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ON ZETA FUNCTIONS ASSOCIATED TO SYMMETRIC MATRICES III

AN EXPLICIT FORM OF L-FUNCTIONS

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Dedicated to Professor Hideo Shimizu on his 60th birthday

Abstract. In [I-S2], we gave an explicit form of zeta functions associated to the space of symmetric matrices. In this paper, the case of L-functions is treated. In the case of definite symmetric matrices, we show the ratinality of special values of these L-functions.

Introduction

This is the third part of the series of our papers [I-S2] on zeta functions associated to the space of symmetric matrices. In the first part, we gave an explicit form of zeta functions, and in the second part, we discussed some analytic properties of them. The purpose of this paper is to give an explicit form of L-functions associated to that space.

For this space, two kinds of L-functions have been introduced by Sato [Sa2], Hashimoto, one of the author [Sai1], and by Arakawa [A].

The first ones are associated to Dirichlet characters, and were introduced as L-functions of the prehomogeneous vector space of symmetric matrices. The others are associated to a symmetric matrix with coefficients in a finite field and appeared in the calculation of the contribution of unipotent elements to the trace of some operators on the space of Siegel cusp forms.

There exists a close relation between these two kinds of L-functions. In fact, the second ones can be written by the first ones by means of the Gauss sums defined in Saito[Sai1]. Between these two kinds of L-functions, the first ones are easy to treat. For example, the analytic continuation and the functional equations of L-functions of the first kind were proved in [Sa2], [Sai1], [Sai2]. Furthermore our procedure of the computation of zeta

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functions in the first part can be easily applied to the case of L-functions of the first kind. The second kind of L-functions seem to have a rather complicated form. For simplicity, we assume $n \geq 3$ in this paper.

In §1, we give the definition of the two kinds of L-functions. Recalling the definition of the Gauss sums in [Sai1], we describe the relation between these two kinds of L-functions. We introduce one more matrix-valued Gauss sum and prove a result on it, which is a complement to the results in [Sai1].

In §2, we give an explicit form of L-functions assuming the result on orbital local series proved in §3. These results contain a generalization of [I-S1].

In the case of positive definite matrices, using these explicit forms, we prove the rationality of the values of the L-fucntions at non-positive integers.

In §3, we determine orbital local series for L-functions and complete the proof of theorems in §2.

§1. L-functions and Gauss sums

For a ring R, we denote by $S_n(R)$ the set of symmetric matrices of degree n with coefficients in R. For a positive integer n, let L_n(resp. L_n^*) be the lattice in $S_n(Q)$ consisiting of integral (resp. half-integral) symmetric matices of degree n, and $L_n^{(i)}$ (resp. $L_n^{*\,(i)}$) its subset consisting of elements with signature $(i, n-i)$. Then $SL_n(Z)$ acts on L_n and L_n^* by $g \cdot x = gx{}^tg$ for $g \in SL_n(Z)$ and $x \in L_n$, L_n^*. We define some functions on L_n or L_n^*, which we call characters in this paper. First, we consider characters defined modulo p^ν for an odd prime p and then those modulo 2^ν. Lastly, we consider general ones. For a prime p and a positive integer ν, we set $R_{p,\nu} = Z/p^\nu Z$.

Let p be an odd prime, and let φ be a Dirichlet character with the conductor $f(\varphi) = p^\nu$ for a positive integer ν. For $x \in L_n$ or L_n^*, set $\tilde{x} = x \bmod p^\nu$ and define

$$\varphi^{(n)}(x) = \begin{cases} \varphi(\det \tilde{x}) & \text{if } \det \tilde{x} \in R_{p,\nu}^\times, \\ 0 & \text{otherwise.} \end{cases}$$

Let χ_p and χ_0 be the quadratic character and the trivial one modulo p respectively. For $\psi = \chi_p$ or χ_0 and and integer r with $1 \leq r \leq n$, we define

$$\psi^{(r)}(x) = \begin{cases} \psi(\det \tilde{x'}) & \text{if } \operatorname{rank}(\tilde{x}) = r, \\ 0 & \text{otherwise,} \end{cases}$$

where $\tilde{x}'$ is an element of $S_r(R_{p,1})$ with $g\tilde{x}{}^t g = \begin{pmatrix} \tilde{x}' & 0 \\ 0 & 0 \end{pmatrix}$ for $g \in GL_n(R_{p,1})$ and $\det \tilde{x}' \neq 0$. The above definition is independent of the choice of $\tilde{x}'$. For $r = n$, the above two definitions are identical. For $r = 0$, we set

$$\chi_p^{(0)}(x) = \chi_0^{(0)}(x) = \begin{cases} 1 & \text{if } rank(\tilde{x}) = 0, \\ 0 & \text{otherwise.} \end{cases}$$

When it is necessary to indicate the prime p, we write $\chi_{0,p}^{(r)}$ instead of $\chi_0^{(r)}$.

Let $p = 2$, and let φ and $\chi_0 = \chi_{0,2}$ be as above. For $x \in L_n$ and r, $0 \leq r \leq n$, we define $\varphi^{(n)}$ and $\chi_0^{(r)}$ in the same way as in the case of p odd. For $x \in L_n^*$, let Q_x be the quadratic form in $t_1, t_2, \cdots, t_n$ associated to x. Then $Q_x \bmod p$ is equivalent to one of

$$t_1 t_2 + \cdots t_{r-2} t_{r-1} + t_r^2,$$

for r odd,

$$t_1 t_2 + \cdots + t_{r-1} t_r, \tag{1.1}$$

$$t_1 t_2 + \cdots + t_{r-3} t_{r-2} + t_{r-1}^2 + t_{r-1} t_r + t_r^2, \tag{1.2}$$

for r even. We define for r even, $2 \leq r \leq n$, and $x \in L_n^*$

$$\chi_p^{*(r)}(x) = \begin{cases} 1 & \text{if } Q_x \bmod p \text{ is equivalent to (1.1),} \\ -1 & \text{if } Q_x \bmod p \text{ is equivalent to (1.2),} \\ 0 & \text{otherwise.} \end{cases}$$

For $r = 0$, we define $\chi_p^{*(0)}(x) = 1$ if $x \in 2L_n^*$, and $\chi_p^{*(0)}(x) = 0$ otherwise.

We consider general characters. Let N_1, N_2, N_3 be three positive square-free integers coprime to each other. For an odd prime $p \mid N_1$, choose a character φ_p defined modulo a power of p with $\varphi_p^2 \neq \chi_0$. When $2 \mid N_1$, we choose a non-trivial character φ_2 defined modulo a power of 2. For each $p \mid N_2$ and $p \mid N_3$, we choose integers r_p, $0 \leq r_p \leq n$. Let N_2^o and N_2^e be the product of primes p dividing N_2 such that r_p is odd or even respectively. We define N_3^o, N_3^e similarly. For L_n, we assume $(2, N_2) = 1$. For $x \in L_n$, we define

$$\psi(x) = \prod_{p|N_1} \varphi_p^{(n)}(x) \prod_{p|N_2} \chi_p^{(r_p)}(x) \prod_{p|N_3} \chi_{0,p}^{(r_p)}(x). \tag{1.3}$$

For L_n^*, we assume $(2, N_1 N_2^o N_3) = 1$, and we define a character ψ by (1.3) replacing $\chi_2^{(r_p)}$ by $\chi_2^{*(r_p)}$ when $2 \mid N_2^e$. It is easy to see ψ is invariant under the action of $SL_n(Z)$.

For ψ, and $L = L_n,\ L_n^*$, we define

$$\zeta_i(s, L, \psi) = c_n \sum_{x \in L^{(i)}/SL_n(\mathbf{Z})} \psi(x)\mu(x)\,|\det x|^{-s},$$

where

$$c_n = \frac{2\prod_{k=1}^n \Gamma(k/2)}{\pi^{n(n+1)/4}},$$

and $\mu(x)$ is the volume attached to x, the definition of which is given in §1 of the part I. This series converges absolutely for $Re(s) > \frac{n+1}{2}$ unless $n = 2$ and $i = 1$.

We give the definition of another kind of L-functions introduced by Arakawa. Let p be an odd prime. For $a \in R$ and a positive integer m, we set $e_m(a) = \exp(2\pi a\sqrt{-1}/m)$. For $S \in S_n(R_{p,1})$ and $x \in L_n^*$, set

$$\tau_S^{(n)}(x) = \sum_{y \sim S} e_p(\mathrm{tr}(xy)),$$

where y is extended over all $y \in S_n(R_{p,1})$ which are equivalent to S. Here we understand $e_p(\mathrm{tr}(z)) = e_p(\mathrm{tr}(\bar{z}))$ for $z \in S_n(R_{p,1})$ with $\bar{z}$ $(\in S_n(Z))$ mod $p = z$. Then Arakawa's L-function is defined by

$$\zeta_i(s, L, S) = c_n \sum_{x \in L^{(i)}/SL_n(\mathbf{Z})} \tau_S^{(n)}(x)\,\mu(x)\,|\det x|^{-s},$$

for $L = L_n,\ L_n^*$. This series converges absolutely also for $Re(s) > \frac{n+1}{2}$ unless $n = 2$ and $i = 1$.

To describe the relation between these L-functions, we recall the Gauss sums introduced in [Sai1]. For $\eta = \chi_p$, or χ_0, and $x \in S_n(R_{p,1})$ of rank r, we define

$$\eta(x) = \eta(\det x'),$$

where x' is an element of $S_r(R_{p,1})$ such that ${}^t gxg = \begin{pmatrix} x' & 0 \\ 0 & 0 \end{pmatrix}$ with $g \in GL_n(R_{p,1})$ and $\det x' \neq 0$. Then $\eta(x)$ is well-defined. If $r = 0$, we set $\eta(x) = 1$. We define

$$W_r^n(x, \eta) = \sum_y \eta(y) e_p(\mathrm{tr}(xy)),$$

where y runs through all elements of $S_n(R_{p,1})$ of rank r. For $x \in L_n$ or L_n^*, we set $W_r^n(x,\eta) = W_r^n(\tilde{x},\eta)$ with $\tilde{x} \equiv x \bmod p$. Then for $S \in S_n(R_{p,1})$ of rank r, we have

$$\tau_S^{(n)}(x) = \frac{1}{2}(W_r^n(x,\chi_0) + \chi_p(S)W_r^n(x,\chi_p)). \tag{1.4}$$

For two integers r, t such that $0 \leq r,\ t \leq n$ and $r \equiv t \bmod 2$, we define $W^n(r,t)$ as follows. When $r \equiv t \equiv 1 \bmod 2$, we set

$$W^n(r,t) = W_r^n(x,\chi_p)\chi_p(x),$$

where x is an element of $S_n(R_{p,1})$ of rank t. Then by Cor. 1.2 of [Sai1], this is independent of the choice of x (denoted by $W_o^n(i,j)$ with $r = 2i-1$, $t = 2j-1$ in [Sai1]). When both of r and t are even, we set

$$W^n(r,t) = W_r^n(x,\chi_p)$$

for $x \in S_n(R_{p,1})$ with rank $x = t$. This is also independent of the choice of x by Cor. 1.14 of [Sai1] (The proof of Cor. 1.14 there is incomplete in the case where n is even and $\operatorname{rank} x = t = n$. But this case can be deduced easily from Prop. 1.12 of [Sai1].). Let $G(\chi_p)$ be the usual Gauss sum for χ_p. In these notations, we can prove

PROPOSITION 1.1. *Let p be an odd prime and let $x \in S_n(R_{p,1})$.*
(1) *If r is odd, then*

$$W_r^n(x,\chi_p) = \sum_{j=0}^{[(n-1)/2]} W^n(r,2j+1)\chi_p^{(2j+1)}(x),$$

$$\begin{aligned}W_r^n(x,\chi_0) = {}& G(\chi_p)^r \sum_{j=1}^{[n/2]} W^{n-1}(r,2j-1)\chi_p^{(2j)}(x) \\ & - G(\chi_p)^{r-1} \sum_{j=1}^{[(n+1)/2]} W^{n-1}(r-1,2j-2)(\chi_0^{(2j)}(x) + chi_0^{(2j-1)}(x)) \\ & + W_r^n(O_n,\chi_0)\chi_0^{(0)}(x).\end{aligned}$$

(2) *Let $r \geq 2$ be even, and let $\tilde{x}$ be an element of $S_{n+1}(R_{p,1})$ such that $\chi_p(\tilde{x}) = \chi_p(x)$ and $rank(\tilde{x}) = rank(x) + 1$. Then one has*

$$W_r^n(x,\chi_p) = \sum_{j=0}^{[n/2]} W^n(r,2j)(\chi_0^{(2j)}(x) + \chi_0^{(2j+1)}(x)),$$

$$W_r^n(x,\chi_0) = G(\chi_p)^{-r-1}W_{r+1}^{n+1}(\tilde{x},\chi_p) - W_{r+1}^n(x,\chi_0).$$

Here we understand that $\chi_p^{(m)}(x) = 0$ *for* $m > n$ *when* n *is even, and* O_n *is the zero matrix of degree* n.

