S. Takayama
Nagoya Math. J.
Vol. 146 (1997), 185–197

ON RELATIVE BASE POINT FREENESS OF ADJOINT BUNDLE

SHIGEHARU TAKAYAMA

Abstract. We give an effective result on the relative base point freeness of an adjoint bundle for a pair of a projective morphism and a relatively ample line bundle.

§1. Introduction

Recently, Angehrn and Siu [AS] and Tsuji [Tj] independently obtained results on the following:

FUJITA'S FREENESS CONJECTURE OF ADJOINT BUNDLES. ([F]) *Let X be an n-dimensional projective manifold defined over $\mathbb{C}$ with an ample line bundle L. Then the adjoint bundle $\mathcal{O}_X(K_X \otimes L^{\otimes m})$ is generated by global sections for every $m > n$.*

Their effective bounds are $m > n(n+1)/2$. The basic ideas of their proofs from [AS] and [Tj] (use of Riemann-Roch theorem, Nadel's vanishing theorem, Ohsawa-Takegoshi's L^2-extension theorem and so on) are extremely simple and can be applied to a variety of contexts. In this note we would like to go into detail about the method and consider the following relative version:

MAIN THEOREM. *Let $f : X \longrightarrow Y$ be a projective morphism from a complex manifold X to a complex space Y, and let L be a relatively ample line bundle on X. Then $\mathcal{O}_X(K_X \otimes L^{\otimes m})$ is f-free, i.e., the natural sheaf homomorphism*

$$f^* f_* \mathcal{O}_X(K_X \otimes L^{\otimes m}) \longrightarrow \mathcal{O}_X(K_X \otimes L^{\otimes m}) \quad \textit{is surjective,}$$

for every

$$m > \frac{1}{2}\, d\, (d+1),$$

here d is the maximum dimension of the fibres of f.

Received September 20, 1995.

Our relative version has some applications to the classification theory of higher dimensional algebraic varieties. For example, we have the following (refer to [KMM] for terminologies):

COROLLARY. *Let X be a projective manifold defined over $\mathbb{C}$ and let $\varphi : X \longrightarrow X'$ be the contraction morphism of an extremal ray R of $\overline{NE}(X)$. Then the $d(d+1)/2$-th anti-pluri-canonical divisor $-d(d+1)/2K_X$ is φ-free, here d is the maximum dimension of the fibres of φ.*

This corollary is much helpful to the classification of the singular fibres of φ and contraction morphisms (cf. [K])

The reader should refer to [D2] (analytic approach) and [L] (algebraic approach) for the recent development of the theory of adjoint bundles.

I would like to express my thanks to Professor Takeo Ohsawa for his suggestions and encouragement during the preparation of this paper. I would also like to express my thanks to Professor Hajime Tsuji who kindly explained his basic ideas.

§2. Singular Hermitian metric and vanishing theorem

Our basic tool is singular Hermitian metrics as in [D1], [D2]. We use vector bundles and the associated locally free sheaves interchangeably. In this section we let X be a complex manifold.

2A. Singular Hermitian metric

Let L be a holomorphic line bundle on X. A metric h on L is called **singular Hermitian**, if there exist a function $\varphi \in L^1_{\text{loc}}(X)$ and a smooth Hermitian metric h_0 on L such that $h = e^{-\varphi}h_0$ holds. This defines a closed current

$$\operatorname{curv} h := \operatorname{curv} h_0 + \sqrt{-1}\partial\bar{\partial}\varphi,$$

where $\operatorname{curv} h_0$ is the curvature form of the Hermitian metric h_0 and $\partial\bar{\partial}$ is taken in the sense of currents. The $(1,1)$-current $\operatorname{curv} h$ is said to be the **curvature current** of the singular Hermitian line bundle (L, h). It is easy to see that $\operatorname{curv} h$ is independent of the choices of h_0 and φ. $\sqrt{-1}\ \bar{\partial}\partial \log h$ is the formal expression of curv h. For a singular Hermitian line bundle (L, h) on a Hermitian manifold (X, ω). The L^2-sheaf $\mathcal{L}^2(L, h)$ is the sheaf defined by

$$\mathcal{L}^2(L,h)(U) = \{s \in \Gamma(U, L)\ ;\ h_0(s,s)e^{-\varphi} \in L^1_{\text{loc}}(U)\},$$

where $h = e^{-\varphi}h_0$ is a local expression of h as above. Similarly, the **multiplier ideal sheaf** $\mathcal{I}(h)$ of the singular Hermitian metric is defined by

$$\mathcal{I}(h)(U) := \{f \in \Gamma(U, \mathcal{O}_M) \ ; \ |f|^2 e^{-\varphi} \in L^1_{\mathrm{loc}}(U)\}.$$

These sheaves do not depend of the choices φ, h_0 and ω, and satisfy the following relation: $\mathcal{L}^2(L,h) = L \otimes \mathcal{I}(h)$.

