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DEGENERATION OF ALGEBRAIC MANIFOLDS AND THE SPECTRUM OF LAPLACIAN

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Abstract. We shall show that the spectrum of Laplacian depends continuously on the parameter for one parameter degeneration of projective algebraic manifolds.

§0. Introduction

In [Gr1], M. Gromov studies the spectral geometry of semi-algebraic sets in the Euclidean space. He treats a family of algebraic manifolds $\{X_t\}$ and studies the continuity of several geometric and analytic quantities. When no degeneration of manifolds happens, most of such quantities depend continuously on the parameter. He states that the spectrum of Laplacian is continuous in the parameter unless serious degeneration happens. As for such degenerations, he mentions the case that the singular fiber has multiple components and that the dimension of the singular fiber is different from that of the general fibers (cf. [Gr1, 4C]).

The purpose of this article is to study the behavior of the spectrum of Laplacian in the case of a degenerating family of projective algebraic manifolds in a fixed complex projective space. We shall show that in the case of a one parameter families, the spectrum is continuous. Therefore discontinuity can happen only for families with many parameters. (Note that the dimension of each fiber is constant in the case of a one parameter families.)

Let $\pi : \mathfrak{X} \to \Delta(1)$ be a one parameter degenerating family of projective algebraic manifolds in $\mathbb{P}^N(\mathbb{C})$ over the unit disc. By this we mean that $\mathfrak{X}$ is a complex submanifold of $\mathbb{P}^N(\mathbb{C}) \times \Delta(1)$ and that $\pi := \mathrm{proj}_2|_{\mathfrak{X}}$ is a proper holomorphic surjection to $\Delta(1) := \{z \in \mathbb{C}; |z| < 1\}$. Then $X_t := \pi^{-1}(t)$ is a pure dimensional projective algebraic variety. Since the discriminant locus of π is a discrete subset in $\Delta(1)$, we may assume that X_t is a smooth

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manifold for $t \neq 0$ and that X_0 is of the form $X_0 = \sum_{\alpha=1}^{a} m_\alpha Y_\alpha$ in the sense of divisor where $m_\alpha \in \mathbb{Z}_+$ and Y_α is an irreducible variety.

Let G be an arbitrary Riemannian metric of $\mathfrak{X}$. Set $g_t := G|_{X_t}$ for $t \neq 0$ and $g_0 := G|_{X_0 - \mathrm{Sing}(X_0)}$. Let Δ_t be the Laplacian of (X_t, g_t), and $\sigma(\Delta_t)$ the spectrum of Δ_t. When $t = 0$, consider the Dirichlet Laplacian of $X_{0,\mathrm{reg}} := \sum_\alpha m_\alpha Y_{\alpha,\mathrm{reg}}$; i.e., $\Delta_0 := \oplus_\alpha m_\alpha \Delta_\alpha$ where Δ_α is the Laplacian whose domain is given by $W_0^{1,2}(Y_{\alpha,\mathrm{reg}}, g_0|_{Y_\alpha})$ where $Y_{\alpha,\mathrm{reg}} := Y_\alpha - \mathrm{Sing}(Y_\alpha)$, and $m_\alpha \Delta_\alpha$ is the m_α-th copy of Δ_α. By the definition, $\sigma(\Delta_0) = \bigcup_{\alpha=1}^{a} m_\alpha \sigma(\Delta_\alpha)$. It is known that $\sigma(\Delta_0)$ consists of discrete eigenvalues (cf. [L-T, §5]). Naively, it seems that $\sigma(\Delta_t)$ converges to $\sigma(\Delta_0)$, since X_t converges to X_0 in the sense of current on $\mathbb{P}^N(\mathbb{C})$ (cf. [F, Proposition 2.3]). But this does not hold in general, unless X_0 is a reduced divisor; i.e., $m_\alpha = 1$ for all α. We shall show that $\sigma(\Delta_t)$ converges to the spectrum of a certain branched covering space of X_0. This covering space is described as follows (cf. [Cl]).

Let $(\Delta(1), s)$ be the unit disc in the s-plane, i.e., the complex plane whose coordinate is given by s, and let F be a holomorphic function defined by $F(s) := s^m$ where $m := \prod_\alpha m_\alpha$. By this maps, we obtain the fiber product

$$F^{-1}\mathfrak{X} := \{(x, s) \in \mathfrak{X} \times \Delta(1); \pi(x) = s^m\} \subset \mathfrak{X} \times (\Delta(1), s).$$

Let $\Pi : F^{-1}\mathfrak{X} \to (\Delta(1), s)$ be the natural projection induced by that of $\mathfrak{X}$. Then its fiber is given by $\Pi^{-1}(t) = X_{t^m}$. In particular, $\Pi^{-1}(0) = X_0$.

Let $\iota : \widehat{F^{-1}\mathfrak{X}} \to F^{-1}\mathfrak{X}$ be the normalization of $F^{-1}\mathfrak{X}$. Finally, set $Z := \iota^{-1}(X_0) = \iota^{-1}(\Pi^{-1}(0))$ and $g_Z := \iota^* g_0$.

MAIN THEOREM. *Let Δ_Z be the Dirichlet Laplacian of (Z_{reg}, g_Z), and $\sigma(\Delta_Z)$ its spectrum. Then,*

$$\lim_{t \to 0} \sigma(\Delta_t) = \sigma(\Delta_Z).$$

In particular, the k-th eigenvalue of the Laplacian is a continuous function on $\Delta(1)$.

In [Fk], continuity of the spectrum is studied relative to the measured Hausdorff topology, and in [K-K1, 2], relative to the spectral distance of Riemannian manifolds. In [J-W], the same problem is studied for a degenerating family of surfaces with respect to several metrics, and in [Y1, 2], for a conic degenerating family of Riemannian manifolds and degenerating family of algebraic curves. Similar to these articles, the proof of Main Theorem is based on the min-max principle developed in [C], [C-F] and [J-W]. To apply these arguments to our situation, uniformity of the Sobolev constant for algebraic varieties in a fixed projective space is crucial, which is due to Li-Tian [L-T]. In the same article, it is also proved that $d_{\max} = d_{\min}$ on the space of functions for singular algebraic varieties ([L-T, Theorem 4.1]). In view of their argument, it is equivalent to the existence of a sequence of cut-off functions approximating the constant function in the $W^{1,2}$-norm. It seems that their construction of such functions contains a gap, because it is not clear whether various formulae and estimates in [G] hold for singular subvarieties in a singular variety. In this article, we shall give a rigorous proof of their theorem.

In view of Theorem 5.1, we can prove that the spectrum of Laplacian is a continuous function of a certain family of Einstein manifolds (cf. [B-K-N], [N]). We conjecture that the same is true for the case treated in [A], and for the case of a degenerating family of minimal submanifolds of S^N, since the Sobolev inequality is uniform in the parameter in both cases (cf. [C-L-Y]).

This article is arranged as follows. In §1, we give a detailed definition of Z. In §2, we recall the results on the Sobolev inequality and the upper bound of the heat kernel. In §3, we discuss the existence of a certain sequence of cut-off functions on arbitrary irreducible algebraic varieties. In §4, we recall the result of Li and Tian (cf. [L-T]) which is the main tool of this paper. In §5, we prove an abstract version of Main Theorem (Theorem 5.1). Here, the Sobolev inequality with uniform Sobolev constant is used essentially in stead of the curvature bound. For simplicity, we prove Theorem 5.1 when $\dim M > 2$. When $\dim M = 2$, we can prove the theorem in the same way, if we use the Sobolev inequality obtained in Corollary 4.2. In §6, we prove Main Theorem. In Appendix, we prove some results concerning dimension and degree of algebraic varieties for the convenience of the reader. The proof given there is due to Y. Namikawa and M. Hashimoto.

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§1. Description of Z

Let $\pi : \mathfrak{X} \to \Delta(1)$ be the same as in the introduction. Set $X_t := \pi^{-1}(t)$. When $t = 0$, we have

$$X_0 = \sum_{\alpha=1}^{a} m_\alpha Y_\alpha \tag{1.1}$$

as a divisor where $m_\alpha \in \mathbb{Z}_+$ and Y_α is an irreducible algebraic variety. Let $(\Delta(1), t_\alpha)$ and $(\Delta(1), s)$ be the unit disc in the t_α-plane and the s-plane respectively. Let f_α and F be holomorphic functions defined by

$$f_\alpha(t_\alpha) := t_\alpha^{m_\alpha}, \quad F(s) := s^m \tag{1.2}$$

where $m := \prod_\alpha m_\alpha$, by which we obtain the fiber product

$$f_\alpha^{-1}\mathfrak{X} := \{(x, t_\alpha) \in \mathfrak{X} \times \Delta(1); \pi(x) = t_\alpha^{m_\alpha}\} \subset \mathfrak{X} \times (\Delta(1), t_\alpha) \tag{1.3}$$

and

$$F^{-1}\mathfrak{X} := \{(x, s) \in \mathfrak{X} \times \Delta(1); \pi(x) = s^m\} \subset \mathfrak{X} \times (\Delta(1), s). \tag{1.4}$$

Let $\pi_\alpha : f_\alpha^{-1}\mathfrak{X} \to (\Delta(1), t_\alpha)$ and $\Pi : F^{-1}\mathfrak{X} \to (\Delta(1), s)$ be the natural projections induced by that of $\mathfrak{X}$ whose fibers are given by $\pi_\alpha^{-1}(t) = X_{t^{m_\alpha}}$ and $\Pi^{-1}(t) = X_{t^m}$. In particular, $\pi_\alpha^{-1}(0) = \Pi^{-1}(0) = X_0$.

