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ON THE SERIES FOR $L(1, \chi)$

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1. Introduction

Let k be a positive integer greater than 1, and let $\chi(n)$ be a real primitive character modulo k, The series

$$L(1, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n}$$

can be divided into groups of k consecutive terms. Let v be any nonnegative integer, j and integer, $0 \leq j \leq k - 1$, and let

$$T(v, j, \chi) = \sum_{n=j+1}^{j+k} \frac{\chi(vk+n)}{vk+n} = \sum_{n=j+1}^{j+k} \frac{\chi(n)}{vk+n}.$$

Then $L(1, \chi) = \sum_{n=1}^{j} \frac{\chi(n)}{n} + \sum_{v=0}^{\infty} T(v, j, \chi)$.

In [3] Davenport proved the following theorem:

THEOREM (H. Davenport). *If $\chi(-1) = 1$, then $T(v, 0, \chi) > 0$ for all v and k. If $\chi(-1) = -1$, then $T(0, 0, \chi) > 0$ for all k, and $T(v, 0, \chi) > 0$ if $v > v(k)$; but for any $r \geq 1$ there exist values of k for which*

$$T(1, 0, \chi) < 0, \ T(2, 0, \chi) < 0, \ldots, T(r, 0, \chi) < 0.$$

In this paper, we will prove

THEOREM 2. *For fixed integers k and j, $0 \leq j \leq k - 1$,*

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$$T(v, j, \chi)T(v+1, j, \chi) > 0$$

for positive integer $v > v(k, j)$.

In the case $j = \left[\frac{k}{2}\right]$, where $[x]$ denotes the greatest integer $\leq x$, we have the following more refined results.

THEOREM 3. *If* $\chi(-1) = 1$, *then* $T\left(v, \left[\frac{k}{2}\right], \chi\right) < 0$ *for all* v *and* k.

THEOREM 6. *Let* $\chi(-1) = -1$.

(1) *If* $k \not\equiv 7 \pmod 8$, *then* $T\left(v, \left[\frac{k}{2}\right], \chi\right) < 0$ *for* $v > k^{\frac{1}{4}}$.

(2) *If* $k \equiv 7 \pmod 8$, *then* $T\left(v, \left[\frac{k}{2}\right], \chi\right) > 0$ *for* $v \geq 0$.

As a consequence of Davenport's theorem [3] and Theorem 3, we have the following inequality for even χ (cf. Corollary 1 (2)):

$$\sum_{n=1}^{k} \frac{\chi(n)}{n} < L(1, \chi) < \sum_{n=1}^{[\frac{k}{2}]} \frac{\chi(n)}{n}.$$

Furthermore, using a result of Davenport [3], we derive a class number formula

$$h = \left[\frac{k^{3/2}}{2\ln \varepsilon} \sum_{n=1}^{[\frac{k}{2}]} \frac{\chi(n)}{n(k-n)}\right] + 1$$

for real quadratic fields, which seems a little more efficient than the class number formulas mentioned in [4] and page 46 of [5]. Also, we give estimates of the class numbers of imaginary quadratic fields (cf. Corollary 2).

We remind the reader that a real primitive character $(\mathrm{mod}\ k)$ exists only when either k or $-k$ is a fundamental discriminant, and that the character is then given by

$$\chi(n) = \left(\frac{d}{n}\right),$$

where d is k or $-k$, and the symbol is that of Kronecker (see, for example, Ayoub [1] for the definition of a Kronecker character).

2. A proof of Theorem 2

PROPOSITION 1. *Let χ be a real primitive character modulo a positive odd integer k. (If $k \equiv 1 \pmod 4$, then $\chi(-1) = 1$, otherwise $\chi(-1) = -1$.) Then*

$$T(0, j, \chi) \neq 0 \quad \text{for} \quad j = 0,1,2,\ldots, k-1.$$

Proof. For any positive odd integer $k > 1$, there exists a unique positive integer α such that $2^{\alpha} < k < 2^{\alpha+1}$. Let γ be the largest power such that $2^{\gamma} \leq j + k$. Then $\gamma = \alpha$ or $\alpha + 1$ depending on j. For integers $i = 1,2,\ldots, k$, we express $j + i = 2^{\beta_i} m_i$ with m_i an odd integer and β_i an integer. Clearly, $j + l = 2^{\gamma}$ for some integer l, $1 \leq l \leq k$, and $\beta_i < \gamma$ for $i \neq l$. Write $\Pi_{i=1}^{k}(j+i) = 2^{t}M$, where $t = \beta_1 + \cdots + \beta_k$ and $M = \Pi_{i=1}^{k} m_i$ is an odd integer. We have

$$T(0, j, \chi) = \sum_{i=1}^{k} \frac{\chi(j+i)}{j+i} = \frac{\sum_{i=1}^{k}\chi(j+i)2^{t-\beta_i}\frac{M}{m_i}}{2^{t}M} =: \frac{N}{2^{t}M}.$$

Write the numerator N as a sum of two parts $\sum_{i\neq l}\chi(j+i)2^{t-\beta_i}\frac{M}{m_i} + \chi(j+l)M2^{t-\gamma}$. Since the modulus k is odd, we know $\chi(2) \neq 0$, and

$$\frac{N}{2^{t-\gamma}} = \sum_{i\neq l}\chi(j+i)2^{\gamma-\beta_i}\frac{M}{m_i} + \chi(2^{\gamma})M \equiv 1 \pmod 2.$$

This implies that $N \neq 0$, and therefore $T(0, j, \chi) = \dfrac{N}{2^{t}M} \neq 0.$ □

Remarks. 1. The above argument actually proves a more general fact, namely, given any two positive integers $M > m$, if there is a positive power of 2 between them, then $\sum_{i=m}^{M} \dfrac{\chi(i)}{i^{r}} \neq 0$ for any positive integer r.