Proof. These assertions follow easily from Cor. 1.2, Cor. 1.14, Prop. 1.13, Prop. 1.11 and Prop. 1.12 of [Sai1].

This shows that the L-functions $\zeta_i(s,L,S)$ can be written as linear combinations of $\zeta_i(s,L,\chi)$. For example, if the rank r of S is odd, then by (1) of Prop. 1.1. and (1.4) we have

$$\begin{aligned}\zeta_i(s,L,S) = \frac{1}{2}\Big(&\chi_p(S)\sum_{j=0}^{[(n-1)/2]} W^n(r,2j+1)\zeta_i(s,L,\chi_p^{(2j+1)})\\ &+ G(\chi_p)^r\sum_{j=1}^{[n/2]} W^{n-1}(r,2j-1)\zeta_i(s,L,\chi_p^{(2j)})\\ &- G(\chi_p)^{r-1}\sum_{j=1}^{[(n+1)/2]} W^{n-1}(r-1,2j-2)(\zeta_i(s,L,\chi_0^{(2j)})+\zeta_i(s,L,\chi_0^{(2j-1)}))\\ &- W_r^n(O_n,\chi_0)\zeta_i(s,L,\chi_0^{(0)})\Big).\end{aligned}$$

We can prove a similar formula for S of rank r even by (2) of Prop. 1.1. Hence the rationality of special values of $\zeta_i(s,L,S)$ follows from that of $\zeta_i(s,L,\psi)$.

Here we insert a result on Gauss sums, which is a complement to Th. 1.15 and Cor. 1.17 of [Sai] (in Th. 1.15, $W_o^n(\chi_p)$ should be read $W_o^n(\chi_p)^2$.) There, for u, v such that $0 \le u$, $v \le [n/2]$, we define

$$W_e^n(u,v) = \begin{cases} W_{2u}^n(x,\chi_0)\chi_p(x) & \text{if } n \text{ is even and } n = 2u,\\ (W_{2u+1}^n(x,\chi_0) & \\ \qquad + W_{2u}^n(x,\chi_0))\chi_p(x) & \text{otherwise,}\end{cases}$$

with $x \in S_n(R_{p,1})$ of rank $x = 2v$, which is independent of the choice of x, and the Gauss sum

$$W_e^n(\chi_p) = (W_e^n(i-1,j-1)).$$

We define one more matrix-valued Gauss sum

$$U_e^n(\chi_p) = (W^n(2i-2,2j-2)),$$

where the (i,j) component of $U_e^n(\chi_p)$ is $W^n(2i-2,2j-2)$.

THEOREM 1.2. *The notation being as above, one has*

$$U_e^n(\chi_0)W_e^n(\chi_p) = p^{n(n+1)/2}E_{[n/2]+1},$$

where $E_{[n/2]+1}$ is the unit matrix of degree $[n/2]+1$.

Proof. We give a proof only for the case n odd, since the other case can be treated in the same way. Then the (i,j) component of the product of matrices on the left hand side is equal to

$$\sum_{k=1}^{[n/2]+1} W_{2i-2}^n(x,\chi_p)(W_{2k-2}^n(y,\chi_0) + W_{2k-1}^n(y,\chi_0))\chi_p(y), \tag{1.5}$$

where $x, y \in S_n(R_{p,1})$ of rank $2k-2$, $2j-2$ respectively. Using the fact that (cf. Prop. 1.13 of [Sai1])

$$W_{2i-2}^n(x,\chi_p) = W_{2i-2}^n(x',\chi_p)$$

for $x,\ x' \in S_n(R_{p,1})$ of rank $2k-1,\ 2k-2$ respectively, we see (1.5) is equal to

$$\begin{aligned}\sum_{z\in S_n(R_{p,1})} &\chi_p(y)e_p(\mathrm{tr}(yz))W_{2i-2}^n(z,\chi_p)\\ &= \sum_w \chi_p(y)\chi_p(w) \sum_{z\in S_n(\mathbf{Z}/p\mathbf{Z})} e_p(\mathrm{tr}((y+w)z))\\ &= p^{n(n+1)/2}\delta_{ij},\end{aligned}$$

where w runs through all elements of $S_n(R_{p,1})$ of rank $2i-2$. Here we used the fact that the rank of y is even. This completes the proof.

§2. Explicit form of L-functions

In this section, we give an explicit form of L-functions assuming the results in §3 and discuss the rationality of the values of L-functions at non-positive integers. As for the calculation of the L-functions, we follow the procedure of [I-S2], and only give an outline.

Let $N_1,\ N_2,\ N_3$ and ψ be as in §1. We set $\psi_p = \varphi_p^{(n)},\ \chi_p^{(r_p)}$, or $\chi_{0,p}^{(r_p)}$ according to whether p divides $N_1,\ N_2$, or N_3, and extend ψ_p to $S_n(R_{p,\nu})$ for a large ν and to $S_n(Z_p)$ naturally. For p prime to $N_1N_2N_3$, let ψ_p be the characteristic functions of $S_n(R_{p,\nu})$ or $S_n(Z_p)$. For $p=2$, let $S_n(Z_p)_e$

be the subset of $S_n(Z_p)$ consisting of elements (x_{ij}) such that $x_{ii} \equiv 0 \bmod p$ for all i, and let $S_n(R_{p,\nu})_e$ be the similar subset of $S_n(R_{p,\nu})$.

If $2 \mid N_2^e$, we set $\psi_p^*(x) = \tilde{\chi}_2^{*(r_p)}(x) = \chi_2^{*(r_p)}(y)$ for $x \in S_n(R_{p,\nu})_e$ or $S_n(Z_p)_e$ taking $y \in L_n^*$ such that $2y \equiv x \bmod p^\nu$. When $(2, N_2^e) = 1$, let ψ_p^* be the characteristic function of $S_n(R_{p,\nu})$ or $S_n(Z_p)$. In the following, we assume $r_p \geq 1$, since the case of $r_p = 0$ can be easily reduced to the case where $p \nmid N_2N_3$.

For i, $0 \leq i \leq n$, let

$$\delta = (-1)^{n-i}, \quad \epsilon = (-1)^{(n-i)(n-i+1)/2},$$

and set

$$\xi_i(s, L_n, \psi) = \sum_{\delta d=1}^{\infty} a_i(d)|d|^{-s}, \quad \xi_i(s, L_n^*, \psi) = \sum_{\delta d=1}^{\infty} a_i^*(d)2^{ns}|d|^{-s}.$$

The first step is to express $a_i(d)$ and $a_i^*(d)$ by local data. For this, we introduce some notations. Let ι_p and ε_p be the constant function with value 1 and the Hasse invariant on $S_n(Z_p)$ or $S_n(R_{p,\nu})$ respectively, and for a, $b \in Q_p^\times$, let $(a, b)_p$ be the Hilbert symbol of a and b. For $\omega_p = \iota_p$ or ε_p and $d \in Z_p$, $d \neq 0$, we define

$$\lambda_p(\psi_p, d, \omega_p) = \lim_{\nu \to \infty} \lambda_{p,\nu}(\psi_p, d, \omega_p)$$
$$(\lambda_p^*(\psi_p, d, \omega_p) = \lim_{\nu \to \infty} \lambda_\nu^*(\psi_p^*, d, \omega_p) \text{ for } p = 2),$$

where

$$\lambda_{p,\nu}(\psi_p, d, \omega_p) = 2^{-\delta_{2,p}} p^{-v(d)+(n(n-1)/2)\nu} |SL_n(R_{p,\nu})|^{-1} \sum_{x \in S_n(R_{p,\nu},d)} \psi_p(x)\omega_p(x)$$
$$(\lambda_{p,\nu}^*(\psi_p^*, d, \omega_p) = 2^{-\delta_{2,p}} p^{-v(d)+(n(n-1)/2)\nu} |SL_n(R_{p,\nu})|^{-1} \times \sum_{x \in S_n(R_{p,\nu},d)_e} \psi_p^*(x)\omega_p(x) \text{for } p = 2).$$

Here v is the additive valuation of Z_p such that $v(p) = 1$, and

$$S_n(R_{p,\nu}, d) = \{\, x \in S_n(R_{p,\nu}) \mid \det x \equiv d \bmod p^\nu \}$$
$$(S_n(R_{p,\nu}, d)_e = S_n(R_{p,\nu}, d) \cap S_n(R_{p,\nu})_e \text{ for } p = 2).$$

For $\omega = \iota$ or ε, and $d \in Z, d \neq 0$, we define

$$\lambda_f(\psi, d, \omega) = \prod_p \lambda_p(\psi_p, d, \omega_p)$$
$$(\lambda_f^*(\psi, d, \omega) = \lambda_2(\psi_p^*, d, \omega_p) \prod_{p \neq 2} \lambda_p(\psi_p, d, \omega_p) \text{ for } p = 2).$$

Then by Siegel's formula and the invariance of ψ in a genus, in the same way as Prop. 2.2 of [I-S], we obtain

$$a_i(d) = c_n(\lambda_f(\psi, d, \iota) + \epsilon\lambda_f(\psi, d, \varepsilon))|d|^{(n+1)/2},$$
$$a_i^*(d) = c_n \prod_{p|N_1} \varphi_p(2^{-n}) \prod_{p|N_2^o} \chi_p(2^{-r_p})(\lambda_f^*(\psi, d, \iota) + \epsilon\lambda_f^*(\psi, d, \varepsilon))|d|^{(n+1)/2}.$$

As in the case of zeta-functions, our L-functions depends only on δ and ϵ, and we set

$$\xi(s, L, \psi, \delta, \epsilon) = \xi_i(s, L, \psi).$$

To sum up the above quantities, we introduce another local data

$$\lambda_p(\psi_p, d, \omega_p, \{n_i\}, \{d_i\}), \quad \lambda_p^*(\psi_p^*, d, \omega_p, \{n_i\}, \{d_i\}),$$

and some power series. Let $n = n_1 + n_2 + \cdots + n_m$ be a partition of n into m positive integers. We denote this by $\{n_i\}$ and call m the length of $\{n_i\}$. A patition is called even if all n_i are even and odd otherwise. A sequence $t_1, t_2, \cdots, t_m$ of integers of the same length m is called a sequence associated to $\{n_i\}$ if it satisfies $t_1 < t_2 < \cdots < t_m$. For $\{n_i\}$, $\{t_i\}$ as above, let $S_n(R_{p,\nu}, d, \{n_i\}, \{t_i\})$ be the subset of $S_n(R_{p,\nu}, d)$ consisting of elements equivalent to

$$\begin{pmatrix} p^{t_1}x_1 & 0\cdots & \cdots & 0 \\ 0 & p^{t_2}x_2 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & p^{t_m}x_m \end{pmatrix} \tag{2.1}$$

with respect to $GL_n(R_{p,\nu})$, for $x_i \in S_{n_i}(R_{p,\nu}, R_{p,\nu}^\times)$, and let

$$S_n(R_{p,\nu}, d, \{n_i\}, \{t_i\})_e = S_n(R_{p,\nu}, d, \{n_i\}, \{t_i\}) \cap S_n(R_{p,\nu})_e.$$

The matrix of the form (2.1) will be denoted by $(\oplus p^{t_i}x_i)$. Similarly as above, for $\{n_i\}$ and $\{t_i\}$, we define

$$\lambda_p(\psi, d, \omega, \{n_i\}, \{t_i\}) = \lim_{\nu \to \infty} \lambda_{p,\nu}(\psi, d, \omega, \{n_i\}, \{t_i\}),$$
$$(\lambda_p^*(\psi_p^*, d, \omega, \{n_i\}, \{t_i\}) = \lim_{\nu \to \infty} \lambda_{p,\nu}^*(\psi_p^*, d, \omega, \{n_i\}, \{t_i\}) \text{ for } p = 2),$$

where

$$\lambda_{p,\nu}(\psi_p, d, \omega_p, \{n_i\}, \{t_i\}) = 2^{-\delta_{2,p}} p^{-v(d)+(n(n-1)/2)\nu} |SL_n(R_{p,\nu})|^{-1} \sum_{x \in S_n(R_\nu, d, \{n_i\}, \{t_i\})} \psi_p(x)\omega_p(x)$$

$$(\lambda^*_{p,\nu}(\psi^*_p, d, \omega_p, \{n_i\}, \{t_i\}) = 2^{-\delta_{2,p}} p^{-v(d)+(n(n-1)/2)\nu} |SL_n(R_{p,\nu})|^{-1} \sum_{x \in S^*_n(R_\nu, d, \{n_i\}, \{t_i\})_e} \psi^*_p(x)\omega_p(x) \quad \text{for } p = 2).$$