We denote the normalization of $f_\alpha^{-1}\mathfrak{X}$ and $F^{-1}\mathfrak{X}$ by

$$\iota_\alpha : \widehat{f_\alpha^{-1}\mathfrak{X}} \longrightarrow f_\alpha^{-1}\mathfrak{X} \tag{1.5}$$

and

$$\iota : \widehat{F^{-1}\mathfrak{X}} \longrightarrow F^{-1}\mathfrak{X}. \tag{1.6}$$

Then, regarding $X_0 = \pi_\alpha^{-1}(0) = \Pi^{-1}(0) \subset F^{-1}\mathfrak{X}$ and $Y_\alpha \subset X_0$, define Z, Z_α and W_α by

$$Z := \iota^{-1}(X_0), \quad Z_\alpha := \iota^{-1}(Y_\alpha), \quad W_\alpha := \iota_\alpha^{-1}(Y_\alpha). \tag{1.7}$$

Since $(\Delta(1), s)$ is a (branched) covering of $(\Delta(1), t_\alpha)$, regarding $t_\alpha^{m_\alpha} = s^{\frac{m}{m_\alpha}}$, we obtain the natural covering map

$$g_\alpha : F^{-1}\mathfrak{X} \ni (x, s) \longrightarrow (x, s^{\frac{m}{m_\alpha}}) \in f_\alpha^{-1}\mathfrak{X}. \tag{1.8}$$

Clearly g_α is a branched m/m_α-sheeted covering (cf. [G-R, pp. 135]). Since g_α is a finite surjective holomorphic map, it naturally induces a finite surjective holomorphic map between $\widehat{F^{-1}\mathfrak{X}}$ and $\widehat{f_\alpha^{-1}\mathfrak{X}}$. We denote this map by $\hat{g}_\alpha$:

$$\hat{g}_\alpha : \widehat{F^{-1}\mathfrak{X}} \longrightarrow \widehat{f_\alpha^{-1}\mathfrak{X}}. \tag{1.9}$$

By the definition, it is clear that $\hat{g}_\alpha$ is a branched m/m_α-sheeted covering. Restricting $\hat{g}_\alpha$ to the fiber at zero, we get the finite surjection

$$\hat{g}_\alpha : Z_\alpha \longrightarrow W_\alpha. \tag{1.10}$$

Set $\mathfrak{X}_\alpha := \widehat{f_\alpha^{-1}\mathfrak{X}}$, $p_\alpha := \pi_\alpha \circ \iota_\alpha$ and define a family

$$p_\alpha : \mathfrak{X}_\alpha \longrightarrow (\Delta(1), t_\alpha) \tag{1.11}$$

whose fiber at $t \neq 0$ is the same one as that of $f_\alpha^{-1}\mathfrak{X}$, and therefore is $X_{t^{m_\alpha}}$, since $f_\alpha^{-1}\mathfrak{X}$ is smooth outside $X_0 = \pi_\alpha^{-1}(0)$. Set

$$\begin{aligned} D_\alpha &:= \operatorname{Sing}(Y_\alpha) \cup \bigcup_{\beta \neq \alpha} Y_\alpha \cap Y_\beta, \\ \Sigma_\alpha &:= \iota_\alpha^{-1}(D_\alpha), \quad S_\alpha := \hat{g}_\alpha^{-1}(\Sigma_\alpha), \quad S := \cup_\alpha S_\alpha. \end{aligned} \tag{1.12}$$

PROPOSITION 1.1. *(1) $\iota : Z_\alpha \to Y_\alpha$ and $\iota_\alpha : W_\alpha \to Y_\alpha$ are m_α-sheeted analytic covering, and $\hat{g}_\alpha : Z_\alpha \to W_\alpha$ is a surjective one sheeted analytic covering. In fact, $\iota : Z_\alpha - S_\alpha \to Y_\alpha - D_\alpha$ and $\iota_\alpha : W_\alpha - \Sigma_\alpha \to Y_\alpha - D_\alpha$ are unramified m_α-sheeted covering, and $\hat{g}_\alpha : Z_\alpha - S_\alpha \to W_\alpha - \Sigma_\alpha$ is an isomorphism.*
(2) $\operatorname{Sing}(W_\alpha) \subset \Sigma_\alpha$ and $\operatorname{Sing}(Z_\alpha) \subset S_\alpha$.
(3) $W_\alpha - \Sigma_\alpha \subset \mathfrak{X}_{\alpha,reg}$ and p_α is of maximal rank on $W_\alpha - \Sigma_\alpha$.

Proof. Let x be an arbitrary point in $Y_\alpha - D_\alpha$. There is a coordinate neighborhood $(U_x, (z_0, \cdots, z_n))$ around x in $\mathfrak{X}$ such that $x = (0, \cdots, 0)$ and for $y = (z_0, \cdots, z_n) \in U_x \cong \Delta(1)^{n+1}$,

$$\pi(y) = z_0^{m_\alpha}, \tag{1.13}$$

since x is a smooth point of Y_α and the multiplicity of Y_α is m_α. Put

$$f_\alpha^{-1}U_x := \{(y, t_\alpha) \in f_\alpha^{-1}\mathfrak{X}; y \in U_x\} \tag{1.14}$$

$$F^{-1}U_x := \{(y, s) \in F^{-1}\mathfrak{X}; y \in U_x\}. \tag{1.15}$$

Using above local coordinates, we can write

$$\begin{aligned} f_\alpha^{-1}U_x &= \{(z, t_\alpha) \in \Delta(1)^{n+1} \times \Delta(1); z_0^{m_\alpha} = t_\alpha^{m_\alpha}\}, \\ &= \{(z, t_\alpha); \prod_{k=1}^{m_\alpha}(z_0 - \zeta_{m_\alpha}^k t_\alpha) = 0\}, \end{aligned} \tag{1.16}$$

$$\begin{aligned} F^{-1}U_x &= \{(z, s) \in \Delta(1)^{n+1} \times \Delta(1); z_0^{m_\alpha} = s^m\} \\ &= \{(z, s); \prod_{k=1}^{m_\alpha}(z_0 - \zeta_{m_\alpha}^k s^{\frac{m}{m_\alpha}}) = 0\} \end{aligned} \tag{1.17}$$

where $\zeta_{m_\alpha} = \exp(2\pi\sqrt{-1}/m_\alpha)$. From the definition of normalization, it follows that

$$\sharp\iota_\alpha^{-1}(x) = \sharp\iota^{-1}(x) = m_\alpha \tag{1.18}$$

and

$$\iota_\alpha^{-1}f_\alpha^{-1}U_x = \cup_{k=1}^{m_\alpha}V_{x,k}, \quad V_{x,k} \cap V_{x,l} = \emptyset \quad (k \neq l) \tag{1.19}$$

$$\iota^{-1}F^{-1}U_x = \cup_{k=1}^{m_\alpha}O_{x,k}, \quad O_{x,k} \cap O_{x,l} = \emptyset \quad (k \neq l) \tag{1.20}$$

where

$$\iota_\alpha : V_{x,k} \longrightarrow \{(z, t_\alpha); z_0 - \zeta_{m_\alpha}^k t_{m_\alpha} = 0\}, \tag{1.21}$$

$$\iota : O_{x,k} \longrightarrow \{(z, s); z_0 - \zeta_{m_\alpha}^k s^{\frac{m}{m_\alpha}} = 0\} \tag{1.22}$$

are isomorphisms. Set $x_{\alpha,k} := \iota_\alpha^{-1}(x) \cap V_{x,k}$ and $x_k := \iota^{-1}(x) \cap O_{x,k}$. Then $V_{x,k}$ (resp. $O_{x,k}$) is a coordinate neighborhood of $\widehat{f_\alpha^{-1}\mathfrak{X}}$ (resp. $\widehat{F^{-1}\mathfrak{X}}$) around $x_{\alpha,k}$ (resp. x_x). Therefore $V_{x,k} \cap W_\alpha$ (resp. $O_{x,k} \cap Z_\alpha$) is a coordinate neighborhood of W_α (resp. Z_α) around $x_{\alpha,k}$ (resp. x_x):

$$\iota_\alpha : V_{x,k} \cap W_\alpha \to \{(z, t_\alpha) \in \Delta(1)^{n+1} \times \Delta(1); z_0 = t_{m_\alpha} = 0\} \cong \Delta(1)^n, \tag{1.23}$$

$$(1.24)\qquad \iota : O_{x,k} \cap Z_\alpha \to \{(z,s) \in \Delta(1)^{n+1} \times \Delta(1); z_0 = s = 0\} \cong \Delta(1)^n,$$

which implies that $W_\alpha - \Sigma_\alpha$ and $Z_\alpha - S_\alpha$ consist of smooth points of W_α and Z_α respectively, since x is an arbitrary point of $Y_\alpha - D_\alpha$. This proves (2).

By (1.19), (1.20), (1.23) and (1.24), ι_α (resp. ι) is a m_α-sheeted covering map between $W_\alpha - \Sigma_\alpha$ and $Y_\alpha - D_\alpha$ (resp. $Z_\alpha - S_\alpha$ and $Y_\alpha - D_\alpha$). This proves (1) for ι_α and ι. Since $g_\alpha(z,s) = (z, s^{\frac{m}{m_\alpha}})$ in the above coordinates, $g_\alpha : \iota(O_{x,k}) \to \iota_\alpha(V_{x,k})$ is an isomorphism. Therefore we obtain an isomorphism $\hat{g}_\alpha : O_{x,k} \to V_{x,k}$. As $\hat{g}_\alpha$ is globally defined on Z_α, it must be an unramified covering resiricted to $Z_\alpha - S_\alpha$. Since $g_\alpha \circ \iota = \iota_\alpha \circ \hat{g}_\alpha$ and g_α is the identity map on Y_α, $\hat{g}_\alpha$ must be an isomorphism between $Z_\alpha - S_\alpha$ and $W_\alpha - \Sigma_\alpha$, comparing the mapping degree. This completes the proof of (1).

By (1.21), there exists a coordinate neighborhood $(V_{x,k}, (\xi_0, \cdots, \xi_n))$ around x_k such that $x_k = (0, \cdots, 0)$ and for $\xi = (\xi_0, \cdots, \xi_n) \in V_{x,k}$,

$$(1.25)\qquad \xi_k := \iota_\alpha^* z_k, \qquad p_\alpha(\xi) = \zeta_{m_\alpha}^{-k} \xi_0.$$

This shows that p_α is of maximal rank at x_k and that p_α is of maximal rank on $W_\alpha - \Sigma_\alpha$. This proves (3). □

Define the ϵ-tubular neighborhood of Σ_α and S_α as follows. By the definition, $f_\alpha^{-1}\mathfrak{X}$ is a subset of $A := \mathbb{P}^N(\mathbb{C}) \times (\Delta(1), t) \times (\Delta(1), t_\alpha)$. Let G_A be a Riemannian metric of A defined by

$$(1.26)\qquad G_A := g_{\mathbb{P}^N} + |dt|^2 + |dt_\alpha|^2$$

where $g_{\mathbb{P}^N}$ is the Fubini-Study metric of $\mathbb{P}^N(\mathbb{C})$. Let $d_A(\cdot,\cdot)$ be the distance of A induced by G_A. Then, define a function of $f_\alpha^{-1}\mathfrak{X} \times f_\alpha^{-1}\mathfrak{X}$ by

$$(1.27)\qquad d_\alpha(x,y) := d_A(i(x), i(y))$$

where $i : f_\alpha^{-1}\mathfrak{X} \to A$ is the natural inclusion. By the definition, it is clear that for any x, $y \in \pi_\alpha^{-1}(t)$,

$$(1.28)\qquad d_\alpha(x,y) = d_{\mathbb{P}^N}(j_t(x), j_t(y))$$

where $j_t : \pi_\alpha^{-1}(t) \hookrightarrow \mathbb{P}^N(\mathbb{C})$ is the natural inclusion.