2B. Vanishing theorem

We recall the following:

NADEL'S COHERENCE AND VANISHING THEOREM 2.1. ([N], [D1, §4]) *Let (X,ω) be a complete Kähler manifold and let (L,h) be a singular Hermitian line bundle on X. Assume that there exists a real number c such that* $\operatorname{curv} h \geq c\,\omega$ *on X. Then*

(1) *the sheaf $\mathcal{I}(h)$ is a coherent ideal sheaf of $\mathcal{O}_X$, and*

(2) *if c is positive, the q-th L^2-cohomology group*

$$H^q_{(2)}(X, K_X \otimes L \otimes \mathcal{I}(h)) = 0$$

for every $q \geq 1$.

As a simple application of the above theorem, we have

PROPOSITION 2.2. *Let (X,ω) be a complete Kähler manifold, x be a point of X, and let L be a holomorphic line bundle on X. Assume that L admits a singular Hermitian metric h_x such that*

(1) *there exists a positive constant c such that* $\operatorname{curv} h_x \geq c\,\omega$*, and that*

(2) *x is isolated in the zero complex space $V\mathcal{I}(h_x)$.*

Then there exists a holomorphic section of $K_X \otimes L$ which does not vanish at x.

We will need the following Serre type vanishing theorem:

PROPOSITION 2.3. ([F, Theorem N′]) *Let L be a positive line bundle on a weakly 1-complete manifold X, i.e., a complex manifold with a smooth plurisubharmonic exhaustion function $\Phi : X \longrightarrow \mathbb{R}$. Then for every coherent analytic sheaf $\mathcal{F}$ on X and for every $c < \sup_X \Phi$, there exists a positive integer m_0 such that*

$$H^q(X_c, \mathcal{F} \otimes L^{\otimes m}) = 0$$

for any $q \geq 1$ and for any $m \geq m_0$, where $X_c := \{x \in X \ ; \ \Phi(x) < c\}$ is a sublevel set of (X,Φ).

2C. Singular Hermitian metric with analytic singularities [AS, §2]

In this subsection we explain how to construct a singular Hermitian metric and how to control the multiplier ideal sheaf. The standard method of the construction is to use holomorphic sections; such metrics are said to be singular Hermitian metrics **with analytic singularities**.

It is convenient to introduce the notion of rational coefficient geometry as follows. We consider a family of local holomorphic functions $s = \{s_\lambda; s_\lambda \in H^0(U_\lambda, \mathcal{O}_X)\}_{\lambda\in\Lambda}$ for some locally finite open cover $\{U_\lambda\}_{\lambda\in\Lambda}$ of X. For a positive rational number q and for any smooth Hermitian holomorphic line bundle (L, h_0) on X, the family of local holomorphic functions s is said to be a **multivalued holomorphic section** of $L^{\otimes q}$ over X, if there exists a positive integer p such that pq being an integer and that $s^p := \{s_\lambda^p\}_{\lambda\in\Lambda}$ defines an element of $H^0(X, L^{\otimes pq})$. We denote

$$\begin{aligned} |s| &:= (h_0^{\otimes pq}(s^p, s^p))^{1/(2p)} : \quad \text{the pointwise length,} \\ (s)_0 &:= \{x \in X \ ; \ s_\lambda(x) = 0 \quad \text{for some } \lambda \in \Lambda\}. \end{aligned}$$

We just consider $(s)_0$ as a set of zeros. We also define a singular Hermitian metric h of $L^{\otimes q}$ by a family of local real valued measurable functions such that h^p defines a singular Hermitian metric of $L^{\otimes pq}$. We can also define the curvature current and the multiplier ideal sheaf.

Let $s_1, \dots, s_k$ be a finite number of multivalued holomorphic sections $L^{\otimes q}$ such that $(s_i)^p$ $(1 \le i \le k)$ is a holomorphic section of $L^{\otimes pq}$ for some positive integer p with pq being an integer. Then we can define a singular Hermitian metric of $L^{\otimes q}$ by

$$h := \frac{h_0^q}{\sum_{i=1}^k |s_i|^2}.$$

The curvature current is a closed positive current on X. Indeed, for local expressions $s_i = \{s_{i\lambda}\}_{\lambda\in\Lambda}$, we see

$$\operatorname{curv} h = \sqrt{-1}\partial\bar\partial \log \sum_{i=1}^k |s_{i\lambda}|^2$$

on every open set U_λ. We note that both the positivity of the curvature current and the multiplier ideal sheaf do not depend on the smooth Hermitian metric h_0. Let $\mathcal{J}$ be the sheaf of ideal of $\mathcal{O}_X$ generated locally by

$\{(s_i)^p\}_{i=1}^k$. We assume that the support of $\mathcal{O}_X/\mathfrak{J}$ is compact. We take a modification $\pi : \widetilde{X} \longrightarrow X$ by a finite number of successive monoidal transforms with nonsingular centers and a family of smooth divisors E_i in $\widetilde{X}$ with only simple normal crossing so that the following three consitions hold:

(0) For every i, $\pi(E_i) \subset$ supp $\mathcal{O}_X/\mathfrak{J}$.