Let $t = \sum_{i=1}^m n_i t_i$. Then we have

$$\lambda_p(\psi_p, d, \omega_p) = \sum_{\{n_i\},\ t=v(d)} \lambda(\psi_p, d, \omega_p, \{n_i\}, \{t_i\})$$

$$(\lambda^*_p(\psi^*_p, d, \omega_p) = \sum_{\{n_i\},\ t=v(d)} \lambda^*_p(\psi^*_p, d, \omega_p, \{n_i\}, \{t_i\}) \text{ for } p = 2).$$

Here $\{n_i\}$ and $\{t_i\}$ run through all partitions of n and all sequences associated to them satisfying $v(d) = t$. By these constants, we define orbital local series with characters as follows. For ψ_p, ψ^*_p, ω_p as above, we set

$$\lambda'_p(\psi_p, d, \omega_p, \{n_i\}, \{t_i\}) = \lambda(\psi_p, d, \omega_p, \{n_i\}, \{t_i\})$$

$$(\lambda^{*\prime}_p(\psi^*_p, d, \omega_p, \{n_i\}, \{t_i\}) = \lambda^*_p(\psi^*_p, d, \omega_p, \{n_i\}, \{t_i\}) \text{ for } p = 2)$$

$$\times \begin{cases} ((-1)^{(n+1)/2}, p)^t_p & \text{if } n \text{ is odd and } \omega_p = \varepsilon_p, \\ 1 & \text{otherwise,} \end{cases}$$

and for $d_0 \in \mathbf{Z}_p^\times$, define

$$Z_n(u, \psi_p, \omega_p, d_0) = \sum_{\{n_i\}\ \{t_i\}} \lambda'_p(\psi_p, d_0 p^t, \omega_p, \{n_i\}, \{t_i\}) p^{((n+1)/2)t} u^t$$

$$(Z^*_n(u, \psi^*_p, \omega_p, d_0) = \sum_{\{n_i\}\ \{t_i\}} \lambda^{*\prime}_p(\psi^*_p, d_0 p^t, \omega_p, \{n_i\}, \{t_i\}) p^{((n+1)/2)t} u^t \text{ for } p = 2),$$

where $\{n_i\}$ runs through all partitions of n and $\{t_i\}$ runs through all sequences associated to $\{n_i\}$ satisfying $0 \le t_1$. In the case n is even, define

$$Z_{n,o}(u, \psi_p, \iota_p, d_0) = \frac{1}{2}(Z_n(u, \psi_p, \iota_p, d_0) - Z_n(-u, \psi_p, \iota_p, d_0)),$$

$$Z_{n,e}(u, \psi_p, \iota_p, d_0) = \frac{1}{2}(Z_n(u, \psi_p, \iota_p, d_0) + Z_n(-u, \psi_p, \iota_p, d_0)),$$

and define $Z^*_{n,o}(u,\psi^*_p,\iota_p,d_0)$ and $Z_{n,e}(u,\psi^*_p,\iota_p,d_0)$ similarly. We denote the series aoociated to the characteristic functions of $S_n(Z_p)$, or $S_n(Z_p)_e$ simply by

$$Z_n(u,\omega_p,d_0),\quad Z_n(u,\omega_p,d_0),\quad Z^*_{n,o}(u,\omega_p,d_0),$$

and so on. These are calculated in §5 of [I-S2]. The other series will be calculated in §3.

We treat the cases of n odd and n even separately. First let n be odd. To state our result, we introduce some notations. For ψ, we define Dirichlet characters $\hat{\psi}$ and $\tilde{\psi}$ by

$$\hat{\psi}=\prod_{p\mid N_1}\varphi_p,\qquad \tilde{\psi}=\prod_{p\mid N_1}\varphi_p\prod_{p\mid N_2^o}\chi_p.$$

For $p\mid N_2N_3$, set

$$\begin{aligned}A_p(u,\psi_p,\iota_p)&=Z_n(u,\psi_p,\iota_p,1)/Z_n(u,\iota_p,1)\\&\qquad\qquad\times\begin{cases}(1-p^{(n-1)/2}u)^{-1} & \text{if } p\mid N_2^o,\\ 1 & \text{otherwise,}\end{cases}\\A_p(u,\psi_p,\varepsilon_p)&=Z_n(u,\psi_p,\varepsilon_p,1)/Z_n(u,\varepsilon_p,1)\\&\qquad\qquad\times\begin{cases}(1-u)^{-1} & \text{if } p\mid N_2^o,\\ 1 & \text{otherwise.}\end{cases}\end{aligned}$$

If $p=2\mid N_2^e$, we set

$$A^*_p(u,\psi^*_p,\omega_p)=Z^*_n(u,\psi^*_p,\omega_p,1)/Z^*_n(u,\omega_p,1)$$

For $\omega=\iota,\ \varepsilon$, and L_n define

$$A(s,\psi,\omega)=\prod_{p\mid N_2N_3}A_p\Big(\Big(\prod_{q\mid N_1}\varphi_q(p)\prod_{q\mid N_2^o\ q\neq p}\chi_q(p)\Big)p^{-s},\psi_p,\omega_p\Big)$$

and for L^*_n, define $A^*(s,\psi,\omega)=A(s,\psi,\omega)$ if $2\nmid N_2^e$ and if $2\mid N_2^e$, define $A^*(s,\psi,\omega)$ replacing $A_2(u,\psi_p,\omega_p)$ by $A^*_2(u,\psi^*_p,\omega_p)$ in the above definition of $A(s,\psi,\omega)$. In these notations, we can prove

THEOREM 2.1. *Let n be an odd integer ≥ 3, and assume $r_p\geq 1$ for*

$p \mid N_2N_3$. Let $A(s,\psi,\omega)$, $A^*(s,\psi,\omega)$, and $\hat{\psi}$, $\tilde{\psi}$ be as above. Then one has

$$\xi(s,L_n,\psi,\delta,\epsilon)=\frac{|\prod_{i=1}^{[n/2]}B_{2i}|}{2^{n-1}(\frac{n-1}{2})!}\tilde{\psi}(\delta)$$

$$\times\Big(2^{(n-1)/2}A(s,\psi,\iota)L(s-\frac{n-1}{2},\tilde{\psi})\prod_{i=1}^{[n/2]}L(2s-(2i-1),\hat{\psi}^2)$$

$$+\epsilon\delta^{(n+1)/2}(-1)^{(n^2-1)/8}A(s,\psi,\varepsilon)L(s,\tilde{\psi})\prod_{i=1}^{[n/2]}L(2s-2i,\hat{\psi}^2)\Big),$$

$$\xi(s,L_n^*,\psi,\delta,\epsilon)=\frac{|\prod_{i=1}^{[n/2]}B_{2i}|}{2^{n-1}(\frac{n-1}{2})!}2^{(n-1)s}\tilde{\psi}(\delta)\prod_{p\mid N_1}\varphi_p(2^{-n})\prod_{p\mid N_2^o}\chi_p(2^{-r_p})$$

$$\times\Big(A^*(s,\psi,\iota)L(s-\frac{n-1}{2},\tilde{\psi})\prod_{i=1}^{[n/2]}L(2s-(2i-1),\hat{\psi}^2)$$

$$+\epsilon\delta^{(n+1)/2}(-1)^{(n^2-1)/8}A^*(s,\psi,\varepsilon)L(s,\tilde{\psi})\prod_{i=1}^{[n/2]}L(2s-2i,\hat{\psi}^2)\Big).$$

Proof. We give a proof only for L_n. We note $(d,N_1)=1$ if $\lambda_f(\psi,d,\omega)\neq 0$. For $\delta d>0$, let

$$d=\delta\prod_p p^{t_p}=p^{t_p}d_{0,p},\quad d_{0,p}\in Z_p^\times.$$

By the results in §3, $Z_n(u,\psi_p,\iota_p,d_0)$ is independent of d_0 if $p\nmid N_1N_2^o$, and we see for $\omega=\iota$

$$\begin{aligned}\lambda_f(\psi,d,\iota)&=\prod_{p\mid N_1}2^{-\delta_{2,p}}\varphi_p(d)(p^{-2})_{[n/2]}^{-1}\prod_{p\mid N_2^o}\chi_p(d_{0,p})\prod_{(p,N_1)=1}\lambda_p(\psi_p,p^{t_p},\iota_p)\\&=\tilde{\psi}(\delta)\prod_{p\mid N_1}2^{-\delta_{2,p}}(p^{-2})_{[n/2]}^{-1}\\&\qquad\times\prod_{(p,N_1)=1}\Big((\prod_{q\mid N_1}\varphi_q(p)\prod_{q\mid N_2^o\ q\neq p}\chi_q(p))^{t_p}\lambda_p(\psi_p,p^{t_p},\iota_p)\Big).\end{aligned}$$

From this we see

$$\sum_{\delta d>0}\lambda_f(\psi,d,\iota)|d|^{(n+1)/2-s}$$

$$= \tilde{\psi}(\delta) \prod_{p|N_1} 2^{-\delta_{2,p}} (p^{-2})^{-1}_{[n/2]}$$

$$\times \prod_{(p,N_1)=1} \Big(\sum_{t_p=0}^{\infty} \big(\prod_{q|N_1} \varphi(p) \prod_{q|N_2^o \ q\neq p} \chi_q(p)\big)^{t_p} \lambda_p(\psi_p, p^{t_p}, \iota_p) p^{(n+1)t_p/2 - t_p s} \Big)$$

$$= \tilde{\psi}(\delta) \prod_{p|N_1} 2^{-\delta_{2,p}} (p^{-2})^{-1}_{[n/2]}$$

$$\times \prod_{(p,N_1)=1} Z_n\big(\big(\prod_{q|N_1} \varphi(p) \prod_{q|N_2^o \ q\neq p} \chi_q(p)\big) p^{-s}, \psi_p, \iota_p, 1\big).$$

From this we easily obtain our formula. The case of $\omega = \varepsilon$ can be treated in the same way, and will be omitted.

From this theorem, we can deduce the following result on the rationality of the values at non-positive integers of L-functions.

COROLLARY 2.2. *Let $Q(\psi)$ be the field generated by the values of ψ over Q. Then for a positive integer m, the values $\xi(1-m, L_n, \psi, \delta, \epsilon)$ and $\xi(1-m, L_n^*, \psi, \delta, \epsilon)$ are contained in $Q(\psi)$.*

Proof. Since

$$\prod_{i=1}^{[n/2]} L(2(1-m) - 2i, \hat{\psi}^2) = 0,$$

we have

$$\xi(1-m, L_n, \psi, \delta, \epsilon) = \frac{|\prod_{i=1}^{[n/2]} B_{2i}|}{2^{n-1}(\frac{n-1}{2})!} \tilde{\psi}(\delta) A(1-m, \psi, \iota) 2^{(n-1)/2}$$

$$\times L(1-m-\frac{n-1}{2}, \tilde{\psi}) \prod_{i=1}^{[n/2]} L(1-2(m+i-1), \hat{\psi}^2).$$

Our assertion for L_n easily follows from this. The case of L_n^* is similar.

Now we turn to the case of n even. We introduce more notations. For a quadratic field K, we denote by d_K the discriminant of K and for $K = Q \oplus Q$, we set $d_K = 1$. For a quadratic field K, we denote by χ_K the Dirichlet character corresponging to K, and for $K = Q \oplus Q$ by χ_K the trivial character.