Regarding $\pi_\alpha^{-1}(0) = X_0$, define the ϵ-neighborhood of D_α by

$$D_{\alpha,\epsilon} := \{x \in f_\alpha^{-1}\mathfrak{X}; d_\alpha(x, D_\alpha) < \epsilon\}, \tag{1.29}$$

and the ϵ-neighborhood of Σ_α, S_α and S by

$$\Sigma_{\alpha,\epsilon} := \iota_\alpha^{-1}(D_{\alpha,\epsilon}), \quad S_{\alpha,\epsilon} := \hat{g}_\alpha^{-1}(\Sigma_{\alpha,\epsilon}), \quad S_\epsilon := \cup S_{\alpha,\epsilon}. \tag{1.30}$$

As $Y_\alpha - D_{\alpha,\epsilon} = \{y \in Y_\alpha; d_{\mathbb{P}^N}(y.D_\alpha) \geq \epsilon\}$ is compact, so are $W_\alpha - \Sigma_{\alpha,\epsilon}$, $Z_\alpha - S_{\alpha,\epsilon}$ and $Z - S_\epsilon$ by Proposition 1.1.

Let G be the same Riemannian metric of $\mathfrak{X}$ as in introduction. Let G_α (resp. H_α) be a Riemannian metric of $f_\alpha^{-1}\mathfrak{X}$ (resp. $\mathfrak{X}_\alpha$), defined by

$$G_\alpha := G + |dt_\alpha|^2|_{f_\alpha^{-1}\mathfrak{X}}, \quad H_\alpha := \iota_\alpha^* G_\alpha. \tag{1.31}$$

By the definition,

$$G_\alpha|_{\pi_\alpha^{-1}(t)} = g_{t^{m_\alpha}}, \quad H_\alpha|_{p_\alpha^{-1}(t)} = \iota_\alpha^* g_{t^{m_\alpha}}, \quad g_{t^{m_\alpha}} := G|_{X_{t^{m_\alpha}}}. \tag{1.32}$$

Finally define a metric of W_α by

$$g_{W_\alpha} := H_\alpha|_{W_\alpha}, \tag{1.33}$$

and metrics of Z_α and Z by

$$g_{Z_\alpha} := \iota^* g_0|_{Y_\alpha} = (\hat{g}_\alpha)^* g_{W_\alpha} = (\iota_\alpha \circ \hat{g}_\alpha)^* g_0, \quad g_Z|_{Z_\alpha} := g_{Z_\alpha}. \tag{1.34}$$

The goal of this section is to prove the following theorem.

THEOREM 1.1. *For every small $0 < \epsilon \ll 1$, there exists $0 < \gamma(\epsilon) < \epsilon$ and a family of into-diffeomorphisms for $|t| < \gamma(\epsilon)$:*

$$f_{\epsilon,t} : Z - S_\epsilon \hookrightarrow X_t$$

by which the following conditions are satisfied.
(1) *On $Z - S_\epsilon$,*

$$\frac{1}{2} g_Z \leq f_{\epsilon,t}^* g_t \leq 2 g_Z.$$

(2) *On every compact subset $K \Subset Z - S\epsilon$,*

$$\lim_{t \to 0} f_{\epsilon,t}^* g_t = g_Z$$

in the C^∞*-topology on* K.
(3) *If* $\mathrm{vol}(\cdot)$ *stands for the volume relative to the induced metric from* G *and* g_Z, *then*

$$\mathrm{vol}(X_t - f_{\epsilon,t}(Z - S_\epsilon)) \leq 3\,\mathrm{vol}(S_\epsilon).$$

For the proof, we need some lemmas and propositions.

LEMMA 1.1. *For every small* $0 < \epsilon \ll 1$, *there exist an open neighborhood* $U_\alpha(\epsilon)$ *of* $W_\alpha - \Sigma_{\alpha,\epsilon}$ *in* $\mathfrak{X}_\alpha$, *and vector fields* $u_\alpha^{(\epsilon)}$, $v_\alpha^{(\epsilon)}$ *on* $U_\alpha(\epsilon)$ *such that*

$$(p_\alpha)_* u_\alpha^{(\epsilon)} = \frac{\partial}{\partial x}, \quad (p_\alpha)_* v_\alpha^{(\epsilon)} = \frac{\partial}{\partial y}$$

where $t_\alpha = x + \sqrt{-1}y$.

Proof. Clear by Proposition 1.1 and the compactness of $W_\alpha - \Sigma_{\alpha,\epsilon}$ in $\mathfrak{X}_\alpha$. □

LEMMA 1.2. *For every small* $0 < \epsilon \ll 1$, *there exist* $0 < \gamma_1(\epsilon) < \epsilon$ *and an into-diffeomorphism*

$$\Phi_\alpha^{(\epsilon)} : (W_\alpha - \Sigma_{\alpha,\epsilon}) \times \Delta(\gamma_1(\epsilon)) \longrightarrow U_\alpha(\epsilon)$$

such that
(1) *For any* $t \in \Delta(\gamma_1(\epsilon))$,

$$\Phi_\alpha^{(\epsilon)}(W_\alpha - \Sigma_{\alpha,\epsilon}, t) \subset p_\alpha^{-1}(t).$$

(2) $\Phi_\alpha^{(\epsilon)}(\cdot, 0)$ *is the identity map on* $W_\alpha - \Sigma_{\alpha,\epsilon}$.

Proof. Integrating the vector fields in Lemma 1.1, desired into-diffeomorphism is obtained. □

PROPOSITION 1.2. *For every small* $0 < \epsilon \ll 1$, *there exist* $0 < \gamma_2(\epsilon) < \epsilon$ *and a family of into-diffeomorphisms for* $|t| < \gamma_2(\epsilon)$

$$\phi_{\alpha,t}^{(\epsilon)} : W_\alpha - \Sigma_{\alpha,\epsilon} \to X_{t^{m_\alpha}}$$

such that
(1) $\phi^{(\epsilon)}_{\alpha,0}$ *is the identity map of* $W_\alpha - \Sigma_{\alpha,\epsilon}$.
(2) *On* $W_\alpha - \Sigma_{\alpha,\epsilon}$,

$$\frac{1}{2} g_{W_\alpha} \leq (\phi^{(\epsilon)}_{\alpha,t})^* g_{t^{m_\alpha}} \leq 2 g_{W_\alpha}.$$

(3) *On* $W_\alpha - \Sigma_{\alpha,\epsilon}$,

$$\lim_{t\to 0} (\phi^{(\epsilon)}_{\alpha,t})^* g_{t^{m_\alpha}} = g_{W_\alpha}$$

in the C^∞*-topology on* $W_\alpha - \Sigma_{\alpha,\epsilon}$.

Proof. For $|t| < \gamma_1(\epsilon)$, set

$$\phi^{(\epsilon)}_{\alpha,t}(x) := \iota_\alpha \circ \Phi^{(\epsilon)}_\alpha(x,t) \tag{1.35}$$

where $\gamma_1(\epsilon)$ and $\Phi^{(\epsilon)}_\alpha$ are the same ones as in Lemma 1.2. Then, $\phi^{(\epsilon)}_{\alpha,t}$ is a map from $W_\alpha - \Sigma_{\alpha,\epsilon}$ to $\iota_\alpha \circ (p_\alpha^{-1}(t)) = \pi_\alpha^{-1}(t) = X_{t^{m_\alpha}}$. By the definition, $\iota_\alpha : p_\alpha^{-1}(t) \to \pi_\alpha(t)$ is an isomorphism for $t \neq 0$. By Lemma 1.2, $\Phi^{(\epsilon)}_\alpha(\cdot, t) : W_\alpha - \Sigma_{\alpha,\epsilon} \to \pi_\alpha^{-1}(t)$ is an into-diffeomorphism for $|t| < \gamma_1(\epsilon)$. Namely, $\phi^{(\epsilon)}_{\alpha,t}$ is an into-diffeomorphism from $W_\alpha - \Sigma_{\alpha,\epsilon}$ to $X_{t^{m_\alpha}}$.

By (1.31) and (1.32), we get

$$\begin{aligned}(\phi^{(\epsilon)}_{\alpha,t})^* g_{t^{m_\alpha}} &= (\iota_\alpha \circ \Phi^{(\epsilon)}_\alpha|_t)^* G_\alpha|_{\pi_\alpha^{-1}(t)} \\ &= (\Phi^{(\epsilon)}_\alpha|_t)^* H_\alpha.\end{aligned} \tag{1.36}$$

Since $\Phi^{(\epsilon)}_\alpha$ depends smoothly on t, it follows that

$$\begin{aligned}\lim_{t\to 0} (\Phi^{(\epsilon)}_\alpha|_t)^* H_\alpha &= (\Phi^{(\epsilon)}_\alpha|_0)^* H_\alpha \\ &= H_\alpha|_{p_\alpha^{-1}(0)} = g_{W_\alpha}.\end{aligned} \tag{1.37}$$

(3) follows from (1.37). Since $W_\alpha - \Sigma_{\alpha,\epsilon}$ is compact, there exists $\gamma_2(\epsilon) \leq \gamma_1(\epsilon)$ such that (2) holds for $|t| < \gamma_2(\epsilon)$. ∎

Proof of Theorem 1.1. For every m_α, let $t^{\frac{1}{m_\alpha}}$ be a fixed branch of the inverse function of t^{m_α} on $\Delta(1)$. Let $\phi^{(\epsilon)}_{\alpha,t}$ be the same map as in Proposition 1.2. Since $Z - S_\epsilon = \cup_\alpha (Z_\alpha - S_{\alpha,\epsilon})$, to define $f_{\epsilon,t} : Z - S_\epsilon \to X_t$, it is sufficient to define it on each $Z_\alpha - S_{\alpha,\epsilon}$. Define $f_{\epsilon,t}$ by

$$f_{\epsilon,t}|_{Z_\alpha - S_{\alpha,\epsilon}} := \phi^{(\epsilon)}_{\alpha,t^{\frac{1}{m_\alpha}}} \circ \hat{g}_\alpha : Z_\alpha - S_{\alpha,\epsilon} \hookrightarrow X_t. \tag{1.38}$$

Since $\hat{g} : Z - S_\epsilon \to \bigcup_\alpha (W_\alpha - S_{\alpha,\epsilon})$ is an isomorphism by Proposition 1.1, $f_{\epsilon,t}$ is an into-diffeomorphism for every small t with $|t| < \gamma_2(\epsilon)$. Then we get (1) and (2) by Proposition 1.2, because $\lim_{t\to 0} t^{\frac{1}{m_\alpha}} = 0$ although $t^{\frac{1}{m_\alpha}} \notin C^0(\Delta(1))$.

