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RANK 2 SYMMETRIC HYPERBOLIC KAC-MOODY ALGEBRAS

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Introduction

Affine Kac-Moody algebras represent a well-trodden and well-understood littoral beyond which stretches the vast, chaotic, and poorly-understood ocean of indefinite Kac-Moody algebras. The simplest indefinite Kac-Moody algebras are the rank 2 Kac-Moody algebras $\mathfrak{g}(a)$ $(a \geq 3)$ with symmetric Cartan matrix $\begin{pmatrix} 2 & -a \\ -a & 2 \end{pmatrix}$, which form part of the class known as hyperbolic Kac-Moody algebras. In this paper, we probe deeply into the structure of those algebras $\mathfrak{g}(a)$, the *e. coli* of indefinite Kac-Moody algebras. Using Berman-Moody's formula ([BM]), we derive a purely combinatorial closed form formula for the root multiplicities of the algebra $\mathfrak{g}(a)$, and illustrate some of the rich relationships that exist among root multiplicities, both within a single algebra and between different algebras in the class. We also give an explicit description of the root system of the algebra $\mathfrak{g}(a)$. As a by-product, we obtain a simple algorithm to find the integral points on certain hyperbolas.

For alternative approaches to the analysis of rank 2 hyperbolic Kac-Moody algebras, the reader should see [BKM, Section 4], which constructs them as $\mathfrak{sl}_2$-modules, and [LM], which shows that the root systems of these algebras coincide with those of quasi-regular cusps on Hilbert modular surfaces defined over certain real quadratic fields.

The structure of the paper is as follows. In Section 1, we introduce Berman-Moody's root multiplicity formula for general symmetrizable Kac-Moody algebras. In Section 2, we specialize to the rank 2 hyperbolic Kac-Moody algebras

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$\mathfrak{g}(a)$ and develop a purely combinatorial formula (2.14) for the root multiplicities. We show how, in many cases, one can obtain the multiplicity of a root in one algebra from the multiplicity of the same root in a different algebra, which may have much simpler calculations. Section 3 extends this notion considerably by proving a stability theorem on the root multiplicities of $\mathfrak{g}(a)$ as a increases and explaining the connections between the algebras $\mathfrak{g}(a)$ and the free Lie algebra of rank 2. The stability theorem holds for arbitrary symmetrizable Kac-Moody algebras.

In Section 4, we consider the root system of the algebra $\mathfrak{g}(a)$. Recalling that real roots and imaginary roots correspond to the integral points on the hyperbolas: $x^2 - axy + y^2 = k$ ($k \in \mathbf{Z}$, $k \leq 1$), we show how all the roots of a given length are Weyl-conjugate to roots in a small and easily defined region. Thus we can easily list all the roots, with multiplicity, by use of some simple recurrence relations. This procedure finds all the integral points on these hyperbolas far more easily than the traditional number-theoretic algorithm.

In Section 5, we analyze some of the monotonic and symmetric relationships between the root multiplicities of the algebra $\mathfrak{g}(a)$ for a fixed value of a. We raise some questions concerning possible relationships among the root multiplicities for given algebras. We conjecture that the multiplicities of roos of a given height t increase monotonically to a maximum at (m, m) for t even, and $(m-1, m+1)$ for t odd. The paper closes with a collection of tables illustrating the main results and the conjecture.

1. Berman-Moody's formula

We first recall some of basic definitions in Kac-Moody theory and Berman-Moody's formula. Let I be an index set. A matrix $A = (a_{ij})_{i,j \in I}$ is called a *generalized Cartan matrix* if it satisfies: (i) $a_{ii} = 2$ for all $i \in I$, (ii) $a_{ij} \in \mathbf{Z}_{\leq 0}$ for $i \neq j$, (iii) $a_{ij} = 0$ if and only if $a_{ji} = 0$. In this paper, we assume that A is *symmetrizable,* i.e., there is an invertible diagonal matrix D such that DA is symmetric. A *realization* of A is a triple $(\mathfrak{h}, \Pi, \Pi^\vee)$, where $\mathfrak{h}$ is a complex vector space of dimension $2|I| - rank\,A$, $\Pi = \{a_i \mid i \in I\}$ and $\Pi^\vee = \{h_i \mid i \in I\}$ are linearly independent subsets of $\mathfrak{h}^*$ and $\mathfrak{h}$, respectively, satisfying $a_j(h_i) = a_{ij}$ for $i, j \in I$.

DEFINITION 1.1. The *Kac-Moody algebra* $\mathfrak{g} = \mathfrak{g}(A)$ with Cartan matrix A is the Lie algebra generated by the elements $e_i, f_i (i \in I)$ and $\mathfrak{h}$ with the following defining relations:

$$(1.1)\quad \begin{aligned} &[h, h'] = 0 \text{ for } h, h' \in \mathfrak{h}, \\ &[h, e_j] = a_j(h)e_j,\ [h, f_j] = -a_j(h)f_j \text{ for } j \in I, \\ &[e_j, f_i] = \delta_{ij}h_i \text{ for } i, j \in I, \\ &(ade_i)^{1-a_{ij}}(e_j) = (adf_i)^{1-a_{ij}}(f_j) = 0 \text{ for } i \neq j. \end{aligned}$$

The elements of Π (resp. $\Pi^{\vee}$) are called the *simple roots* (resp. *simple coroots*) of $\mathfrak{g}$. For each $i \in I$, let $r_i \in \mathrm{Aut}(\mathfrak{h}^*)$ be the *simple reflection* on $\mathfrak{h}^*$ defined by $r_i(\lambda) = \lambda - \lambda(h_i)\alpha_i$. The subgroup W of $GL(\mathfrak{h}^*)$ generated by the r_i's $(i \in I)$ is called the *Weyl group* of $\mathfrak{g}$.

Let $Q = \bigoplus_{i \in I} \mathbf{Z}\alpha_i$, $Q_+ = \bigoplus_{i \in I} \mathbf{Z}_{\geq 0}\alpha_i$, and $Q_- = -Q_+$. We define a partial ordering $\geq$ on $\mathfrak{h}^*$ by $\lambda \geq \mu$ if and only if $\lambda - \mu \in Q_+$. The Kac-Moody algebra $\mathfrak{g} = \mathfrak{g}(A)$ has the *root space decomposition* $\mathfrak{g} = \bigoplus_{\alpha \in Q} \mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x$ for $h \in \mathfrak{h}\}$ is the *α-root space*. An element $\alpha \in Q$ is called a *root* if $a \neq 0$ and $\mathfrak{g}_\alpha \neq 0$. The number $\mathrm{mult}(\alpha) := \dim \mathfrak{g}_\alpha$ is called the *multiplicity* of the root α. A root $\alpha > 0$ (resp. $\alpha < 0$) is called *positive* (resp. *negative*). It is known that all the roots are either positive or negative. We denote by Δ, Δ^+, and Δ^- the set of all roots, and negative roots, respectively. For $\alpha = \sum_{i \in I} k_i\alpha_i \in Q$, the number $\mathrm{ht}(\alpha) := \sum_{i \in I} k_i$ is called the *height* of α.

A $\mathfrak{g}$-module V is *$\mathfrak{h}$-diagonalizable* if $V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda$, where $V_\lambda = \{v \in V \mid h \cdot v = \lambda(h)v$ for all $h \in \mathfrak{h}\}$ is the *λ-weight space*. If $V_\lambda \neq 0$, then λ is called a *weight* of V. The number $\mathrm{mult}_V(\lambda) := \dim V_\lambda$ is called the *multiplicity* of λ. When all the weight spaces are finite-dimensional, we define the *character* of V to be

$$(1.2)\qquad \mathrm{ch}V = \sum_{\lambda \in \mathfrak{h}^*} (\dim V_\lambda) e^\lambda.$$

An $\mathfrak{h}$-diagonalizable module V is *integrable* if all the f_i $(i \in I)$ are locally nilpotent on V. A $\mathfrak{g}$-module V is called a *highest weight module* with highest weight $\lambda \in \mathfrak{h}^*$ if there is nonzero vector $v_0 \in V$ such that (i) $e_i \cdot v_0 = 0$ for all $i \in I$, (ii) $h \cdot v_0 = \lambda(h)v_0$ for all $h \in \mathfrak{h}$, (iii) $U(\mathfrak{g}) \cdot v_0 = V$. The vector v_0 is called a *highest weight vector*. We denote by $V(\lambda)$ the irreducible highest weight module over $\mathfrak{g}$.

Let $S \subset I$ and $\mathfrak{g}_S = \mathfrak{g}(A_S)$ be the Kac-Moody algebra with Cartan matrix $A_S = (a_{ij})_{i,j \in S}$. We denote by Δ_S, $\Delta_S^{\pm}$, and W_S the set of roots, the set of positive (resp. negative) roots, and the Weyl group of $\mathfrak{g}_S$, respectively. Let $\Delta^{\pm}(S) = \Delta^{\pm} \setminus \Delta_S^{\pm}$, and $W(S) = \{w \in W \mid \Phi_w \subset \Delta^+(S)\}$, where $\Phi_w = \{\alpha \in \Delta^+ \mid w^{-1}(\alpha) < 0\}$. We also define $\mathfrak{g}_0^{(S)} = \mathfrak{g}_S + \mathfrak{h}$, and $\mathfrak{g}_{\pm}^{(S)} = \bigoplus_{\alpha \in \Delta^{\pm}(S)} \mathfrak{g}_\alpha$. Then we have the *triangular de-*

composition:

$$\mathfrak{g} = \mathfrak{g}_-^{(S)} \oplus \mathfrak{g}_0^{(S)} \oplus \mathfrak{g}_+^{(S)}. \tag{1.3}$$

Let $\mathbf{C}$ be the trivial $\mathfrak{g}$-module. The homology modules $H_k(\mathfrak{g}_-^{(S)}, \mathbf{C})$ are obtained from the $\mathfrak{g}_0^{(S)}$-module complex

$$\begin{aligned} \cdots \to \Lambda^k(\mathfrak{g}_-^{(S)}) \xrightarrow{d_k} \Lambda^{k-1}(\mathfrak{g}_-^{(S)}) \to \cdots \\ \to \Lambda^1(\mathfrak{g}_-^{(S)}) \xrightarrow{d_1} \Lambda^0(\mathfrak{g}_-^{(S)}) \xrightarrow{d_0} \mathbf{C} \to 0, \end{aligned} \tag{1.4}$$

with the differentials $d_k : \Lambda^k(\mathfrak{g}_-^{(S)}) \to \Lambda^{k-1}(\mathfrak{g}_-^{(S)})$ defined by

$$d_k(x_1 \wedge \cdots \wedge x_k) = \sum_{s<t} (-1)^{s+t}([x_s, x_t] \wedge x_1 \wedge \cdots \wedge \widehat{x_s} \wedge \cdots \wedge \widehat{x_t} \wedge \cdots \wedge x_k)$$

for $x_i \in \mathfrak{g}_-^{(S)}$. For simplicity, we write $H_k(\mathfrak{g}_-^{(S)})$ for $H_k(\mathfrak{g}_-^{(S)}, \mathbf{C})$. The $\mathfrak{g}_0^{(S)}$-module structure of the homology modules $H_k(\mathfrak{g}_-^{(S)})$ is determined by the following formula known as *Kostant's formula.*

PROPOSITION 1.2 ([GL], [Li]).

$$H_k(\mathfrak{g}_-^{(S)}) \cong \bigoplus_{\substack{w \in W(S) \\ l(w)=k}} V_S(w\rho - \rho), \tag{1.5}$$

where $V_S(\lambda)$ denotes the irreducible highest weight $\mathfrak{g}_0^{(S)}$-module with highest weight λ, and ρ is an element of $\mathfrak{h}^*$ satisfying $\rho(h_i) = 1$ for all $i \in I$. □