To describe the ι-part of the L-function, we define two Dirichlet series $D(s,\psi,\delta)$ and $D^*(s,\psi,\delta)$. If $N_2^o \neq 1$, we set $D(s,\psi,\delta) = D^*(s,\psi,\delta) = 0$. For K as above and an odd prime $p \mid N_2N_3$, we set

$$B_p(u,\psi_p,\iota_p,K) = \begin{cases} Z_{n,o}(u,\psi_p,\iota_p,d_K/p)/Z_{n,o}(u,\iota_p,d_K/p) & \text{if } p \mid N_2N_3,\ p \mid d_K, \\ Z_{n,e}(u,\psi_p,\iota_p,d_K)/Z_{n,e}(u,\iota_p,d_K) & \text{if } p \mid N_2N_3,\ p \nmid d_K, \end{cases}$$

for $p = 2 \mid N_3$

$$B_p(u,\psi_p,\iota_p,K) = \begin{cases} Z_{n,o}(u,\psi_p,\iota_p,d_K/p^3)/Z_{n,o}(u,\iota_p,d_K/p^3) & \text{if } p^3 \mid d_K, \\ Z_{n,e}(u,\psi_p,\iota_p,d_K/p^2)/Z_{n,e}(u,\iota_p,d_K/p^2) & \text{if } p^2 \parallel d_K, \\ Z_{n,e}(u,\psi_p,\iota_p,d_K)/Z_{n,e}(u,\iota_p,d_K) & \text{if } p \nmid d_K, \end{cases}$$

and for K with $(d_K,2) = 1$

$$\tilde{B}_2(u,K) = \begin{cases} (1-(1-u^2)(1-2^{n-1}u^2)(1-\chi_K(2)2^{n/2-1}u^2)^{-1}) & \text{if } 2 \nmid N_1, \\ (1+\chi_K(2)2^{-n/2})^{-1}(1+2^{-n}+2\chi_K(2)2^{-n-n/2}) & \text{if } 2 \mid N_1. \end{cases}$$

If $2 \mid N_2^e$ for $L = L_n^*$, we define $B_2^*(u,\psi_2^*,\iota_2,K)$ in the same way as above taking $Z_{n,o}^*(u,\psi_2^*,\ \iota_2,d_0)$, $Z_{n,e}^*(u,\psi_2^*,\iota_2,d_0)$ instead of $Z_{n,o}(u,\psi_2,\iota_2,d_0)$,[1] $Z_{n,e}(u,\psi_2,\iota_2,d_0)$. Using these functions, we define

$$B(s,\psi,\iota,K) = \prod_{p \mid N_2N_3} B_p(\hat{\psi}(p)p^{-s},\psi_p,\iota_p,K) \times \begin{cases} \hat{\psi}(d_K)\tilde{B}_2(\hat{\psi}(2)2^{-s},K) & \text{if } 2 \nmid d_K, \\ \hat{\psi}(d_K/4) & \text{if } 2 \mid d_K, \end{cases}$$

$$B^*(s,\psi,\iota,K) = \hat{\psi}(d_K) \begin{cases} \prod_{p \mid N_2N_3} B_p(\hat{\psi}(p)p^{-s},\psi_p,\iota_p,K) & \text{if } 2 \nmid N_2^e, \\ B_2^*(\hat{\psi}(2)2^{-s},\psi_2^*,\iota_2,K) \\ \qquad \times \prod_{p \mid N_2N_3/2} B_p(\hat{\psi}(p)p^{-s},\psi_p,\iota_p,K) & \text{if } 2 \mid N_2^e. \end{cases}$$

and the Dirichlet series by

$$D^*(s,\psi,\delta) = (-1)^{[n/4]} \sum_{\substack{(-1)^{n/2}\delta d_K > 0 \\ (d_K,N_1)=1}} 2(2\pi)^{-n/2}(\frac{n}{2}-1)!|d_K|^{(n-1)/2}$$

$$\times B^*(s,\psi,\iota,K)\hat{\psi}(d_K)L(\frac{n}{2},\chi_K)\frac{L(2s,\hat{\psi}^2)L(2s-n+1,\hat{\psi}^2)}{L(2s-(n/2-1),\hat{\psi}^2\chi_K)}|d_K|^{-s},$$

$$D(s,\psi,\delta)=(-1)^{[n/4]}2^{2s}\sum_{\substack{(-1)^{n/2}\delta d_K>0\\ (d'_K,N_1)=1}} 2(2\pi)^{-n/2}(\frac{n}{2}-1)!|d_K|^{(n-1)/2}$$

$$\times B(s,\psi,\iota,K)L(\frac{n}{2},\chi_K)\frac{L(2s,\hat{\psi}^2)L(2s-n+1,\hat{\psi}^2)}{L(2s-(n/2-1),\hat{\psi}^2\chi_K)}|d_K|^{-s}.$$

Here K runs through quadratic fields or $Q\oplus Q$ such that $(-1)^{n/2}\delta d_K>0$, and $d'_K=d_K/(d_K,4)$. In the above definition, we understand $\hat{\psi}^2\chi_K$ as a character modulo $N_1 d_K$.

Next we introduce notations necessary to describe the ε-part of the L-function. We set

$$\kappa(n,\delta,\psi)=\begin{cases}1 & \text{if } (-1)^{n/2}\delta N_2^o\equiv 1 \bmod 4,\\ 0 & \text{otherwise,}\end{cases}$$

and denote by H the quadratic field or $Q\oplus Q$ such that $d_H=(-1)^{n/2}\delta N_2^o$ when $\kappa(n,\delta,\psi)=1$. For a prime $p\mid N_2N_3$, we set

$$B_p(u,\psi_p,\varepsilon_p)=Z_n(u,\psi_p,\varepsilon_p,d_H/p)/Z_n(u,\varepsilon_p,d_H/p)$$
$$\times\begin{cases}(1-((-1)^{n/2}d_H/p,p)_p p^{-n/2})^{-1} & \text{if } p\mid N_2^o,\\ 1 & \text{otherwise,}\end{cases}$$

and when $2\mid N_2^e$ for L_n^*, for $p=2$, we set

$$B_p^*(u,\psi_p^*,\varepsilon_p)=Z_n^*(u,\psi_p^*,\varepsilon_p,d_H)/Z_n^*(u,\varepsilon_p,d_H).$$

Finally we define

$$B(s,\psi,\varepsilon)=\prod_{p\mid N_2N_3}B_p(\hat{\psi}(p)p^{-s},\psi_p,\varepsilon_p),$$

$$B^*(s,\psi,\varepsilon)=\begin{cases}\prod_{p\mid N_2N_3}B_p(\hat{\psi}(p)p^{-s},\psi_p,\varepsilon_p) & \text{if } 2\not\mid N_2^e,\\ B_2^*(\hat{\psi}(p)p^{-s},\psi_2^*,\varepsilon_2) & \\ \qquad\times\prod_{p\mid N_2N_3/2}B_p(\hat{\psi}(p)p^{-s},\psi_p,\varepsilon_p) & \text{if } 2\mid N_2^e.\end{cases}$$

In these notation, we can prove

THEOREM 2.3. *Let n be even ≥ 4, and assume $r_p\geq 1$ for $p\mid N_2N_3$.*

Let the notation be as above. Then one has

$$\begin{aligned}
&\xi(s, L_n, \psi, \delta, \epsilon) \\
&= \frac{|\prod_{i=1}^{n/2-1} B_{2i}|}{2^{n-1}(\frac{n-2}{2})!}\hat{\psi}((-1)^{n/2})\Big((-1)^{[n/4]}D(s,\psi,\delta)\prod_{i=1}^{[n/2]-1} L(2s-2i,\hat{\psi}^2) \\
&\qquad + \epsilon\kappa(n,\delta,\psi)(-1)^{n(n+2)/8}\hat{\psi}(d_H)\frac{2^{(n+2)/2}B'_{n/2,\psi}}{n} \\
&\qquad \times B(s,\psi,\varepsilon)\prod_{i=1}^{n/2} L(2s-(2i-1),\hat{\psi}^2)\Big),
\end{aligned}$$

$$\begin{aligned}
\xi(s, L_n^*, \psi, \delta, \epsilon) &= 2^{ns}\frac{|\prod_{i=1}^{n/2-1} B_{2i}|}{2^{n-1}(\frac{n-2}{2})!}\hat{\psi}((-1)^{n/2})\prod_{p|N_1}\varphi_p(2^{-n})\prod_{p|N_2^o}\chi_p(2^{-r_p}) \\
&\qquad \times \Big((-1)^{[n/4]}D^*(s,\psi,\delta)\prod_{i=1}^{[n/2]-1} L(2s-2i,\hat{\psi}^2) \\
&\qquad + \epsilon\kappa(n,\delta,\psi)(-1)^{n(n+2)/8}\hat{\psi}(d_H)\frac{2B'_{n/2,\psi}}{n} \\
&\qquad \times B^*(s,\psi,\varepsilon)\prod_{i=1}^{n/2} L(2s-(2i-1),\hat{\psi}^2)\Big),
\end{aligned}$$

where

$$B'_{n/2,\psi} = 2(\frac{n}{2})!\,(2\pi)^{-n/2}L(\frac{n}{2},\chi_H).$$

Proof. We treat the case of L_n^*. In this case, $(N_1N_2^oN_3, 2) = 1$. As in the case of n odd, the L-function is, up to the factor

$$2^{ns}\prod_{p|N_1}\varphi_p(2^{-n})\prod_{p|N_2^o}\chi_p(2^{-r_p}),$$

the sum of the following two series

$$c_n\sum_{\delta d=1}^{\infty}\lambda_f^*(d,\psi,\iota)|d|^{(n+1)/2-s}, \tag{2.1}$$

$$c_n\sum_{\delta d=1}^{\infty}\lambda_f^*(d,\psi,\varepsilon)|d|^{(n+1)/2-s}. \tag{2.2}$$

We note $(d, N_1) = 1$ if $\lambda_f^*(d, \psi, \omega) \neq 0$.

First we calculate the series (2.1). Let $d = \det 2x$ for $x \in L_n^*$. Then there exists K and a positive integer f such that $(-1)^{n/2} d = d_K f^2$. Let $d_{0,p}$ be as in the proof of Th. 2.1. Then by the results in §3, we have

$$\lambda_f^*(\psi, d, \iota) = \prod_{p|N_1} \varphi_p(d)(p^{-2})_{n/2}^{-1}(1 + ((-1)^{n/2}d, p))_p p^{-n/2})$$
$$\times \lambda_2^*(\psi_2^*, 2^{t_2} d_{0,2}, \iota_2) \prod_{(p,2N_1)=1} \lambda_p(\psi_p, p^{t_p} d_{0,p}, \iota_p).$$

If $N_2^o \neq 1$, this vanishes by (2) of Th. 3.1. Hence we assume $N_2^o = 1$. Let $L_n^{*(i)}(K)$ be the subset of $L_n^{*(i)}$ consisting of all the elements x such that $(-1)^{n/2} \det 2x = d_K f^2$ for a positive integer f, and set

$$\zeta_K^{(i)}(s, \psi, \iota) = (\prod_{p|N_1} \varphi_p(2^{-n}))^{-1} 2^{-ns} c_n \sum_{L_n^{*(i)}(K)/SL_n(\mathbf{Z})} \psi(x)\mu(x)|\det x|^{-s}.$$

Then the series (2.1) is the sum of these series over K such that $(d_K, N_1) = 1$ and $(-1)^{n/2}\delta d_K > 0$. Since d_K is fixed, we see as in the case of n odd

$$\begin{aligned}
&\zeta_K^{(i)}(s, \psi, \iota) \\
&\quad = c_n \hat{\psi}((-1)^{n/2} d_K)|d_K|^{(n+1)/2-s} \prod_{p|N_1} (p^{-2})_{n/2-1}^{-1}(1 - \chi_K(p)p^{-n/2})^{-1} \\
&\qquad \times \prod_{(p,N_1)=1,\ p\neq 2,\ p|d_K} Z_{n,o}(\hat{\psi}(p)p^{-s}, \psi_p, \iota_p, d_K/p) \\
&\qquad \times \prod_{(p,N_1 d_K)=1,\ p\neq 2} Z_{n,e}(\hat{\psi}(p)p^{-s}, \psi_p, \iota_p, d_K) \\
&\qquad \times \begin{cases} Z_{n,o}^*(\hat{\psi}(2)2^{-s}, \psi_2^*, \iota_2, d_K/2^3) & \text{if } 8 \mid d_K, \\ Z_{n,e}^*(\hat{\psi}(2)2^{-s}, \psi_2^*, \iota_2, d_K/2^2) & \text{if } 4 \parallel d_K, \\ Z_{n,e}^*(\hat{\psi}(2)2^{-s}, \psi_2^*, \iota_2, d_K) & \text{if } 2 \not| d_K. \end{cases}
\end{aligned}$$

Then we see

$$\zeta_K^{(i)}(s, \psi, \iota) = 2^{-1} c_n \hat{\psi}((-1)^{n/2} d_K)(\prod_{i=1}^{n/2-1} \zeta(2i))|d_K|^{(n+1)/2-s} L(\frac{n}{2}, \chi_K)$$
$$\times B_p^*(s, \psi, \iota, K) \frac{L(2s, \hat{\psi}^2)L(2s - n + 1, \hat{\psi}^2)}{L(2s - n/2 + 1, \hat{\psi}^2 \chi_K)} \prod_{i=1}^{n/2-1} L(2s - 2i, \hat{\psi}^2).$$

From this we obtain our result for ι.

To compute the series associated to ε, first we note d contributing to (2.2) is of the form $\delta N_2^o f^2$ for a positive integer f by the results in §5 of [I-S2] and Th. 3.1 and Th. 3.2. At $p=2$, $d_{0,2}/\delta N_2^o \in Q_2^{\times 2}$. Taking account of the factor $2^{-1}(1+((-1)^{n/2}d_{0,2}-1)_2)$ of $Z_n^*(u,\iota_2,d_{0,2})$(cf. Th. 5.3 of [I-S2]), we see (2.2) vanishes unless $(-1)^{n/2}\delta N_2^o \equiv 1 \bmod 4$. Let H be the quadratic field or $Q\oplus Q$ such that $d_H=(-1)^{n/2}\delta N_2^o$. Then we see under this condition that (2.2) is equal to

$$
\begin{aligned}
&c_n\hat{\psi}((-1)^{n/2}d_H) \\
&\quad\times\Big(\prod_{p|N_1}((p^{-2})^{-1}_{[n/2]-1}(1-\chi_H(p)p^{-n/2})^{-1})\Big)Z_n^*(\hat{\psi}(2)2^{-s},\psi_2^*,\varepsilon_2,d_H) \\
&\quad\times\prod_{(p,2N_1N_2^o)=1} Z_n(\hat{\psi}(p)p^{-s},\psi_p,\varepsilon_p,d_H)\prod_{p|N_2^o} Z_n(\hat{\psi}(p)p^{-s},\psi_p,\varepsilon_p,d_H/p),
\end{aligned}
$$

and hence is equal to

$$
\begin{aligned}
&\frac{|\prod_{i=1}^{n/2-1}B_{2i}|}{2^{n-1}(\frac{n-2}{2})!}(-1)^{n(n+2)/8}\hat{\psi}((-1)^{n/2}d_H)\chi_H(2) \\
&\qquad\times 2(2\pi)^{-n}(n/2-1)!L(\frac{n}{2},\chi_H)B^*(s,\psi,\varepsilon)\prod_{i=1}^{n/2}L(2s-(2i-1),\hat{\psi}^2).
\end{aligned}
$$

This completes the proof. The case of L_n can be treated in the same way and will be omitted.