By Proposition 1.2, for every small $0 < \epsilon \ll 1$, there exists $\gamma_3(\epsilon)$ such that

$$|\operatorname{vol}(f_{\epsilon,t}(Z - S_\epsilon)) - \operatorname{vol}(Z - S_\epsilon)| < \operatorname{vol}(S_\epsilon) \tag{1.39}$$

for every t with $|t| < \gamma_3(\epsilon)$. Since $\operatorname{vol}(Z - S_\epsilon) = \operatorname{vol}(Z) - \operatorname{vol}(S_\epsilon) = \operatorname{vol}(X_0) - \operatorname{vol}(S_\epsilon)$, it follows from (1.39)

$$\operatorname{vol}(f_{\epsilon,t}(Z - S_\epsilon)) > \operatorname{vol}(X_0) - 2\operatorname{vol}(S_\epsilon) \tag{1.40}$$

for t with $|t| < \gamma_3(\epsilon)$. Let ω_G be the Kähler form of G. Then,

$$\operatorname{vol}(X_t) := \int_{X_t} \omega_G^n. \tag{1.41}$$

By [F, Proposition 2.3], as $\operatorname{vol}(X_t)$ is a continious function in t, for every small $0 < \epsilon \ll 1$, there exists $\gamma_4(\epsilon)$ such that

$$|\operatorname{vol}(X_t) - \operatorname{vol}(X_0)| < \operatorname{vol}(S_\epsilon) \tag{1.42}$$

for every t with $|t| < \gamma_4(\epsilon)$. By (1.40) and (1.42), for t with $|t| < \min\{\gamma_3(\epsilon), \gamma_4(\epsilon)\}$,

$$\begin{aligned} \operatorname{vol}(X_t - f_{\epsilon,t}(Z - S_\epsilon)) &= \operatorname{vol}(X_t) - \operatorname{vol}(f_{\epsilon,t}(Z - S_\epsilon)) \\ &< \operatorname{vol}(X_t) - \operatorname{vol}(X_0) + 2\operatorname{vol}(S_\epsilon) \\ &< 3\operatorname{vol}(S_\epsilon). \end{aligned} \tag{1.43}$$

Set $\gamma(\epsilon) := \min\{\gamma_2(\epsilon), \gamma_3(\epsilon), \gamma_4(\epsilon)\}$. Then, Theorem 1.1 is proved for this $\gamma(\epsilon)$. □

§2. Sobolev inequality and upper bound of the heat kernel

In this section, we recall some basic results on the Sobolev inequality and the heat kernel. For the reference, see [D].

Let (M, g) be a compact Riemannian manifold of dimension m with possibly smooth boundary. We denote by Δ the Laplacian and by $k(t, x, y)$ the heat kernel. When M has boundary, consider the Dirichlet Laplacian and the Dirichlet heat kernel. As an application of the logarithmic Sobolev inequality, the following result is well known.

THEOREM 2.1. ([C-K-S] and [D, Theorem 2.4.2]) *For $\mu > 2$, the following two inequalities are equivalent:*
(1) *For $0 < t \leq 1$ and $(x, y) \in M \times M$,*

$$k(t, x, y) \leq C_1 t^{-\frac{\mu}{2}}.$$

(2) *For every $f \in C^\infty(M)$,*

$$\|f\|_{\frac{2\mu}{\mu-2}} \leq C_2(\|df\|_2 + \|f\|_2).$$

Here C_1 and C_2 depend continuously on each other.

Let $\{0 = \lambda_0 < \lambda_1 \leq \lambda_2 \cdots\}$ be the eigenvalue of the Laplacian and ϕ_i be the normalized eigenfunction such that

$$\Delta\phi_i = \lambda_i \phi_i, \quad (\phi_i, \phi_j) = \delta_{ij}. \tag{2.1}$$

By [C-L, Corollary 1], the following estimate holds.

PROPOSITION 2.1. *Under the two inequalities of Theorem 2.1,*

$$\|\phi_i\|_\infty \leq C_3 \lambda_i^{\frac{\mu}{2}}$$

where C_3 depends on C_1 and C_2.

In the sequel of this article (in §5), we shall use Theorem 2.1 and Proposition 2.1 for $\mu = m$ when $m > 2$ and $\mu = 4$ when $m = 2$.

§3. The space of functions on algebraic varieties

Let (X, g_X) be an irreducible projective algebraic variety of dimension n in $\mathbb{P}^N(\mathbb{C})$ with the Bergmann metric, i.e. the restriction of the standard Fubini-Study metric of $\mathbb{P}^N(\mathbb{C})$. We denote by Σ_X the singular set of X.

Let $C^{1/2}(X)$ (resp. $C_0^{1/2}(X)$) be the space of all Lipschitz functions (resp. with compact support on $X - \Sigma_X$) on X. Define $W^{1,2}(X)$ and $W_0^{1,2}(X)$ by

$$W^{1,2}(X) := \overline{\{f \in C^{1/2}(X); f \in L^2(X), df \in L^2(X)\}} \tag{3.1}$$

$$W_0^{1,2}(X) := \overline{C_0^{1/2}(X)} \tag{3.2}$$

where the completion is taken with respect to the norm $\|\cdot\|_{L^2_1}$:

$$\|f\|_{L^2_1} = \|f\|_2 + \|df\|_2.$$

In [L-T], it is proved that $W^{1,2}(X) = W_0^{1,2}(X)$. Since the proof seems to be rough, we shall give a detailed proof in this section.

Let us fix notations. For any closed subset $S \subset \mathbb{P}^N(\mathbb{C})$, set

$$r_S(x) := d(x, S) = \inf_{y \in S} d(x, y), \tag{3.3}$$

$$S_\epsilon := \{x \in \mathbb{P}^N(\mathbb{C}); r_S(x) < \epsilon\} \tag{3.4}$$

where $d(\cdot, \cdot)$ is the distance of $\mathbb{P}^N(\mathbb{C})$.

THEOREM 3.1. *Let Y be a subvariety of pure dimension d $(< n)$ in X. Then, for the pair (X, Y), there exist functions $\gamma(\cdot)$, $\delta(\cdot)$ (≥ 0) $\in C^0([0, 1/16])$ and $\eta_\epsilon \in C_0^0(X - Y) \cap W^{1,2}(X)$ for any $\epsilon \in (0, 1/16]$ which satisfy the following conditions:*

(1)
$$0 \leq \eta_\epsilon \leq 1, \quad \|\eta_\epsilon - 1\|_2 + \|d\eta_\epsilon\|_2 \leq \gamma(\epsilon), \quad \lim_{\epsilon \to 0} \gamma(\epsilon) = 0.$$

(2)
$$X - Y_{\delta(\epsilon)} \subset \operatorname{supp} \eta_\epsilon \subset X - Y_\epsilon, \quad \operatorname{supp}(1 - \eta_\epsilon) \subset Y_{2\delta(\epsilon)},$$
$$\epsilon \leq \delta(\epsilon), \quad \lim_{\epsilon \to 0} \delta(\epsilon) = 0.$$

THEOREM 3.2. *There exist functions $\delta(\cdot)(\geq 0) \in C^0([0, 1/16])$ and $\bar{\eta}_\epsilon \in C_0^0(X - \Sigma_X) \cap W^{1,2}(X)$ for any $\epsilon \in (0, 1/16]$ which satisfy the following conditions:*

(1)
$$\operatorname{supp} \bar{\eta}_\epsilon \subset X - \Sigma_{\frac{\epsilon}{2}}, \quad 0 \leq \bar{\eta}_\epsilon \leq 1 + \epsilon.$$

(2)
$$\lim_{\epsilon \to 0} \|\bar{\eta}_\epsilon - 1\|_2 + \|d\bar{\eta}_\epsilon\|_2 = 0.$$

(3) *If we set $S(\epsilon) := \{x \in X; \bar{\eta}_\epsilon(x) = 0\}$, then $\partial S(\epsilon)$ consists of finitely*

many smooth manifolds.

(4)

$$\Sigma_{\frac{\epsilon}{2}} \subset S(\epsilon) \subset \Sigma_{\delta(\epsilon)}, \quad \lim_{\epsilon \to 0} \delta(\epsilon) = 0.$$

In view of [L-T, Theorem 4.1], we get the following corollary.

COROLLARY 3.1. *For any projective algebraic variety of pure dimension, $W^{1,2}(X) = W_0^{1,2}(X)$. Equivalently, $d_{\max} = d_{\min}$ on $L^2(X)$.*

For the proof, we prepare several lemmas and propositions.

We denote by $Y_{\mathrm{reg}} := Y - \Sigma_Y$ the regular part of Y. For $y \in Y_{\mathrm{reg}}$, set

$$N_y := \exp_y(T_y Y)^{\perp} \subset \mathbb{P}^N(\mathbb{C}) \tag{3.5}$$

where the exponential map is considered in $\mathbb{P}^N(\mathbb{C})$ relative to the Fubini-Study metric. Then N_y is isometric to $\mathbb{P}^{N-d}(\mathbb{C})$ and intersect Y transversally at y. Therefore there is a neighborhood U of y such that $Y \cap N_y \cap U = \{y\}$. Let

$$N_y \cap X = \cup_\alpha (N_y \cap X)_\alpha \tag{3.6}$$

be the irreducible decomposition. Set

$$(N_y \cap X)_y := \bigcup_{\beta; y \in (N_y \cap X)_\beta} (N_y \cap X)_\beta. \tag{3.7}$$

At first let us study the dimension of $(N_y \cap X)_y$. Since each $(N_y \cap X)_\alpha$ is a pure dimensional space, it follows that

$$\begin{aligned} \dim(N_y \cap X)_\alpha &= \dim_y(N_y \cap X)_\alpha, \\ \dim(N_y \cap X)_y &= \max_\alpha \dim_y(N_y \cap X)_\alpha \end{aligned} \tag{3.8}$$

where $\dim_y W$ stands for the analytic dimension of W at y for algebraic variety W. See [G-R, Chap. 5] for the definition of dimension.

PROPOSITION 3.1. *There exists a nonempty Zariski open subset U_Y of Y such that if $y \in U_Y$,*

$$\dim_y N_y \cap X = n - d.$$

Proof. See Appendix Proposition A.1.

Since $\dim(N_y \cap X)_\alpha \geq n - d$ by the intersection inequality ([G-R, p. 102]), we get the following proposition.

PROPOSITION 3.1′. *If $y \in U_Y$,*

$$\dim(N_y \cap X)_y = n - d.$$

Next let us study the volume of $(N_y \cap X)_y$. For an algebraic variety $W \subset \mathbb{P}^N(\mathbb{C})$, its volume with respect to the Fubini-Study metric is given by

$$\mathrm{vol}(W) = \int_W (\frac{\sqrt{-1}}{2}\omega)^l|_W, \tag{3.9}$$

where ω is the Kähler form of the ambient projective space and $l = \dim W$. Since $\frac{\sqrt{-1}}{2\pi}\omega$ represents the same current as H, a hyperplane in the ambient projective space (cf. [G-H]), we have

$$\mathrm{vol}(W) = (\pi)^l \deg(W), \quad \deg(W) = \sharp(W \cap H_1 \cap \cdots \cap H_l), \tag{3.10}$$

where $H_1, \cdots, H_l$ are generic hyperplanes.