(1) The sheaf $\pi^{-1}\mathfrak{J}\cdot\mathcal{O}_{\widetilde{X}}$ which is the image of $\pi^*\mathfrak{J}$ under the natural map $\pi^*\mathfrak{J} \longrightarrow \mathcal{O}_{\widetilde{X}}$ is equal to the ideal sheaf $\mathcal{O}(-\sum r_i' E_i)$ for some non-negative integers r_i'.

(2) $K_{\widetilde{X}} = \pi^* K_X \otimes \mathcal{O}(\sum b_i E_i)$ for some non-negative integers b_i. In other words, the holomorphic Jacobian determinant of the map $\pi : \widetilde{X} \longrightarrow X$ vanishes precisely of order b_i along E_i and vanishes nowhere on $\widetilde{X} - \bigcup_i E_i$. Let $r_i := r_i'/p$. For every $t \geq 0$, we set

$$\mathcal{I}(t) := \mathcal{L}^2(\mathcal{O}_X, (\sum_{i=1}^{k} |s_i|^2)^{-t}).$$

We see that every $\mathcal{I}(t)$ is a coherent ideal sheaf by 2.1(1) and that $\mathcal{I}(1) = \mathcal{I}(h)$. Then a point x on X belongs to the zero complex subspace $V\mathcal{I}(t)$ if and only if there exists an index i such that E_i intersects $\pi^{-1}(x)$ and that $tr_i - b_i \geq 1$. For every point $x \in V\mathcal{I}(1)$, we see that

$$\begin{aligned}&\sup\{t \geq 0 \ ; \ \mathcal{I}(t)_x = \mathcal{O}_{X,x}\} \\ &\qquad = \min\{t \geq 0 \ ; \ tr_i - b_i \geq 1 \text{ for } i \text{ such that } E_i \text{ intersects } \pi^{-1}(x)\}.\end{aligned}$$

Note that the quantity, say $\alpha(x)$, is always a rational number and $0 < \alpha(x) \leq 1$.

§3. Preliminary lemma

3A. Reduction and non-vanishing with a parameter space

Let $f : X \longrightarrow Y$ and L be as in Main Theorem. We fix a point x on X. The situation is local on Y, so we may assume that Y is a closed complex subspace of the unit ball $\mathbb{B}^M$ in $\mathbb{C}^M$ with the global coordinate $(y_1, \ldots, y_M)$ and that $f(x) = 0$. Since L is f-ample, restricting Y on a smaller ball $\mathbb{B}^M$ if necessary, we may assume L admits a smooth Hermitian metric h whose curvature form is positive on X.

In the local situation as above, we show the following non-vanishing lemma with a parameter space. It is important to handle the case that the zero complex subspace of the multiplier ideal sheaf has singularities.

LEMMA 3.1. ([AS, Lemma 4.1]) *Let Z be a closed subvariety (reduced and irreducible) in X of positive dimension d such that $x \in Z$ and $f(Z) = f(x)$, B_0 and B_1 be smaller balls centered at $0 \in \mathbb{B}^M$ with $B_0 \subsetneqq B_1 \subsetneqq \mathbb{B}^M$, $Y_i := Y \cap B_i$ and $X_i := f^{-1}(Y_i)$ for $i = 0,\ 1$, and let N be a positive integer. Let Δ' be a local holomorphic curve in Z passing through x with x as the only singularity such that the normalization $\sigma : \Delta \longrightarrow \Delta'$ is a one to one holomorphic map from the open unit disk Δ in $\mathbb{C}$ with $\sigma(0) = x$. Then, replace Δ with a smaller disk if necessary, there exist a positive integer m and a finite number of holomorphic sections $\{\widetilde{\tau}_j\}_{j=1}^K \subset H^0(X_1 \times \Delta, pr_X^* L^{\otimes m(N+1)})$, where $pr_X : X \times \Delta \to X$ is the first projection, such that*

$$\widetilde{\tau}_j|_{Z\times u} \in H^0(Z, L^{\otimes m(N+1)} \otimes \mathcal{M}_{Z,\sigma(u)}^{mN}) \quad \textit{for every } u \in \Delta - 0$$

and for every j, and that their common zeros satisty

$$x \in X_0 \cap \bigcap_{j=1}^{K} (\widetilde{\tau}_j|_{X_1\times 0})_0 \subsetneqq Z.$$