We now recall the root multiplicity formula for $\mathfrak{g}$ obtained in [Ka2]. Applying the Euler-Poincaré principle to (1.4) yields:

$$\sum_{k=0}^{\infty} (-1)^k \mathrm{ch}\Lambda^k(\mathfrak{g}_-^{(S)}) = \sum_{k=0}^{\infty} (-1)^k \mathrm{ch} H_k(\mathfrak{g}_-^{(S)}). \tag{1.6}$$

Let

$$\begin{aligned} H &= \sum_{k=0}^{\infty} (-1)^{k+1} H_k(\mathfrak{g}_-^{(S)}) = \sum_{k=0}^{\infty} (-1)^{k+1} \sum_{\substack{w \in W(S) \\ l(w)=k}} V_S(w\rho - \rho) \\ &= \sum_{\substack{w \in W(S) \\ l(w) \geq 1}} (-1)^{l(w)+1} V_S(w\rho - \rho), \end{aligned} \tag{1.7}$$

an alternating direct sum of $\mathfrak{g}_0^{(S)}$-modules. For $\tau \in Q_-$, we define the *dimension* of the τ-weight space of H to be

$$\dim H_\tau = \sum_{k=1}^{\infty} (-1)^{k+1} \dim H_k(\mathfrak{g}_-^{(S)})_\tau. \tag{1.8}$$

Let $P(H) = \{\tau \in Q_- \mid \dim H_\tau \neq 0\}$ and let $\{\tau_i \mid i \geq 1\}$ be the enumeration of $P(H)$ given by the height and lexicographical ordering. For $\tau \in Q_-$, we define a set

(1.9) $$T(\tau) = \{(n) = (n_i)_{i \geq 1} \mid n_i \in \mathbf{Z}_{\geq 0}, \sum n_i \tau_i = \tau\},$$

and a function

(1.10) $$B(\tau) = \sum_{(n) \in T(\tau)} \frac{(\sum n_i - 1)!}{\Pi(n_i!)} \Pi (\dim H_{\tau_i})^{n_i}.$$

Then we have:

THEOREM 1.3 ([Ka2]). *Let α be a root in $\Delta^-(S)$. Then*

(1.11) $$\dim \mathfrak{g}_\alpha = \sum_{\tau \mid \alpha} \mu\left(\frac{\alpha}{\tau}\right) \frac{\tau}{\alpha} B(\tau),$$

where μ is the classical Möbius function and $\tau \mid \alpha$ if $\alpha = k\tau$ for some positive integer k, in which case $\frac{\alpha}{\tau} = k$ and $\frac{\tau}{\alpha} = \frac{1}{k}$. □

If $S = \phi$, then we have $W(S) = W$, and

(1.12) $$H = \sum_{\substack{w \in W \\ l(w) \geq 1}} (-1)^{l(w)+1} \mathbf{C}_{w\rho - \rho}.$$

Hence $P(H) = \{w\rho - \rho \mid w \in W\}$. Let $\{\tau_i = w_i\rho - \rho \mid i \geq 1\}$ be the enumeration of the set $P(H)$ given by the height and lexicographical ordering. Then we have

(1.13) $$\dim H_{\tau_i} = \sum_{\substack{w \in W \\ l(w) \geq 1}} (-1)^{l(w)+1} \delta_{w\rho - \rho, \tau_i} = (-1)^{l(w_i)+1}.$$

Therefore the formula (1.11) reduces to Berman-Moody's formula:

COROLLARY 1.4 ([BM]).

(1.14) $$\dim \mathfrak{g}_\alpha = \sum_{\tau \mid \alpha} \mu\left(\frac{\alpha}{\tau}\right) \frac{\tau}{\alpha} \sum_{(n) \in T(\tau)} \frac{(\sum n_i - 1)!}{\Pi(n_i!)} \Pi((-1)^{l(w_i)+1})^{n_i}.$$ □

2. The hyperbolic Kac-Moody algebras $\mathfrak{g}(a)$

In this section, we study the structure of rank 2 hyperbolic Kac-Moody algebras $\mathfrak{g}(a)$ with symmetric Cartan matrix $\begin{pmatrix} 2 & -a \\ -a & 2 \end{pmatrix}$ using Berman-Moody's formula. Let $I = \{0,1\}$ be the index set for the simple roots of $\mathfrak{g}(a)$, and take $S = \phi$. Then $\mathfrak{g}(a)_0^{(S)} = \mathfrak{h} = \mathbf{C}h_0 \oplus \mathbf{C}h_1$, the Cartan subalgebra, and $\Delta^-(S) = \Delta^-$. Moreover, $W(S) = W$, the full Weyl group, and we have an explicit description of W:

$$W = \{1,\ r_0(r_1r_0)^j,\ r_1(r_0r_1)^j,\ (r_0r_1)^{j+1},\ (r_1r_0)^{j+1} \mid j \geq 0\}. \tag{2.1}$$

Hence, by (1.12), we have

$$\begin{aligned} H &= \sum_{\substack{w\in W \\ l(w)\geq 1}} (-1)^{l(w)+1} \mathbf{C}_{w\rho-\rho} \\ &= \frac{\sum_{j\geq 0}(\mathbf{C}_{r_0(r_1r_0)^j\rho-\rho} \oplus \mathbf{C}_{r_1(r_0r_1)^j\rho-\rho})}{\sum_{j\geq 0}(\mathbf{C}_{(r_0r_1)^{j+1}\rho-\rho} \oplus \mathbf{C}_{(r_1r_0)^{j+1}\rho-\rho})}, \end{aligned} \tag{2.2}$$

where we let A/B denote $A \ominus B$.

We introduce a sequence $\{A_n\}_{n\geq 0}$ defined as follows:

$$\begin{aligned} &A_0 = 0, \quad A_1 = 1, \\ &A_{n+2} = aA_{n+1} - A_n + 1 \text{ for } n \geq 0. \end{aligned} \tag{2.3}$$

When we want to emphasize that the sequence $\{A_n\}_{n\geq 0}$ depends on a, we will write $\{A_n(a)\}_{n\geq 0}$. Since the sequence $\{A_n\}_{n\geq 0}$ will play a crucial role in writing the root multiplicity formula for $\mathfrak{g}(a)$, we investigate some of the basic properties of the sequence $\{A_n\}_{n\geq 0}$.

The first few terms of the sequence $\{A_n\}_{n\geq 0}$ are

$$\begin{aligned} &A_0 = 0,\ A_1 = 1,\ A_2 = 1 + a,\ A_3 = a + a^2, \\ &A_4 = -a + a^2 + a^3, \cdots. \end{aligned}$$

When $a = 3$, we have $A_n = F_{2n} - 1$, where $\{F_n\}_{n\geq 0}$ is the Fibonacci sequence defined by

$$F_0 = F_1 = 1,\ F_{n+2} = F_{n+1} + F_n \ (n \geq 0). \tag{2.4}$$

To see this, let $A'_n = A_n + 1$. Then $A'_0 = 1$, $A'_1 = 2$, and for $n \geq 0$

$$\begin{aligned} A'_{n+2} &= A_{n+2} + 1 = 3A_{n+1} - A_n + 2 \\ &= 3(A_{n+1} + 1) - (A_n + 1) = 3A'_{n+1} - A'_n. \end{aligned}$$

On the other hand, (2.4) yields

$$\begin{aligned} F_{2n+4} &= F_{2n+3} + F_{2n+2} = 2F_{2n+2} + F_{2n+1} \\ &= 3F_{2n+2} - F_{2n}. \end{aligned}$$

Since $F_0 = 1$ and $F_2 = 2$, we have $A'_n = F_{2n}$ for all $n \geq 0$, which proves our assertion. The relation between the Fibonacci sequence and the hyperbolic Kac-Moody algebra $\mathfrak{g}(a)$ was first noticed by Feingold ([F]).

Let us find a closed form expression for the sequence $\{A_n\}_{n\geq 0}$. Let $F(x) = \sum_{n=0}^{\infty} A_n x^n$ be the generating function for the sequence $\{A_n\}_{n\geq 0}$. Then multiplying (2.3) by x^{n+2} yields

$$A_{n+2}x^{n+2} = axA_{n+1}x^{n+1} - x^2 A_n x^n + x^{n+2} \ (n \geq 0).$$

Summing up over $n \geq 0$ gives

$$F(x) - A_0 - A_1 x = ax(F(x) - A_0) - x^2 F(x) + (x^2 + x^3 + \cdots),$$

which yields

$$(1 - ax + x^2)F(x) = x + x^2 + x^3 + \cdots = \frac{x}{1-x}.$$

Let

$$\gamma = \frac{a + \sqrt{a^2 - 4}}{2} \tag{2.5}$$

be a zero of $1 - ax + x^2$. Then

$$\begin{aligned} F(x) &= \frac{x}{(1-x)(1-ax+x^2)} \\ &= \frac{-\gamma}{(1-\gamma)^2}\frac{1}{1-x} + \frac{\gamma^2}{(1+\gamma)(1-\gamma)^2}\frac{1}{\gamma - x} \\ &\quad + \frac{\gamma}{(1+\gamma)(1-\gamma)^2}\frac{1}{\frac{1}{\gamma} - x} \\ &= \frac{-\gamma}{(1-\gamma)^2}\sum_{n=0}^{\infty} x^n + \frac{\gamma}{(1+\gamma)(1-\gamma)^2}\sum_{n=0}^{\infty}\left(\frac{x}{\gamma}\right)^n \\ &\quad + \frac{\gamma^2}{(1+\gamma)(1-\gamma)^2}\sum_{n=0}^{\infty}(\gamma x)^n \\ &= \sum_{n=0}^{\infty}\left(\frac{-\gamma}{(1-\gamma)^2} + \frac{\gamma}{\gamma^n(1+\gamma)(1-\gamma)^2} + \frac{\gamma^{n+2}}{(1+\gamma)(1-\gamma)^2}\right)x^n \end{aligned}$$

$$= \sum_{n=0}^{\infty} \frac{1 - \gamma^n(1+\gamma) + \gamma^{2n+1}}{\gamma^{n-1}(1+\gamma)(1-\gamma)^2} x^n.$$

Therefore, we obtain

$$A_n = \frac{1 - \gamma^n(1+\gamma) + \gamma^{2n+1}}{\gamma^{n-1}(1+\gamma)(1-\gamma)^2} \ (n \geq 0). \tag{2.6}$$

Now we derive the root multiplicity formula for $\mathfrak{g}(a)$. For $j \geq 0$, by induction

$$\begin{aligned} r_0(r_1r_0)^j\rho - \rho &= -A_{2j+1}\alpha_0 - A_{2j}\alpha_1, \\ r_1(r_0r_1)^j\rho - \rho &= -A_{2j}\alpha_0 - A_{2j+1}\alpha_1, \\ (r_0r_1)^{j+1}\rho - \rho &= -A_{2j+2}\alpha_0 - A_{2j+1}\alpha_1, \\ (r_1r_0)^{j+1}\rho - \rho &= -A_{2j+1}\alpha_0 - A_{2j+2}\alpha_1. \end{aligned} \tag{2.7}$$

It follows from (2.2) and (2.7) that

$$H = \frac{\sum_{j\geq 0} (\mathbf{C}_{-A_{2j+1}\alpha_0 - A_{2j}\alpha_1} \oplus \mathbf{C}_{-A_{2j}\alpha_0 - A_{2j+1}\alpha_1})}{\sum_{j\geq 0} (\mathbf{C}_{-A_{2j+2}\alpha_0 - A_{2j+1}\alpha_1} \oplus \mathbf{C}_{-A_{2j+1}\alpha_0 - A_{2j+2}\alpha_1})}, \tag{2.8}$$

and hence for $\tau = -m\alpha_0 - n\alpha_1$, we have

$$\dim H_\tau = \begin{cases} 1 & \text{if } (m, n) = (A_{2j+1}, A_{2j}) \text{ or } (A_{2j}, A_{2j+1}), \\ -1 & \text{if } (m, n) = (A_{2j+2}, A_{2j+1}) \text{ or } (A_{2j+1}, A_{2j+2}), \\ 0 & \text{otherwise.} \end{cases} \tag{2.9}$$