PROPOSITION 3.2. *Let $H_1, \cdots, H_r$ be arbitrary $r(< n)$-th hyperplanes in $\mathbb{P}^N(\mathbb{C})$. Let*

$$X \cap H_1 \cap \cdots \cap H_r = \sum_{i=0}^{r} X_r^{(n-i)} \qquad (\dim X_r^{(n-i)} = n - i)$$

be the decomposition into the pure dimensional components. Then,

$$\deg(X) \geq \deg(X_r^{(n-r)}).$$

Proof. See Appendix Proposition A.2.

Since N_y is a linear subspace of dimension $N - d$ in the ambient projective space, there are hyperplanes $H_1, \cdots, H_d$ such that

$$N_y = H_1 \cap \cdots \cap H_d. \tag{3.11}$$

By Proposition 3.2, we get the following proposition.

PROPOSITION 3.2$'$. *If $y \in U_Y$,*

$$\deg(N_y \cap X)_y \leq \deg(X).$$

Let us study the tubular neighborhood of Y in $\mathbb{P}^N(\mathbb{C})$. If Y is smooth, it is diffeomorphic to the normal bundle of Y. Define the normal projective bundle N_Y by

$$N_Y := \{(y, w) \in Y_{\mathrm{reg}} \times \mathbb{P}^N(\mathbb{C}); w \in N_y\}. \tag{3.12}$$

We denote by $i : N_Y \hookrightarrow Y_{\mathrm{reg}} \times \mathbb{P}^N(\mathbb{C})$ the inclusion. Define projections by

$$\pi := p_1 \circ i : N_Y \longrightarrow Y_{\mathrm{reg}}, \qquad p := p_2 \circ i : N_Y \longrightarrow \mathbb{P}^N(\mathbb{C}) \tag{3.13}$$

where p_i stands for the projection to the i-th factor. It is clear that $\pi : N_Y \to Y_{\mathrm{reg}}$ is a fiber bundle with fiber $\mathbb{P}^{N-d}(\mathbb{C})$. Define a subbundle of the tangent bundle of N_Y by

$$TN := \mathrm{Ker}(\pi_* : TN_Y \to TY_{\mathrm{reg}}). \tag{3.14}$$

Then for $(y, w) \in N_Y$, it follows that $TN_w = T_w N_y$. By this identification, define a Riemannian metric g_N of TN by

$$g_N := \bigcup_{y \in Y_{\mathrm{reg}}} g_{N_y} \tag{3.15}$$

where $g_{N_y} := g_{\mathbb{P}^N}|_{N_y}$ is a Riemannian metric of N_y. Define a ball bundle $N_Y(\epsilon)$ by

$$N_Y(\epsilon) := \{(y, w) \in N_Y; d_{\mathbb{P}^N}(y, w) < \epsilon\}. \tag{3.16}$$

LEMMA 3.1. *Let S be a subvariety of Y such that $S \supset \mathrm{Sing}(Y)$ and $\dim S < \dim Y$. Then, for the pair (Y, S), there exists an increasing function $a(\epsilon) \in C^0([0, 1])$ which satisfies the following conditions:*
(1)

$$\lim_{\epsilon \longrightarrow 0} a(\epsilon) = 0, \quad a(\epsilon) \geq \sqrt{\epsilon}.$$

(2) *$p : \pi^{-1}(Y - S_{a(\epsilon)}) \cap N_Y(\epsilon) \to \mathbb{P}^N(\mathbb{C})$ is an into-diffeomorphism such that*

$$\frac{1}{2} p^* g_{\mathbb{P}^N} \leq \pi^* g_Y + g_N \leq 2 p^* g_{\mathbb{P}^N}$$

on $\pi^{-1}(Y - S_{a(\epsilon)}) \cap N_Y(\epsilon)$ where $g_Y := g_{\mathbb{P}^N}|_Y$.

Proof. By the definition of p, it is the identity map restricted to Y_{reg}. Choose a point $y \in Y_{\text{reg}}$. By the definition of N_y, we get the following decomposition of the tangent space of $\mathbb{P}^N(\mathbb{C})$ and N_Y at y:

$$T_y\mathbb{P}^N(\mathbb{C}) = T_yN_Y = T_yY \oplus T_yN_y, \quad T_yY \perp T_yN_y. \tag{3.17}$$

Choose tangent vectors $u \in T_yY$ and $v \in T_yN_y$ arbitrarily. Since $p|_Y = \text{id}$ as above, we know

$$(p_*)_y u = u. \tag{3.18}$$

Choose a curve $\gamma(t) := (y, \exp_y(tv))$ in N_Y. By the definition,

$$(p_*)_y v = \frac{d}{dt}|_{t=0} p(\gamma(t)) = \frac{d}{dt}|_{t=0} \exp_y(tv) = v. \tag{3.19}$$

This shows $(p_*)_y = 1$. Since y is an arbitrary point of Y_{reg}, we get $p_* = 1$ along Y_{reg}. Set

$$\begin{aligned} b(\epsilon) := \sup\{b \in (0,1]\,;\\ p : \pi^{-1}(Y - S_\epsilon) \cap N_Y(b) \to \mathbb{P}^N(\mathbb{C}) \text{ is an embedding}\}. \end{aligned} \tag{3.20}$$

By (3.18) and (3.19), we get on Y_{reg}

$$p^* g_{\mathbb{P}^N} = \pi^* g_Y + g_N. \tag{3.21}$$

In the same way as the definition of $b(\epsilon)$, set

$$\begin{aligned} c(\epsilon) := \sup\{c \in (0,1];\\ \frac{1}{2} p^* g_{\mathbb{P}^N} \le \pi^* g_Y + g_N \le 2 p^* g_{\mathbb{P}^N} \text{ on } \pi^{-1}(Y - S_\epsilon) \cap N_Y(c)\}. \end{aligned} \tag{3.22}$$

By the definition, $b(\epsilon)$ and $c(\epsilon)$ are increasing functions and satisfy $b(\epsilon)$, $c(\epsilon) < 1$. Since $Y - S_\epsilon$ is compact, $b(\epsilon)$, $c(\epsilon) > 0$.

Define $B(\epsilon)$ by

$$B(\epsilon) := \int_0^\epsilon \min\{b(t), c(t)\} t dt \le \frac{\epsilon^2}{2} \min\{b(\epsilon), c(\epsilon)\}. \tag{3.23}$$

Then $p : \pi^{-1}(Y - S_\epsilon) \cap N_Y(B(\epsilon)) \to \mathbb{P}^N(\mathbb{C})$ is an into-diffeomorphism and the inequality of Lemma 3.1 (2) holds on $\pi^{-1}(Y - S_\epsilon) \cap N_Y(B(\epsilon))$. Furthermore, $B(\epsilon)$ is a continuous increasing function satisfying

$$\lim_{\epsilon \to 0} B(\epsilon) = 0. \tag{3.24}$$

Finally, define $a(\epsilon)$ by the inverse function of $B(\epsilon)$:

$$a(\epsilon) := B^{-1}(\epsilon)(\geq \sqrt{\epsilon}). \tag{3.25}$$

By (3.23) and (3.24), $a(\epsilon)$ is a continuous increasing function satisfying Lemma 3.1 (1).

By (3.22) and (3.23), the inequality of Lemma 3.1 (2) is satisfied by $a(\epsilon)$. □

LEMMA 3.2. *Let $\mathbb{B}(r) := \{z = (z_1, \cdots, z_N) \in \mathbb{C}^N; \|z\|^2 := \sum_{i=1}^N |z_i|^2 < r\}$ be the ball of radius r in $\mathbb{C}^N$. Then the following inequality holds:*

$$-\partial\bar{\partial}\log(-\log\|z\|^2) \geq \frac{\partial\bar{\partial}\|z\|^2}{\|z\|^2(-\log\|z\|^2)^2}$$

for $z \in \mathbb{B}(1/2)$ where $\partial\bar{\partial}\phi$ is the complex Hessian:

$$\partial\bar{\partial}\phi := \sum_{i,j=1}^N \frac{\partial^2\phi}{\partial z_i \partial\bar{z}_j} dz_i d\bar{z}_j.$$

Proof. By computations, we get

$$-\partial\bar{\partial}\log(-\log\|z\|^2) = \sum_{i,j=1}^N G_{i\bar{j}}(z) dz_i d\bar{z}_j, \tag{3.26}$$

$$G_{i\bar{j}}(z) = \frac{\delta_{ij}}{\|z\|^2(-\log\|z\|^2)} - \frac{\bar{z}_i z_j}{\|z\|^4(-\log\|z\|^2)} + \frac{\bar{z}_i z_j}{\|z\|^4(-\log\|z\|^2)^2}, \tag{3.27}$$

which implies

$$G(z) := (G_{i\bar{j}}) \geq \frac{1}{\|z\|^2(-\log\|z\|^2)^2} I. \tag{3.28}$$

□

LEMMA 3.3. *Let $V \subset \mathbb{P}^N(\mathbb{C})$ be a projective algebraic variety of pure dimension d. Let U be the standard affine open subset of $\mathbb{P}^N(\mathbb{C})$; i.e., $U = \mathbb{C}^N$. Let $(z_1, \cdots, z_N)$ be the coordinate of U. Then the following inequality holds for any $\epsilon \in (0,1)$:*

$$\int_{\mathbb{B}(\epsilon)\cap V} dv_V \leq C(N)\epsilon^{2d}\deg(V)$$

where $C(N)$ is a constant which depends only on N, and dv_V is the volume form of V with respect to the Bergmann metric of V.