Proof. We denote the ideal sheaf of the graph $\sigma \times 1 : \Delta \longrightarrow Z \times \Delta$ by $\mathcal{I}_\Gamma \subset \mathcal{O}_{Z\times\Delta}$. Since Z is compact, the direct image sheaf $pr_{\Delta *}(pr_X^* L^{\otimes m(N+1)} \otimes \mathcal{O}_{Z\times\Delta} \otimes \mathcal{I}_\Gamma^{mN})$ is a coherent sheaf on Δ, where $pr_\Delta : Z \times \Delta \longrightarrow \Delta$ be the projection. The sheaf $pr_{\Delta *}(pr_X^* L^{\otimes m(N+1)} \otimes \mathcal{O}_{Z\times\Delta} \otimes \mathcal{I}_\Gamma^{mN})$ is generically locally free with $H^0(Z \times u, L^{\otimes m(N+1)} \otimes \mathcal{M}_{Z\times u,\sigma(u)\times u}^{mN})$ as the generic fibre. We see that the latter space is non-zero for every large m and for every $u \in \Delta - 0$ by the following asymptotic dimension compairing:

$$\begin{aligned}\dim H^0(Z, L^{\otimes m(N+1)}) &= (N+1)^d (L^d \cdot Z)(d!)^{-1} m^d + O(m^{d-1});\\ \operatorname{rank} \mathcal{O}_Z/\mathcal{M}_{Z,\sigma(u)}^{mN} &= \binom{mN+d-1}{d} = N^d (d!)^{-1} m^d + O(m^{d-1}).\end{aligned}$$

Then by Theorem A of Cartan-Serre we see that, for every large m, there exists a section $\widetilde{\tau} \in H^0(Z \times \Delta, pr_X^* L^{\otimes m(N+1)} \otimes \mathcal{O}_{Z\times\Delta} \otimes \mathcal{I}_\Gamma^{mN})$ such that $\widetilde{\tau}|_{Z\times 0}$ is not identically zero.

For a smaller disk Δ_1, we take a sublevel set W of a weakly 1-complete manifold $X \times \Delta$ for an appropriate smooth plurisubharmonic exhaustion function which contains $X_1 \times \Delta_1$. By Proposition 2.3, for every $x_0 \times u_0 \in X \times \Delta - Z \times \Delta$, there exists a positive integer m_0 such that

$$H^1(W, pr_X^* L^{\otimes m} \otimes \mathcal{I}_{Z\times\Delta} \otimes \mathcal{M}_{X\times\Delta, x_0\times u_0}) = 0$$

for any $m \geq m_0$. Hence we can extend $\widetilde{\tau}$ as sections $\widetilde{\tau}_1, \dots, \widetilde{\tau}_K \in H^0(X_1 \times \Delta_1, pr_X^* L^{\otimes m(N+1)})$ which satisfy the desired properties. □

3B. Calculus Lemma

The following simple calculus lemma on non-integrability will be used later to locate the zero-set of the multiplier ideal sheaf of a singular metric.

LEMMA 3.2. ([AS, Lemma 3.1]) *Let m and N be positive integers and $0 < a < 1$. Let $f_1, \ldots, f_k$ be holomorphic functions on the unit polydisk Δ^n on $\mathbb{C}^n$ with coordinates $z, w_1, \ldots, w_{n-1}$. Let $H := \{z = 0\}$ and let V be the subset of $H \cap \Delta^n$ where the vanishing order of $f_j|_{H\cap\Delta^n}$ is at least mN for any j. Let d be the codimension of V in $H \cap \Delta^n$ at the origin. Then $|z|^{-2a}(\sum |f_j|^2)^{-t/(mN)}$ is not locally integrable at the origin for $t \geq d + mN(1-a)$.*

Proof. By slicing and Fubini's theorem, we may assume $d = n - 1$. Then $\sum |f_j|^2 \leq C_1(|z|^2 + |w|^{2mN})$ for some positive constant C_1, where we set $|w|^2 := \sum_{i=1}^{n-1} |w_i|^2$. The non-integrability of $|z|^{-2a}(\sum |f_j|^2)^{-t/(mN)}$ follows from that of $|z|^{-2a}(|z|^2 + |w|^{2mN})^{-t/(mN)}$. Then we see the non-integability by direct calculation by using polar coordinates for z and w with $x = |z|^2$ and $y = |w|$. □

§4. Proof of Theorem

We fix a point x on X. We let $f : X \longrightarrow Y \subset \mathbb{B}^M \subset (\mathbb{C}^M; y_1, \ldots, y_M)$ and (L, h) be the local reduction around $f(x)$ as in 3A. We take smaller balls B_0 and B_1 centered at $0 \in \mathbb{B}^M$ with $B_0 \subsetneqq B_1 \subsetneqq \mathbb{B}^M$. We set $Y_i := Y \cap B_i$ and $X_i := f^{-1}(Y_i)$ for $i = 0, 1$. With these notations, our Main Theorem follows from the following