2. The sign of $T(0, j, \chi)$ is known for the following cases: When $j = 0$, it is positive for any modulus k (cf. [3]); when $j = \left[\dfrac{k}{2}\right]$, it is negative for any k such that $\chi(-1) = 1$ (cf. Theorem 3), and it is positive for $k \equiv 7 \pmod 8$ which implies $\chi(-1) = -1$ (cf. Theorem 6).

Instead of proving Theorem 2 directly we shall prove a more general statement first.

For each positive integer d, let f_d be a function on the integers such that $f_d(j+1), \ldots, f_d(j+d)$ are not all zero for some integer j. Let $C(l, j, f_d) =$

$\sum_{m=1}^{d} f_d(j+m)m^l$, where l is any integer. Then we have the following result:

THEOREM 1. *For some integer l, $0 \le l \le d-1$, one has $C(l, j, f_d) \neq 0$.*

Proof. Express the system of equations

$$C(l, j, f_d) = \sum_{m=1}^{d} f_d(j+m)m^l, \quad l = 0, 1, \ldots, d-1,$$

in matrix form:

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 2 & \cdots & d \\ \cdot & \cdot & \cdot & \cdot \\ 1^{d-1} & 2^{d-1} & \cdots & d^{d-1} \end{pmatrix} \begin{pmatrix} f_d(j+1) \\ f_d(j+2) \\ \cdot \\ \cdot \\ \cdot \\ f_d(j+d) \end{pmatrix} = \begin{pmatrix} C(0, j, f_d) \\ C(1, j, f_d) \\ \cdot \\ \cdot \\ \cdot \\ C(d-1, j, f_d) \end{pmatrix}.$$

Since the Vandermonde matrix is invertible, and $f_d(j+1), \ldots, f_d(j+d)$ are not all zero, so $C(l, j, f_d) \neq 0$ for some l, $0 \le l \le d-1$. □

For integers $v \ge 1$ and $0 \le j \le k-1$, we have

$$\begin{aligned} T(v, j, \chi) &= \sum_{m=1}^{k} \frac{\chi(j+m)}{vk+j+m} \\ &= \frac{1}{vk+j} \sum_{m=1}^{k} \frac{\chi(j+m)}{1 + \frac{m}{vk+j}} \\ &= \frac{1}{vk+j} \sum_{m=1}^{k} \chi(j+m) \sum_{l=0}^{\infty} (-1)^l \frac{m^l}{(vk+j)^l} \\ &= \frac{1}{vk+j} \sum_{l=0}^{\infty} \left(\sum_{m=1}^{k} \chi(j+m)m^l \right) \left(\frac{-1}{vk+j} \right)^l. \end{aligned}$$

(In the above expansion, $m = vk + j$ occurs only when $j = 0$, $v = 1$ and $m = k$, in which case $\chi(j+m) = 0$ and there is no need to consider such a term.) As a corollary of Theorem 1, we have:

THEOREM 2. *For any fixed integers k and j, $0 \le j \le k-1$, one has*

$$T(v, j, \chi)T(v+1, j, \chi) > 0$$

for positive integer $v > v(k, j)$.

Proof. Applying Theorem 1 to the case $d = k$ and $f_d = \chi$, we have $\sum_{m=1}^{k} \chi(j+m)m^l = C(l, j, \chi) \neq 0$ for some integer l, $0 \leq l \leq k-1$. Let l_0 be the smallest nonnegative integer such that $C(l_0, j, \chi) \neq 0$. Then there exists a positive integer $v(k, j)$ such that

$$(-1)^{l_0} C(l_0, j, \chi) T(v, j, \chi) > 0$$

for $v > v(k, j)$. □

Remark. From the proof of Theorem 2, we know that, for integer v large enough, the sign of $T(v, j, \chi)$ and the sign of $(-1)^{l_0} C(l_0, j, \chi)$ are the same, where l_0 is the smallest nonnegative integer such that $C(l_0, j, \chi) \neq 0$. Moreover, we may choose $v(k, j)$ in the proof above to be $\frac{1}{k}((k+1)^{l_0+2} - j)$. In general, the sign of $T(v, j, \chi)$, with fixed χ, j and varying v, changes sometimes, but our computer data never showed these partial sums equal to zero.[1]

3. The real quadratic fields

From the definition of Kronecker character we know that $\chi(n) = \chi(-n) \cdot \mathrm{sgn}(d)$, where d is the fundamental discriminant equal to k or $-k$ (cf. [1, page 292]). If both k and $-k$ are fundamental discriminants (which happens if and only if $k = 8k'$, where k' is odd and squarefree) there are two real primitive characters (Kronecker character) $(\bmod k)$, otherwise only one. Clearly, we have that $\chi(-1) = 1$ if and only if $d > 0$. In this section we restrict ourselves to the case $d = k$. Fix such an integer k, let χ be a real primitive character attached to the real quadratic field $\mathbf{Q}(\sqrt{k})$ with $\chi(-1) = 1$.

THEOREM 3. *For any integer* $v \geq 0$, $T\left(v, \left[\frac{k}{2}\right], \chi\right) < 0$.

Proof. Write $T(v, j, \chi) = \sum_{n=j+1}^{j+k} \frac{\chi(n)}{vk+n} = \frac{1}{k} \sum_{n=j+1}^{j+k} \frac{\chi(n)}{v + \frac{n}{k}}$ and keep in

[1] After this paper was written, the first author showed in [7] that the sums $T(v, j, \chi)$ are indeed nonzero for any odd prime k.

mind that j is equal to $\left[\frac{k}{2}\right]$ in this proof.