For $i, j \geq 0$, define

$$\tau_i = \begin{cases} -A_{2j+1}\alpha_0 - A_{2j}\alpha_1 & \text{if } i = 4j, \\ -A_{2j}\alpha_0 - A_{2j+1}\alpha_1 & \text{if } i = 4j+1, \\ -A_{2j+2}\alpha_0 - A_{2j+1}\alpha_1 & \text{if } i = 4j+2, \\ -A_{2j+1}\alpha_0 - A_{2j+2}\alpha_1 & \text{if } i = 4j+3. \end{cases} \tag{2.10}$$

Then we have an enumeration of all the weights of $H : P(H) = \{\tau_i \mid i \geq 0\}$, where

$$\dim H_{\tau_i} = \begin{cases} 1 & \text{if } i \equiv 0,1 \pmod 4, \\ -1 & \text{if } i \equiv 2,3 \pmod 4. \end{cases} \tag{2.11}$$

For $\tau \in Q_-$, set

$$T_a(\tau) = \{(n) = (n_i)_{i\geq 0} \mid n_i \in \mathbf{Z}_{\geq 0}, \textstyle\sum n_i\tau_i = \tau\}, \tag{2.12}$$

and define

$$B_a(\tau) = \sum_{(n)\in T_a(\tau)} \frac{(\sum n_i - 1)!}{\Pi(n_i!)} (-1)^{\sum_{i\equiv 2,3 \pmod 4} n_i}. \tag{2.13}$$

Then, by Berman-Moody's formula, we obtain

PROPOSITION 2.1

$$\dim \mathfrak{g}(a)_\alpha = \sum_{\tau|\alpha} \mu\left(\frac{\alpha}{\tau}\right) \frac{\tau}{\alpha} B_a(\tau). \tag{2.14}$$ □

Next we examine more closely the formula (2.13) for $B_a(\tau)$. Let $\alpha = -n\alpha_0 - (n+j)\alpha_1$ be a root of $\mathfrak{g}(a)$ and write $B_a(n, n+j)$ for $B_a(\alpha)$. By symmetry of the root system, we may suppose without loss of generatity that $j \geq 0$. We noted above that any weight $\tau_k \in P(H)$ is of the form

$$\tau_k = -A_i\alpha_0 - A_{i+1}\alpha_1$$
$$\text{or } \tau_k = -A_{i+1}\alpha_0 - A_i\alpha_1.$$

Let
$\mathscr{C} = \{\mathbf{c} = (c_0^0, c_0^1, c_1^0, c_1^1, \ldots) \mid c_i^j \text{ are non-negative integers}, j \in \{0,1\}, j \geq 0\}$.
For given a, n, and j, define

$$\mathscr{C}(a, n, j) = \{\mathbf{c} \in \mathscr{C} \mid \sum_{i=0}^{\infty} (c_i^0 A_{i+1} + c_i^1 A_i) = n, \sum_{i=0}^{\infty} (c_i^0 A_i + c_i^1 A_{i+1}) = n + j\}.$$

Then all partitions of α in $T_a(\alpha)$ are of the form

$$\alpha = \sum_{i=0}^{\infty} [c_i^0(-A_{i+1}\alpha_0 - A_i\alpha_1) + c_i^1(-A_i\alpha_0 - A_{i+1}\alpha_1)]. \tag{2.15}$$

Weights in the partition count with negative multiplicity precisely when i is odd, and so we have

PROPOSITION 2.2

$$\begin{aligned} B_a(n, n+j) &= \sum_{\mathscr{C}(a,n,j)} (-1)^{\sum_{i \text{ odd}}(c_i^0+c_i^1)} \frac{(\sum_{i\geq 0}(c_i^0 + c_i^1) - 1)!}{\Pi_{i\geq 0} c_i^0! c_i^1!} \\ &= \sum_{\mathscr{C}(a,n,j)} (-1)^{\sum_{i \text{ odd}}(c_i^0+c_i^1)} \frac{1}{\sum_{i\geq 0}(c_i^0 + c_i^1)} \frac{(\sum_{i\geq 0}(c_i^0 + c_i^1))!}{c_0^0! c_0^1! c_1^0! c_1^1! c_2^0! \cdots} \\ &= \sum_{\mathscr{C}(a,n,j)} (-1)^{\sum_{i \text{ odd}}(c_i^0+c_i^1)} \frac{1}{\sum_{i\geq 0}(c_i^0 + c_i^1)} \binom{\sum_{i\geq 0}(c_i^0 + c_i^1)}{c_0^0, c_0^1, c_1^0, c_1^1, \ldots}. \end{aligned} \tag{2.16}$$ □

EXAMPLE 2.3. Let $a = 3$ and consider $\alpha = -4\alpha_0 - 5\alpha_1$. That is, $n = 4$, $j = 1$. For $a = 3$, $A_0 = 0$, $A_1 = 1$, $A_2 = 4$ and $A_3 = 12 > 5$. Thus, for any $\mathbf{c} \in \mathscr{C}(3,4,1)$, $c_k^0 = c_k^1 = 0$ for all $k \geq 2$, and we have the additional two conditions that

$$c_0^0 + 4c_1^0 + c_1^1 = 4,$$
$$c_0^1 + c_1^0 + 4c_1^1 = 5. \tag{2.17}$$

Hence,

$$\mathscr{C}(3,4,1) = \{(4,5,0,0,\ldots), (0,4,1,0,\ldots), (3,1,0,1,\ldots)\}$$

corresponding to the partitions

$$\begin{aligned} -4\alpha_0 - 5\alpha_1 &= 4(-\alpha_0) + 5(-\alpha_1) \\ &= 4(-\alpha_1) + (-4\alpha_0 - \alpha_1) \\ &= 3(-\alpha_0) + (-\alpha_1) + (-\alpha_0 - 4\alpha_1). \end{aligned} \tag{2.18}$$

Thus, suppressing trailing zeroes in the multinomials, we have

$$\begin{aligned} B_3(4,5) &= \frac{1}{9}\binom{9}{4,5} + (-1)^1\frac{1}{5}\binom{5}{0,4,1} + (-1)^1\frac{1}{5}\binom{5}{3,1,0,1} \\ &= 14 - 1 - 4 = 9. \end{aligned} \tag{2.19}$$

Since no other root τ divides $\alpha = -4\alpha_0 - 5\alpha_1$, we have immediately from (2.14) that $\dim \mathfrak{g}_\alpha = 9$. □

EXAMPLE 2.4. This example gives a foretaste of the stability theory in the following section. Let $\alpha = -n\alpha_0 - (n+j)\alpha_1$. Suppose $a \geq n + j$. Then $A_2 = a + 1 > n + j$. Thus, any partition can involve only A_0 and A_1. Hence,

$$\mathscr{C}(a, n, j) = \{(n, n+j, 0,0,\ldots)\},$$

and

$$B_a(n, n+j) = \frac{1}{2n+j}\binom{2n+j}{n}.$$

For $j \neq 0$, if $\gcd(n, n+j) = 1$, then $\alpha = -n\alpha_0 - (n+j)\alpha_1$ has no divisors other than itself and

$$\dim \mathfrak{g}_\alpha = \frac{1}{2n+j}\binom{2n+j}{n}. \tag{2.20}$$

For example, if $j = 1$, and $a \geq n + 1$, than

$$\dim \mathfrak{g}_\alpha = \frac{1}{2n+1}\binom{2n+1}{n}. \tag{2.21}$$

Thus the multiplicity of, say, $\alpha = -5\alpha_0 - 6\alpha_1$ in the Kac-Moody algebra $\mathfrak{g}(a)$ $(a \geq 6)$ is $\dim \mathfrak{g}(a)_\alpha = \frac{1}{11}\binom{11}{5} = 42$ (see Table 5 in Section 6). In the next section, we will see that this is actually the same as the multiplicity of $5\alpha_0 + 6\alpha_1$ in the free Lie algebra of rank 2.

If $j = 0$ and n is an odd prime p, (2.14) reduces to

$$\dim \mathfrak{g}_\alpha = \frac{1}{2p}\binom{2p}{p} - \frac{1}{p}. \tag{2.22}$$

Since $\dim \mathfrak{g}_\alpha$ is an integer, this shows that $\binom{2p}{p} = 2 \pmod{2p}$. □

EXAMPLE 2.5. Suppose $j = 0$, $n \geq 3$ and $a = n - 1$. Then $A_0 = 0$, $A_1 = 1$, $A_2 = a + 1 = n$ and $A_k > n$ for $k \geq 3$. Thus, any partition of $\alpha = -n\alpha_0 - n\alpha_1$ can involve only A_0, A_1, and A_2. Hence, if $\mathbf{c} \in \mathscr{C}(n-1, n, 0)$, $c_k^0 = c_k^1 = 0$ for $k \geq 2$. The corresponding partitions of α are:

$$\begin{aligned} -n\alpha_0 - n\alpha_1 &= n(-\alpha_0) + n(-\alpha_1) \\ &= (n-1)(-\alpha_0) + (-\alpha_0 - n\alpha_1) \\ &= (n-1)(-\alpha_1) + (-n\alpha_0 - \alpha_1). \end{aligned} \tag{2.23}$$

Thus,

$$\mathscr{C}(n-1, n, 0) = \{(n, n, 0,0,\ldots), (n-1,0,0,1,\ldots), (0, n-1,1,0,\ldots)\},$$

and

$$\begin{aligned} B_{n-1}(n, n) &= \frac{1}{2n}\binom{2n}{n} + (-1)^1\frac{1}{n}\binom{n}{1} + (-1)^1\frac{1}{n}\binom{n}{1} \\ &= \frac{1}{2n}\binom{2n}{n} - 2 \\ &= B_n(n, n) - 2. \end{aligned} \tag{2.24}$$

For example, if $n = 5$, we have $B_4(5,5) = \frac{252}{10} - 2 = 23\frac{1}{5}$. Hence, the multiplicity of the root $\alpha = -5\alpha_0 - 5\alpha_1$ in $\mathfrak{g}(4)$ is $\dim\mathfrak{g}_\alpha = 23\frac{1}{5} - \frac{1}{5} = 23$.

By a similar argument, it is easy to see that, for n sufficiently large, we obtain formulas relating $B_{n-k}(n, n)$ and $B_n(n, n)$. The first few are:

$$\begin{aligned}
&\text{For } n \geq 3, \quad B_{n-1}(n, n) = B_n(n, n) - 2\\
&\text{For } n \geq 4, \quad B_{n-2}(n, n) = B_n(n, n) - 2n + 1\\
&\text{For } n \geq 5, \quad B_{n-3}(n, n) = B_n(n, n) - n(n+1) + 6\\
&\text{For } n \geq 7, \quad B_{n-4}(n, n) = B_n(n, n) - \frac{n(n+1)(n+2)}{3} + 30\\
&\text{For } n \geq 9, \quad B_{n-5}(n, n) = B_n(n, n) - \frac{n(n+1)(n+2)(n+3)}{12}\\
&\qquad\qquad + 140.
\end{aligned} \tag{2.25}$$

□

3. Stability of root multipicities

In this section, we prove the stability of the root multiplicities of the Kac-Moody algebras $\mathfrak{g}(a)$, and discuss the relation with the free Lie algebra with 2 generators. We start with the discussion on free Lie algebras. Let $X = \{x \mid i = 1,2,3,\ldots\}$ be a totally ordered set (possibly countably infinite) and let R be an (additive) partially ordered abelian semigroup with a countable basis such that each element α of R can be expressed as a sum of elements of R which are less than or equal to α in only finitely many ways. Let G be the free Lie algebra on the set X. We make G an R-graded Lie algebra as follows. Let $\mathcal{S} = \{\mu_i \mid i = 1,2,3,\ldots\}$ be a collection of elements in R such that $\mu_i \leq \mu_j$ for $i < j$. We allow only finitely many repetitions. Define $\deg x_i = \mu_i$, and

$$\deg[[\cdots[x_{i_1}, x_{i_2}],\cdots]x_{i_r}] = \mu_{i_1} + \mu_{i_2} + \cdots + \mu_{i_r}.$$

Then G becomes an R-graded Lie algebra $G = \bigoplus_{\alpha\in R} G_\alpha$, where G_α is the subspace of G spanned by all the brackets $[[\cdots[x_{i_1}, x_{i_2}],\cdots]x_{i_r}]$ such that $\mu_{i_1} + \mu_{i_2} + \cdots + \mu_{i_r} = \alpha$.