THEOREM 4.1. *For every $m > d_0(d_0+1)/2$, there exists a holomorphic section $\tau \in H^0(X_1, \mathcal{O}_X(K_X \otimes L^{\otimes m}))$ such that $\tau(x) \neq 0$, where d_0 is the dimension of a maximum dimensional irreducible component of the fibre $f^{-1}(f(x))$ which contains x.*

By Proposition 2.2, all we have to do is to show the following

PROPOSITION 4.2. *For every $m > d_0(d_0+1)/2$, $L|_{X_1}^{\otimes m}$ admits a singular Hermitian metric H_m such that*

(1) *the curvature current dominants a complete Kähler form on X_1, and that*

(2) *x is isolated in the zero subspace $V\mathcal{I}(H_m)$.*

If f is constant, then X is a projective manifold with an ample line bundle L, that is (a part of) the statement of [AS], [Tj]. Hence we assume that f is non-constant.

4A. Statement of the induction step

We formulate an induction statement for the proof of Proposition 4.2. Let $m_d := \sum_{n=d+1}^{d_0} n$ for $0 \le d < d_0$ and let $m_{d_0} := 0$. We take rational numbers $0 = \varepsilon(d_0+1) < \varepsilon(d_0) < \varepsilon(d_0-1) \cdots < \varepsilon(0) < 1$. For every positive rational number q and for every multivalued holomorphic section s of $L^{\otimes q}$ on an open set of X, we denote $|s|$ the length with respect to the smooth Hermitian metric h. For every d with $0 \le d \le d_0$, we consider the following

INDUCTION STATEMENT $(*)_d$. There exist a rational number $\varepsilon(d+1) < \varepsilon_d < \varepsilon(d)$ and a finite number of multivalued holomorphic sections $s_1^{(d)}, \dots, s_{k_d}^{(d)}$ of $L^{\otimes(m_d+\varepsilon_d)}$ on X_1 such that

(i) $(\bigcap_{i=1}^{k_d}(s_i^{(d)})_0 \cap X_0) \subset f^{-1}(f(x))$,

(ii) $x \in (Z_d(1) \cap X_0) \subset f^{-1}(f(x))$,

(iii) $x \notin Z_d(t)$ for $t < 1$, and that

(iv) The dimension of $Z_d(1)$ at x is at most d,

where

$$\mathcal{I}_d(t) := \mathcal{L}^2(\mathcal{O}_{X_1}, (\sum_{i=1}^{k_d} |s_i^{(d)}|^2)^{-t}) \quad \text{for every } t \ge 0$$

be the multiplier ideal sheaf and where $Z_d(t) := V\mathcal{I}_d(t)$ be the complex subspace of X_1 defined by the ideal sheaf $\mathcal{I}_d(t)$. □

We note that, by the vanishing theorem: Proposition 2.3, there exist a finite number of multivalued holomorphic sections $\{t_i\}_{i=1}^{K}$ of L on X_1 such that $X_0 \cap \bigcap_{i=1}^{K}(t_i)_0$ is empty. We verify the first step:

LEMMA 4.3. *$(*)_{d_0}$ holds.*

Proof. We set

$$\mathcal{I}_*(t) := \mathcal{L}^2(\mathcal{O}_X, (\sum_{i=1}^{M} |f^*y_i|^2)^{-t}) \quad \text{for every } t \ge 0;$$
$$\alpha_* := \sup\{t \ge 0 \ ; \ \mathcal{I}_*(t)_x = \mathcal{O}_{X,x}\}.$$

We see that every $\mathcal{I}_*(t)$ is a coherent ideal sheaf and that α_* is a positive rational number. We consider the complex subspace $V\mathcal{I}_*(\alpha_*)$ defined by the ideal sheaf $\mathcal{I}_*(\alpha_*)$. This space $V\mathcal{I}_*(\alpha_*)$ is compact and $x \in V\mathcal{I}_*(\alpha_*) \subset f^{-1}(f(x))$. We choose a positive rational number $0 < \varepsilon_{d_0} < \varepsilon(d_0)$ and set the multivalued holomorphic sections

$$\{s_i^{(d_0)}\}_{i=1}^{k_{d_0}} := \{f^*y_i^{\alpha_*} \times t_j^{\varepsilon_{d_0}}\}_{i,j}$$

of $L^{\otimes\varepsilon_{d_0}}$ on X_1. Then we can verify $(*)_{d_0}$ by the following relation on X_0:

$$\mathcal{I}_{d_0}(t) := \mathcal{L}^2(\mathcal{O}_{X_1}, (\sum_{i=1}^{k_{d_0}} |s_i^{(d_0)}|^2)^{-t}) = \mathcal{I}_*(t\alpha_*).$$

