\le C_2 \Big(C_1 \int_M |\nabla^2 u|^2 dv_0 + C_2 \int_M |\nabla u|^2\Big)^2 \\
&\le 4C_1 \Big[C_1 \int_M |\nabla^2 u|^4 dv_0 + C_2 \int_M |\nabla u|^2 dv_0\Big] C_{17}^{\frac{1}{2}} \\
&\quad + 2C_2 \Big[C_1^2 \Big(\int_M |\nabla^2 u|^2 dv_0\Big)^2 + C_2^2 \Big(\int_M |\nabla u|^2 dv_0\Big)^2\Big] \\
&\le C_{18}
\end{aligned}
$$

where we have used the fact that

$$
\int_M |\nabla u|^2 dv_0 = -\int_M (\Delta u) u dv_0 \le \left[\int_M (\Delta u)^2 dv_0\right]^{\frac{1}{2}} \left[\int_M u^2 dv_0\right]^{\frac{1}{2}}
$$

is bounded. Thus we obtain

$$
\int_M |\nabla u|^8 dv_0 \le C_6^2. \tag{40}
$$

Now since

$$
\sum B_{ij}^2 u^4 = \sum B_{0ij}^2 - 4 \sum B_{0ij} h_{ij} + 4 \sum h_{ij}^2,
$$

where

$$h_{ij} = \frac{u_{ij}}{u} - \frac{2u_i u_j}{u^2} - \frac{1}{4}\left(\frac{\Delta u}{u} - 2\frac{|\nabla u|^2}{u^2}\right) g_{0ij},$$

from (37) and (38), it is easy to see that

$$\int_M |B|^2 dv = \int_M (\sum B_{ij}^2)^2 u^4 dv_0 \leq C_{19}$$

for some constant C_{19}.

But

$$\begin{aligned} \int_M |W|^4 u^4 dv_0 &\leq C_\alpha^{-4} \int_M |W_0|^4 u^4 dv_0 \\ &\leq C_\alpha^{-4} C_\beta^4 \int_M |W_0|^4 u^4 dv_0 \\ &\equiv C_{20}. \end{aligned} \tag{41}$$

Therefore we have obtained

$$\begin{aligned} \int_M R^4 u^4 dv_0 &\leq C \int_M (|W_0|^4 + |B|^4 + |s_g|^4) u^4 dv_0 \\ &\leq C(C_{20} + C_{19} + C_0) \\ &\equiv C_{15}. \end{aligned}$$

This finishes the proof of Proposition 2.

§5. Proof of Main Theorem

In this section we will prove the theorems stated in Section 1. To get C^∞ compactness, the very common means is to use Gromov's compactness theorem. Let $\mathcal{M}$ denote the space of smooth Riemannian metrics on a fixed smooth manifold M, modulo the action of the diffeomorphism group. We define the C^k, or $C^{k,\alpha}$, topology on $\mathcal{M}$ via convergence of the sequences. Thus a sequence $\{g_i\}$ converges in the C^k topology on M if and only if there are diffeomorphisms $f_i : M \to M$, such that the metrics $f_i^* g_i$, when expressed as metrics in a smooth atlas for M, converge in the C^k topology on functions on domains in R^n. In the same way, we define the Hölder $C^{k,\alpha}$ topology.

The C^k version of the Cheeger-Gromov compactness theorem then states that the space of n dimensional Riemannian manifolds satisfying the bounds

$$|\nabla^j R|_{C^0} \leq \Lambda(j), j \leq k, \tag{42}$$

$$\mathrm{vol}(M, g) \geq V > 0, \tag{43}$$

$$\mathrm{diam}_M(g) \leq D \tag{44}$$

is (pre)compact in the $C^{k+1,\alpha}$ topology on $\mathcal{M}$. More precisely, given any $\alpha < 1$ sequence of metrics $\{g_i\}$ on M satisfying bounds (42), (43) and (44) has a subsequence converging in the $C^{k+1,\alpha'}$ topology, for $\alpha' < \alpha$, to a limit $C^{k+1,\alpha}$ Riemannian metric g on M.

Thus one would like to use the specreum to control the quantities in (42), (43) and (44). To this end, the main tool one could use is the heat invariant, i.e., the coefficients a_i in the asymptotic expansion of the trace of the heat kernel

$$Z(t) = \sum e^{-\lambda_i t} \simeq \frac{1}{(4\pi t)^{\frac{n}{2}}} \sum a_i t^i$$

as $t \to 0$. The coefficients a_t are spectral invariants with the first few given by

LEMMA 5.

$$a_0(g) = \mathrm{vol}(M, g)\,;$$
$$a_1(g) = \frac{1}{6}\int_M S_g dv_g\,;$$
$$a_2(g) = \frac{1}{180}\int_M \left[|W|^2 + \frac{6-n}{n-2}|B|^2 + \frac{5n^2-7n+6}{2n(n-1)} s_g^2\right] dv_g.$$

Proof. It is well known.