We recall the dimension formula for G_α obtained in [Ka1]. Let $P = \{\tau_i \mid i = 1,2,3,\ldots\}$ be the set of distinct elements in $\mathcal{S}$, and let H be the subspace of G spanned by the elements of X. Then H has the decomposition $H = \bigoplus_{i=1}^{\infty} H_{\tau_i}$. For $\tau \in R$, let

$$T_0(\tau) = \{(n) = (n_i)_{i\geq 1} \mid n_i \in \mathbf{Z}_{\geq 0}, \textstyle\sum n_i\tau_i = \tau\}, \tag{3.1}$$

and define

$$B_0(\tau) = \sum_{(n)\in T(\tau)} \frac{(\sum n_i - 1)!}{\prod n_i!} \prod (\dim H_{\tau_i})^{n_i}. \tag{3.2}$$

Then by generalizing the proof of the Witt formula given in [Se], we obtain

PROPOSITION 3.1 ([Ka1]).

$$\dim G_\alpha = \sum_{\tau|\alpha} \mu\left(\frac{\alpha}{\tau}\right) \frac{\tau}{\alpha} B_0(\tau). \tag{3.3}$$ □

Fix the index set $I = \{1,2,,\cdots, n\}$ and let $\mathfrak{F}$ be the free Lie algebra generated by the elements e_i $(i = 1,\cdots, n)$. Since $\deg e_i = \alpha_i$ for $i = 1,\cdots, n$, $\mathfrak{F}$ is a Q_+-graded Lie algebra $\mathfrak{F} = \bigoplus_{\alpha\in Q_+} \mathfrak{F}_\alpha$, where $\mathfrak{F}_\alpha$ is the subspace of $\mathfrak{F}$ spanned by all the brackets $[[\cdots[e_{i_1}, e_{i_2}],\cdots]e_{i_r}]$ such that $\alpha_{i_1} + \alpha_{i_2} + \cdots + \alpha_{i_r} = \alpha$. For $\tau = \sum_{i=1}^n t_i\alpha_i \in Q_+$, let $ht(\tau) = \sum_{i=1}^n t_i$, and define

$$B_\infty(\tau) = \frac{(\sum t_i - 1)!}{\prod t_i!} = \frac{1}{\mathrm{ht}(\tau)} \binom{\mathrm{ht}(\tau)}{t_1,\cdots, t_n}. \tag{3.4}$$

Then Proposition 3.1 implies

COROLLARY 3.2

$$\dim \mathfrak{F}_\alpha = \sum_{\tau|\alpha} \mu\left(\frac{\alpha}{\tau}\right) \frac{\tau}{\alpha} B_\infty(\tau). \tag{3.5}$$

Proof. For $\tau = \sum_{i=1}^n t_i\alpha_i \in Q_+$, the only partition of τ is

$$\tau = t_1(\alpha_1) + \cdots + t_n(\alpha_n).$$

Now the result follows immediately from Proposition 3.1. □

EXAMPLE 3.3 Let $\mathfrak{F}$ be the free Lie algebra generated by the elements e_0 and e_1. For $\tau = m\alpha_0 + m\alpha_1$, we write $\tau = (m, n)$. By (3.4), we have

$$B_\infty(m, n) = \frac{1}{m+n} \binom{m+n}{m}. \tag{3.6}$$

Therefore, if $\alpha = 5\alpha_0 + 6\alpha_1$, then

$$\dim \mathfrak{F}_\alpha = B_\infty(5,6) = \frac{1}{11}\binom{11}{5} = 42.$$

If $\alpha = 4\alpha_0 + 6\alpha_0$, then

$$\dim \mathfrak{F}_\alpha = B_\infty(4,6) - \frac{1}{2} B_\infty(2,3) = \frac{1}{10}\binom{10}{4} - \frac{1}{2}\frac{1}{5}\binom{5}{2} = 20. \qquad \square$$

We now discuss the stability of root multiplicities for general symmetrizable Kac-Moody algebras. Let $A = (a_{ij})_{i,j\in I}$ and $A' = (a'_{ij})_{i,j\in I}$ be generalized Cartan matrices of the same size. We define $A \leq A'$ if $|a_{ij}| \leq |a'_{ij}|$ for all $i \neq j$. We write $A > 0$ if $|a_{ij}| > 0$ for all $i \neq j$.

PROPOSITION 3.4. *Suppose $A \leq A'$, and let $\alpha \in Q$ be a root of the Kac-Moody algebra $\mathfrak{g} = \mathfrak{g}(A)$. Then α is also a root of the Kac-Moody algebra $\mathfrak{g}' = \mathfrak{g}(A')$, and we have* $\dim \mathfrak{g}_\alpha \leq \dim \mathfrak{g}'_\alpha$.

Proof. We may assume that $\alpha \in Q_+$. We denote by $e'_i, f'_i\,(i \in I)$ the Chevalley generators of $\mathfrak{g}'$. Let $\mathfrak{g}_+$ (resp. $\mathfrak{g}'_+$) be the subalgebra of $\mathfrak{g}$ (resp. e') generated by the elements e_i (resp. e'_i) for $i \in I$. Since $|a_{ij}| \leq |a'_{ij}|$ for all $i \neq j$ $(A \leq A')$, by Gabber-Kac Theorem ([GK]), there is a surjective Lie algebra homomorphism $\phi_{A',A} : \mathfrak{g}'_+ \to \mathfrak{g}_+$ defined by $e'_i \mapsto e_i$. Therefore $\dim \mathfrak{g}_\alpha \leq \dim \mathfrak{g}'_\alpha$ for all $\alpha \in Q_+$. $\square$

COROLLARY 3.5. *If $a \leq a'$, them* $\dim \mathfrak{g}(a)_\alpha \leq \dim \mathfrak{g}(a')_\alpha$. $\square$

THEOREM 3.6. *Let $\alpha = \sum_{i\in I} k_i\alpha_i \in Q_+$, and let $\mathfrak{g} = \mathfrak{g}(A)$ be the Kac-Moody algebra associated with a generalized Cartan matrix $A = (a_{ij})_{i,j\in I}$. Then* $\dim \mathfrak{g}_\alpha$ *is the same for all $A \gg 0$. Moreover, if $A \gg 0$, we have*

$$\dim \mathfrak{g}_\alpha = \sum_{\tau|\alpha} \mu\left(\frac{\alpha}{\tau}\right) \frac{\tau}{\alpha} B_\infty(\tau), \tag{3.7}$$

where $B_\infty(\tau)$ is defined by (3.4).

Proof. We will consider $\dim \mathfrak{g}_{-\alpha}$. We take $S = \phi$ and apply Berman-Moody's formula. By Kostant's formula, we have

$$H_1(\mathfrak{g}_-) = \bigoplus_{i\in I} \mathbf{C}_{r_i\rho-\rho} = \bigoplus \mathbf{C}_{-\alpha_i},$$
$$H_2(\mathfrak{g}_-) = \bigoplus_{\substack{i,j\in I\\ i\neq j}} \mathbf{C}_{r_ir_j\rho-\rho} = \bigoplus_{\substack{i,j\in I\\ i\neq j}} \mathbf{C}_{-\alpha_j-(1-a_{ij})\alpha_i}$$
$$\cdots\cdots.$$

Therefore

$$P(H) = \{-\alpha_i \mid i \in I\} \cup \{-\alpha_j - (1 - a_{ij})\alpha_i \mid i, j \in I, i \neq j\} \cup \cdots\cdots.$$

Since $A \gg 0$, for any $\tau = \sum_{i \in I} t_i \alpha_i$ that divides α, there is only one partition of $-\tau$ into a sum of the elements in $P(H)$:

$$-\tau = t_1(-\alpha_1) + \cdots + t_n(-\alpha_n),$$

and $\dim H_{-\alpha_i} = 1$ for all $i \in I$. Hence we obtain

$$\dim \mathfrak{g}_\alpha = \sum_{\tau \mid \alpha} \mu\left(\frac{\alpha}{\tau}\right) \frac{\tau}{\alpha} \frac{(\sum t_i - 1)!}{\prod t_i!} = \sum_{\tau \mid \alpha} \mu\left(\frac{\alpha}{\tau}\right) \frac{\tau}{\alpha} B_\infty(\tau). \qquad \square$$

Remark. We can summarize the above discussion as follows. By Proposition 3.4, we have a projective system $\{\mathfrak{g}_+(A), \phi_{A',A}\}$. It follows from Corollary 3.2 and Theorem 3.6 that the projective limit of the above system is the free Lie algebra $\mathfrak{F}$ generated by the elements e_i $(i = 1, \ldots, n)$. We may consider $\mathfrak{F}$ as the subalgebra of the Kac-Moody algebra $\mathfrak{g}(A)$ generated by the elements e_i $(i = 1, \ldots, n)$, where the Cartan matrix $A = (a_{ij})$ is given by $a_{ij} = -\infty$ for all $i \neq j$. That is, in the rank 2 case, we may consider $\mathfrak{F}$ as $\mathfrak{g}_+(\infty)$.

In the following example, we illustrate the stability of root multiplicities for the root $\alpha = 5\alpha_0 + 6\alpha_1$. Compare with Example 2.4 and Example 3.3.

EXAMPLE 3.7.
(a) Let $a = 3$. Then by (2.3) we have

$$A_0(3) = 0,\ A_1(3) = 1,\ A_2(3) = 4,\ A_3(3) = 12, \cdots\cdots,$$

and hence

$$\mathscr{C}(3,5,6) = \{(5,6,0,0,\cdots),\ (1,5,1,0,\cdots),\ (4,2,0,1,\cdots),\ (0,1,1,1,\cdots)\},$$

corresponding to the partitions

$$\begin{aligned} -5\alpha_0 - 6\alpha_1 &= 5(-\alpha_0) + 6(-\alpha_1) \\ &= (-\alpha_0) + 5(-\alpha_1) + (-4\alpha_0 - \alpha_1) \\ &= 4(-\alpha_0) + 2(-\alpha_1) + (-\alpha_0 - 4\alpha_1) \\ &= (-\alpha_0) + (-4\alpha_0 - \alpha_1) + (-\alpha_0 - 4\alpha_1). \end{aligned}$$

It follows that

$$B_3(5,6) = \frac{1}{11}\binom{11}{5,6} + (-1)^1 \frac{1}{7}\binom{7}{1,5,1} + (-1)^1 \frac{1}{7}\binom{7}{4,2,0,1}$$

$$+ (-1)^2 \frac{1}{3} \binom{3}{1,1,1} = 23.$$

Therefore $\dim \mathfrak{g}(3)_\alpha = B_3(5,6) = 23$.
(b) If $a = 4$, then we have

$$A_0(4) = 0,\ A_1(4) = 1,\ A_2(4) = 5,\ A_3(4) = 20, \cdots\cdots,$$

and

$$\mathscr{C}(4,5,6) = \{(5,6,0,0,\cdots),\ (0,5,1,0,\cdots),\ (4,1,0,1,\cdots)\}.$$

Therefore

$$\dim \mathfrak{g}(4)_\alpha = B_4(5,6) = \frac{1}{11} \binom{11}{5,6} + (-1)^1 \frac{1}{6} \binom{6}{5,1} + (-1)^1 \frac{1}{6} \binom{6}{4,1,0,1} = 36.$$

(c) If $a = 5$, then we have

$$A_0(5) = 0,\ A_1(5) = 1,\ A_2(5) = 6,\ A_3(5) = 30, \cdots\cdots,$$

and

$$\mathscr{C}(5,5,6) = \{(5,6,0,0,\cdots),\ (4,0,0,1,\cdots)\}.$$

Therefore

$$\dim \mathfrak{g}(5)_\alpha = B_5(5,6) = \frac{1}{11} \binom{11}{5,6} + (-1)^1 \frac{1}{5} \binom{5}{4,0,0,1} = 41.$$

(d) If $a = 6$, then we have

$$A_0(6) = 0,\ A_1(6) = 1,\ A_2(6) = 7,\ A_3(6) = 42, \cdots\cdots,$$

and

$$\mathscr{C}(6,5,6) = \{(5,6,0,0,\cdots)\}.$$

Therefore

$$\dim \mathfrak{g}(6)_\alpha = B_4(5,6) = \frac{1}{11} \binom{11}{5,6} = 42.$$

Clearly, for $\alpha \geq 6$, we have

$$\mathscr{C}(a,\ 5,6) = \{(5,6,0,0,\cdots)\},$$

and hence $\dim \mathfrak{g}(a)_\alpha = B_a(5,6) = 42$ (see Table 7 in Section 6). □

4. The root system of the algebra $\mathfrak{g}(a)$

In this section, we study the root system of the algebra $\mathfrak{g}(a)$, and give an explicit description of the real roots and imaginary roots. As a by-product, we obtain a simple algorithm to find the integral points on certain hyperbolas.