For integer $v \geq 0$, consider the function

$$g(x) = \frac{1}{v+x} \text{ defined for } \frac{1}{2} \leq x \leq \frac{3}{2}.$$

Over the interval $\left(\frac{1}{2}, \frac{3}{2}\right)$, it has Fourier expansion

$$g(x) = \frac{1}{2} a_0 + \sum_{m=1}^{\infty} (a_m \cos 2\pi m x + b_m \sin 2\pi m x),$$

where

$$a_m = 2 \int_{\frac{1}{2}}^{\frac{3}{2}} \frac{\cos 2\pi m x}{v+x} dx \text{ and } b_m = 2 \int_{\frac{1}{2}}^{\frac{3}{2}} \frac{\sin 2\pi m x}{v+x} dx.$$

Using integration by parts, we have, for $m \geq 1$,

$$a_m = \frac{-2\cos 2\pi m x}{(2\pi m)^2 (v+x)^2} \Bigg|_{1/2}^{3/2} + \frac{12 \cos 2\pi m x}{(2\pi m)^4 (v+x)^4} \Bigg|_{1/2}^{3/2} + \frac{48}{(2\pi m)^4} \int_{\frac{1}{2}}^{\frac{3}{2}} \frac{\cos 2\pi m x}{(v+x)^5} dx.$$

Let

$$X = \frac{12 \cos 2\pi m x}{(2\pi m)^4 (v+x)^4} \Bigg|_{1/2}^{3/2} \text{ and } Y = \frac{48}{(2\pi m)^4} \int_{\frac{1}{2}}^{\frac{3}{2}} \frac{\cos 2\pi m x}{(v+x)^5} dx.$$

Then $|Y| < |X|$ and $XY < 0$. We have

$$a_m = (-1)^m \frac{2}{(2\pi m)^2} \left\{ \frac{1}{\left(v+\frac{1}{2}\right)^2} - \frac{1}{\left(v+\frac{3}{2}\right)^2} \right\}$$
$$+ (-1)^{m+1} \frac{12}{(2\pi m)^4} \left\{ \frac{1}{\left(v+\frac{1}{2}\right)^4} - \frac{1}{\left(v+\frac{3}{2}\right)^4} \right\} \theta_m,$$

where $\theta_m = \frac{X+Y}{X}$ depending on v and $0 < \theta_m < 1$. Now

$$T(v, j, \chi) = \frac{1}{k} \sum_{n=j+1}^{j+k} \chi(n) g\left(\frac{n}{k}\right)$$
$$= \frac{1}{k} \sum_{n=j+1}^{j+k} \chi(n) \left\{ \sum_{m=1}^{\infty} \left(a_m \cos 2\pi m \frac{n}{k} + b_m \sin 2\pi m \frac{n}{k} \right) \right\} \quad \left(\text{since } \sum_{n=j+1}^{j+k} \chi(n) = 0 \right)$$

$$= \frac{1}{k} \sum_{m=1}^{\infty} \left\{ a_m \sum_{n=j+1}^{j+k} \chi(n) \cos 2\pi m \frac{n}{k} + b_m \sum_{n=j+1}^{j+k} \chi(n) \sin 2\pi m \frac{n}{k} \right\}$$
$$= \frac{1}{k} \sum_{m=1}^{\infty} a_m \chi(m) \sqrt{k}.$$

Here we used the fact that Gauss sum $\sum_{n=1}^{k} \chi(n) \exp \frac{2\pi i m n}{k} = \chi(m)\sqrt{k}$ since $\chi(-1) = 1$. Rigorously speaking, the above expression for $T(v, j, \chi)$ is valid for k odd; when k is even, we have $k \equiv 0 \pmod 4$, hence $\chi(j + k) = \chi\left(\left[\frac{k}{2}\right] + k\right) = 0$ and $T(v, j, \chi)$ is really summing over $j + 1 \le n \le j + k - 1$ so that we may replace g by its Fourier expansion. After interchanging the sum over m and n, we may change the limit for n back to $j + 1 \le n \le j + k$ since $\chi(j + k) = 0$. The final conclusion for $T(v, j, \chi)$ remains the same. Hence

$$\sqrt{k}\, T\left(v, \left[\frac{k}{2}\right], \chi\right) = \sum_{m=1}^{\infty} a_m \chi(m)$$
$$= \frac{1}{2\pi^2} \left\{ \frac{1}{\left(v + \frac{1}{2}\right)^2} - \frac{1}{\left(v + \frac{3}{2}\right)^2} \right\} \sum_{m=1}^{\infty} \frac{(-1)^m \chi(m)}{m^2}$$
$$+ \frac{3}{4\pi^4} \left\{ \frac{1}{\left(v + \frac{1}{2}\right)^4} - \frac{1}{\left(v + \frac{3}{2}\right)^4} \right\} \sum_{m=1}^{\infty} \frac{(-1)^{m+1} \chi(m) \theta_m}{m^4}.$$

We divide the argument into two cases:

Case 1. $v \ge 1$.

Since

$$\sum_{m=1}^{\infty} \frac{(-1)^m \chi(m)}{m^2} = -1 + \sum_{m=2}^{\infty} \frac{(-1)^m \chi(m)}{m^2}$$
$$< -2 + \sum_{m=1}^{\infty} \frac{1}{m^2} = -2 + \frac{\pi^2}{6} < 0$$

and $\zeta(4) = \frac{\pi^4}{90}$, we have

$$\sqrt{k}\, T\left(v, \left[\frac{k}{2}\right], \chi\right)$$

$$< \frac{1}{2\pi^2}\left\{\frac{1}{\left(v+\frac{1}{2}\right)^2}-\frac{1}{\left(v+\frac{3}{2}\right)^2}\right\}\left(-2+\frac{\pi^2}{6}\right)+\frac{3}{4\pi^4}\left\{\frac{1}{\left(v+\frac{1}{2}\right)^4}-\frac{1}{\left(v+\frac{3}{2}\right)^4}\right\}\zeta(4)$$

$$= \left(\frac{1}{12}-\frac{1}{\pi^2}\right)\left\{\frac{1}{\left(v+\frac{1}{2}\right)^2}-\frac{1}{\left(v+\frac{3}{2}\right)^2}\right\}+\frac{1}{120}\left\{\frac{1}{\left(v+\frac{1}{2}\right)^4}-\frac{1}{\left(v+\frac{3}{2}\right)^4}\right\}$$

$$= \left\{\frac{1}{\left(v+\frac{1}{2}\right)^2}-\frac{1}{\left(v+\frac{3}{2}\right)^2}\right\}\left\{\frac{1}{12}-\frac{1}{\pi^2}+\frac{1}{120}\left(\frac{1}{\left(v+\frac{1}{2}\right)^2}+\frac{1}{\left(v+\frac{3}{2}\right)^2}\right)\right\}.$$