From Lemma 5 and Proposition 1, it can be easily seen that (43) and (44) hold.

Now our theorems stated in Section 1 have been reduced to the following main result in this section:

PROPOSITION 3. *Suppose* (M, g_0) *is a compact* 4*-dimensional Riemannian manifold without boundary. If* $\int_M |R|^4 dv_g \leq C_{21}$ *and*

$$|a_k| \leq b_k, \quad k = 3,4,\ldots$$

and there is a constant $\lambda > 0$ *such that* $0 < \lambda^{-1} g_0 \leq g \leq \lambda g_0$. *Then*

$$\int_M |\nabla^k R|^2 dv \leq C(k)$$

for some constant $C(k)$ *depending* k, b_k, C_{21}, λ *and geometry of* g_0.

Proof. The exact form for a_3 is also know ([G1], [S], [T]), but we do not need it, we will not copy it here. The higher coefficients $a_i(g)$ become rapid increasingly complex and difficult to compute. However, the exact forms of the heat invariants a_i are not so important for our purpose. What is important is that they have the general leading coefficients [G2]

$$a_k(g) = (-1)^k \int_M (c_k |\nabla^{k-2}R|^2 + d_k |\nabla^{k-3}s_g|^2) dv_g + \int_M Q_k dv_g \tag{45}$$

where c_k and d_k are positive constants and Q_k is a lower order term involving covariant derivatives of R and its contractions of order at most $k-3$. More precisely, Q_k is a polynomial of weight $2k$ in contractions of $R_{ijkl,I}$ with $|I| \leq k-3$, with coefficients depending only on the metric g. Each monomial in Q_k is a product of contraction of $R_{ijkl,I}$ of weight $2k$, where the weight of $R_{ijkl,I}$ is defined to be $|I| + 2$ and the weight of the monomial is the sum of the weights of the factors.

First of all, since the coefficients c_k, d_k in (45) are positive, the bound on a_3 gives a bound

$$\int_M |\nabla R|^2 dv \leq h_{33} + h_{23} \int_M |R|^3 dv. \tag{46}$$

By Hölder inequality, one sees that $\int_M |R|^3 dv_g \leq \left(\int_M |R|^4 dv_g\right)^{\frac{3}{4}} \mathrm{vol}(M)^{\frac{1}{4}}{}_0$, i.e.,

$$\int_M |\nabla R|^2 dv_g \leq C(3)$$

where $C(3) = h_{33} + h_{23}(C_{21}^{\frac{3}{4}}) \mathrm{vol}(M)^{\frac{1}{4}}$.

Next, bound on a_4 gives a bound

$$\int_M |\nabla^2 R|^2 dv_g \leq h_{24} \int_M |R|^4 dv_g + h_{34} \int_M |\nabla R|^2 |R| dv_g + h_{44}. \tag{47}$$

By assumption, the first term on the right hand side of (47) is bounded. To bound the second term, choose $\eta = 2(C_{21})^{\frac{1}{2}} C_s h_{34} > 0$ where C_s is Sobolev constant with respect to metric g which can be chosen only depend on the metric g_0 and λ since g is equivalent to g_0. Now let $\Omega = \{x \in M, |R|(x) \geq \eta\}$ and β_s a constant. Then we have

$$
\begin{aligned}
h_{34}\int_M |\nabla R|^2 |R|\, dv_g &= h_{34}\int_\Omega |\nabla R|^2 |R|\, dv_g + h_{34}\int_{M\setminus\Omega} |\nabla R|^2 |R|\, dv_g \\
&\le h_{34}\Big(\int_\Omega |R|^2|\, dv_g\Big)^{\frac{1}{2}}\Big(\int_\Omega |\nabla R|^4\, dv_g\Big)^{\frac{1}{2}} + h_{34}\eta\int_M |\nabla R|^2 dv_g \\
&\le h_{34}\Big(\int_\Omega |R|^2|\, dv_g\Big)^{\frac{1}{2}}\Big[C_s\int_M |\nabla^2 R|^2\, dv_g + \beta_s\int_M |\nabla R|^2 dv_g\Big] \\
&\quad + h_{34}\eta\int_M |\nabla R|^2 dv_g \\
&\le \frac{h_{34}C_s}{\eta}\Big(\int_\Omega |R|^4\, dv_g\Big)^{\frac{1}{2}}\int_M |\nabla^2 R|^2 dv_g \\
&\quad + h_{34}\beta_s\Big(\int_M |R|^2 dv_g\Big)^{\frac{1}{2}}\Big(\int_M |\nabla R|^2 dv_g\Big) + h_{34}\eta\int_M |\nabla R|^2\, dv_g \\
&\le h_{34}C_s C_{21}^{\frac{1}{2}}\eta\int_M |\nabla^2 R|^2\, dv_g + h_{45}
\end{aligned}
$$