□

4B. Concentration of the singularity

In this subsection we verify the induction step. We assume $(*)_d$ with $d > 0$. Let p be a positive integer such that $p(m_d + \varepsilon_d)$ being integer and that $(s_i^{(d)})^p$ $(1 \leq i \leq k_d)$ is a holomorphic section of $L^{\otimes p(m_d+\varepsilon_d)}$ on X_1. Let $\mathfrak{J}'$ be the sheaf of ideal of $\mathcal{O}_{X_0}$ generated locally by $\{(s_i^{(d)})^p|_{X_0}\}_{i=1}^{k_d}$. By the assumption $(*)_d$ (i), we can extend $\mathfrak{J}'$ as a coherent ideal sheaf $\mathfrak{J}$ of $\mathcal{O}_X$ by setting $\mathfrak{J} = \mathcal{O}_X$ on $X - X_0$. We take a modification $\pi : \widetilde{X} \longrightarrow X$ by a finite number of successive monoidal transforms with nonsingular centers and a family of smooth divisors E_i in $\widetilde{X}$ with only simple normal crossing so that the following three consitions hold:

(0) $\pi(E_i) \subset f^{-1}(f(x))$ for every i.

(1) The sheaf $\pi^{-1}\mathfrak{J}\cdot\mathcal{O}_{\widetilde{X}}$ which is the image of $\pi^*\mathfrak{J}$ under the natural map $\pi^*\mathfrak{J} \longrightarrow \mathcal{O}_{\widetilde{X}}$ is equal to the ideal sheaf $\mathcal{O}(-\sum r_i' E_i)$ for some non-negative integers r_i'.

(2) $K_{\widetilde{X}} = \pi^* K_X \otimes \mathcal{O}(\sum b_i E_i)$ for some non-negative integers b_i.

Let $r_i := r_i'/p$. The three conditions (ii)–(iv) in the statement $(*)_d$ can now be rewritten as the condition (ii)′–(iv)′ below. Let Λ be the set of all i so that E_i intersects $\pi^{-1}(x)$ and $r_i - b_i \geq 1$.

(ii)′ Λ is not empty;

(iii)′ $i \in \Lambda$ then $r_i - b_i = 1$;

(iv)′ $i \in \Lambda$ then $\dim \pi(E_i) \leq d$.

We may assume that the index $i = 0$ is an element of Λ and that $\dim \pi(E_0) = \max\{\dim \pi(E_i)\ ;\ i \in \Lambda\}$.

We choose ε_{d-1} such that $\varepsilon(d) < \varepsilon_{d-1} < \varepsilon(d-1)$. Let $Z := \pi(E_0) \subset f^{-1}(f(x))$. If the dimension of Z is less than d, then for $(*)_{d-1}$ we simply choose multivalued holomorphic sections

$$\{s_i^{(d-1)}\}_{i=1}^{k_{d-1}} := \{s_i^{(d)} \times t_j^{(m_{d-1}+\varepsilon_{d-1})-(m_d+\varepsilon_d)}\}_{i,j}$$

of $L^{\otimes(m_{d-1}+\varepsilon_{d-1})}$ on X_1. We now assume without loss of generality that the dimension of Z is precisely d.

We take a local smooth holomorphic curve Γ in $E_0 - \bigcup\{E_i\ ;\ i \notin \Lambda\}$ with the following three properties:

(3) Γ intersects $\pi^{-1}(x)$ at one point.

(4) Γ either does not intersect $\bigcup_{i\neq 0} E_i$ or intersects $\bigcup_{i\neq 0} E_i$ only at the point $\Gamma \cap \pi^{-1}(x)$.

(5) Γ either does not intersect $\pi^{-1}(\mathrm{Sing}\ Z)$ or intersects $\pi^{-1}(\mathrm{Sing}\ Z)$ only at the point $\Gamma \cap \pi^{-1}(x)$.

The image $\pi(\Gamma)$ is a local holomorphic curve in Z. Let Δ be the unit disk in $\mathbb{C}$. By replacing Γ by a relatively compact open neighborhood of $\Gamma \cap \pi^{-1}(x)$ in Γ, we may assume that there is a nomalization $\sigma : \Delta \longrightarrow \pi(\Gamma)$ of $\pi(\Gamma)$ which is one to one and $\sigma(0) = x$.

We take a positive integer N such that $d/N < \varepsilon_{d-1} - \varepsilon(d)$. By Lemma 3.1, replace Δ with a smaller disk if necessary, there exist a positive integer m and a finite number of holomorphic sections $\{\widetilde{\tau}_j\}_{j=1}^{K_d} \subset H^0(X_1 \times \Delta,\ pr_X^* L^{\otimes m(N+1)})$ such that

$$\tau_{j,u}|_Z \in H^0(Z, L^{\otimes m(N+1)} \otimes \mathcal{M}_{Z,\sigma(u)}^{mN}) \quad \text{for every } u \in \Delta - 0$$

and for every j, and that

$$x \in X_0 \cap \bigcap_{j=1}^{K_d} (\tau_j)_0 \subsetneqq Z,$$

where $\tau_{j,u} := \widetilde{\tau}_j|_{X_1 \times u}$ (we regard $\tau_{j,u} \in H^0(X_1, L^{\otimes m(N+1)})$) and $\tau_j := \tau_{j,0}$.