For integer $v \geq 1$, we have

$$120\left(\frac{1}{\pi^2}-\frac{1}{12}\right) > \frac{1}{\left(\frac{3}{2}\right)^2}+\frac{1}{\left(\frac{5}{2}\right)^2} \geq \frac{1}{\left(v+\frac{1}{2}\right)^2}+\frac{1}{\left(v+\frac{3}{2}\right)^2}.$$

This gives

$$\frac{1}{12}-\frac{1}{\pi^2}+\frac{1}{120}\left\{\frac{1}{\left(v+\frac{1}{2}\right)^2}+\frac{1}{\left(v+\frac{3}{2}\right)^2}\right\} < 0.$$

Hence $T\left(v, \left[\frac{k}{2}\right], \chi\right) < 0$ for integer $v \geq 1$.

Case 2. $v = 0$.

We have

$$\begin{aligned}\sqrt{k}\,T\left(0, \left[\frac{k}{2}\right], \chi\right) &= \frac{32}{18\pi^2}\left\{\sum_{m=1}^{\infty}\frac{(-1)^m\chi(m)}{m^2}+\frac{20}{3\pi^2}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}\chi(m)\theta_m}{m^4}\right\}\\ &= \frac{32}{18\pi^2}\left\{\sum_{m=1}^{\infty}\frac{(-1)^m\chi(m)(m^2-\alpha\theta_m)}{m^4}\right\}\left(\text{where } \alpha=\frac{20}{3\pi^2}\right)\\ &= \frac{16}{9\pi^2}\left\{-1+\alpha\theta_1+\sum_{m=2}^{\infty}\frac{(-1)^m\chi(m)(m^2-\alpha\theta_m)}{m^4}\right\}\\ &< \frac{16}{9\pi^2}\left\{-1+\alpha\theta_1+\sum_{m=2}^{\infty}\frac{1}{m^2}\right\}\\ &= \frac{16}{9\pi^2}\{-2+\alpha\theta_1+\zeta(2)\}\\ &= \frac{16}{9\pi^2}\left\{-2+\alpha\theta_1+\frac{\pi^2}{6}\right\}.\end{aligned}$$

To estimate $-2 + \alpha\theta_1 + \frac{\pi^2}{6}$, write

$$a_1 = \frac{-2}{(2\pi)^2}\left(4 - \frac{4}{9}\right) + \frac{12}{(2\pi)^4}\left(16 - \frac{16}{81}\right) + \frac{48}{(2\pi)^4}\int_{\frac{1}{2}}^{\frac{3}{2}} \frac{\cos 2\pi x}{x^5}\, dx.$$

We have $\theta_1 = 1 - \beta$, where

$$\beta = -\left\{\frac{48}{(2\pi)^4}\int_{\frac{1}{2}}^{\frac{3}{2}} \frac{\cos 2\pi x}{x^5}\, dx\right\} / \left\{\frac{12}{(2\pi)^4}\left(16 - \frac{16}{81}\right)\right\}$$

$$= \frac{-81}{320}\int_{\frac{1}{2}}^{\frac{3}{2}} \frac{\cos 2\pi x}{x^5}\, dx.$$

By using computing software *Mathematica*, we have $\beta \approx 0.555924$, so $\beta > 0.555$. Since $\theta_1 = 1 - \beta < 0.445$ and $\alpha = \frac{20}{3\pi^2} < \frac{20}{3(3.14)^2}$, we have

$$-2 + \alpha\theta_1 + \frac{\pi^2}{6} < -2 + \frac{20}{3(3.14)^2}(0.445) + \frac{(3.15)^2}{6} < -0.04.$$

Hence $T\left(0, \left[\frac{k}{2}\right], \chi\right) < 0$. □

To give bounds for $L(1, \chi)$, define, for integer $v \geq 0$,

$$A(v) = \sum_{n=1}^{\left[\frac{k}{2}\right]} \frac{\chi(n)}{vk + n} \quad \text{and} \quad B(v) = \sum_{n=\left[\frac{k}{2}\right]+1}^{k} \frac{\chi(n)}{vk + n}.$$

Then

$$T(v, 0, \chi) = A(v) + B(v) \quad \text{and} \quad T\left(v, \left[\frac{k}{2}\right], \chi\right) = B(v) + A(v + 1).$$

Combining Davenport's theorem [3], Theorem 3 and the fact $L(1, \chi) > 0$, we obtain the following bounds for $L(1, \chi)$.

PROPOSITION 2. *For any integers m, $n \geq 0$,*

$$\sum_{v=0}^{n} (A(v) + B(v)) < L(1, \chi) < A(0) + \sum_{v=0}^{m} (B(v) + A(v + 1)).$$

COROLLARY 1. (1) *For integer $v \geq 0$, $A(v) > 0$ and $B(v) < 0$.*

(2) $A(0) + B(0) < L(1, \chi) < A(0)$.

(3) *For* $k > 1000$, $0 < A(0) - L(1, \chi) < 0.12$.