Proof. By the definition,

$$dv_V = (\omega^d)|_V \tag{3.29}$$

where ω is the Kähler form of $\mathbb{P}^N(\mathbb{C})$. We denote by Ω the Kähler form of $\mathbb{C}^N$ with respect to the Euclidean metric. On $\mathbb{B}(1)$,

$$\frac{1}{2}\Omega \leq \omega \leq \Omega. \tag{3.30}$$

Let T_ϵ be a linear transform of $\mathbb{C}^N$ defined by

$$T_\epsilon(z) := \epsilon z. \tag{3.31}$$

We denote by $\tilde{T}_\epsilon$ the extension of T_ϵ to $\mathbb{P}^N(\mathbb{C})$. Since

$$\int_{\mathbb{B}(\epsilon)\cap V} \Omega^d = \epsilon^{2d} \int_{\mathbb{B}(1)\cap \tilde{T}_{\epsilon^{-1}}(V)} \Omega^d, \tag{3.32}$$

it follows from (3.9), (3.10) and (3.30)

$$\int_{\mathbb{B}(\epsilon)\cap V} dv_V \leq (4\pi)^d \epsilon^{2d} \deg(\tilde{T}_{\epsilon^{-1}}(V)). \tag{3.33}$$

Since $\tilde{T}_{\epsilon^{-1}} \in \mathrm{Aut}(\mathbb{P}^N(\mathbb{C}))$, we find

$$\deg(V) = \deg(\tilde{T}_{\epsilon^{-1}}(V)), \tag{3.34}$$

which combined with (3.33) yields the assertion. □

LEMMA 3.4. *Let C be an algebraic curve in $\mathbb{P}^N(\mathbb{C})$. Let U be the affine open set and $z = (z_1, \cdots, z_N)$ be the coordinate of U as in the previous lemma. Set*

$$\rho(z) := d_{\mathbb{P}^N}(0, z) = \int_0^{\|z\|} \frac{dr}{1+r^2}.$$

Then, for any $\epsilon \in (0, 1/16)$, the following inequality holds:

$$\int_{C\cap\{\epsilon\leq\rho\leq\sqrt{\epsilon}\}} \frac{dv_C}{\rho^2(-\log\epsilon)^2} \leq C(N)\frac{\deg(C)}{-\log\epsilon}.$$

Proof. In this proof, we consider $\partial\bar{\partial}\phi$ as a 2-form for any function ϕ. By the definition,

$$\frac{1}{2}\|z\| \leq \rho(z) \leq \|z\| \quad \text{for } \rho \leq \sqrt{\epsilon}. \tag{3.35}$$

Therefore by Lemma 3.2,

$$\begin{aligned}
&\int_{C\cap\{\epsilon\leq\rho\leq\sqrt{\epsilon}\}} \frac{dv_C}{\rho^2(-\log\epsilon)^2} \\
&\qquad \leq 2\int_{C\cap\{\epsilon\leq r\leq 2\sqrt{\epsilon}\}} \frac{\sqrt{-1}}{2}\frac{\partial\bar{\partial}\|z\|^2}{\|z\|^2(-\log\|z\|^2)^2} \\
&\qquad \leq 2\int_{C\cap\{\epsilon\leq r\leq 2\sqrt{\epsilon}\}} -\frac{\sqrt{-1}}{2}\partial\bar{\partial}\log(-\log\|z\|^2)
\end{aligned} \tag{3.36}$$

where $r(z) := \|z\|$. Set

$$d^c := -\sqrt{-1}(\partial - \bar{\partial}), \quad dd^c = 2\sqrt{-1}\partial\bar{\partial}. \tag{3.37}$$

By computation,

$$\begin{aligned}
I(\epsilon) &:= \int_{C\cap\{\epsilon\leq r\leq 2\sqrt{\epsilon}\}} -\frac{\sqrt{-1}}{2}\partial\bar{\partial}\log(-\log\|z\|^2) \\
&= \frac{1}{4}\int_{C\cap\{\epsilon\leq r\leq 2\sqrt{\epsilon}\}} -dd^c\log(-\log\|z\|^2) \\
&= \frac{1}{4}\int_{C\cap\{r=2\sqrt{\epsilon}\}} -d^c\log(-\log\|z\|^2) - \frac{1}{4}\int_{C\cap\{r=\epsilon\}} -d^c\log(-\log\|z\|^2) \\
&= \frac{1}{4}\int_{C\cap\{r=2\sqrt{\epsilon}\}} \frac{d^c\|z\|^2}{\|z\|^2(-\log\|z\|^2)} - \frac{1}{4}\int_{C\cap\{r=\epsilon\}} \frac{d^c\|z\|^2}{\|z\|^2(-\log\|z\|^2)} \\
&= \frac{1}{4}\int_{C\cap\{r=2\sqrt{\epsilon}\}} \frac{d^c\|z\|^2}{(2\sqrt{\epsilon})^2(-\log(2\sqrt{\epsilon})^2)} - \frac{1}{4}\int_{C\cap\{r=\epsilon\}} \frac{d^c\|z\|^2}{\epsilon^2(-\log\epsilon^2)} \\
&= (2\sqrt{\epsilon})^{-2}(-\log(2\sqrt{\epsilon})^2)^{-1}\int_{C\cap\mathbb{B}(2\sqrt{\epsilon})} \frac{\sqrt{-1}}{2}\partial\bar{\partial}\|z\|^2 \\
&\qquad\qquad - \epsilon^{-2}(-\log\epsilon^2)^{-1}\int_{C\cap\mathbb{B}(\epsilon)} \frac{\sqrt{-1}}{2}\partial\bar{\partial}\|z\|^2.
\end{aligned} \tag{3.38}$$

Applying Lemma 3.3 to the integrands of the last equality of (3.38), we get

$$\begin{aligned}
&\int_{C\cap\{\epsilon\leq r\leq 2\sqrt{\epsilon}\}} -\frac{\sqrt{-1}}{2}\partial\bar{\partial}\log(-\log\|z\|^2) \\
&\qquad\qquad \leq C(N)\deg(C)\left(\frac{1}{-\log 4\epsilon} + \frac{1}{-\log\epsilon^2}\right).
\end{aligned} \tag{3.39}$$

When $\epsilon \in (0, 1/16)$, it is easy to verify the inequality:

$$\frac{1}{-\log 4\epsilon} + \frac{1}{-\log \epsilon^2} \leq (\log \frac{1}{\epsilon})^{-1}. \tag{3.40}$$

By (3.36), (3.38), (3.39) and (3.40), we get the desired inequality. □

Proof of Theorem 3.1. We prove the theorem by induction. When $\dim Y \leq \dim X - 2$, define a function $\rho_{Y,\epsilon}$ by

$$\rho_{Y,\epsilon}(x) := \begin{cases} 0 & (r_Y(x) \leq \epsilon) \\ \epsilon^{-1}(r_Y(x) - \epsilon) & (\epsilon \leq r_Y(x) \leq 2\epsilon) \\ 1 & (r_Y(x) \geq 2\epsilon), \end{cases} \tag{3.41}$$

where r_Y is the same function as (3.3). When $\dim Y = \dim X - 1$, define $\rho_{Y,\epsilon}$ by

$$\rho_{Y,\epsilon}(x) := \begin{cases} 0 & (r_Y(x) \leq \epsilon) \\ \frac{2}{\log \epsilon^{-1}} \int_\epsilon^{r_Y(x)} \frac{d\rho}{\rho} & (\epsilon \leq r_Y(x) \leq \sqrt{\epsilon}) \\ 1 & (r_Y(x) \geq \sqrt{\epsilon}). \end{cases} \tag{3.42}$$

It is clear by the definition that $0 \leq \rho_{Y,\epsilon} \leq 1$.

First let us prove the theorem when $\dim X > 1$ and $\dim Y = 0$. Set $\gamma(\epsilon) = C\epsilon$, $\delta(\epsilon) = 2\epsilon$ and $\eta_\epsilon(x) := \rho_{Y,\epsilon}(x)$. By the definition (3.41), the condition (2) of the theorem is satisfied. Since Y is a discrete set of X, we may assume $Y = \{p\}$. By computation,

$$\begin{aligned} \|1 - \eta_\epsilon\|_2^2 + \|d\eta_\epsilon\|_2^2 &\leq \mathrm{vol}(X \cap Y_\epsilon) + \epsilon^{-2} \int_{\epsilon \leq r_Y \leq 2\epsilon} |dr_Y|^2 dv_X \\ &\leq C(N) \deg(X) \epsilon^{2n} + \sup_X |dr_Y|^2 \epsilon^{-2} \mathrm{vol}(X \cap Y_{2\epsilon}) \\ &\leq C(N) \deg(X) \epsilon^{2(n-1)} (1 + \sup_X |dr_Y|^2) \end{aligned} \tag{3.43}$$

where Lemma 3.3 is used. By the definition of r_Y, we find

$$|r_Y(x) - r_Y(y)| \leq d_{\mathbb{P}^N}(x, y) \leq d_X(x, y) \tag{3.44}$$

for any $x, y \in X$ which implies

$$\sup_X |dr_Y| \leq \sqrt{2n}. \tag{3.45}$$

Commbining (3.43) and (3.44), we get

$$\|1-\eta_\epsilon\|_2^2+\|d\eta_\epsilon\|_2^2 \le C(N)\deg(X)\epsilon^{2(n-1)}, \tag{3.46}$$

which proves the theorem when $\dim Y=0$.

Next assume the theorem for any subvariety of pure dimension $k-1$ and prove it for an arbitrary subvariety Y of pure dimension k. Let U_Y be the Zariski open subset of Y considered in Proposition 3.1. Set

$$Z=Z_Y:=(Y-U_Y)\cup \operatorname{Sing}(Y). \tag{3.47}$$

Then Z is a subvariety of Y whose dimension is strictly smaller than k.

Let $Z=\cup_i Z_i$ be the irreducible decomposition of Z. By the induction hypothesis, there exists a function $\eta_{i,\epsilon}$ which satisfies the conditions of Theorem 3.1 for (X,Z_i). Set

$$\xi_\epsilon(x):=\prod_i \eta_{i,\epsilon}(x). \tag{3.48}$$

Let $\gamma_i(\epsilon)$ and $\delta_i(\epsilon)$ be the functions of Thorem 3.1 for (X,Z_i). Set

$$\gamma_Z(\epsilon):=\sharp\{i\}\max_i \gamma_i(\epsilon), \quad \delta_Z(\epsilon):=\max_i \delta_i(\epsilon). \tag{3.49}$$

Then it is easily verified that $\gamma_Z(\epsilon)$, $\delta_Z(\epsilon)$ and ξ_ϵ satisfy the conditions of Theorem 3.1 for the pair (X,Z). Therefore $\{\xi_\epsilon\}$ is a family of functions which satisfies the conditions of Theorem 3.1 for (X,Z).

Since $Z\supset \operatorname{Sing}(Y)$, we can apply Lemma 3.1 to the pair (Y,Z). Let $a(\epsilon)$ be the function considered in the lemma. Define η_ϵ by

$$\eta_\epsilon(x):=\rho_{Y,\epsilon}(x)\,\xi_{2a(2\epsilon)}(x). \tag{3.50}$$

In the sequel, we shall verify the conditions of Theorem 3.1 for $\{\eta_\epsilon\}$.