A root α of a Kac-Moody algebra $\mathfrak{g}(A)$ is called *real* if $\alpha = w(\alpha_i)$ for some $w \in W$ and $i \in I$. A root that is not real is called *imaginary*. We denote by Δ^{re}, Δ^{re}_+, Δ^{im}, and Δ^{im}_+ the set of real roots, positive real roots, imaginary roots, and positive imaginary roots, respectively. We recall some of the fundamental properties of the imaginary roots.

PROPOSITION 4.1 ([K]).

(a) *The set* Δ^{im}_+ *is* W*-invariant.*

(b) *For* $\alpha \in \Delta^{\mathrm{im}}_+$, *there exists a unique* $\beta \in \Delta^{\mathrm{im}}_+$ *such that* $\beta = w(\alpha)$ *for some* $w \in W$ *and* $\beta(h_i) \leq 0$ *for all* $i \in I$. □

For a Kac-Moody algebra $\mathfrak{g}(A)$ with a symmetrizable generalized Cartan matrix $A = (a_{ij})_{i,j \in I}$, there is a nondegenerate symmetric bilinear form $(\,|\,)$ on $\mathfrak{h}^*$ such that $a_{ij} = \dfrac{2(\alpha_i \mid \alpha_j)}{(\alpha_i \mid \alpha_i)}$ for all $i, j \in I$. If A is symmetric, we take $(\alpha_i \mid \alpha_i) = 2$ for all $i \in I$ so that $a_{ij} = (\alpha_i \mid \alpha_j)$ for all $i, j \in I$.

A generalized Cartan matrix A is said to be of *finite type* if all of its principal minors are positive, of *affine type* if all of its proper principal submatrices are of finite type and $\det A = 0$, and of *indefinite type* otherwise. A is of *hyperbolic type* if it is of indefinite type and all of its proper principal submatrices are either of finite type or of affine type. The corresponding Kac-Moody algebra $\mathfrak{g}(A)$ is called a finite, affine, indefinite, or hyperbolic Kac-Moody algebra, respectively. The following proposition gives a nice description of the root system of hyperbolic Kac-Moody algebras.

PROPOSITION 4.2 ([K],[M]). Let $\mathfrak{g}(A)$ *be a hyperbolic Kac-Moody algebra with a symmetric generalized Cartan matrix. Then we have:*

(a) $\Delta^{\mathrm{re}} = \{\alpha \in Q \mid (\alpha \mid \alpha) = 2\}$,

(b) $\Delta^{\mathrm{im}} = \{\alpha \in Q \setminus \{0\} \mid (\alpha \mid \alpha) \leq 0\}$. □

We now focus on the structure of the algebra $\mathfrak{g}(a)$. We assume $a \geq 3$. Then the algebra $\mathfrak{g}(a)$ is of hyperbolic type. We identify an element $\alpha = x\alpha_0 + y\alpha_1 \in Q$ with an integral point $(x, y) \in \mathbf{Z} \times \mathbf{Z}$. We call $\alpha = (x, y) \in \mathbf{Z} \times \mathbf{Z}$ a *positive integral point if* $x, y \in \mathbf{Z}_{\geq 0}$. Define a symmetric bilinear form $(\,|\,)$ on $\mathfrak{h}^*$ by

$$(4.1) \qquad \begin{aligned} (\alpha_0 \,|\, \alpha_0) &= (\alpha_1 \,|\, \alpha_1) = 2, \\ (\alpha_0 \,|\, \alpha_1) &= -\, a. \end{aligned}$$

Then for $\alpha = (x, y) \in \mathbf{Z} \times \mathbf{Z}$, we have $(\alpha \,|\, \alpha) = 2(x^2 - axy + y^2)$. Therefore, as an immediate consequence of Proposition 4.2, we obtain:

COROLLARY 4.3. *For $a \geq 3$, the root system of the algebra $\mathfrak{g}(a)$ is given by*
(a) $\Delta^{\mathrm{re}} = \{(x, y) \in \mathbf{Z} \times \mathbf{Z} \,|\, x^2 - axy + y^2 = 1\}$,
(b) $\Delta^{\mathrm{im}} = \{(x, y) \in \mathbf{Z} \times \mathbf{Z} \,|\, x^2 - axy + y^2 \leq 0\}$.

In particular, there is a one-to-one correspondence between the set of real roots of $\mathfrak{g}(a)$ and the set of all integral points on the hyperbola $\mathscr{H}^1 : x^2 - axy + y^2 = 1$. Since there are no integral points on the union of lines $x^2 - axy + y^2 = 0$, the imaginary roots of $\mathfrak{g}(a)$ correspond to the integral points on the hyperbolas $\mathscr{H}_k : x^2 - axy + y^2 = -\, k$ for $k \geq 1$. In other words, for each $k \geq 1$, there is a one-to-one correspondence between the set of all imaginary roots α with lenghth $(\alpha \,|\, \alpha) = -\, 2k$ and the set of all integral points on the hyperbola $\mathscr{H}_k$.

To describe the root system of the algebra $\mathfrak{g}(a)$, we introduce a sequence $\{B_n\}_{n\geq 0}$ defined by

$$(4.2) \qquad \begin{aligned} &B_0 = 0, \quad B_1 = 1, \\ &B_{n+2} = aB_{n+1} - B_n \text{ for } n \geq 0. \end{aligned}$$

If $a = 3$, then we have $B_n = F_{2n-1}$ for all $n \geq 1$, where $\{F_n\}$ is the Fibonacci sequence defined by (2.4). By a similar method used in Section 2, we obtain a closed form expression for the sequence $\{B_n\}_{n\geq 0}$:

$$(4.3) \qquad B_n = \frac{1 - \gamma^{2n}}{\gamma^{n-1}(1 - \gamma^2)} \; (n \geq 0),$$

where $\gamma = \dfrac{a + \sqrt{a^2 - 4}}{2}$ is a zero of $1 - ax + x^2$. We can directly check that the points (B_n, B_{n+1}) lie on the hyperbola $\mathscr{H}^1 : x^2 - axy + y^2 = 1$ for all $n \geq 0$. Since the hyperbola $\mathscr{H}^1$ is asymptotic to the line $y = \gamma x$, it follows that

(4.4) $$B_{n+1} = \lceil \gamma B_n \rceil \text{ for all } n \geq 0,$$

where $\lceil x \rceil$ denotes the smallest integer $\geq x$.

For $j \geq 0$, induction, we have

(4.5) $$\begin{aligned} r_0(r_1r_0)^j(\alpha_0) &= -B_{2j+1}\alpha_0 - B_{2j}\alpha_1, \\ r_1(r_0r_1)^j(\alpha_0) &= B_{2j+1}\alpha_0 + B_{2j+2}\alpha_1, \\ (r_0r_1)^{j+1}(\alpha_0) &= B_{2j+3}\alpha_0 + B_{2j+2}\alpha_1, \\ (r_1r_0)^{j+1}(\alpha_0) &= -B_{2j+1}\alpha_0 - B_{2j+2}\alpha_1, \end{aligned}$$

and

(4.6) $$\begin{aligned} r_0(r_1r_0)^j(\alpha_1) &= B_{2j+2}\alpha_0 + B_{2j+1}\alpha_1, \\ r_1(r_0r_1)^j(\alpha_1) &= -B_{2j}\alpha_0 - B_{2j+1}\alpha_1, \\ (r_0r_1)^{j+1}(\alpha_1) &= -B_{2j+2}\alpha_0 - B_{2j+1}\alpha_1, \\ (r_1r_0)^{j+1}(\alpha_1) &= B_{2j+2}\alpha_0 + B_{2j+3}\alpha_1. \end{aligned}$$

Hence we obtain:

PROPOSITION 4.4 (cf. [BKM], [F]). *For $a \geq 3$, the set of all positive real roots of the algebra $\mathfrak{g}(a)$ is*

$$\Delta_+^{\mathrm{re}} = \{(B_n, B_{n+1}), (B_{n+1}, B_n) \mid n \geq 0\}. \qquad \square$$

The correspondence established in [LM] between the root the systems of rank 2 Kac-Moody algebras and quasi-regular cusps reveals more of the geometric nature of the root system. In particular, one should see [LM, Theorem 4.1], which characterizes the real roots as a support polygon.

We now consider the set of imaginary roots of $\mathfrak{g}(a)$. For a positive integer k, let $\Delta_{+,k}^{\mathrm{im}}$ be the set of all positive imaginary roots α of $\mathfrak{g}(a)$ with length $(\alpha \mid \alpha) = -2k$. That is, $\Delta_{+,k}^{\mathrm{im}}$ is the set of all positive integral points on the hyperbola $\mathscr{H}_k$: $x^2 - axy + y^2 = -k$. For any $\alpha \in \Delta_{+,k}^{\mathrm{im}}$, by Proposition 4.1, there is a unique $\beta = (m, n) \in \Delta_+^{\mathrm{im}}$ such that $\beta = w(\alpha)$ for some we $W \in W$ and

(4.7) $$\begin{aligned} \beta(h_0) &= 2m - an \leq 0, \\ \beta(h_1) &= -am + 2n \leq 0. \end{aligned}$$

Since the bilinear form $(\,|\,)$ is W-invariant, we also have $\beta \in \Delta_{+,k}^{\mathrm{im}}$. Let Ω_k be the set of all positive integral points $\beta = (m, n)$ on the hyperbola $\mathscr{H}_k$ that lie between the lines $y = x$ and $y = \frac{ax}{2}$, and let $\overline{\Omega_k}$ be the mirror image of the set Ω_k with respect to the line $y = x$. Then, by Proposition 4.1 and the symmetry of the root system, we have

$$\Delta^{\mathrm{im}}_{+,k} = (W\cdot\Omega_k) \cup (W\cdot\overline{\Omega_k}). \tag{4.8}$$

The hyperbola $\mathscr{H}_k$ and the line $y = \frac{ax}{2}$ meet at the point $P_k = \left(\frac{2\sqrt{k}}{\sqrt{a^2-4}}, \frac{a\sqrt{k}}{\sqrt{a^2-4}}\right)$. Also, the tangent line to the hyperbola $\mathscr{H}_k$ at the point P_k is the line $x = \frac{2\sqrt{k}}{\sqrt{a^2-4}}$. On the other hand, the line $y = x$ meets the hyperbola $\mathscr{H}_k$ at the point $\left(\sqrt{\frac{k}{a-2}}, \sqrt{\frac{k}{a-2}}\right)$. Hence we obtain

$$\Omega_k = \Big\{(m, n)\, \mathbf{Z}_{\geq 0} \times \mathbf{Z}_{\geq 0} \,\Big|\, \frac{2\sqrt{k}}{\sqrt{a^2-4}} \leq m \leq \sqrt{\frac{k}{a-2}}, \quad n = \frac{am - \sqrt{(a^2-4)m^2 - 4k}}{2}\Big\}. \tag{4.9}$$

THEOREM 4.5. *For $a \geq 3$, the set of all positive imaginary roots of the algebra $\mathfrak{g}(a)$ with length $-2k$ is*

$$\begin{aligned}\Delta^{\mathrm{im}}_{+,k} = \{&(mB_{j+1} - nB_j, mB_{j+2} - nB_{j+1}),\\ &(mB_{j+2} - nB_{j+1}, mB_{j+1} - nB_j), (nB_{j+1} - mB_j, nB_{j+2} - mB_{j+1}),\\ &(nB_{j+2} - mB_{j+1}, nB_{j+1} - mB_j) \mid (m,n) \in \Omega_k\}.\end{aligned}$$ □

Remark. Theorem 4.5 provides us with a simple algorithm to find all the integral points on the hyperbola $\mathscr{H}_k : x^2 - axy + y^2 = -k \ (k \geq 1)$, as is illustrated in the following example.