Proof. (1) Since $B(v) + A(v+1) = T\left(v, \left[\frac{k}{2}\right], \chi\right) < 0$ for integer $v \geq 0$ and $L(1, \chi) > 0$, so $A(0) > 0$. On the other hand, by Proposition 2, we have

$$\sum_{v=0}^{n} (A(v) + B(v)) < A(0) + \sum_{v=0}^{n} (B(v) + A(v+1))$$

for any integer $n \geq 0$, which implies $A(n+1) > 0$. Hence $B(n) < 0$.

(2) The inequalities holds by putting $m = n = 0$ in Proposition 2 and the fact $B(0) + A(1) < 0$.

(3) The proofs for the case $k \equiv 0 \pmod 4$ and the case $k \equiv 1 \pmod 4$ are the same, here we consider the case $k \equiv 0 \pmod 4$. By (2), we know that

$$A(0) + B(0) < L(1, \chi) < A(0).$$

Since

$$\begin{aligned} A(0) + B(0) &= \sum_{n=1}^{\frac{k}{2}} \frac{\chi(n)}{n} + \sum_{n=\frac{k}{2}+1}^{k} \frac{\chi(n)}{n} \\ &> \sum_{n=1}^{\frac{k}{2}} \frac{\chi(n)}{n} - \sum_{n=\frac{k}{2}+1}^{\frac{3k}{4}} \frac{1}{n} + \sum_{n=\frac{3k}{4}+1}^{k-1} \frac{1}{n} \\ &> A(0) - \int_{\frac{k}{2}}^{\frac{3k}{4}} \frac{1}{x}\,dx + \int_{\frac{3k}{4}+1}^{k} \frac{1}{x}\,dx \\ &> A(0) - 0.12 \quad \text{for } k > 1000, \end{aligned}$$

we have $0 < A(0) - L(1, \chi) < 0.12$ for $k > 1000$. □

Dirichlet's class number formula asserts that

$$h = \frac{\sqrt{k}}{2\ln \varepsilon} L(1, \chi),$$

where h is the class number, and ε (> 1) is the fundamental unit of $\mathbf{Q}(\sqrt{k})$. Thus the estimates on $L(1, \chi)$ in Corollary 1 above yields the following results on the class number of $\mathbf{Q}(\sqrt{k})$.

- If $\frac{\sqrt{k}}{2\ln \varepsilon} A(0) \leq 2$, then $h = 1$.

• If $\frac{\sqrt{k}}{2\ln \varepsilon}(A(0)+B(0)) \geq 1$, then $h \neq 1$.

In fact, the class number h for the real quadratic field $\mathbf{Q}(\sqrt{k})$ can be expressed explicitly as follows.

THEOREM 4. *We have*

$$h = \left[\frac{\sqrt{k}}{2\ln \varepsilon}(A(0)+B(0))\right]+1 = \left[\frac{k^{3/2}}{2\ln \varepsilon}\sum_{n=1}^{[\frac{k}{2}]}\frac{\chi(n)}{n(k-n)}\right]+1,$$

where $[x]$ *denotes the greatest integer* $\leq x$.

Proof. Since $\varepsilon = \frac{1}{2}(t+u\sqrt{k}) > 1$ is the fundamental unit of $\mathbf{Q}(\sqrt{k})$, we have $\varepsilon \geq \frac{1+\sqrt{5}}{2}$. Due to Davenport [3], we have the following inequality.

$$(L(1, \chi)-(A(0)+B(0)))\sqrt{k} < \frac{11}{120}.$$

From this inequality and $A(0)+B(0) < L(1, \chi)$, we obtain

$$\begin{aligned}\frac{\sqrt{k}}{2\ln \varepsilon}(A(0)+B(0)) < h &= \frac{\sqrt{k}}{2\ln \varepsilon}L(1, \chi)\\ &< \frac{\sqrt{k}}{2\ln \varepsilon}(A(0)+B(0)) + \frac{11}{120}\frac{1}{2\ln b},\end{aligned}$$

where $b = \frac{1+\sqrt{5}}{2}$. Since $\frac{11}{120}\frac{1}{2\ln b} < 1$, so we have

$$h = \left[\frac{\sqrt{k}}{2\ln \varepsilon}(A(0)+B(0))\right]+1. \qquad \square$$

Remarks. 1. By Theorem 4, the following two conjectures are equivalent:

(1) (Gauss conjecture) There exist infinitely many real quadratic fields $\mathbf{Q}(\sqrt{p})$ of class number one, where p is a prime congruent to 1 modulo 4.

(2) There exist infinitely many real quadratic fields $\mathbf{Q}(\sqrt{p})$ with $\frac{p^{3/2}}{2\ln \varepsilon}\sum_{n=1}^{[\frac{p}{2}]}\frac{\chi(n)}{n(p-n)} < 1$, where p is a prime congruent to 1 modulo 4 and $\varepsilon > 1$ is the fundamental unit of $\mathbf{Q}(\sqrt{p})$.

2. For an evaluation of the regulator $\ln \varepsilon$ in the class number formula, see, for

example, Williams and Broere [6].

As a corollary of Theorem 4 and the class number formula of Ono [4], we can get the following interesting inequality without involving the class number h and the fundamental unit ε.

THEOREM 5. *Let $p \equiv 1 \pmod 4$ be a prime. Then*

$$\ln\left(\frac{2}{\sqrt{p}}\sum_{n=1}^{N-1} d_n + \frac{d_N}{\sqrt{p}}\right) > \frac{p^{3/2}}{2}\sum_{n=1}^{[\frac{p}{2}]}\frac{\chi(n)}{n(p-n)},$$

where $N = \frac{p-1}{4}$, $d_0 = 1$ and $2nd_n = \sum_{v=1}^{n}\left(1 + \left(\frac{v}{p}\right)\sqrt{p}\right)d_{n-v}$, $1 \le n \le N$. (Here $\left(\frac{x}{y}\right)$ denotes the Legendre symbol.)