By the definition, it is clear

$$0\le \eta_\epsilon \le 1. \tag{3.51}$$

Next we verify (2). Since

$$\begin{gathered}\operatorname{supp}\eta_\epsilon=\operatorname{supp}\rho_\epsilon\cap\operatorname{supp}\xi_{2a(2\epsilon)}, \quad \operatorname{supp}\rho_\epsilon=X-Y_\epsilon,\\ \bigcup_i(X-Z_{i,\delta_i(\epsilon)})\subset\operatorname{supp}\xi_\epsilon\subset\bigcup_i(X-Z_{i,\epsilon}),\end{gathered} \tag{3.52}$$

we have

$$(3.53)\quad (X - Y_\epsilon) \cap \cup_i (X - Z_{i,\delta_i(2a(2\epsilon))}) \subset \operatorname{supp} \eta_\epsilon \subset (X - Y_\epsilon) \cap \cup_i (X - Z_{i,\epsilon}).$$

As Z is a subset of Y and $\delta_Z(2a(2\epsilon)) \geq \max_i\{\delta_i(2a(2\epsilon)), \epsilon\}$, it is clear

$$(3.54)\qquad X - Y_{\delta_Z(2a(2\epsilon))} \subset \operatorname{supp} \eta_\epsilon \subset X - Y_\epsilon.$$

Since

$$(3.55)\qquad \begin{aligned} \operatorname{supp}(1 - \eta_\epsilon) &\subset \operatorname{supp}(1 - \rho_\epsilon) \cup \operatorname{supp}(1 - \xi_{2a(2\epsilon)}) \\ &\subset \operatorname{supp}(1 - \rho_\epsilon) \cup \bigcup_i \operatorname{supp}(1 - \eta_{i,2a(2\epsilon)}), \end{aligned}$$

when $k = \dim Y < n - 1$, we get

$$(3.56)\qquad \operatorname{supp}(1 - \eta_\epsilon) \subset Y_{2\epsilon} \cup \bigcup_i Z_{i,2a(2\epsilon)} \subset Y_{2\delta_Z(2a(2\epsilon))}$$

and when $k = \dim Y = n - 1$,

$$(3.57)\qquad \operatorname{supp}(1 - \eta_\epsilon) \subset Y_{\sqrt{\epsilon}} \cup Z_{\delta_Z(2a(2\epsilon))}.$$

Set

$$(3.58)\qquad \tilde{\delta}(\epsilon) := \max\{\sqrt{\epsilon}, \delta_Z(2a(2\epsilon))\} (\geq \epsilon).$$

By (3.54), (3.57) and (3.58), Theorem 3.1 (2) is satisfied by η_ϵ and $\tilde{\delta}(\epsilon)$.

Next verify the latter part of (1).

Case 1 ($k = \dim Y < n - 1$)

From (3.50) and the induction hypothesis, it follows that

$$(3.59)\qquad \begin{aligned} \|1 - \eta_\epsilon\|_2 &\leq \|1 - \rho_\epsilon\|_2 + \|1 - \xi_{2a(2\epsilon)}\|_2 \\ &\leq \sqrt{\operatorname{vol}(Y_{2\epsilon})} + \gamma_Z(2a(2\epsilon)) \end{aligned}$$

and

$$(3.60)\qquad \begin{aligned} \|d\eta_\epsilon\|_2 &= \|\xi_{2a(2\epsilon)} d\rho_\epsilon + \rho_\epsilon d\xi_{2a(2\epsilon)}\|_2 \\ &\leq \|\xi_{2a(2\epsilon)} d\rho_\epsilon\|_2 + \|d\xi_{2a(2\epsilon)}\|_2 \\ &\leq \|\xi_{2a(2\epsilon)} d\rho_\epsilon\|_2 + \gamma_Z(2a(2\epsilon)). \end{aligned}$$

It follows from (3.41),

$$\|\xi_{2a(2\epsilon)}d\rho_\epsilon\|_2^2 = \epsilon^{-2}\int_{X\cap(Y_{2\epsilon}-Y_\epsilon)\cap\operatorname{supp}\xi_{2a(2\epsilon)}} |dr_Y|^2 dv_X \le 2n\epsilon^{-2}\operatorname{vol}((Y_{2\epsilon}-Y_\epsilon)\cap(X-Z_{2a(2\epsilon)})). \tag{3.61}$$

When $x \in (Y_{2\epsilon}) \cap (X - Z_{2a(2\epsilon)})$, let y be a point of Y such that $r_Y(x) = d_{\mathbb{P}^N}(x,y)$. By the trigonometrical inequality and (3.25), we get

$$\begin{aligned} d_{\mathbb{P}^N}(y,Z) &\ge d_{\mathbb{P}^N}(x,Z) - d_{\mathbb{P}^N}(x,y) \\ &\ge 2a(2\epsilon) - 2\epsilon \\ &\ge a(2\epsilon), \end{aligned} \tag{3.62}$$

which implies $y \in Y - Z_{a(2\epsilon)}$ and $x \in N_y(2\epsilon)$. Therefore we have

$$(Y_{2\epsilon} - Y_\epsilon) \cap (X - Z_{2a(2\epsilon)}) \subset p(\pi^{-1}(Y - Z_{a(2\epsilon)}) \cap N_Y(2\epsilon)) \tag{3.63}$$

where $N_Y(\epsilon)$ is the same as in (3.16) and p and π are the same as in (3.13).

From (3.61), (3.63), Lemma 3.1 and Fubini's theorem, it follows that

$$\begin{aligned} \|\xi_{2a(2\epsilon)}d\rho_\epsilon\|_2^2 &\le 2n\epsilon^{-2}\operatorname{vol}(X \cap p(\pi^{-1}(Y - Z_{a(2\epsilon)}) \cap N_Y(2\epsilon))) \\ &\le 4^n 2n\epsilon^{-2}\int_{Y-Z_{a(2\epsilon)}} dv_Y \int_{X\cap N_y(2\epsilon)} dv_{X\cap N_y(2\epsilon)}. \end{aligned} \tag{3.64}$$

Since $y \in U_Y$, $(X \cap N_y)_y$ is a pure dimensional space and thus Lemma 3.3 is applicable. Set $W_y := (X \cap N_y)_y$ and consider y as the origin of U in Lemma 3.3. Applying Lemma 3.3 and Proposition 3.2, we get

$$\begin{aligned} \int_{X\cap N_y(2\epsilon)} dv_{X\cap N_y(2\epsilon)} &= \int_{W_y\cap\{\rho\le 4\epsilon\}} dv_{\mathbb{P}^{n-k}} \\ &\le C(n-k)\deg(W_y)(8\epsilon)^{2(n-k)} \\ &\le C(N)\deg(X)\epsilon^{2(n-k)}, \end{aligned} \tag{3.65}$$

which combined with (3.60), (3.64) yields

$$\|d\eta_\epsilon\|_2^2 \le C(N)\deg(X)\epsilon^{2(n-k-1)}\operatorname{vol}(Y) + \gamma_Z(2a(2\epsilon)). \tag{3.66}$$

By (3.60) and (3.66), Theorem 3.1 (1) is proved for η_ϵ.

Set

$$\tilde{\gamma}(\epsilon) := C(N)\deg(X)\epsilon^{2(n-k-1)} + \gamma_Z(2a(2\epsilon)). \tag{3.67}$$

From (3.49), (3.58) and the induction hypothesis, it follows that

$$\lim_{\epsilon\to 0}\tilde{\gamma}(\epsilon) = 0, \tag{3.68}$$

which combined with (3.66) yields, the latter part of (1).

Case 2 ($k = \dim Y = n-1$)

By (3.59) and (3.60), it is sufficient to show

$$\lim_{\epsilon\to 0}\|\xi_{2a(2\epsilon)}d\rho_\epsilon\|_2 = 0. \tag{3.69}$$

It follows from (3.42),

$$\begin{aligned}\|\xi_{2a(2\epsilon)}d\rho_\epsilon\|_2^2 &= 4(\log\epsilon^{-1})^{-2}\int_{X\cap(Y_{\sqrt{\epsilon}}-Y_\epsilon)\cap\operatorname{supp}\xi_{2a(2\epsilon)}} r_Y^{-2}|dr_Y|^2 dv_X \\ &\le 8n(\log\epsilon^{-1})^{-2}\int_{X\cap(Y_{\sqrt{\epsilon}}-Y_\epsilon)\cap\operatorname{supp}\xi_{2a(2\epsilon)}} r_Y^{-2}dv_X.\end{aligned} \tag{3.70}$$

When $x \in (Y_{\sqrt{\epsilon}}) \cap (X - Z_{2a(2\epsilon)})$, let y be a point of Y such that $r_Y(x) = d_{\mathbb{P}^N}(x,y)$. By the trigonometrical inequality and (3.25), we get

$$\begin{aligned} d_{\mathbb{P}^N}(y,Z) &\ge d_{\mathbb{P}^N}(x,Z) - d_{\mathbb{P}^N}(x,y) \\ &\ge 2a(2\epsilon) - \sqrt{\epsilon} \\ &\ge a(\epsilon),\end{aligned} \tag{3.71}$$

which implies

$$X\cap(Y_{\sqrt{\epsilon}}-Y_\epsilon)\cap\operatorname{supp}\xi_{3a(\epsilon)} \subset p(\pi^{-1}(Y-Z_{a(\epsilon)})\cap\{N_Y(\sqrt{\epsilon})-N_Y(\epsilon)\}). \tag{3.72}$$

From (3.70), (3.72), Lemma 3.1 and Fubini's theorem, it follows that

(3.73)

$$\begin{aligned} &\|\xi_{3a(\epsilon)}d\rho_\epsilon\|_2^2 \\ &\le 8n(\log\epsilon^{-1})^{-2}\int_{X\cap p(\pi^{-1}(Y-Z_{a(\epsilon)})\cap\{N_Y(\sqrt{\epsilon})-N_Y(\epsilon)\})} r_Y^{-2}dv_X \\ &\le 4^n 8n(\log\epsilon^{-1})^{-2}\int_{p^{-1}(X)\cap\pi^{-1}(Y-Z_{a(\epsilon)})\cap(N_Y(\sqrt{\epsilon})-N_Y(\epsilon))} r_Y^{-2}\pi^* dv_Y dv_{X\cap N_Y} \\ &= 4^n 8n(\log\epsilon^{-1})^{-2}\int_{Y-Z_{a(\epsilon)}} dv_Y(y)\int_{(X\cap N_y)_y\cap(N_y(\sqrt{\epsilon})-N_y(\epsilon))} r_Y^{-2}dv_{X\cap N_y}.\end{aligned}$$

Set $C_y := (X \cap N_y)_y$. By (3.73), Lemma 3.4 and Proposition 3.2,

(3.74)

$$\begin{aligned}\|\xi_{2a(2\epsilon)} d\rho_\epsilon\|_2^2 &\leq 4^n 8n \int_{Y - Z_{a(\epsilon)}} dv_Y(y) \int_{C_y \cap \{\epsilon \leq \rho_y \leq \sqrt{\epsilon}\}} (\log \epsilon^{-1})^{-2} \rho_y^{-2} dv_{C_y} \\ &\leq \int_{Y - Z_{a(\epsilon)}} C(N) \deg(C_y)(\log \epsilon^{-1})^{-1} dv_Y(y) \\ &\leq C(N) \deg(X) \operatorname{vol}(Y)(\log \epsilon^{-1})^{-1},\end{aligned}$$

which yields (3.69). Set

$$\tilde{\gamma}(\epsilon) := C(N) \deg(X) \operatorname{vol}(Y)(\log \epsilon^{-1})^{-1} + \gamma_Z(2a(2\epsilon)). \tag{3.75}$$

Then

$$\|1 - \eta_\epsilon\|_2^2 + \|d\eta_\epsilon\|_2^2 \leq \tilde{\gamma}(\epsilon), \quad \lim_{\epsilon \to 0} \tilde{\gamma}(\epsilon) = 0, \tag{3.76}$$

which proves the latter part of (1). As in the proof of Case 2, we can prove the theorem when $n = 1$ and leave it to the reader. $\square$

Proof of Theorem 3.2. Let $\Sigma_X = \cup_\alpha \Sigma_\alpha$ be the irreducible decomposition. By Theorem 3.1, we may assume that for the pair (X, Σ_α), there exists a family of cut-off functions $\{\eta_{\alpha,\epsilon}\}_{\epsilon \in (0,1/16)}$ which satisfies the conditions of Theorem 3.1. Fix such a family and set

$$\eta_\epsilon := \prod_\alpha \eta_{\alpha,\epsilon}. \tag{3.77}$$

Then we find $\operatorname{supp} \eta_\epsilon \subset X - \Sigma_\epsilon$ by Theorem 3.1 (3). Since the boundaries of the zero set of each η_ϵ are not smooth by the construction, we must regularize the cut-off functions. For simplicity, set $\Sigma = \Sigma_X$.