Then we take a positive rational number ε such that $\varepsilon m r_0(1+N) + d/N < \varepsilon_{d-1} - \varepsilon(d)$. For every $u \in \Delta$ and for every $t \geq 0$, we set

$$\begin{aligned} \mathcal{I}(u,t) &:= \mathcal{L}^2 \left(\mathcal{O}_{X_1}, \left(\sum |s_i^{(d)}|^2\right)^{-(1-\varepsilon)} \left(\sum |\tau_{j,u}|^2\right)^{-t/(mN)} \right); \\ \alpha(u) &:= \sup\{t \geq 0\ ;\ \mathcal{I}(u,t)_{\sigma(u)} = \mathcal{O}_{X,\sigma(u)}\}. \end{aligned}$$

We see that every $\alpha(u)$ is a positive rational number and that

$$x \in (V\mathcal{I}(0,\alpha(0)) \cap X_0) \subsetneqq Z \subset f^{-1}(f(x)).$$

We would like to estimate $\alpha(0)$ by d and $O(\varepsilon)$ (Lemma 4.6 below). The following semicontinuity lemma of multiplier ideal sheaves due to Angehrn and Siu is the key step to reduce the case $x \in \mathrm{Sing}\ Z$ to the case $x \in \mathrm{Reg}\ Z$.

LEMMA 4.4. ([AS Lemma 6.1]) *Let t_0 be a positive number. Assume that $\alpha(u) < t_0$ for almost all $u \in \Delta - 0$ with respect to the 2-dimensional Lebesgue measure on Δ. Then $\alpha(0) \leq t_0$ holds.*

The outline of the proof is as follows: Assume that $\alpha(0) > t_0$. Then the following theorem of Ohsawa and Takegoshi shows that $\alpha(u) \geq t_0$ for almost all $u \in \Delta - 0$ which is a contradiction.

OHSAWA-TAKEGOSHI'S L^2-EXTENSION THEOREM 4.5. ([OT]) *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^{n+1}$ with coordinate $z_1, \ldots, z_n, w$. Let H be a complex hyperplane defined by $w = 0$, and let ϕ be a plurisubharmonic function on Ω. Then there exists a constant C_Ω depending only on the diameter of Ω such that; for any holomorphic function f on $\Omega \cap H$ satisfying*

$$\int_{\Omega\cap H} |f|^2 e^{-\phi} dV_n < \infty,$$

where dV_n denotes the $2n$-dimensional Lebesgue measure, there exists a holomorphic function F on Ω satisfying $F|_{\Omega\cap H} = f$ and

$$\int_\Omega |F|^2 e^{-\phi} dV_{n+1} \leq C_\Omega \int_{\Omega\cap H} |f|^2 e^{-\phi} dV_n.$$

Then we have

LEMMA 4.6. *$\alpha(0) \leq d + mNr_0\varepsilon$.*

Proof. By Lemma 4.4, it is enough to estimate $\alpha(u)$ for $u \in \Delta - 0$. We show $(\sum |s_i^{(d)}|^2)^{-(1-\varepsilon)}(\sum |\tau_{j,u}|^2)^{-t/(mN)}$ is not locally integrable at $\sigma(u)$ for $t \geq d + mNr_0\varepsilon$.

We take $u \in \Delta - 0$ and a point $\widetilde{x}$ in $\pi^{-1}(\sigma(u)) \cap \Gamma$. We see $\widetilde{x} \in E_0$ and $\widetilde{x} \notin \bigcup_{i\neq 0} E_i$. Let W be an open neighborhood of $\widetilde{x}$ in $\widetilde{X} - \bigcup_{i\neq 0} E_i$ so that a local coordinate $z, w_1, \ldots, w_{n-1}$ on W with $E_0 \cap W$ defined by $\{z = 0\}$, where n is the dimension of X. Since π maps the $(n-1)$-dimensional manifold E_0 onto the irreducible d-dimensional subvariety Z, it follows that the codimension of $\pi^{-1}(x) \cap E_0$ in E_0 is at most d at $\widetilde{x}$. The restriction $(\pi^*\tau_{j,u})|_{E_0}$ vanishes to order at least mN at $\widetilde{x}$. Let $\mathrm{Jac}(\pi)$ be the holomorphic Jacobian determinant of the map π. On W the divisor of $\mathrm{Jac}(\pi)$ is presicely b_0E_0. To conclude the local non-integrability of $(\sum |s_i^{(d)}|^2)^{-(1-\varepsilon)}(\sum |\tau_{j,u}|^2)^{-t/(mN)}$ at $\sigma(u)$, it suffices to prove the local non-integrability of

$$|z|^{-2(1-\varepsilon)r_0}\left(\sum |\pi^*\tau_{j,u}|^2\right)^{-t/(mN)}|z|^{2b_0} \quad \text{at } \widetilde{x}.$$