Proof. By [4], we have

$$h \ln \varepsilon = \ln\left(\frac{2}{\sqrt{p}}\sum_{n=1}^{N-1} d_n + \frac{d_N}{\sqrt{p}}\right).$$

On the other hand, by Theorem 4, we have

$$h = \left[\frac{p^{3/2}}{2\ln \varepsilon}\sum_{n=1}^{[\frac{p}{2}]}\frac{\chi(n)}{n(p-n)}\right] + 1$$

which gives

$$h > \frac{p^{3/2}}{2\ln \varepsilon}\sum_{n=1}^{[\frac{p}{2}]}\frac{\chi(n)}{n(p-n)}, \text{ or equivalently, } h \ln \varepsilon > \frac{p^{3/2}}{2}\sum_{n=1}^{[\frac{p}{2}]}\frac{\chi(n)}{n(p-n)},$$

hence Theorem follows. □

4. The imaginary quadratic fields

In this section we restrict ourselves to the case $d = -k$. Fix such an integer k, let χ be a real primitive character attached to the imaginary quadratic field $\mathbf{Q}(\sqrt{-k})$ with $\chi(-1) = -1$. Let $L = \sum_{m=1}^{\infty}\frac{(-1)^m\chi(m)}{m}$, $L_1 = \sum_{m=1}^{\infty}\frac{\chi(2m-1)}{2m-1}$ and $L_2 = \sum_{m=1}^{\infty}\frac{\chi(2m)}{2m}$. Then $L_2 = \sum_{m=1}^{\infty}\frac{\chi(2m)}{2m} = \frac{\chi(2)}{2}L(1, \chi)$ and $L_1 = \sum_{m=1}^{\infty}\frac{\chi(m)}{m} - \sum_{m=1}^{\infty}\frac{\chi(2m)}{2m} = \left(1 - \frac{\chi(2)}{2}\right)L(1, \chi)$. Furthermore, we have $L = L_2 - L_1 = (\chi(2) - 1)L(1, \chi)$ which gives the following lemma.

LEMMA 1.

$$L = \begin{cases} 0, & \text{if } -k \equiv 1 \pmod 8; \\ -L(1, \chi) & \text{if } -k \equiv 0 \pmod 4; \\ -2L(1, \chi) & \text{if } -k \equiv 5 \pmod 8. \end{cases}$$

Now we are ready to prove Theorem 6.

THEOREM 6. (1) *If* $k \not\equiv 7 \pmod 8$, *then* $T\left(v, \left[\frac{k}{2}\right], \chi\right) < 0$ *for integer* $v > k^{\frac{1}{4}}$.
(2) *If* $k \equiv 7 \pmod 8$, *then* $T\left(v, \left[\frac{k}{2}\right], \chi\right) > 0$ *for integer* $v \geq 0$.

Proof. Express $T(v, j, \chi) = \frac{1}{k} \sum_{n=j+1}^{j+k} \frac{\chi(n)}{v + \frac{n}{k}}$ and keep in mind that $j = \left[\frac{k}{2}\right]$ in this proof.

For integer $v \geq 0$, as in the proof of Theorem 3, consider the Fourier expansion of

$$g(x) = \frac{1}{v+x} \quad \text{for} \quad \frac{1}{2} < x < \frac{3}{2}.$$

Proceeding as before and applying Gauss's sum $\sum_{n=j+1}^{j+k} \chi(n) \exp(2\pi imn/k) = i\chi(m)\sqrt{k}$ for $\chi(-1) = -1$, we have

$$\sqrt{k}\, T\left(v, \left[\frac{k}{2}\right], \chi\right) = \sum_{m=1}^{\infty} \chi(m) b_m,$$

where $b_m = 2\int_{\frac{1}{2}}^{\frac{3}{2}} \frac{\sin 2\pi mx}{v+x} dx$. By integration by parts, we obtain

$$b_m = \frac{(-1)^m}{\pi m}\left(\frac{1}{v+\frac{1}{2}} - \frac{1}{v+\frac{3}{2}}\right) - \frac{(-1)^m}{2(\pi m)^3}\left(\frac{1}{\left(v+\frac{1}{2}\right)^3} - \frac{1}{\left(v+\frac{3}{2}\right)^3}\right)\phi_m,$$

where $\phi_m = \phi_m(v)$ depending on v and $0 < \phi_m < 1$. Now we have

$$\sqrt{k}\, T\left(v, \left[\frac{k}{2}\right], \chi\right) = \sum_{m=1}^{\infty} \chi(m) b_m$$

$$= \frac{1}{\pi}\left(\frac{1}{v+\frac{1}{2}} - \frac{1}{v+\frac{3}{2}}\right)\sum_{m=1}^{\infty}\frac{(-1)^m\chi(m)}{m}$$
$$-\frac{4}{(2\pi)^3}\left(\frac{1}{\left(v+\frac{1}{2}\right)^3} - \frac{1}{\left(v+\frac{3}{2}\right)^3}\right)\sum_{m=1}^{\infty}\frac{(-1)^m\chi(m)\phi_m}{m^3}.$$

Let $J = \sum_{m=1}^{\infty}(-1)^m\chi(m)\phi_m m^{-3}$, then, independent of v, we have

$$|J+\phi_1| = |\sum_{m=2}^{\infty}(-1)^m\chi(m)\phi_m m^{-3}| < \sum_{m=2}^{\infty}\frac{1}{m^3} < 0.21.$$