We give a special case of the above result as a corollary, which is a generalization of Th. 1 of [I-S1].

COROLLARY 2.4. *Let $L=L_n^*$, and $\psi=\chi_p^{(r_p)}$ for an odd prime p and an odd integer r_p. Then one has*

$$
\xi(s,L_n^*,\psi,\delta,\epsilon)=\frac{2^{ns}|\prod_{i=1}^{n/2-1}B_{2i}|}{2^{n-1}(\frac{n-2}{2})!}(-1)^{n(n+2)/8}
$$
$$
=\begin{cases} B^*(s,\psi,\varepsilon)\chi_p(2^{-r_p})\chi_H(2) & \\ \quad\times 2(2\pi)^{-n}(n/2-1)!L(\frac{n}{2},\chi_H)\prod_{i=1}^{n/2}\zeta(2s-(2i-1)) & \\ & \text{if } (-1)^{n/2}\delta p\equiv 1 \bmod 4, \\ 0 & \text{otherwise.}\end{cases}
$$

In the above corollary, $B^*(s,\psi,\varepsilon)$ can be given explicitly by

$$
C_{r_p}(p^{-s})((-1)^{(n-r_p+1)/2},p)_p(p^{r_p-2s},p^2)^{-1}_{[(n-r_p+1)/2]}
$$

in the notation of §3.

Next we discuss the rationality of the values of the L-functions at non-positive integers. For a Dirichlet character φ and a Dirichlet series $A(s) = \sum_{n=1}^{\infty} a_n n^{-s}$, we set

$$(A \mid R_{\varphi})(s) = \sum_{n=1}^{\infty} a_n \varphi(n) n^{-s}.$$

For the trivial character $\chi_{0,p}$ modulo p, we set

$$(A \mid R_{\chi_{0,p}})(s) = \sum_{(n,p)=1} a_n n^{-s},$$

and

$$(A \mid U_2)(s) = \sum_{8 \mid n} a_n n^{-s}.$$

Lastly we set

$$(A \mid I)(s) = A(s).$$

Let $D^*(s,\delta)$ be the Dirichlet series introduced in §1 of [I-S2], which is $D(s,\psi,\delta)$ for the trivial ψ and is associated to Eisenstein series of weight $(n+1)/2$. For abbreviation, we set $D^*(s) = D^*(s,\delta)$. We show our series $D(s,\psi,\delta)$ and $D^*(s,\psi,\delta)$ can be written as a linear combination of Dirichlet series of the form

(2.3)

$$B(s)(A \mid R_{\hat{\psi}} \prod_{p \in S} R_{\chi_p} \prod_{p \in T} R_{\chi_{0.p}})(s), \quad B(s)(A \mid R_{\hat{\psi}} \prod_{p \in S} R_{\chi_p} \prod_{p \in T} R_{\chi_{o.p}} U_2)(s)$$

for $A(s) = D^*(s)$, where $B(s)$ is a rational function in p^{-s} for $p \mid N$. First we note

$$\begin{aligned}
&(D^* \mid R_{\hat{\psi}})(s) \\
&\quad = (-1)^{[n/4]} \sum_{(-1)^{n/2}\delta d_K > 0} 2(2\pi)^{-n}(n/2-1)!|d_K|^{(n-1)/2}\hat{\psi}(d_K)L(\frac{n}{2},\chi_K) \\
&\qquad\qquad \times \frac{L(2s,\hat{\psi}^2)L(2s-n+1,\hat{\psi}^2)}{L(2s-(n/2-1),\hat{\psi}^2\chi_K)}|d_K|^{-s} \\
&\quad = (-1)^{[n/4]} \sum_{(-1)^{n/2}\delta d_K > 0,\ (d_K,N_1)=1} (\eta_K \mid R_{\hat{\psi}})(s),
\end{aligned}$$

where

$$\eta_K(s) = 2(2\pi)^{-n}(n/2-1)!|d_K|^{(n-1)/2} \times L(\frac{n}{2},\chi_K)\frac{L(2s,\hat{\psi}^2)L(2s-n+1,\hat{\psi}^2)}{L(2s-(n/2-1),\hat{\psi}^2\chi_K)}|d_K|^{-s}.$$

Let p be an odd prime with $(p,N_1)=1$. Then we see

$$(D^* \mid R_{\hat{\psi}}R_{\chi_{0,p}})(s) = (-1)^{[n/4]} \sum_{\substack{(d_K,pN_1)=1,\\ (-1)^{n/2}\delta d_K>0}} \frac{(1-\hat{\psi}^2(p)p^{-2s})(1-\hat{\psi}^2(p)p^{n-1-2s})}{(1-\hat{\psi}^2(p)\chi_K(p)p^{n/2-1-2s})}(\eta_K \mid R_{\hat{\psi}})(s).$$

Hence for $\varepsilon=\pm1$ we have

$$(D^* \mid R_{\hat{\psi}}R_{\chi_{0,p}}(\varepsilon R_{\chi_p}+I)/2)(s) = (-1)^{[n/4]}\frac{(1-\hat{\psi}^2(p)p^{-2s})(1-\hat{\psi}^2(p)p^{n-1-2s})}{(1-\hat{\psi}^2(p)\chi_K(p)p^{n/2-1-2s})} \times \sum_{(-1)^{n/2}\delta d_K>0,\ (d_K,N_1p)=1,\ \chi_p(d_K)=\varepsilon} (\eta_K \mid R_{\hat{\psi}})(s),$$

and

$$\sum_{(-1)^{n/2}\delta d_K>0,\ (d_K,N_1p)=1,\ \chi_p(d_K)=\varepsilon} (\eta_K \mid R_{\hat{\psi}})(s) = (-1)^{[n/4]}\frac{(1-\hat{\psi}^2(p)\chi_K(p)p^{n/2-1-2s})}{(1-\hat{\psi}^2(p)p^{-2s})(1-\hat{\psi}^2(p)p^{n-1-2s})} \times (D^* \mid R_{\hat{\psi}}R_{\chi_{0,p}}(\varepsilon R_{\chi_p}+I)/2)(s).$$

Subtracting the above series for $\varepsilon=\pm1$ from $(D^* \mid R_{\hat{\psi}})(s)$, we obtain an expression for

$$\sum_{p|d_K}(\eta_K \mid R_{\hat{\psi}})(s),$$

by the series of type (2.3).

Now let $p=2$ and let χ_2 be the character modulo 8 such that $\chi_2(m)=(-1)^{(m^2-1)/8}$ for an odd integer m. Then by the above procedure we obtain an expression of

$$\sum_{(-1)^{n/2}\delta d_K>0,\ (N_1p,d_K)=1,\ \chi_2(d_K)=\varepsilon} (\eta_K \mid R_{\hat{\psi}})(s)$$

for $\varepsilon = \pm 1$ and

$$\sum_{(-1)^{n/2}\delta d_K>0,\ p|d_K} (\eta_K \mid R_{\hat\psi})(s)$$

as a linear combination of Dirichlet series of the form (2.3). Now we see

$$\sum_{(-1)^{n/2}\delta d_K>0,\ p|d_K} (\eta_K \mid R_{\hat\psi}(I-U_2))(s)$$
$$= \sum_{(-1)^{n/2}\delta d_K>0,\ p^2\|d_K} (1-\hat\psi^2(p)p^{-2s})(1-\hat\psi^2(p)p^{n-1-2s})(\eta_K \mid R_{\hat\psi})(s).$$

This shows that

$$\sum_{(-1)^{n/2}\delta d_K>0,\ p^2\|d_K} (\eta_K \mid R_{\hat\psi})(s), \qquad \sum_{(-1)^{n/2}\delta d_K>0,\ p^3|d_K} (\eta_K \mid R_{\hat\psi})(s)$$

can be written as linear combinations of Dirichlet series of type (2.3). Since the rational functions $B_p(u,\psi_p,\iota_p,K)$ and $B_p^*(u,\psi_p,\iota_p,K)$ depend only on $\chi_K(p)$ for an odd prime p (whether $(p,d_K)=1$, $\chi_K(2)=\pm 1$, $p^2 \parallel d_K$, or $p^3 \mid d_K$ for $p=2$), combining the above results for odd primes and for $p=2$, our assertion can be easily verified. If $A(s)$ is a Dirichlet series associated to a holomorphic modular form of weght $(n+1)/2$, by a result of [S], the series of type (2.3) for $B(s)=1$ are also Dirichlet series associated to holomorphic modular forms of weight $(n+1)/2$.

Therefore in the case of $\delta = 1$, for $A(s) = D^*(s)$ the Dirichlet series of type (2.3) for $B(s)=1$ are holomorphic at non-positive integers, as in the case of zeta functions. By the results in §5 of [I-S2], Th. 3.1 and Th. 3.2, we can check that in the expression of $D(d,\psi,\delta)$, or $D^*(s,\psi,\delta)$ as a linear combination of functions of the form (2.3), the denominators of $B(s)$'s do not vanish at non-positive integers. Hence we obtain

$$\begin{aligned}
&\zeta(1-m,L,\psi,1,\epsilon)\\
&= \epsilon\kappa(n,\delta,\psi)\frac{|\prod_{i=1}^{n/2-1}B_{2i}|}{2^{n-1}(\frac{n-2}{2})!}\hat\psi((-1)^{n/2})2(2\pi)^{-n}(\frac{n}{2}-1)!\,L(\frac{n}{2},\chi_H)\\
&\quad\times\prod_{i=1}^{n/2-1} L(1-2(m+i-1),\hat\psi^2)\\
&\quad\times\begin{cases} 2^{n/2}B(1-m,\psi,\varepsilon) & \text{if } L=L_n,\\ 2^{n(1-m)}\prod_{p|N_1}\varphi_p(2^{-n}) & \\ \qquad\qquad \times\prod_{p|N_2^o}\chi_p(2^{-r_p})B^*(1-m,\psi,\varepsilon) & \text{if } L=L_n^*.\end{cases}
\end{aligned}$$

From this we obtain the following result.

PROPOSITION 2.5. *Let n be an even integer ≥ 4, and let $Q(\psi)$ be as in Cor. 2.2. Assume $\delta = 1$. Then one has*

$$\xi(1-m, L, \psi, 1, \epsilon) \in Q(\psi)$$

for a positive integer m and $L = L_n,\ L_n^*$.

As a special case, we can prove

COROLLARY 2.6. *Let n be as in Prop. 2.5. Let $L = L_n^*$ and assume $N = N_1 = p$ for an odd prime p, or $N = N_2 = p$ for an odd prime p and $r_p = n$. Let φ be a character modulo p such that $\varphi^2 \neq \chi_{0,p}$ in the first case and $\varphi = \chi_p$ in the second case. Let ψ be the character defined by these data as in (1.3). Then one has*

$$\begin{aligned}
&\xi(1-m, L_n^*, \psi, 1, \epsilon) \\
&\quad = 2^{n(1-m)} \frac{|\prod_{i=1}^{n/2-1} B_{2i}|}{2^{n-1}(\frac{n-2}{2})!} \epsilon\varphi(2^{-n})\kappa(n,1,\epsilon)(-1)^{n(n+2)/8} \\
&\qquad\qquad \times \hat{\psi}((-1)^{n/2}) 2(2\pi)^{-n}(\frac{n}{2}-1)!\zeta(\frac{n}{2}) \\
&\qquad\qquad \times \prod_{i=1}^{n/2-1} L(1-2(m+i-1), \varphi^2),
\end{aligned}$$

for a positive integer m. Here we understand $L(s, \varphi^2) = \zeta(s)$ in the second case.

This is a generalization of Th. 2 of [A]. The case of $n = 2$ can be treated by the functional equation in [Sai2].

§3. Orbital local series

In this section, we determine the orbital local series for L-fuctions and comopletes our calculation. Throughout this section, we fix a prime p and sometimes we abbreviate the suffix p, for example, $R_{p,\nu} = R_\nu,\ \varphi^{(n)} = \varphi_p^{(n)}$.