EXAMPLE 4.6. Let $a = 3$. Thus we have $B_n = F_{2n-1}\ (n \geq 1)$, where $\{F_n\}$ is the Fibonacci sequence defined by (2.4).

(a) If $k = 1$, then we have

$$\frac{2\sqrt{k}}{\sqrt{a^2-4}} = \frac{2}{\sqrt{5}} = 0.89\cdots, \text{ and } \sqrt{\frac{k}{a-2}} = 1.$$

Hence $m = 1$ is the only integer in the interval $\frac{2}{\sqrt{5}} \le m \le 1$. For $m = 1$, we have $n = 1$. Therefore $\Omega_1 = \{(1,1)\}$, and hence the set of all integral points on the hyperbola $x^2 - 3xy + y^2 = -1$ is

$$\{(B_{j+1} - B_j, B_{j+2} - B_{j+1}), (B_{j+2} - B_{j+1}, B_{j+1} - B_j),$$
$$(B_j - B_{j+1}, B_{j+1} - B_{j+2}), (B_{j+1} - B_{j+2}, B_j - B_{j+1}) \mid j \ge 0\}.$$

(b) If $k = 2$, then

$$\frac{2\sqrt{k}}{\sqrt{a^2-4}} = \frac{2\sqrt{2}}{\sqrt{5}} = 1.26\cdots, \text{ and } \sqrt{\frac{k}{a-2}} = \sqrt{2} = 1.14\cdots.$$

Thus $\Omega_2 = \phi$, and hence there is no integral point on the hyperbola $x^2 - 3xy + y^2 = -2$.

(c) If $k = 100$, then

$$\frac{2\sqrt{k}}{\sqrt{a^2-4}} = \frac{20}{\sqrt{5}} = 8.94\cdots, \text{ and } \sqrt{\frac{k}{a-2}} = 10.$$

Hence $m = 9$ and $m = 10$ are all the integers in the interval $\frac{20}{\sqrt{5}} \le m \le 10$. If $m = 9$, then $n = \frac{27 - \sqrt{5}}{2} \notin \mathbf{Z}_+$, and if $m = 10$, then $n = 10$. Therefore $\Omega_{100} = \{(10,10)\}$, and hence the set of all integral points on the hyperbola $x^2 - 3xy + y^2 = -100$ is

$$\{(10B_{j+1} - 10B_j, 10B_{j+2} - 10B_{j+1}), (10B_{j+2} - 10B_{j+1}, 10B_{j+1} - 10B_j),$$
$$(10B_j - 10B_{j+1}, 10B_{j+1} - 10B_{j+2}), (10B_{j+1} - 10B_{j+2}, 10B_j - 10B_{j+1}) \mid j \ge 0\}.$$

(d) If $k = 121$, then

$$\frac{2\sqrt{k}}{\sqrt{a^2-4}} = \frac{22}{\sqrt{5}} = 9.83\cdots, \text{ and } \sqrt{\frac{k}{a-2}} = 11.$$

Hence $m = 10$ and $m = 11$ are all the integers in the interval $\frac{22}{\sqrt{5}} \le m \le 11$. If $m = 10$, then $n = 13$, and if $m = 11$, then $n = 11$. Therefore $\Omega_{121} = \{(10,13), (11,11)\}$, and hence the set of all integral points on the hyperbola $x^2 - 3xy - y^2 = -121$ is

$\{(10B_{j+1} - 13B_j, 10B_{j+2} - 13B_{j+1}), (10B_{j+2} - 13B_{j+1}, 10B_{j+1} - 13B_j),$
$(13B_j - 10B_{j+1}, 13B_{j+1} - 10B_{j+2}), (13B_{j+1} - 10B_{j+2}, 13B_j - 10B_{j+1}),$
$(11B_{j+1} - 11B_j, 11B_{j+2} - 11B_{j+1}), (11B_{j+2} - 11B_{j+1}, 11B_{j+1} - 11B_j),$
$(11B_j - 11B_{j+1}, 11B_{j+1} - 11B_{j+2}), (11B_{j+1} - 11B_{j+2}, 11B_j - 11B_{j+1}) \mid$
$j \geq 0\}$. □

5. Root multiplicity relationships

In this section, we explain some of the relationships among the root multiplicities of the hyperbolic Kac-Moody algebras $\mathfrak{g}(a)$. In Section 3, we considered how the multiplicity of a given root varied with the algebra; here, we will restrict to a single algebra at a time and examine relationships between the multiplicities of different roots of that algebra.

§5.1. Column symmetry

We look first at the symmetry apparent in the columns of Tables 1-4. That is, fix an algebra $\mathfrak{g} = \mathfrak{g}(a)$, fix n and consider the multiplicities of the roots $-n\alpha_0 - (n+j)\alpha_1$ as j varies in $\mathbf{Z}$. We begin with a result of Kac [K, Proposition 3.6].

Proposition 5.1 ([K]). *Let V be a finite-dimensional module over $\mathfrak{sl}_2$, and let λ be a weight of V. Denote by M the set of all $t \in \mathbf{Z}$ such that $\lambda + t\alpha$ is a weight of V, where α is the simple root of $\mathfrak{sl}_2$. Let $m_t = \mathrm{mult}_V(\lambda + t\alpha)$. Then*

(a) *M is the closed interval of integers $[-p, q]$, where $p, q \in \mathbf{Z}_+$ and $p - q = \lambda(h)$.*

(b) *The function $t \mapsto m_t$ is increasing on the interval $\left[-p, \frac{1}{2}\lambda(h)\right]$ and is symmetric with respect to $t = \frac{1}{2}\lambda(h)$.* □

With a and n fixed, let $V = \bigoplus_{j\in\mathbf{Z}} \mathfrak{g}_{-(n\alpha_0+(n+j)\alpha_1)}$. Then V is a finite-dimensional module over the subalgebra $\mathfrak{g}_0$ of $\mathfrak{g}(a)$ generated by e_1, f_1, h_1. That is, $\mathfrak{g}_0 \cong \mathfrak{sl}_2$. Let $\lambda = -n\alpha_0 - n\alpha_1$. Then $\lambda(h_1) = n(a-2)$ and we can use part (b) of Proposition 5.1 to find the location of the maximal values of $\mathrm{mult}(\lambda + t\alpha_1)$. The result divides into three cases according as a and n are even or odd.

(1) If a is even, then $-\frac{1}{2}\lambda(h_1) = -\frac{1}{2}n(a-2) \in \mathbf{Z}$ and hence the maximal

root multiplicity occurs for the root $\lambda - \frac{1}{2}n(a-2)\alpha_1 = -n\alpha_0 - \frac{1}{2}an\alpha_1$.

(2) If a is odd and $n = 2m$ is even, then $-\frac{1}{2}\lambda(h_1) = -m(a-2) \in \mathbf{Z}$ and the maximum root multiplicity occurs for the root $\lambda - m(a-2)\alpha_1 = -2m\alpha_0 - am\alpha_1$.

(3) If a is odd and $n = 2m+1$ is odd, then $-\frac{1}{2}\lambda(h_1) = -\frac{1}{2}(2m+1)\cdot(a-2) \notin \mathbf{Z}$, and the closest integers to $-\frac{1}{2}\lambda(h_1)$ are $-\frac{1}{2}(2m+1)\cdot(a-2) \pm \frac{1}{2}$. Therefore the maximum multiplicity occurs for the roots

$$\lambda - \frac{1}{2}\{(2m+1)(a-2) \pm 1\}\alpha_1 = -(2m+1)\alpha_0 - \frac{1}{2}\{(2m+1)a \pm 1\}\alpha_1.$$

Also by part (b) of Proposition 5.1, we obtain the corresponding symmetries for the root multiplicities. That is,

(1) $\text{mult}\left(n, \frac{1}{2}na + j\right) = \text{mult}\left(n, \frac{1}{2}na - j\right)$,

(2) $\text{mult}(2m, ma + j) = \text{mult}(2m, ma - j)$,

(3) $\text{mult}\left(2m+1, \frac{1}{2}\{(2m+1)a + 1\} + j\right)$

$= \text{mult}\left(2m+1, \frac{1}{2}\{(2m+1)a - 1\} - j\right)$.

EXAMPLE 5.2 Let $a = 3$, $n = 2m+1 = 5$. Then the maximum multiplicity occurs for the roots $-5\alpha_0 - \frac{1}{2}\{5\cdot 3 \pm 1\}\alpha_1$, which are $-5\alpha_0 - 8\alpha_1$ and $-5\alpha_0 - 7\alpha_1$. This can be seen in Table 1 where the corresponding values of j are $j = 2$ and $j = 3$. The column of the table is symmetric about these maxima. □

Another view of the column symmetry is provided by considering directly the action of the Weyl group. Let r_i, $i = 0,1$ be the simple reflections defined by $r_i(\alpha) = \alpha - \alpha(h_i)\alpha_i$ where $\alpha_j(h_i) = a_{ij}$. Let $\alpha = k_0\alpha_0 + k_1\alpha_1$ in $\mathfrak{g}(a)$, for some fixed a. Then

(5.1) $$r_1(\alpha) = \alpha - (2k_1 - ak_0)\alpha_1.$$

In particular, for $\alpha = -n\alpha_0 - (n+j)\alpha_1$, as the roots are arranged in Tables 1-4, we have

$$\begin{aligned} r_1(\alpha) &= \alpha - (2(-n-j) - a(-n))\alpha_1 \\ &= -n\alpha_0 - ((a-1)n - j)\alpha_1. \end{aligned} \tag{5.2}$$

Hence $\mathrm{mult}(n, n+j) = \mathrm{mult}(n, (a-1)n - j)$.

For example, for $a = 3$, $n = 5$, we have

$$\mathrm{mult}(5,5+j) = \mathrm{mult}(5,10-j) = \mathrm{mult}(5,5+(5-j)).$$

as can be seen in Table 1.

Proposition 5.1 gives us that $p - q = \lambda(h)$, but does not specify the integers themselves. In order to determine the actual values of p and q, and thus the length of the root-chain, we consider root lengths.