On the other hand,

$$b_1 = 2\int_{\frac{1}{2}}^{\frac{3}{2}}\frac{\sin 2\pi x}{v+x}\,dx$$
$$= \frac{-\cos 2\pi x}{\pi(v+x)}\Big|_{1/2}^{3/2} + \frac{4\cos 2\pi x}{(2\pi)^3(v+x)^3}\Big|_{1/2}^{3/2} + \frac{12}{(2\pi)^3}\int_{\frac{1}{2}}^{\frac{3}{2}}\frac{\cos 2\pi x}{(v+x)^4}\,dx$$
$$= \frac{-\cos 2\pi x}{\pi(v+x)}\Big|_{1/2}^{3/2} + \frac{4\cos 2\pi x}{(2\pi)^3(v+x)^3}\Big|_{1/2}^{3/2}\phi_1,$$

which gives

$$(4.1) \qquad \phi_1 - 1 = \left\{3\int_{\frac{1}{2}}^{\frac{3}{2}}\frac{\cos 2\pi x}{(v+x)^4}\,dx\right\}\Big/\left\{\frac{1}{\left(v+\frac{1}{2}\right)^3} - \frac{1}{\left(v+\frac{3}{2}\right)^3}\right\}.$$

Let $g_\nu(x) = \dfrac{1}{(v+x)^4} - \dfrac{1}{\left(v+\frac{3}{2}-x\right)^4} - \dfrac{1}{\left(\nu+\frac{1}{2}+x\right)^4} + \dfrac{1}{(v+2-x)^4}$ for $\frac{1}{2} \le x \le \frac{3}{4}$. Then

$$(4.2) \qquad \int_{\frac{1}{2}}^{\frac{3}{2}}\frac{\cos 2\pi x}{(v+x)^4}\,dx = \int_{\frac{1}{2}}^{\frac{3}{4}}g_v(x)\cos 2\pi x\,dx.$$

Since $g_v'(x) < 0$ for $\frac{1}{2} \le x \le \frac{3}{4}$ and integer $v \ge 0$, also $g_v\left(\frac{3}{4}\right) = 0$, so $g_v(x) \ge 0$ for $\frac{1}{2} \le x \le \frac{3}{4}$ and integer $v \ge 0$. Hence, by (4.2),

$$\frac{2}{\left(v+\frac{5}{4}\right)^3} - \frac{2}{\left(v+\frac{3}{4}\right)^3} - \frac{1}{\left(v+\frac{3}{2}\right)^3} + \frac{1}{\left(v+\frac{1}{2}\right)^3} = 3\int_{\frac{1}{2}}^{\frac{3}{4}}g_v(x)\,dx$$

$$\geq 3\int_{\frac{1}{2}}^{\frac{3}{2}} \frac{\cos 2\pi x}{(v+x)^4}\,dx$$

$$\geq -3\int_{\frac{1}{2}}^{\frac{3}{4}} g_v(x)\,dx$$

$$= \frac{2}{\left(v+\frac{3}{4}\right)^3} - \frac{2}{\left(v+\frac{5}{4}\right)^3} - \frac{1}{\left(v+\frac{1}{2}\right)^3} + \frac{1}{\left(v+\frac{3}{2}\right)^3}.$$

Substituting into (4.1), we obtain

$$(4.3)\qquad \frac{2\left\{\frac{1}{\left(v+\frac{5}{4}\right)^3} - \frac{1}{\left(v+\frac{3}{4}\right)^3}\right\}}{\left\{\frac{1}{\left(v+\frac{1}{2}\right)^3} - \frac{1}{\left(v+\frac{3}{2}\right)^3}\right\}} + 2 \geq \phi_1(v) \geq \frac{2\left\{\frac{1}{\left(v+\frac{3}{4}\right)^3} - \frac{1}{\left(v+\frac{5}{4}\right)^3}\right\}}{\left\{\frac{1}{\left(v+\frac{1}{2}\right)^3} - \frac{1}{\left(v+\frac{3}{2}\right)^3}\right\}}.$$

Let

$$F(v) = 2\left\{\frac{1}{\left(v+\frac{3}{4}\right)^3} - \frac{1}{\left(v+\frac{5}{4}\right)^3}\right\} \Big/ \left\{\frac{1}{\left(v+\frac{1}{2}\right)^3} - \frac{1}{\left(v+\frac{3}{2}\right)^3}\right\}$$

$$= \frac{3(v+1)^2 + \frac{1}{16}}{3(v+1)^2 + \frac{1}{4}}\left(\frac{(v+1)^2 - \frac{1}{4}}{(v+1)^2 - \frac{1}{16}}\right)^3 \quad \text{for } v \geq 0.$$

Then $F(v)$ is increasing as v increases. We have $1.52 > 2 - F(0) \geq 2 - F(v) \geq \phi_1(v) \geq F(v) \geq F(0) > 0.48$ which implies $F(v) - 2.21 \leq -\phi_1(v) - 0.21 < J < 0.21 - \phi_1(v) \leq 0.21 - F(v)$ for integer $v \geq 0$. Now we have

$$\frac{1}{\pi}\left(\frac{1}{v+\frac{1}{2}} - \frac{1}{v+\frac{3}{2}}\right)L + \frac{8.84 - 4F(v)}{(2\pi)^3}\left\{\frac{1}{\left(v+\frac{1}{2}\right)^3} - \frac{1}{\left(v+\frac{3}{2}\right)^3}\right\}$$

$$(4.4)\qquad > \sqrt{k}\,T\left(v, \left[\frac{k}{2}\right], \chi\right)$$

$$= \frac{1}{\pi}\left(\frac{1}{v+\frac{1}{2}} - \frac{1}{v+\frac{3}{2}}\right)L - \frac{4}{(2\pi)^3}\left\{\frac{1}{\left(v+\frac{1}{2}\right)^3} - \frac{1}{\left(v+\frac{3}{2}\right)^3}\right\}J$$