For non-negative integers $m,\ n$, and $d \in Z_p$, and indeterminates $U,\ q$, we set

$$\beta(n,d) = \begin{cases} 0 & \text{if } n \text{ is odd}, \\ ((-1)^{n/2}d, p) & \text{if } n \text{ is even}, \end{cases}$$

$$(U,q)_m = \begin{cases} \prod_{i=1}^{m}(1-q^{(i-1)}U) & \text{if } m \geq 1, \\ 1 & \text{if } m = 0, \end{cases}$$

$$(p^{-2})_m = (p^{-2},p^{-2})_m = \begin{cases} \prod_{i=1}^{m}(1-p^{-2i}) & \text{if } m \geq 1, \\ 1 & \text{if } m = 0. \end{cases}$$

For an integer r, $0 \leq r \leq n$ we define a polynomial $C_r(u)$ in u by

$$C_r(u) = (p^{-2})^{-1}_{[r/2]}(p^{-2})^{-1}_{[(n-r)/2]}p^{r(n-r)/2}u^{n-r}.$$

We recall some formal power series introduced in [I-S2]. For a partition $\{n_i\}$ of length m and a sequence $\{t_i\}$ associated to it, we set

$$Q(\{n_i\},\{t_i\}) = -\sum_{i=1}^{m}\frac{n_i(n_i+1)}{2}t_i - \sum_{j<i}n_in_jt_j,$$
$$\tilde{Q}(\{n_i\},\{t_i\}) = Q(\{n_i\},\{t_i\}) + \frac{n+1}{2}\sum_{i=1}^{m}n_it_i.$$

We define

$$X_n(u,\iota) = \sum_{\{n_i\}}\sum_{1\leq t_1}\Big(\prod_{i=1}^{m}(p^{-2})^{-1}_{[n_i/2]}\Big)p^{\tilde{Q}(\{n_i\},\{t_i\})}u^{\sum_{i=1}^{m}n_it_i},$$
$$Y_n(u,\iota) = \sum_{\{n_i\}\text{even}}\sum_{1\leq t_1}\Big(\prod_{i=1}^{m}(p^{-2})^{-1}_{[n_i/2]}\Big)p^{\tilde{Q}(\{n_i\},\{t_i\})}u^{\sum_{i=1}^{m}n_it_i},$$
$$X_n(u,\varepsilon) = \sum_{\{n_i\}}\sum_{1\leq t_1,\ t_i\equiv 0 \bmod 2 \text{ for } n_i \text{ odd}}\Big(\prod_{i=1}^{m}(p^{-2})^{-1}_{[n_i/2]}\Big)p^{\tilde{Q}(\{n_i\},\{t_i\})}$$
$$\times\Big(\prod_{n_i \text{ even},\ t_i \text{ odd}}p^{-n_i/2}\Big)u^{\sum_{i=1}^{m}n_it_i},$$
$$Y_n(u,\varepsilon) = \sum_{\{n_i\}}\sum_{1\leq t_1,\ t_i\equiv 0 \bmod 2 \text{ for } n_i \text{ odd}}\Big(\prod_{i=1}^{m}(p^{-2})^{-1}_{[n_i/2]}\Big)p^{\tilde{Q}(\{n_i\},\{t_i\})}$$
$$\times\Big(\prod_{n_i \text{ even } t_i \text{ even}}p^{-n_i/2}\Big)u^{\sum_{i=1}^{m}u_it_i}.$$

Here $\{n_i\}$ runs through all partitions of n, even ones in $Y(u,\iota)$ and $\{t_i\}$ runs through all sequences associated to $\{n_i\}$ with $t_1 \geq 1$, satisfying $t_i \equiv 0 \bmod 2$ for n_i odd in the latter two cases.

By Prop. 5.6 and Prop. 5.9 of [I-S2], we have explicitly

$$X_n(u,\iota) = (p^{-2})_{[n/2]}^{-1} u^n (1-p^{(n-1)/2}u)^{-1} \times \begin{cases} (pu^2,p^2)_{[n/2]}^{-1} & \text{if } n \text{ is odd,} \\ (1-p^{-(n+1)/2}u)(u^2,p^2)_{[n/2]}^{-1} & \text{if } n \text{ is even,} \end{cases}$$

$$Y_n(u,\iota) = (p^{-2})_{[n/2]}^{-1} u^n (u^2,p^2)_{n/2}^{-1},$$

$$X_n(u,\varepsilon) = (p^{-2})_{[n/2]}^{-1} \begin{cases} u^{n+1}(u^2,p^2)_{(n+1)/2}^{-1} & \text{if } n \text{ is odd,} \\ p^{-n/2}u^n(pu^2,p^2)_{n/2}^{-1} & \text{if } n \text{ is even,} \end{cases}$$

$$Y_n(u,\varepsilon) = (p^{-2})_{[n/2]}^{-1} \begin{cases} u^{n}(u^2,p^2)_{(n+1)/2}^{-1} & \text{if } n \text{ is odd,} \\ u^n(pu^2,p^2)_{n/2}^{-1} & \text{if } n \text{ is even.} \end{cases}$$

First we treat the case of odd primes.

THEOREM 3.1. *Let p be an odd prime, and let $d_0 \in Z_p^\times$.*
(1) *Let $\psi = \varphi^{(n)}$. Then, one has for $\omega = \iota$ or ε*

$$Z_n(u,\varphi^{(n)},\omega,d_0) = (p^{-2})_{[n/2]}^{-1}\varphi(d_0)(1+\beta(n,d_0)p^{-n/2}).$$

(2) *Let $\psi = \chi_p^{(r)}$.*

(a) *Let $\omega = \iota$. If n is odd, then*

$$Z_n(u,\chi_p^{(r)},\iota,d_0) = C_r(u) \times \begin{cases} ((-1)^{(n-r)/2}d_0,p)p^{-(n-r)/2}(p^ru^2,p^2)_{[(n-r)/2]}^{-1} & \text{if } r \text{ are odd,} \\ ((-1)^{r/2},p)p^{-r/2} \\ \qquad \times (1-p^{(n-1)/2}u)^{-1}(p^{r+1}u^2,p^2)_{[(n-r)/2]}^{-1} & \text{if } r \text{ is even.} \end{cases}$$

Let n be even. If r is odd, then

$$Z_{n,o}(u,\chi_p^{(r)},\iota,d_0) = Z_{n,e}(u,\chi_p^{(r)},\iota,d_0) = 0,$$

and if r is even, then

$$\begin{aligned} &Z_{n,o}(u,\chi_p^{(r)},\iota,d_0) \\ &\quad = C_r(u)((-1)^{r/2},p)(1-p^{n-1}u^2)^{-1}(p^ru^2,p^2)_{(n-r)/2}^{-1} \\ &\qquad\qquad \times (-(1-p^{-r})+(1-p^{-n}))p^{(n+r-1)/2}u, \end{aligned}$$

$$Z_{n,e}(u,\chi_p^{(r)},\iota,d_0) = C_r(u)((-1)^{r/2},p)(1-p^{n-1}u^2)^{-1}(p^ru^2,p^2)^{-1}_{(n-r)/2} \\ \times p^{r/2}\Big(-(1-p^{-r}) \\ +(1+((-1)^{n/2}d_0,p)p^{-n/2})(1-((-1)^{n/2}d_0,p)p^{n/2-1}u^2)\Big).$$

(b) *Let $\omega=\varepsilon$. Then one has*

$$Z_n(u,\chi_p^{(r)},\varepsilon,d_0) = C_r(u) \\ \times\begin{cases} ((-1)^{(n-r)/2}d_0,p)(p^{r+1}u^2,p^2)^{-1}_{[(n-r)/2]} & \text{if } n \text{ is odd and } r \text{ is odd,} \\ ((-1)^{r/2},p)(1+u)(p^ru^2,p^2)^{-1}_{[(n-r+1)/2]} & \text{if } n \text{ is odd and } r \text{ is even,} \\ ((-1)^{(n-r+1)/2},p)(p^ru^2,p^2)^{-1}_{[(n-r+1)/2]} & \text{if } n \text{ is even and } r \text{ is odd,} \\ ((-1)^{r/2},p)(((-1)^{n/2}d_0,p)+p^{-n/2}) & \\ \qquad\qquad \times(p^{r+1}u^2,p^2)^{-1}_{[(n-r)/2]} & \text{if } n \text{ is even and } r \text{ is even.} \end{cases}$$

(3) *Let $\psi=\chi_0^{(r)}$.*

(a) *Let $\omega=\iota$. If n is odd, then*

$$Z_n(u,\chi_0^{(r)},\iota,d_0) = C_r(u)(1-p^{(n-1)/2}u)^{-1} \\ \times\begin{cases} (1-p^{-(n-2r+1)/2}u)(p^ru^2,p^2)^{-1}_{[(n-r)/2]} & \text{if } r \text{ is odd,} \\ (p^{r+1}u^2,p^2)^{-1}_{[(n-r)/2]} & \text{if } r \text{ is even,} \end{cases}$$

If n is even, then

$$Z_{n,o}(u,\chi_0^{(r)},\iota,d_0) = C_r(u)(1-p^{n-1}u^2)^{-1} \\ \times\begin{cases} (p^{r+1}u^2,p^2)^{-1}_{[(n-r)/2]} & \text{if } r \text{ is odd,} \\ (1-p^{-(n-r)})p^{(n-1)/2}u(p^ru^2,p^2)^{-1}_{[(n-r)/2]} & \text{if } r \text{ is even,} \end{cases}$$

$$Z_{n,e}(u,\chi_0^{(r)},\iota,d_0) = C_r(u)(1-p^{n-1}u^2)^{-1} \\ \times\begin{cases} p^{(n-1)/2}u(p^{r+1}u^2,p^2)^{-1}_{[(n-r)/2]} & \text{if } r \text{ is odd,} \\ ((1+((-1)^{n/2}d_0,p)p^{-n/2}) & \\ \qquad\times(1-((-1)^{n/2}d_0,p)p^{n/2-1}u^2) & \\ \qquad+(p^{-1}-p^{r-1})u^2)(p^ru^2,p^2)^{-1}_{[(n-r)/2]} & \text{if } r \text{ is even.} \end{cases}$$

(b) *Let $\omega = \varepsilon$. Then one has*

$$Z_n(u, \chi_0^{(r)}, \varepsilon, d_0) = C_r(u) \times \begin{cases} p^{-(n-r)/2}(p^{r+1}u^2, p^2)^{-1}_{[(n-r)/2]} & \text{if } n \text{ is odd and } r \text{ is odd}, \\ (p^{-r/2} + p^{r/2}u)(p^r u^2, p^2)^{-1}_{[(n-r+1)/2]} & \text{if } n \text{ is odd and } r \text{ is even}, \\ p^{r/2}u(p^r u^2, p^2)^{-1}_{[(n-r+1)/2]} & \text{if } n \text{ is even and } r \text{ is odd}, \\ (p^{-(n-r)/2} + ((-1)^{n/2}d_0, p)p^{-r/2}) & \\ \qquad \times (p^{r+1}u^2, p^2)^{-1}_{[(n-r)/2]} & \text{if } n \text{ is even and } r \text{ is even}. \end{cases}$$

Proof. These formulas can be proved in the same way as in the case of zeta functions, and we give a proof only for formulas in (2). Let $d = d_0 p^t$ with $d_0 \in Z_p^\times$. If $\chi_p^{(r)}(x) \neq 0$ for $x \in S_n(R_\nu, d_0 p^t, \{n_i\}, \{t_i\})$, then $n_1 = r$ and $t_1 = 0$. In the following we assume this condition. Let $x = (\oplus p^{t_i} x_i)$. Then $\chi_p^{(r)}(x) = \chi_p(\det x_1) = (\det x_1, p)$.

Let $\omega = \iota$. Then in the same way as in §3 of the part I, we see

$$\lambda(\chi_p^{(r)}, d, \iota, \{n_i\}, \{t_i\}) = p^{Q(\{n_i\},\{t_i\})}\Big(\prod_{i=1}^m (p^{-2})^{-1}_{[n_i/2]}\Big)\Lambda(\chi_p^{(r)}, \iota, \{n_i\}),$$

where

$$\Lambda(\chi_p^{(r)}, \iota, \{n_i\}) = (p-1)^{-(m-1)} \sum_{d_1 d_2 \cdots d_m \equiv d_0 \bmod p} (d_1, p) \prod_{i=1}^m (1 + \beta(n_i, d_i)p^{-n_i/2}).$$

The summation is extended over $d_1, d_2, \cdots, d_m \in R_1$ such that $d_1 d_2 \cdots d_m \equiv d_0 \bmod p$. For i, $1 \le i \le m-1$, let $n_i' = n_{i+1}$. Then $\{n_i'\}$ gives a partition of $n - r$.