Since $r_0 - b_0 = 1$, by Lemma 3.2, $t \geq d + mNr_0\varepsilon$ implies the non-integrability. Hence $\alpha(u) < d + mNr_0\varepsilon$ for $u \in \Delta - 0$. ∎

We set $\mathcal{I}(t) := \mathcal{I}(0,t)$ and $\alpha := \alpha(0)$. Then by Lemma 4.6, we see $(d + \varepsilon_{d-1}) - \alpha(1 + 1/N) > \varepsilon(d)$. We set the multivalued holomorphic section

$$\{s_i^{(d-1)}\}_{i=1}^{k_{d-1}} := \{s_i^{(d)(1-\varepsilon)} \times \tau_j^{\alpha/(mN)} \times t_k^{\varepsilon(m_d+\varepsilon_d)+(d+\varepsilon_{d-1})-\alpha(1+1/N)}\}_{i,j,k}$$

of $L^{\otimes(m_{d-1}+\varepsilon_{d-1})}$ on X_1. We can verify $(*)_{d-1}$ by noting that $\mathcal{I}_{d-1}(t) \supset \mathcal{I}(t\alpha)$ for $0 \le t \le 1$ and that $\mathcal{I}_{d-1}(1) = \mathcal{I}(\alpha)$, where

$$\mathcal{I}_{d-1}(t) := \mathcal{L}^2(\mathcal{O}_{X_1}, (\sum |s_i^{(d-1)}|^2)^{-t}).$$

4C. Completion of the proof

In 4A and 4B, we showed that $(*)_0$ hold. Therefore there exist a rational number ε_0 with $0 < \varepsilon_0 < 1$ and a finite number of multivalued holomorphic sections $\{s_i^{(0)}\}_{i=1}^{k_0}$ of $L^{\otimes(m_0+\varepsilon_0)}$ on X_1 such that x is isolated in $V\mathcal{L}^2(\mathcal{O}_{X_1}, (\sum |s_i^{(0)}|^2)^{-1})$. We take a smooth Hermitian metric h_1 of $L|_{X_1}$ such that the curvature form gives a complete Kähler form on X_1. Then we get the desired singular Hermitian metric

$$H_m := h_1^{m-(m_0+\varepsilon_0)} \frac{h_1^{m_0+\varepsilon_0}}{\sum |s_i^{(0)}|_1^2}$$

for every integer $m > m_0 = d_0(d_0 + 1)/2$, where $|s_i^{(0)}|_1$ is the length with respect to h_1.

References

[AS] U. Angehrn, Y. T. Siu, *Effective freeness and point separation for adjoint bundles*, Invent. math., **122** (1995), 291–308.

[D1] J. P. Demailly, *A numerical criterion for very ample line bundles*, Jour. of Diff. Geom., **37** (1993), 323–374.

[D2] J. P. Demailly, *L^2 vanishing theorem for positive line bundles and adjunction theory*, to appear, CIME Session, Transcendental Methods in Algebraic Geometry, Cetraro, Italy (1994).

[F] T. Fujita, *On polarized manifolds whose adjoint bundles are not semipositive,* Algebraic Geometry, Sendai, 1985, Advanced Studies in Pure Math., 10, North-Holland, Amsterdam (1987), pp. 167–178.

[K] Y. Kachi, *Extremal contractions from 4-dimensional manifolds to 3-folds*, preprint (1995).

[KMM] Y. Kawamata, K. Matsuda, K. Matsuki, *Introduction to the Minimal Model Problem,* Algebraic Geometry, Sendai, 1985, Advanced Studies in Pure Math., 10, North-Holland, Amsterdam (1987), pp. 283–360.

[L] Lazarsfeld R., *Lectures on Linear Series*, to appear (1994).

[N] A. M. Nadel, *Multiplier ideal sheaves and existence of Kähler-Einstein metrics of positive scalar curvature*, Ann. of Math., **132** (1990), 549–596.

[OT] T. Ohsawa, K. Takegoshi, *On the Extension of L^2 Holomorphic Functions*, Math. Z., **195** (1987), 197–204.

[Tj] H. Tsuji, *Global generation of adjoint bundles*, Nagoya Math. J., **142** (1996), 5–16.

Department of Mathematics
Naruto University of Education
Takashima, Naruto-cho, Naruto-shi, 772
Japan
takayama@naruto-u.ac.jp