Let $\alpha = -n\alpha_0 - (n+j)\alpha_1$ be a root in $\mathfrak{g} = \mathfrak{g}(a)$, for some fixed $a \geq 3$. Then

$$\begin{aligned} \frac{(\alpha, \alpha)}{2} &= n^2 + (n+j)^2 - an(n+j) \\ &= (2-a)n(n+j) + j^2. \end{aligned} \tag{5.3}$$

As discussed in Section 4, Moody ([M]) showed that for hyperbolic Kac-Moody algebras α is an imaginary root if and only if $\frac{(\alpha, \alpha)}{2} \leq 0$, and, for $\mathfrak{g}(a)$, α is a real root if and only if $\frac{(\alpha, \alpha)}{2} = 1$.

We may suppose without loss of generality, that $j \geq 0$. Then α is an imaginary root (and exists, i.e. has non-zero multiplicity) if

$$j \leq n(\gamma - 1), \text{ where } \gamma = \frac{a + \sqrt{a^2 - 4}}{2} \text{ (as in Section 2).} \tag{5.4}$$

We also have that α is a real root if and only if

$$\frac{(\alpha, \alpha)}{2} = (2-a)n(n+j) + j^2 = 1. \tag{5.5}$$

This is equivalent to the condition that

$$j = n\left[\frac{(a-2) + \sqrt{(a^2-4) + \frac{4}{n^2}}}{2}\right] \in \mathbf{Z}. \tag{5.6}$$

If the right-hand side is integral, then there is a real root (with multiplcity 1) $-n\alpha_0 - (n+j)\alpha_1$, for $j = \lfloor n(\gamma - 1)\rfloor + 1$ (where $\lfloor x \rfloor$ indicates the greatest integer less than or equal to x) and so $p = \lfloor n(\gamma - 1)\rfloor + 1$.

If the right-hand side of (5.6) is not integral, then the last root in the chain is imaginary and $p = \lfloor n(\gamma - 1)\rfloor$.

EXAMPLE 5.3 Fix $a = 3$. Then $\gamma = \frac{3+\sqrt{5}}{2}$, and the right hand side of (5.6) becomes $n\left[\frac{1 + \sqrt{5 + \frac{4}{n^2}}}{2}\right]$. Thus, for $n = 3$,

$$3\left[\frac{1+\sqrt{5+\frac{4}{9}}}{2}\right] = 5 \in \mathbf{Z},$$

and there is a (real) root with $j = \lfloor 3\left(\frac{1+\sqrt{5}}{2}\right)\rfloor + 1 = 5$. That is, $\alpha = -3\alpha_0 - 8\alpha_1$ is a real root of $\mathfrak{g}(3)$. For $n = 5$,

$$5\left[\frac{1+\sqrt{5+\frac{4}{25}}}{2}\right] \notin \mathbf{Z},$$

and so the last root has $j = \lfloor n(\gamma - 1)\rfloor = \lfloor 5\left(\frac{1+\sqrt{5}}{2}\right)\rfloor = 8$. That is, $\alpha = -5\alpha_0 - 13\alpha_1$ is a root, but $-5\alpha_0 - 14\alpha_1$ is not. □

§5.2. Multiplicity monotonicity

As consequence of the symmetry displayed above, we show that, as roots get "larger," their multiplicity increases. More precisely, for the roots $\alpha = (m, n)$ and $\beta = (k, l)$, we define $\alpha \le \beta$ if and only if $m \le k$ and $n \le l$. Then, in the fundamental chamber, we have that if $\alpha \le \beta$, then mult $\alpha \le$ mult β. In general, we have $\text{mult}(m, n) \le \text{mult}(m+1, n+1)$.

As in the previous subsection,we fix $a \ge 3$, and consider the roots of the algebra $\mathfrak{g}(a)$. Making the identification of the point (m, n) with the root $m\alpha_0 +$

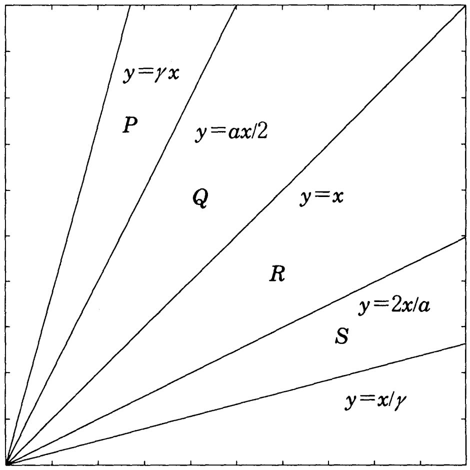

Fig 1.

$n\alpha_1$, from Section 4 we have that the imaginary roots of $\mathfrak{g}(a)$ are the integral points inside the cone $P \cup Q \cup R \cup S$ of Fig 1. (See also Tables 8 and 9 for examples.)

Further, the analysis of the previous subsection showed that the root multiplicities on a vertical line (that is, for a fixed m) are symmetric about the line $y = \frac{a}{2} x$, increasing monotonically as they approach this line. By symmetry of the root system, the root multiplicities in the horizontal direction are therefore monotonic and symmetric about the line $y = \frac{2}{a}x$. A consequence of this observation is that, if $\alpha = (m, n)$ is a root in the fundamental chamber $Q \cup R$, then any root in the region defined by the intersection of $Q \cup R$, $x \geq m$ and $y \geq n$, has multiplicity at least equal to that of α. Thus we have

Proposition 5.4. *Suppose α, β are in $Q \cup R$, with $\alpha \leq \beta$. Then* $\operatorname{mult} \alpha \leq \operatorname{mult} \beta$. □

Notice that Proposition 5.4 implies that if $\alpha = (n_1, kn_1)$ and $\beta = (n_2, kn_2)$ with $n_2 > n_1$, (i.e., α and β both lie on the line $y = kx$), and $\frac{2}{a} \leq k \leq \frac{a}{2}$, then $\operatorname{mult} \alpha \leq \operatorname{mult} \beta$. The Weyl reflection r_1 rotates (integral points on) a line $y = mx$ to (integral points on) a line $y = (a - m)x$. In particular, if $\frac{1}{\gamma} \leq k \leq \frac{2}{a}$, so that a root $\alpha = (n, kn) \in S$, then $r_1\alpha \in Q \cup R$. The remaining case is similar, and we obtain

COROLLARY 5.5 *Let* $\alpha = (n_1, kn_1)$, $\beta = (n_2, kn_2)$ *with* $n_2 > n_1$. *Then* $\operatorname{mult}\alpha \leq \operatorname{mult}\beta$. □

This observation was pointed out to us by V. Kac.

Now we may prove our general result.

PROPOSITION 5.6. *If* $\alpha = (m, n)$ *and* $\beta = (m+1, n+1)$ *are roots of* $\mathfrak{g}(a)$, $a \geq 3$, *then* $\operatorname{mult}\alpha \leq \operatorname{mult}\beta$.

Proof. We may suppose without loss of generality than $m \geq n$. From above, if $m > n\gamma$, we must that $m = \lfloor n\gamma \rfloor + 1$ and α is real root. Thus α has multiplicity 1 and $\operatorname{mult}\beta \geq \operatorname{mult}\alpha = 1$. From now on we will suppose that α (and hence β) is an imaginary root.

If $\alpha \in Q \cup R$, then $\beta \in Q \cup R$, and by Proposition 5.4 we are done. Hence, we may suppose $\alpha \notin Q \cup R$. By the hypothesis that $m \geq n$, $\alpha \in S$.

Consider the Weyl reflection r_1 and the root permutation σ defined for $\alpha = (m, n)$ by:

$$r_1(\alpha) = (m, am - n), \qquad \sigma(\alpha) = (n, m). \tag{5.7}$$

Both σ and r_1 preserve the root multiplicities. Geometrically, σ is the reflection in the line $y = x$, and r_1 reflects a root vertically about the line $y = \frac{a}{2}x$.

Define the following sequences:

$$\begin{aligned} &\alpha_0 = \alpha, \ \beta_0 = \beta, \\ &\alpha_j = (r_1\sigma)^j\alpha, \quad \beta_j = (r_1\sigma)^j\beta \quad (j \geq 1). \end{aligned} \tag{5.8}$$

Recall the sequence $\{B_n\}$ from (4.2) and introduce the sequence $\{C_n\}$ defined by:

$$\begin{aligned} B_0 = 0, \quad B_1 = 1, \quad B_{n+2} = aB_{n+1} - B_n \ (n \geq 0), \\ C_0 = 1, \quad C_1 = 1, \quad C_{n+2} = aC_{n+1} - C_n \ (n \geq 0). \end{aligned} \tag{5.9}$$

Then

$$\text{(5.10)}\qquad \begin{aligned} \alpha_j &= (B_j n - B_{j-1}m,\ B_{j+1}n - B_j m), \\ \beta_j &= (B_j n - B_{j-1}m + C_j,\ B_{j+1}n - B_j m + C_{j+1}), \quad j \geq 1. \end{aligned}$$

Note that both $\{B_n\}$ and $\{C_n\}$ are positive and increasing for $a \geq 3$. Hence, $\alpha_j \leq \beta_j$ for all j (and so $\sigma\alpha_j \leq \sigma\beta_j$).

Let $\eta = (p,q) \in S$, and consider $r_1\sigma\eta = (q,\ aq - p)$. Since $\eta \in S$, $q \leq \frac{2}{a}p$. Hence $\frac{a}{2}q \leq p$, $aq - \frac{a}{2}q \leq p$ and $aq - p \leq \frac{a}{2}q$. That is, $r_1\sigma\eta \in Q \cup R \cup S$. In fact, if $q > \frac{1}{a-1}p$, then $aq - p > q$ and $r_1\sigma\eta \in Q$, and, if $\frac{1}{a-1}p \leq q \leq \frac{a}{a^2-2}p$, then $aq - p \geq \frac{2}{a}q$ and $r_1\sigma\eta \in R$.

Let s be the smallest non-negative integer such that $\beta_s \in Q \cup R$. Note that, although $\alpha_0 \in S$, it is possible that $\beta_0 \in R$. Then, for $j < s$, we have $\beta_j \in S$.

CLAIM. *If $\beta_j \in S$, then $\alpha_j \in S$.*

Proof. Geometrically, $\beta_j = \alpha_j + (C_j,\ C_{j+1})$. Thus the slope of the line joining α_j to β_j (recall $\alpha_j \leq \beta_j$) is $\frac{C_{j+1}}{C_j} > 1 > \frac{2}{a}$ for $a \geq 3$. Hence β_j is closer to the line $y = \frac{2}{a}x$ than α_j, and, if $\beta_j \in S$, then so is α_j. □

If $\alpha_s \in Q \cup R$, then we have $\alpha_s \leq \beta_s$, $\alpha_s, \beta_s \in Q \cup R$, and, by Proposition 5.4, we are done.

Now suppose $\alpha_s \notin Q \cup R$. That is, $\alpha_s \in S$. Note that, in general, for $\eta = (p,\ q) \in S$, $r_1\sigma\eta \not\leq \eta$, but $\sigma r_1\sigma\eta \leq \eta$, or, equivalently, $r_1\sigma\eta \leq \sigma\eta$. To see this, recall that since $r_1\sigma\eta = (q,\ aq - p)$, we have $q \leq p$, but $aq - p$ may be greater than q, However, $aq - p \leq p$, since $q \leq \frac{2}{a}p$. Hence, $(r_1\sigma)^j\eta \leq \sigma^j\eta$.

Let r be the smallest such that $\alpha_r \in Q \cup R$. Then

$$\alpha_r \leq (r_1\sigma)^{r-s}\alpha_s \leq \sigma^{r-s}\alpha_s \leq \sigma^{r-s}\beta_s.$$

That is, either $\alpha_r \leq \beta_s$, or $\sigma\alpha_r \leq \beta_s$. In either case, we have $\operatorname{mult}\alpha \leq \operatorname{mult}\beta$. □

COROLLARY 5.7. *If* $\alpha = (m, n)$ *and* $\beta = (m + j, n + j)$ *are roots of* $\mathfrak{g}(a)$, $a \geq 3$ *with* j *a non-negative integer, then* $\text{mult}\,\alpha \leq \text{mult}\,\beta$. □

§5.3. Multiplicity questions

In this final section we raise some questions concerning further possible relationships among the root multiplicities for given algebras.