If n is odd, then we see easily

$$\Lambda(\chi_p^{(r)}, \iota, \{n_i\}) = \begin{cases} 0 & \text{if } r \text{ is odd, and } \{n_i'\} \text{ is odd}, \\ ((-1)^{(n-r)/2}d_0, p)p^{-(n-r)/2} & \text{if } r \text{ is odd, and } \{n_i'\} \text{ is even}, \\ ((-1)^{r/2}, p)p^{-r/2} & \text{if } r \text{ is even}. \end{cases}$$

Let n be even. If r is odd, $\{n_i'\}$ is odd and as above $\Lambda(\chi_p^{(r)}, \iota, \{n_i\}) = 0$. If r is even, then

$$\Lambda(\chi_p^{(r)}, \iota, \{n_i\}) = \begin{cases} ((-1)^{r/2}, p)p^{-r/2} & \text{if } \{n_i'\} \text{ is odd}, \\ ((-1)^{r/2}, p)p^{-r/2} & \\ \qquad + ((-1)^{(n-r)/2}d_0, p)p^{-(n-r)/2} & \text{if } \{n_i'\} \text{ is even}. \end{cases}$$

By the same calculation as in the proof of Lemma 5.7 of [I-S2], we see

$$Z_n(u, \chi_p^{(r)}, \iota, d_0)$$
$$= (p^{-2})_{[r/2]}^{-1} \begin{cases} ((-1)^{(n-r)/2} d_0, p) p^{-(n-r)/2} Y_{n-r}(p^{r/2} u, \iota) & \text{if } r \text{ is odd}, \\ ((-1)^{r/2}, p) p^{-r/2} X_{n-r}(p^{r/2} u, \iota) & \text{if } r \text{ is even}, \end{cases}$$

if n is odd and

$$Z_n(u, \chi_p^{(r)}, \iota, d_0) = (p^{-2})_{[r/2]}^{-1} \Big(((-1)^{r/2}, p) p^{-r/2} X_{n-r}(p^{r/2} u, \iota) + ((-1)^{(n-r)/2} d_0, p)\, p^{-(n-r)/2} Y_{n-r}(p^{r/2} u, \iota) \Big)$$

if n and r are even. This proves (a).

Next assume $\omega = \varepsilon$. In this case, as above we have

$$\lambda(\chi_p^{(r)}, d_0 p^t, \varepsilon, \{n_i\}, \{t_i\}) = p^{Q(\{n_i\},\{t_i\})} \Big(\prod_{i=1}^{m} (p^{-2})_{[n_i/2]}^{-1} \times \varepsilon(d_0, \{n_i\}, \{t_i\})\, \Lambda(\chi_p, \varepsilon, \{n_i\}, \{t_i\}) \Big),$$

where

$$\Lambda(\chi_p^{(r)}, \varepsilon, \{n_i\}, \{t_i\}) = (p-1)^{-(m-1)} \sum_{d_1 d_2 \cdots d_m \equiv d_0 \bmod p} (d_1, p) \prod_{i=1}^{m} (1 + \beta(n_i, d_i)\, p^{-n_i/2}) (d_i, p)^{t_i},$$

and

$$\varepsilon(d_0, \{n_i\}, \{t_i\}) = (p^t, d_0) \prod_{i<j} (p^{n_i}, p^{n_j})^{t_i t_j} \prod_{i=1}^{m} ((-1)^{n_i(n_i+1)/2}, p)^{t_i}.$$

Let n be odd. If r is odd, by substituting

$$d_1 = d_0\, d_2^{-1} \cdots d_m^{-1}$$

we see easily $\Lambda(\chi_p^{(r)}, \varepsilon, \{n_i\}, \{t_i\})$ vanishes unless $t_i \equiv 1 \bmod 2$ for n_i odd, $i > 1$. If this condition is satisfied, then

$$\Lambda(\chi_p^{(r)}, \varepsilon, \{n_i\}, \{t_i\}) = (d_0, p) \prod_{n_i \text{ even},\, t_i \text{ even}} ((-1)^{n_i/2}, p)\, p^{-n_i/2}.$$

We see $t \equiv n - r \equiv 0 \bmod 2$ and

$$\begin{aligned}\prod_{i<j}(p^{n_i},p^{n_j})^{t_it_j}\prod_{i=1}^{m}((-1)^{n_i(n_i+1)/2},p)^{t_i}\prod_{n_i\text{ even},\,t_i\text{ even}}((-1)^{n_i/2},p)\\ &= ((-1)^{r(r+1)/2},p)\prod_{1<j}(p^r,p^{n_j})^{t_j}\prod_{i<j}(p^{n_i},p^{n_j})\prod_{i=1}^{m}((-1)^{n_i(n_i+1)/2},p)\\ &= ((-1)^{r(r+1)/2},p)((-1)^{n(n+1)/2},p)\\ &= ((-1)^{(n-r)/2},p).\end{aligned}$$

Hence we have

$$\varepsilon(d_0,\{n_i\},\{t_i\})\Lambda(\chi_p^{(r)},\varepsilon,\{n_i\},\{t_i\}) = ((-1)^{(n-r)/2}d_0,p)\prod_{n_i\text{ even},\,t_i\text{ even}}p^{-n_i/2}.$$

Since $t \equiv 0 \bmod 2$ in this case, in the same way as in Lemma 5.8 of the part I by the above formula we see

$$Z_n(u,\chi_p^{(r)},\varepsilon,d_0) = ((-1)^{(n-r)/2}d_0,p)(p^{-2})^{-1}_{[r/2]}Y_{n-r}(p^{r/2}u,\varepsilon).$$

If r is even, we see easily $\Lambda(\chi_p^{(r)},\varepsilon,\{n_i\},\{t_i\})$ vanishes unless there exists t_0 such that $t_i \equiv t_0 \bmod 2$ for n_i odd. Assume this condition. Then we have

$$\begin{aligned}\Lambda(\chi_p^{(r)},\varepsilon,\{n_i\},\{t_i\}) &= (d_0,p)^{t_0}\prod_{n_i\text{ even},\ t_i\not\equiv t_0 \bmod 2,\ 2\le i}((-1)^{n_i/2},p)p^{-n_i/2}\\ &\quad\times\begin{cases}((-1)^{r/2},p)p^{-r/2} & \text{if } t_1(=0)\equiv t_0 \bmod 2,\\ 1 & \text{if } t_1(=0)\not\equiv t_0 \bmod 2.\end{cases}\end{aligned}$$

Hence by a similar calculation, we obtain

$$\begin{aligned}\varepsilon(d_0,\{n_i\},\{t_i\})\,\Lambda(\chi_p,\varepsilon,\{n_i\},\{t_i\}) &= ((-1)^{(n+1)/2},p)^{t_0}((-1)^{r/2},p)\\ &\quad\times\prod_{n_i\text{ even},\ t_i\not\equiv t_0 \bmod 2,\ 2\le i}p^{-n_i/2}\times\begin{cases}p^{-r/2} & \text{if } t_0\equiv 0 \bmod 2,\\ 1 & \text{if } t_0\not\equiv 0 \bmod 2.\end{cases}\end{aligned}$$

Since $t \equiv t_0 \bmod 2$, from this we see

$$\begin{aligned}Z_n(u,&\chi_p^{(r)},\varepsilon,d_0)\\ &= (p^{-2})^{-1}_{[r/2]}((-1)^{r/2},p)(p^{-r/2}X_{n-r}(p^{r/2}u,\varepsilon)+Y_{n-r}(p^{r/2}u,\varepsilon)).\end{aligned}$$

Let n be even. If r is odd, then $\Lambda(\chi_p^{(r)}, \varepsilon, \{n_i\}, \{t_i\})$ vanishes unless $t_i \equiv 1 \bmod 2$ for n_i odd, $i > 1$. If this condition is satisfied, then

$$\Lambda(\chi_p^{(r)}, \varepsilon, \{n_i\}, \{t_i\}) = (d_0, p) \prod_{n_i \text{ even}, t_i \text{ even}} ((-1)^{n_i/2}, p)p^{-n_i/2}.$$

In this case we see $t \equiv n - r \equiv 1 \bmod 2$ and

$$\begin{aligned}\prod_{i<j}(p^{n_i}, p^{n_j})^{t_i t_j} \prod_{i=1}^{m}((-1)^{n_i(n_i+1)/2}, p)^{t_i} \prod_{n_i \text{ even}, t_i \text{ even}} ((-1)^{n_i/2}, p)\\ = ((-1)^{r(r+1)/2}, p) \prod_{1<j}(p^r, p^{n_j})^{t_j} \prod_{i<j}(p^{n_i}, p^{n_j}) \prod_{i=1}^{m}((-1)^{n_i(n_i+1)/2}, p)\\ = ((-1)^{(n-r+1)/2}, p).\end{aligned}$$

Hence we have

$$Z_n(u, \chi_p^{(r)}, \varepsilon, d_0) = ((-1)^{(n-r+1)/2}, p)(p^{-2})_{r/2}^{-1} Y_{n-r}(p^{r/2}u, \varepsilon)$$

If r is even, then we see easily $\Lambda(\chi_p^{(r)}, \varepsilon, \{n_i\}, \{t_i\})$ vanishes unless there exists t_0 such that $t_i \equiv t_0 \bmod 2$ for n_i odd. If this is satisfied, in the same way as above we have

$$\begin{aligned}\varepsilon(d_0, \{n_i\}, \{t_i\})\Lambda(\chi_p^{(r)}, \varepsilon, \{n_i\}, \{t_i\})\\ = ((-1)^{n/2}d_0, p)^{t_0}((-1)^{r/2}, p) \prod_{n_i \text{ even}, \; t_i \not\equiv t_0 \bmod 2, \; 2 \le i} p^{-n_i/2}\\ \times \begin{cases} p^{-r/2} & \text{if } t_0 \equiv 0 \bmod 2, \\ 1 & \text{if } t_0 \not\equiv 0 \bmod 2, \end{cases}\end{aligned}$$

for $\{n_i\}$ odd, and

$$\begin{aligned}\varepsilon(d_0, \{n_i\}, \{t_i\})\,\Lambda(\chi_p^{(r)}, \varepsilon, \{n_i\}, \{t_i\})\\ = ((-1)^{(n-r)/2}d_0, p) \prod_{n_i \text{ even}, \; t_i \not\equiv 1 \bmod 2, \; 2 \le i} p^{-n_i/2}\\ + ((-1)^{r/2}, p)p^{-r/2} \prod_{n_i \text{ even}, \; t_i \not\equiv 0 \bmod 2, \; 2 \le i} p^{-n_i/2},\end{aligned}$$

for $\{n_i\}$ even. This shows

$$\begin{aligned}Z_n(u, \chi_p^{(r)}, \varepsilon, d_0) = (p^{-2})_{[r/2]}^{-1}((-1)^{r/2}, p)\\ \times \Big(((-1)^{n/2}d_0, p)Y_{n-r}(p^{r/2}u, \varepsilon) + p^{-r/2}X_{n-r}(p^{r/2}u, \varepsilon)\Big).\end{aligned}$$

This completes the proof.

For $p=2$, in the same way as above, we can prove

THEOREM 3.2. *Let $p=2$, and let $d_0 \in Z_p^{\times}$.*
(1) *Let $\psi=\varphi^{(n)}$ for a non-trivial character φ. Then one has*

$$Z_n(u,\varphi^{(n)},\iota,d_0)=2^{-1}(p^{-2})_{[n/2]}^{-1}\varphi(d_0)$$
$$\times\begin{cases}1 & \{\text{if } n \text{ is odd,}\\ 1+((-1)^{n/2}d_0,-1)p^{-n} & \\ \quad +(((-1)^{n/2}d_0,p)+((-1)^{n/2}d_0,-p))p^{-n-n/2} & \text{if } n \text{ is even,}\end{cases}$$

and

$$Z_n(u,\varphi^{(n)},\varepsilon,d_0)=p^{-[(n+1)/2]}\varphi(d_0)(p^{-2})_{[n/2]}^{-1}$$
$$\times\begin{cases}(-1)^{(n^2-1)/8}(d_0,(-1)^{(n+1)/2}) & \text{if } n \text{ is odd,}\\ (-1)^{n(n+2)/8}2^{-1}(1+((-1)^{n/2}d_0,-1)) & \\ \quad \times(1+((-1)^{n/2}d_0,-1)p^{-n/2}) & \text{if } n \text{ is even.}\end{cases}$$

(2) *Let $\psi=\tilde{\chi}_p^{*(r)}$ with r even.*
(a) *Let $\omega=\iota$. Then one has*

$$Z_n^*(u,\tilde{\chi}_p^{*(r)},\iota,d_0)=2^{-1}p^{-3r/2}C_r(u)(1-p^{(n-1)/2}u)^{-1}(p^{r+1}u^2,p^2)_{[(n-r)/2]}^{-1}$$

if n is odd, and if n is even

$$Z_{n,o}^*(u,\tilde{\chi}_p^{*(r)},\iota,d_0)=2^{-1}C_r(u)p^{-r/2}(-(1-p^{-r})$$
$$+(1-p^{-n}))p^{(n-1)/2}u\times(1-p^{n-1}u^2)^{-1}(p^ru^2,p^2)_{(n-r)/2}^{-1},$$

$$Z_{n,e}^*(u,\tilde{\chi}_p^{*(r)},\iota,d_0)=2^{-1}C_r(u)(1-p^{n-1}u^2)^{-1}p^{-r/2}$$
$$\times\Big(p^{-r}(1-p^{r-1}u^2)+(1-p^{n-1}u^2)(p^{-n}((-1)^{n/2}d_0,-1)$$
$$+p^{-(3n/2-r)}(((-1)^{n/2}d_0,2)+((-1)^{n/2}d_0,-2))\Big)$$
$$\times(p^ru^2,p^2)_{(n-r)/2}^{-1}.$$