$$> \frac{1}{\pi}\left(\frac{1}{v+\frac{1}{2}} - \frac{1}{v+\frac{3}{2}}\right)L + \frac{4F(v) - 0.84}{(2\pi)^3}\left\{\frac{1}{\left(v+\frac{1}{2}\right)^3} - \frac{1}{\left(v+\frac{3}{2}\right)^3}\right\}$$

for integer $v \geq 0$. For simplicity, write $T(v)$, a and b for $T\left(v, \left[\frac{k}{2}\right], \chi\right)$, $\frac{1}{v+\frac{1}{2}}$ and $\frac{1}{v+\frac{3}{2}}$ respectively, then dividing each term in (4.4) by $\frac{a-b}{\sqrt{k}}$, we obtain

$$\frac{\sqrt{k}}{\pi} L + \frac{\sqrt{k}(8.84 - 4F(v))}{(2\pi)^3}(a^2 + ab + b^2) > \frac{kT(v)}{a-b}$$
$$> \frac{\sqrt{k}}{\pi} L + \frac{\sqrt{k}(4F(v) - 0.84)}{(2\pi)^3}(a^2 + ab + b^2),$$

which gives

$$(4.5) \quad \frac{\sqrt{k}(8.84 - 4F(v))}{(2\pi)^3}(a^2 + ab + b^2) > \frac{kT(v)}{a-b} - \frac{\sqrt{k}}{\pi} L$$
$$> \frac{\sqrt{k}(4F(v) - 0.84)}{(2\pi)^3}(a^2 + ab + b^2).$$

By applying Dirichlet's class number formula for imaginary quadratic fields, Lemma 1, the inequality $1 > \frac{\sqrt{k}}{v^2} > \frac{\sqrt{k}(8.84 - 4F(v))}{(2\pi)^3}(a^2 + ab + b^2)$ for integer $v > k^{\frac{1}{4}}$ and (4.5), if $k \not\equiv 7 \pmod 8$, then $T\left(v, \left[\frac{k}{2}\right], \chi\right) < 0$ for integer $v > k^{\frac{1}{4}}$ (since the class number $h \geq 1$ is a positive integer), if $k \equiv 7 \pmod 8$, then $T\left(v, \left[\frac{k}{2}\right], \chi\right) > 0$ for integer $v \geq 0$. □

Let $T(v)$, a and b be the ones defined in the proof of Theorem 6, then we have the following estimates of the class number h of $\mathbf{Q}(\sqrt{-k})$.

COROLLARY 2. *Suppose* $k > 4$.

(1) $h < \frac{k}{\pi\sqrt{k} - 1} \sum_{n=1}^{k} \frac{\chi(n)}{n}$.

(2) *If* $k \equiv 0 \pmod 4$, *then*

$$h = \left[\frac{-kT(v)}{a-b}\right] + 1 \text{ for any integer } v > k^{\frac{1}{4}}.$$

(3) *If* $k \equiv 3 \pmod 8$, *then*

$$h = \left[\frac{-kT(v)}{2(a-b)}\right] + 1 \text{ for any integer } v > k^{\frac{1}{4}}.$$

(4) *If* $k \equiv 7 \pmod 8$, *then*

$$h > \frac{\sqrt{k}}{\pi} \sum_{n=1}^{[\frac{k}{2}]} \frac{\chi(n)}{n} + \frac{28.08}{27\pi^4}.$$

The symbol $[x]$ *denotes the greatest integer* $\leq x$.

Proof. In [3], we have

$$\left(L(1, \chi) - \sum_{n=1}^{k} \frac{\chi(n)}{n}\right)\sqrt{k} < \frac{1}{\pi} L(1, \chi).$$

Applying class number formula for imaginary quadratic fields $h = \frac{\sqrt{k}}{\pi} L(1, \chi)$ $(k > 4)$, we have statement (1).

The statements (2) and (3) are consequences of (4.5).

For statement (4), we write

$$L(1, \chi) = \sum_{n=1}^{[\frac{k}{2}]} \frac{\chi(n)}{n} + \sum_{v=0}^{\infty} T\left(v, \left[\frac{k}{2}\right], \chi\right)$$

which implies, by Theorem 6 (2), that

$$L(1, \chi) > \sum_{n=1}^{[\frac{k}{2}]} \frac{\chi(n)}{n} + T\left(0, \left[\frac{k}{2}\right], \chi\right).$$

Hence, by taking $v = 0$ in (4.4), we have

$$h = \frac{\sqrt{k}}{\pi} L(1, \chi) > \frac{\sqrt{k}}{\pi} \sum_{n=1}^{[\frac{k}{2}]} \frac{\chi(n)}{n} + \frac{28.08}{27\pi^4}. \qquad \square$$

Remark. It is proved in [2] that, for k sufficiently large, one has $\sum_{n=1}^{[\frac{k}{2}]} \frac{\chi(n)}{n} > 0$ for any real character modulo k.

REFERENCES

[1] R. Ayoub, An Introduction to the Analytic Theory of Numbers, Mathematical surveys, No. 10, Amer. Math. Soc., Providence, 1963.

[2] P. T. Bateman and S. Chowla, The equivalence of two conjectures in the theory of

numbers, J. Indian Math. Soc. (N. S.) **17** (1954), 177–181.
[3] H. Davenport, On the series for $L(1)$, J. London Math. Soc. **24** (1949), 229–233.
[4] T. Ono, A deformation of Dirichlet's class number formula, Algebraic Analysis **2** (1988), 659–666.
[5] L. C. Washington, Introduction to cyclotomic fields, Springer-Verlag, New York, 1982.
[6] H. C. Williams and J. Broere, A computational technique for evaluating $L(1, \chi)$ and the class number of a real quadratic field, Math. Comp., **30** (1976), 887–893.
[7] M.-G. Leu, On a problem of Davenport and Erdös concerning the series for $L(1, \chi)$ (1995), submitted.

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