The analysis of Section 4 revealed a connection between the root multiplicity and the root length. Specifically, we showed that an imaginary root of $\mathfrak{g}(a)$ of length $-2k$ lies on the hyperbola $\mathscr{H}_k$ and is W-conjugate to some root in Ω_k. If $|\Omega_k| = 1$, all roots of length $-2k$ must have the same multiplicity. If $|\Omega_k| > 1$, this need not be the case and there may be roots of the same length, but with different multiplicities. For example, the roots (9,17) and (11,11) of $\mathfrak{g}(4)$ both have length -484, but have multiplicities 18900 and 18901, respectively.

A natural question is whether relationship between length and multiplicity is monotonic. That is, does the multiplicity always increase as (the magnitude of) the length increases? Geometrically, this would imply that the multiplicity of any root "inside" a hyperbola is greater than or equal to the multiplicity of any root on the hyperbola. In general, this is not true. We do not know of any counterexamples for the algebras $\mathfrak{g}(3)$ and $\mathfrak{g}(4)$, but in $\mathfrak{g}(5)$, the root $\alpha = (3,7)$ has length $-2k = -94$ and multiplicity 9, while the root $\beta = (4,4)$ has length -96 and multiplicity 8. There are similar examples for $a = 6$ and 7. In view of the apparent exponential increase in multiplicity with respect to length, it would be very interesting to discover precisely the conditions required for this to happen.

Note for comparison the mysterious formulas of [FF] and [FFR] for $HA_1^{(1)}$, [KMW] for E_{10}, and [KM] for $HA_n^{(1)}$, which showed that, for roots of low level in certain hyperbolic Kac-Moody algebras, the multiplicity depends only on the length of the root and increases monotonically with the root length.

In the previous section, we showed that root multiplicities increase monotonically along the lines $y = x + j$, $j \in \mathbf{Z}$, $j \geq 0$. It is also interesting to consider the lines perpendicular to these, the integral points of which represent roots of a given height t. It can easily be seen that, for the roots in the tables, the multiplicities increase monotonically to maximum at (m, m) for t even, and $(m - 1, m + 1)$ for t odd. We believe that this may be true in general. That is, we have the following conjecture:

CONJECTURE 5.8. *Let* m, n *be non-negative integers with* $m \leq n-2$. *Then, for any* $\mathfrak{g}(a)$, $a \geq 3$, $\text{mult}(m, n) \leq \text{mult}(m+1, n-1)$. □

6. Root multiplicity tables

In this section we present some tables of root multiplicities of the rank 2 Kac-Moody algebras $\mathfrak{g}(a)$ to illustrate the relationships explained in the text.

In Tables 1-7, we consider the roots of the form $n\alpha_0 + (n+j)\alpha_1$ in various different settings. First, in Tables 1-4, we consider the root multiplicities in a given algebra as n and j vary. That is, we have a fixed $a = 3, \dots, 6$ in each table. For comparison, Table 5 then gives the root multiplicities for the same roots in the free Lie algebra of rank 2. Tables 6 and 7 rearrange the roots to illustrate the stability theorem of Section 3. Here we fix $j = 0$ and $j = 1$, respectively, and allow a and n to vary. The stability theorem and the precise relationships indicated in (2.24) and (2.25) can be seen clearly.

Tables 8 and 9 present a different view of the algebras. In these tables we give the multiplicities for roots written in the form $m\alpha_0 + n\alpha_0$ for $\mathfrak{g}(a)$, $a = 3$ and 4, with the regions P, Q, R, S from Fig 1, and some of the hyperbolas $\mathscr{H}_k$ superimposed. The reader will easily be able to see Proposition 5.6 and the force of Conjecture 5.8.

The root multiplicities were calculated using the *Kacmoody* algorithm of A.J. Coleman of Queen's University. The program itself was written by R. McCann and I. Wilmott, and modified by R. McCann and M. Roth. The algorithm is based upon Berman-Moody's formula (1.14).

Root multiplicities for $a = 3$

$j\backslash n$	1	2	3	4	5	6	7	8	9	10
0	1	1	3	6	16	39	107	288	808	2278
1	1	2	4	9	23	60	162	449	1267	3630
2	1	1	4	9	27	73	211	600	1754	5130
3	0	1	3	9	27	80	240	720	2167	6555
4	0	0	2	6	23	73	240	758	2407	7554
5	0	0	1	4	16	60	211	720	2407	7936
6	0	0	0	1	9	39	162	600	2167	7554
7	0	0	0	0	4	23	107	449	1754	6555
8	0	0	0	0	1	9	60	288	1267	5130
9	0	0	0	0	0	3	27	162	808	3630
10	0	0	0	0	0	0	9	73	449	2278

Table 1

Root multiplicities for $a = 4$

$j\backslash n$	1	2	3	4	5	6	7	8	9	10
0	1	1	3	8	23	64	195	590	1850	5861
1	1	2	5	13	36	106	321	995	3144	10088
2	1	2	6	16	50	151	480	1521	4928	16070
3	1	2	6	20	63	202	660	2169	7185	23990
4	0	1	6	20	72	243	840	2860	9810	33605
5	0	1	5	20	75	276	995	3550	12590	44498
6	0	0	3	16	72	283	1100	4115	15238	55717
7	0	0	2	13	63	276	1137	4510	17441	66284
8	0	0	1	8	50	243	1100	4635	18900	74886
9	0	0	0	5	36	202	995	4510	19409	80600
10	0	0	0	2	23	151	840	4115	18900	82543

Table 2

Root multiplicities for $a = 5$

$j \backslash n$	1	2	3	4	5	6	7	8	9	10
0	1	1	3	8	25	73	232	734	2400	7935
1	1	2	5	14	41	125	395	1277	4207	14073
2	1	2	7	19	61	190	625	2059	6930	23511
3	1	3	8	26	84	276	928	3150	10800	37361
4	1	2	9	30	108	370	1300	4540	16005	56536
5	0	2	9	35	130	475	1725	6250	22628	81950
6	0	1	8	35	147	566	2175	8177	30612	113869
7	0	1	7	35	156	650	2612	10262	39725	152243
8	0	0	5	30	156	698	2993	12300	49525	195928
9	0	0	3	26	147	720	3275	14175	59400	243339
10	0	0	2	19	130	698	3425	15645	68625	291685

Table 3

Root multiplicities for $a = 6$

$j \backslash n$	1	2	3	4	5	6	7	8	9	10
0	1	1	3	8	25	75	243	785	2616	8815
1	1	2	5	14	42	131	421	1387	4654	15855
2	1	2	7	20	65	206	686	2297	7857	27111
3	1	3	9	29	94	312	1059	3641	12660	44450
4	1	3	11	36	128	441	1557	5489	19558	69943
5	1	3	12	45	165	602	2189	7967	29050	106162
6	0	2	12	50	203	774	2951	11085	41594	155545
7	0	2	12	56	238	966	3283	14904	57540	220728
8	0	1	11	56	266	1143	4769	19305	77028	303534
9	0	1	9	56	284	1311	5737	24228	99932	405426
10	0	0	7	50	290	1436	6664	29389	125804	526177

Table 4

Root multiplicities for free Lie algebra of rank 2

$j\backslash n$	1	2	3	4	5	6	7	8	9	10
0	1	1	3	8	25	75	245	800	2700	9225
1	1	2	5	14	42	132	429	1430	4862	16796
2	1	2	7	20	66	212	715	2424	8398	29372
3	1	3	9	30	99	333	1144	3978	13995	49742
4	1	3	12	40	143	497	1768	6288	22610	81686
5	1	4	15	55	200	728	2652	9690	35530	130750
6	1	4	18	70	273	1026	3876	14520	54477	204248
7	1	5	22	91	364	1428	5537	21318	81719	312455
8	1	5	26	112	476	1932	7752	30624	120175	468611
9	1	6	30	140	612	2583	10659	43263	173583	690690
10	1	6	35	168	775	3384	14421	60060	246675	1001400

Table 5

Root multiplicities of $\mathfrak{g}(a)$, for $j = 0$

$a\backslash n$	1	2	3	4	5	6	7	8	9	10
3	1	1	3	6	16	39	107	288	808	2278
4	1	1	3	8	23	64	195	590	1850	5861
5	1	1	3	8	25	73	232	734	2400	7935
6	1	1	3	8	25	75	243	785	2616	8815
7	1	1	3	8	25	75	245	798	2683	9121
8	1	1	3	8	25	75	245	800	2698	9206
9	1	1	3	8	25	75	245	800	2700	9223
10	1	1	3	8	25	75	245	800	2700	9225

Table 6

Root multiplicities of $\mathfrak{g}(a)$, for $j = 1$

$a\backslash n$	1	2	3	4	5	6	7	8	9	10
3	1	2	4	9	23	60	162	449	1267	3630
4	1	2	5	13	36	106	321	995	3144	10088
5	1	2	5	14	41	125	395	1277	4207	14073
6	1	2	5	14	42	131	421	1387	4654	15855
7	1	2	5	14	42	132	428	1421	4809	16522
8	1	2	5	14	42	132	429	1429	4852	16732
9	1	2	5	14	42	132	429	1430	4861	16785
10	1	2	5	14	42	132	429	1430	4862	16795

Table 7

Roots of $\mathfrak{g}(3)$

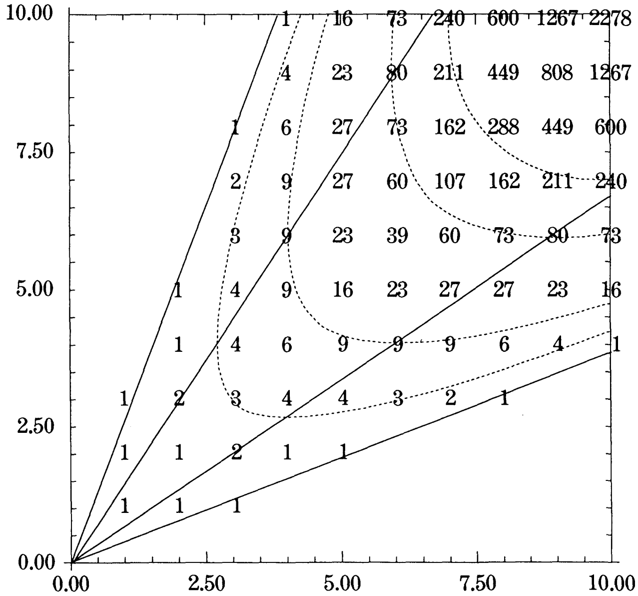

Table 8

Roots of $\mathfrak{g}(4)$

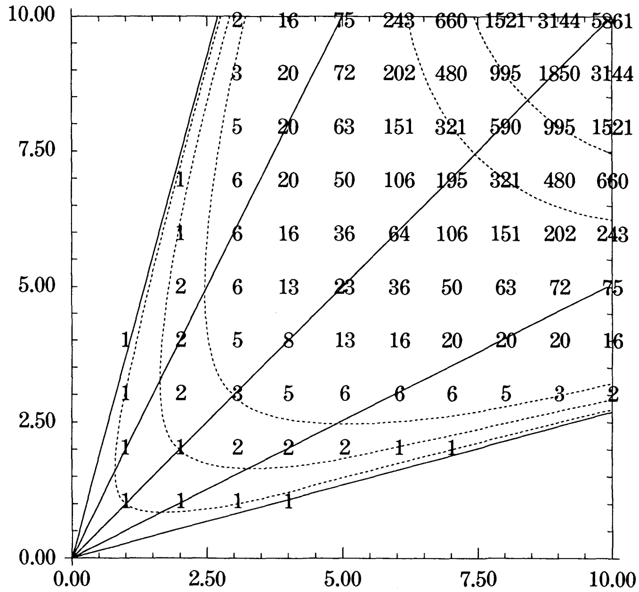

Table 9

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