(b) *Let $\omega=\varepsilon$.*
If n is odd, then

$$Z_n^*(u,\tilde{\chi}_p^{*(r)},\varepsilon,d_0)=p^{-(n+r+1)/2}(-1)^{(n^2-1)/8}((-1)^{(n+1)/2},d_0)C_r(u)$$
$$\times(p^{-r/2}+p^{r/2}u)(p^ru^2,u^2)_{[(n-r)/2]}^{-1}$$

If n is even, then

$$Z_n^*(u, \tilde{\chi}_p^{*(r)}, \varepsilon, d_0) = p^{-(n+r)/2} 2^{-1}(1 + ((-1)^{n/2} d_0, -1))(-1)^{n(n+2)/8} C_r(u) \times (p^{-(n-r)/2} + ((-1)^{n/2} d_0, p) p^{-r/2})(p^{r+1}u^2, p^2)^{-1}_{[(n-r)/2]}$$

(3) *Let $\psi = \chi_0^{(r)}$.*

(a) *Let $\omega = \iota$.*

If n is odd, then

$$Z_n(u, \chi_0^{(r)}, \iota, d_0) = 2^{-1} C_r(u)(1 - p^{(n-1)/2}u)^{-1} \times \begin{cases} (1 - p^{-(n-2r+1)/2}u)(p^r u^2, p^2)_{[(n-r)/2]} & \text{if } r \text{ is odd,} \\ (p^{r+1}u^2, p^2)_{[(n-r)/2]} & \text{if } r \text{ is even.} \end{cases}$$

If n is even, then

$$Z_{n,o}(u, \chi_0^{(r)}, \iota, d_0) = 2^{-1} C_r(u)(1 - p^{n-1}u^2)^{-1} \times \begin{cases} (p^{r+1}u^2, p^2)^{-1}_{[(n-r)/2]} & \text{if } r \text{ is odd,} \\ (1 - p^{-(n-r)}) p^{(n-1)/2} u (p^r u^2, p^2)^{-1}_{[(n-r)/2]} & \text{if } r \text{ is even,} \end{cases}$$

and

$$Z_{n,e}(u, \chi_0^{(r)}, \iota, d_0) = 2^{-1} C_r(u)(1 - p^{n-1}u^2)^{-1}$$
$$\begin{cases} p^{(n-1)/2} u (p^{r+1}u^2, p^2)^{-1}_{[(n-r)/2]} & \text{if } r \text{ is odd,} \\ \Big(1 - p^{r-1}u^2 + (1 - p^{n-1}u^2)(((-1)^{n/2} d_0, -1)p^{-n} \\ \quad + (((-1)^{n/2} d_0, 2) + ((-1)^{n/2} d_0, -2)) p^{-n-n/2})\Big) \\ \qquad \times (p^r u^2, p^2)^{-1}_{[(n-r)/2]} & \text{if } r \text{ is even.} \end{cases}$$

(b) *Let $\omega = \varepsilon$.*

If n is odd, then

$$Z_n(u, \chi_0^{(r)}, \varepsilon, d_0) = p^{-(n+1)/2}(-1)^{(n^2-1)/8}(d_0, (-1)^{(n+1)/2}) C_r(u) \times \begin{cases} p^{-(n-r)/2}(p^{r+1}u^2, p^2)^{-1}_{[(n-r)/2]} & \text{if } r \text{ is odd,} \\ (p^{-r/2} + p^{r/2}u)(p^r u^2, p^2)^{-1}_{[(n-r+1)/2]} & \text{if } r \text{ is even.} \end{cases}$$

If n is even, then

$$Z_n(u, \chi_0^{(r)}, \varepsilon, d_0) = (-1)^{n(n+2)/8} p^{-n/2} 2^{-1}(1 + ((-1)^{n/2} d_0, -1)) C_r(u)$$
$$\begin{cases} p^{r/2} u (p^r u^2, p^2)^{-1}_{[(n-r+1)/2]} & \text{if } r \text{ is odd,} \\ (p^{-(n-r)/2} + ((-1)^{n/2} d_0, p) p^{-r/2})(p^{r+1}u^2, p^2)^{-1}_{[(n-r)/2]} & \text{if } r \text{ is even.} \end{cases}$$

Proof. We give a proof for (2), since the other case can be treated in the same way, and give a proof for the case $r \geq 1$. The case $r = 0$ can be treated in a similar way. Let $x \in S_n^*(R_\nu, d, \{n_i\}, \{t_i\})$ and $2y \equiv x \bmod p^\nu$ for $y \in L_n^*$. Let x be equivalent to $(\oplus p^{t_i} x_i)$, with $x_i \in S_{n_i}(R_\nu)$. We see Q_y is equivalent to (1.1) or (1.2) if and only if $n_1 = r$, $t_1 = 0$ and $((-1)^{r/2} \det x_1, 2) = ((-1)^{r/2} d_1, 2) = 1$ or -1 respectively, and $\tilde{\chi}_p^{*(r)}(x) = ((-1)^{r/2} d_1, 2)$. Hence we have as in the proof of Prop. 3.6 of the part I

$$\lambda^*(\tilde{\chi}_p^{*(r)}, d_0, \{n_i\}, \{t_i\}) = p^{Q(\{n_i\},\{t_i\})}(\prod_{i=1}^{m}(p^{-2})^{-1}_{[n_i/2]})\Lambda^*(\tilde{\chi}_p^{*(r)}, \iota, \{n_i\}, \{t_i\}),$$

where

$$\begin{aligned}\Lambda^*(\tilde{\chi}_p^{*(r)}, \iota, \{n_i\}, \{t_i\}) &= 2^{-1}p^{-2(m-1)} \\ \times \sum_{d_1 d_2 \cdots d_m \equiv d_0 \bmod p^3} & ((-1)^{r/2}d_1, 2)(((-1)^{r/2}d_1, -1)+1)(1+((-1)^{r/2}d_1, 2)p^{-r/2})p^{-r} \\ \times \prod_{n_i \text{ even},\ 2 \leq i} & (1 + ((-1)^{n_i/2}d_i, -1)p^{-n_i} + ((-1)^{n_i/2}d_i, 2)p^{-n_i - n_i/2} \\ & + ((-1)^{n_i/2}d_i, -2)p^{-n_i-n_i/2}),\end{aligned}$$

Here the summation is extended over all $d_i \in R_3$ such that

$$d_1 d_2 \cdots d_m \equiv d_0 \bmod 2^3.$$

If n is odd, then we see

$$\Lambda^*(\tilde{\chi}_p^{*(r)}, \iota, \{n_i\}, \{t_i\}) = 2^{-1}p^{-3r/2}$$

This shows that

$$Z_n^*(u, \tilde{\chi}_p^{*(r)}, \iota, d_0) = 2^{-1}p^{-3r/2}(p^{-2})^{-1}_{r/2} X_{n-r}(p^{r/2}u, \iota).$$

If n is even, then we see

$$\begin{aligned}&\Lambda^*(\tilde{\chi}_p^{*(r)}, \iota, \{n_i\}, \{t_i\}) \\ &= \frac{1}{2}\begin{cases} p^{-3r/2} & \text{if } \{n_i\} \text{ is odd,} \\ (p^{-3r/2} + p^{-3r/2}((-1)^{n/2}d_0, -1)p^{-(n-r)} & \\ \quad + p^{-r}(((-1)^{n/2}d_0, 2) & \\ \quad + ((-1)^{n/2}d_0, -2))p^{-3(n-r)/2}) & \text{if } \{n_i\} \text{ is even.}\end{cases}\end{aligned}$$

This shows that

$$\begin{aligned}
&Z_n^*(u, \tilde{\chi}_p^{*(r)}, \iota, d_0) \\
&\quad = 2^{-1}(p^{-2})_{r/2}^{-1}\Big(p^{-3r/2}X_{n-r}(p^{r/2}u, \iota) + (((-1)^{n/2}d_0, -1)p^{-n-r/2} \\
&\qquad + (((-1)^{n/2}d_0, 2) + ((-1)^{n/2}d_0, -2))p^{-(3n/2-r/2)})Y_{n-r}(p^{r/2}u, \iota)\Big).
\end{aligned}$$

Let $\omega = \varepsilon$. Then by means of the remark after Prop.3.9 of the part I we have

$$\begin{aligned}
&\lambda^*(\tilde{\chi}_p^{*(r)}, d, \varepsilon, \{n_i\}, \{t_i\}) \\
&\qquad = p^{Q(\{n_i\},\{t_i\})}\Big(\prod_{i=1}^{m}(p^{-2})_{[n_i/2]}^{-1}\Big)\Lambda^*(\chi_p^{*(r)}, \varepsilon, \{n_i\}, \{t_i\}),
\end{aligned}$$

where

$$\begin{aligned}
&\Lambda^*(\tilde{\chi}_p^{*(r)}, \varepsilon, \{n_i\}, \{t_i\}) \\
&\quad = 2^{-1}p^{-2(m-1)} \sum_{d_1 d_2 \cdots d_m \equiv d_0 \bmod p^3} (p^t, d_0) \prod_{i=1}^{m}((-1)^{n_i(n_i+1)/2}d_i, p)^{t_i} \\
&\qquad \times \prod_{i<j}(d_i, d_j) \times \prod_{i<j}(p^{n_i}, p^{n_j})^{t_i t_j} \\
&\qquad \times \Big(p^{-r/2}(1 + ((-1)^{r/2}d_1, p)p^{-r/2})\varepsilon_r(d_1)\Big) \times \prod_{2\le i}\varepsilon'_{n_i}(d_i).
\end{aligned}$$

Here $\varepsilon_{n_i}(d_i)$ is as in the proof of Lemma 3.8 of the part I, and

$$\varepsilon'_{n_i}(d_i) = (1 + ((-1)^{n_i/2}d_i, p)\, p^{-n_i/2})\, \varepsilon_{n_i}(d_i).$$

If $\{n_i\}$ is odd, $\Lambda^*(\tilde{\chi}_p^{*(r)}, \varepsilon, \{n_i\}, \{t_i\})$ vanishes unless there exists t_0 such that $t_i \equiv t_0 \bmod 2$ for n_i odd. We note this implies $t \equiv t_0 \bmod 2$ if n is odd. Under this condition, by the same calculation as in the proof of Prop. 3.8 of the part I we obtain

$$\begin{aligned}
&\Lambda^*(\chi_p^{*(r)}, \varepsilon, \{n_i\}, \{t_i\}) \\
&\quad = p^{-[(n+1)/2]}((-1)^{[(n+1)/2]}d_0^{n+1}, p)^{t_0}p^{-r/2} \prod_{n_i \text{ even},\ t_i \not\equiv t_0 \bmod 2} p^{-n_i/2} \\
&\qquad \times \begin{cases} (-1)^{(n^2-1)/8}(d_0, (-1)^{(n+1)/2}) & \text{if n is odd,} \\ (-1)^{n(n+2)/8}2^{-1}(1 + ((-1)^{n/2}d_0, -1)) & \text{if n is even.} \end{cases}
\end{aligned}$$

If $\{n_i\}$ is even, then

$$\Lambda^*(\tilde{\chi}_p^{*(r)}, \varepsilon, \{n_i\}, \{t_i\}) = p^{-(n+r)/2} 2^{-1}(1 + ((-1)^{n/2} d_0, -1))(-1)^{n(n+2)/8} \times (((-1)^{n/2} d_0, p) \prod_{t_i \text{ even}} p^{-n_i/2} + \prod_{t_i \text{ odd}} p^{-n_i/2}).$$

This shows that $Z_n^*(u, \tilde{\chi}_p^{*(r)}, \varepsilon, d_0)$ is equal to

$$= p^{-(n+r+1)/2}(-1)^{(n^2-1)/8}(d_0, (-1)^{(n+1)/2})(p^{-2})^{-1}_{[r/2]} \times (X_{n-r}(p^{r/2}u, \varepsilon) + p^{-r/2} Y_{n-r}(p^{r/2}u, \varepsilon)),$$

if n is odd, and is equal to

$$2^{-1}(1 + ((-1)^{n/2} d_0, -1))p^{-(n+r)/2}(-1)^{n(n+2)/8}(p^{-2})^{-1}_{r/2} \times (((-1)^{n/2} d_0, p)p^{-r/2} Y_{n-r}(p^{r/2}u, \varepsilon) + X_{n-r}(p^{r/2}u, \varepsilon)),$$

if n is even. This completes the proof.

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