which implies, by combining (47),

$$
\int_M |\nabla^2 R|^2\, dv_g \le C(4). \tag{48}
$$

Now apply (48) to get

$$
\begin{aligned}
\Big(\int_M R^8 dv_g\Big)^{\frac{1}{2}} &\le C_s\int_M |\nabla R^2|^2 dv_g + \beta_s\int_M R^4 dv_g \\
&\le 4C_s\int_M R^2|\nabla R|^2 dv_g + \beta_s\int_M R^4 dv_g \\
&\le 4C_s\Big(\int_M R^4 dv_g\Big)^{\frac{1}{2}}\Big(\int_M |\nabla R|^4 dv_g\Big)^{\frac{1}{2}} + \beta_s\int_M R^4 dv_g \\
&\le 4C_s\Big(\int_M R^4 dv_g\Big)^{\frac{1}{2}}\Big(C_s\int_M |\nabla^2 R|^2 dv_g + \beta_s\int_M |\nabla R|^2\Big) + \beta_s\int_M R^4 dv_g \\
&\equiv C_{22}.
\end{aligned}
$$

Next we bound $\int_M |\nabla^3 R|^2 dv_g$ and $\int_M |\nabla R|^8 dv_g$ in essentially the same way. Namely, as above, the bound on a_5 gives a bound

$$
\begin{aligned}
\int_M |\nabla^3 R|^2 dv_g &\le h_{15} + h_{25}\int_M |\nabla^2 R|^2 |R|\, dv_g \\
&\quad + h_{35}\int_M |\nabla^2 R|\,|\nabla R|^2 dv_g + h_{45}\int_M |\nabla R|^2 |R|^2 dv_g + h_{55}\int_M |R|^5 dv_g.
\end{aligned} \tag{49}
$$

The last three terms on the right side of (49) are bounded from above estimates, Hölder inequality and Sobolev inequality. The second term can be bounded as above. Repeat above argument to have

$$\int_M |\nabla^2 R|^4 dv_g \leq C_{23} \tag{50}$$

and

$$\int_M |\nabla R|^8 dv_g \leq C_{24}. \tag{51}$$

The proof is now completed by induction in a similar fashion. Thus suppose we have bounded

$$\int_M |\nabla^l R|^2 \, dv_g \leq C(l+2), \; l \leq k-1, \tag{52}$$

with $k \geq 4$. We claim that

$$\int_M |\nabla^k R|^2 \, dv_g \leq C(k+2). \tag{53}$$

To see this, note first by Sobolev embedding that (52) implies the bounds:

$$\int_M |\nabla^{k-3} R|^8 \, dv_g \leq C_{25}, \tag{54}$$

$$|\nabla^m R|_{C_0} \leq C_{26} \quad \text{for} \quad m \leq k-4, \tag{55}$$

and

$$\int_M |\nabla^{k-2} R|^4 \, dv_g \leq C_{27}. \tag{56}$$

Since the heat invariant a_{k+2} is bounded, the bounds (54) follows from the expression (45) and a bound on the terms containing Q_{k+2} in terms of (55), (56), (54) and (52). Now recall that Q_{k+2} is a polynomial of weight $2k+4$, each monomial being a product of terms which are contractions of $R_{ijkl,I}$ with $|I| \leq k-1$, the weight of $R_{ijkl,I}$ being $|I|+2$. Thus modulo terms of the form $R_{ijkl,I}$ with $|I| \leq k-4$, which are bounded by (55), Q_{k+2} at most contains terms of the form:

(i) $$|\nabla^{k-1} R|^2 |R|$$

(ii) $$|\nabla^{k-1} R| |\nabla^{k-2}| |\nabla R|$$

(iii) $$|\nabla^{k-1} R| |\nabla^{k-3} R| |R|^2$$

(iv) $$|\nabla^{k-2}R||\nabla^{k-3}R|(|\nabla R||R|+|\nabla^3 R|) \text{ or } |\nabla^{k-2}R|^2|R|^2$$

(v) $$|\nabla^{k-3}R|^2 \text{ if } k=4$$

(vi) $|\nabla^{k-3}R|^p$. (some terms with derivatives of order $\leq k-4$) with $p \leq 3$.

Then it is not hard to see that terms (iii)-(vi) are easy to bound in terms of (52), (54), (55) and (56). For (ii), we have

$$\begin{aligned}
&\int_M |\nabla^{k-1}R||\nabla^{k-2}R||\nabla R|\,dv_g \\
&\leq \int_M |\nabla^{k-1}R|^2\,dv_g + \int_M |\nabla^{k-2}R|^4 + \int_M |\nabla R|^4 dv_g
\end{aligned}$$

which is bounded by (52), (54) and (56). Now for term (i), it follows from (52), (54) and (55) since $|R|_{C^0} \leq C_{26}$.

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