Let $i(x) > 0$ be the injectivity radius at $x \in X - \Sigma$. Put

$$i(\epsilon) := \frac{1}{4} \min\{\inf_{x \in X - \Sigma_\epsilon} i(x), \epsilon\}. \tag{3.78}$$

Since $X - \Sigma_\epsilon$ is compact, $i(\epsilon) > 0$ for $\epsilon > 0$. Let $\{U_i\}_{i=1}^L$ be an open covering of $X - \Sigma_\epsilon$ chosen in such a way that $\bigcup_i U_i \subset X - \Sigma_{\epsilon/2}$ and for each U_i, there is a metric ball $B_{i(\epsilon)}(x_i)$ of radius $i(\epsilon)$ centered at x_i such that $U_i = B_{i(\epsilon)}(x_i)$ and $x_i \in X - \Sigma_\epsilon$. By the Gauss lemma, identify U_i

with the ball of radius $i(\epsilon)$ in the Eucilidean space of dimension $2n$ via the exponential map

$$\phi_i : \mathbb{B}(r) \to U_i = B_r(x_i) \tag{3.79}$$

for all $r \leq 2i(\epsilon)$. Since $i(\epsilon) \leq i(x_i)$, there is a constant $C_i > 1$ such that, on $\mathbb{B}(i(\epsilon))$,

$$C_i^{-1} g_E \leq \phi_i^* g_X \leq C_i g_E. \tag{3.80}$$

where g_E is the standard Euclidean metric. Let $\{h_i\}$ be a partition of unity subject to the covering $\{U_i\}$ of $X - \Sigma_\epsilon$. Since $\eta_\epsilon = \sum_i h_i \eta_\epsilon$, first consider the regularization of $h_i \eta_\epsilon$. Set $\eta_{i,\epsilon} := \phi_i^*(h_i \eta_\epsilon)$. Since $\eta_\epsilon \in C_0^0(X - \Sigma_\epsilon) \cap W^{1,2}(X)$, it follows that $\eta_{i,\epsilon} \in C_0^0(\mathbb{B}(i(\epsilon)) \cap W^{1,2}(\mathbb{B}(i(\epsilon))$.

Following [G-H, Chap. 3, §1], let $\chi \in C_0^\infty(\mathbb{R}^{2n})$ be a nonnegative function supported in a neighborfood of the origin with

$$\int_{\mathbb{R}^{2n}} \chi(x) dv = 1. \tag{3.81}$$

Put

$$\chi_\delta(x) := \delta^{-2n} \chi(\frac{x}{\delta}). \tag{3.82}$$

If $\operatorname{supp} \chi = K$, then $\operatorname{supp} \chi_\delta = \delta K$ and

$$\int_{\mathbb{R}^{2n}} \chi_\delta(x) dv = 1. \tag{3.83}$$

For any function $f \in L^1_{\text{loc}}(\mathbb{R}^{2n})$, set

$$f_\delta(x) := \int_{\mathbb{R}^{2n}} \chi_\delta(x-y) f(y)\, dy = \int_{\mathbb{R}^{2n}} f(x-y) \chi_\delta(y)\, dy. \tag{3.84}$$

By [G-H, pp. 374] and [Na, Lemma 14.1], if $f \in C_0^0(\mathbb{B}(r)) \cap W^{1,2}(\mathbb{R}^{2n})$, then $\{f_\delta\}$ satisfies the following conditions:
(1) f_δ is a smooth function and $\operatorname{supp} f_\delta \subset \mathbb{B}(r+\delta)$.
(2)

$$\lim_{\delta \to 0} \sup_{\mathbb{B}(2r)} |f_\delta - f| = 0. \tag{3.85}$$

(3)

$$\lim_{\delta\to 0} \|f_\delta - f\|_2 + \|df_\delta - df\|_2 = 0. \tag{3.86}$$

Set

$$\eta_{\epsilon,\delta} := \sum_i (\phi^{-1})^* (\eta_{i,\epsilon})_\delta. \tag{3.87}$$

From (3.79), Theorem 3.1 (1) and (1) above, if $\delta < i(\epsilon)$, it follows that $\eta_{\epsilon,\delta} \in C_0^\infty(X - \Sigma_{\epsilon/2})$. Since

$$\sup_{X-\Sigma_{\epsilon/2}} |\eta_{\epsilon,\delta} - \eta_\epsilon| \leq \sum_i \sup_{U_i} |(\eta_{i,\epsilon})_\delta - \eta_{i,\epsilon}|, \tag{3.88}$$

$$\|\eta_{\epsilon,\delta} - \eta_\epsilon\|_2 + \|d\eta_{\epsilon,\delta} - d\eta_\epsilon\|_2 \leq \sum_i \|(\eta_{i,\epsilon})_\delta - \eta_{i,\epsilon}\|_2 + \|d(\eta_{i,\epsilon})_\delta - d\eta_{i,\epsilon}\|_2, \tag{3.89}$$

using (3.79), (3.85) and (3.86), there exists $j(\epsilon)$ such that $0 < j(\epsilon) < i(\epsilon)$ and that for any $\delta \leq j(\epsilon)$,

$$\sup_{X-\Sigma_{\epsilon/2}} |\eta_{\epsilon,\delta} - \eta_\epsilon| < \epsilon, \tag{3.90}$$

$$\|\eta_{\epsilon,\delta} - \eta_\epsilon\|_2 + \|d\eta_{\epsilon,\delta} - d\eta_\epsilon\|_2 < \epsilon. \tag{3.91}$$

By the definition, $\eta_{\epsilon,j(\epsilon)}$ is a smooth function which takes zero and is not identically equal to zero. By Sard's theorem, there exists a regular value $A(\epsilon)$ of $\eta_{\epsilon,j(\epsilon)}$ such that $0 < A(\epsilon) < \epsilon$. Finally, set

$$\bar{\eta}_\epsilon(x) := \begin{cases} \eta_{\epsilon,j(\epsilon)}(x) - A(\epsilon) & (\eta_{\epsilon,j(\epsilon)}(x) \geq A(\epsilon)) \\ 0 & (\eta_{\epsilon,j(\epsilon)}(x) \leq A(\epsilon)). \end{cases} \tag{3.92}$$

Let us verify that $\bar{\eta}_\epsilon$ satisfies the conditions of Theorem 3.2.

It is clear that $\bar{\eta}_\epsilon \geq 0$, and that

$$\operatorname{supp} \bar{\eta}_\epsilon \subset \operatorname{supp} \eta_{\epsilon,j(\epsilon)} \subset X - \Sigma_{\frac{\epsilon}{2}}, \tag{3.93}$$

as $j(\epsilon) < i(\epsilon)$ and $\operatorname{supp} \eta_{\epsilon,\delta} \subset X - \Sigma_{\epsilon/2}$ for $\delta < i(\epsilon)$. From (3.90) and Theorem 3.1 (1), it follows that

$$\bar{\eta}_\epsilon \leq \eta_{\epsilon,j(\epsilon)} \leq 1 + \epsilon, \tag{3.94}$$

which proves Theorem 3.2 (1). By (3.91) and (3.92),

$$(3.95)\quad \begin{aligned}\|\bar{\eta}_\epsilon - 1\|_2 + \|d\bar{\eta}_\epsilon\|_2 &\leq \|\eta_{\epsilon,j(\epsilon)} - 1\|_2 + \|d\eta_{\epsilon,j(\epsilon)}\|_2 + \epsilon\sqrt{\mathrm{vol}(X)}\\ &\leq \|\eta_\epsilon - 1\|_2 + \|d\eta_\epsilon\|_2 + \epsilon(1+\sqrt{\mathrm{vol}(X)}).\end{aligned}$$

By (3.77) and Theorem 3.1 (1), we get

$$(3.96)\qquad \|\eta_\epsilon - 1\|_2 + \|d\eta_\epsilon\|_2 \leq \sum_\alpha \|\eta_{\alpha,\epsilon} - 1\|_2 + \|d\eta_{\alpha,\epsilon}\|_2 \leq \sum_\alpha \gamma_\alpha(\epsilon)$$

where $\gamma_\alpha(\epsilon)$ satisfies Theorem 3.1 (1) for the pair (X, Σ_α), which combined with (3.95) yields Theorem 3.2 (2). By (3.92), it is clear that

$$(3.97)\qquad S(\epsilon) = \{x \in X; \eta_{\epsilon,j(\epsilon)}(x) \leq A(\epsilon)\},\quad \partial S(\epsilon) = \eta^{-1}_{\epsilon,j(\epsilon)}(A(\epsilon)).$$

As $A(\epsilon)$ is a regular value of $\eta_{\epsilon,j(\epsilon)}$, $\partial S(\epsilon)$ is a smooth $2n-1$-dimensional manifold and consists of finitely many connected components by the compactness of X. This proves (3).

When $x \in \Sigma_{\frac{\epsilon}{2}}$, then $\eta_{\epsilon,j(\epsilon)}(x) = 0$ and therefore $x \in S(\epsilon)$ by (3.92), which implies $\Sigma_{\frac{\epsilon}{2}} \subset S(\epsilon)$. When $x \in S(\epsilon)$, it follows that $\eta_{\epsilon,j(\epsilon)}(x) \leq A(\epsilon) \leq \epsilon$ which combined with (3.90) yields $\eta_\epsilon(x) < 2\epsilon < 1$. This implies $x \in \mathrm{supp}(1-\eta_\epsilon)$. By (3.77), one of α satisfies $\eta_{\alpha,\epsilon}(x) < 1$. Therefore we have

$$(3.98)\qquad S(\epsilon) \subset \bigcup_\alpha \mathrm{supp}(1-\eta_{\alpha,\epsilon}) \subset \bigcup_\alpha \Sigma_{\alpha,2\delta_\alpha(\epsilon)} \subset \Sigma_{\delta(\epsilon)}$$

where $\delta(\epsilon) := 2\max_\alpha\{\delta_\alpha(\epsilon)\}$ and $\delta_\alpha(\epsilon)$ satisfies Theorem 3.1 (2) for the pair (X, Σ_α).

Since $\lim_{\epsilon\to 0}\delta(\epsilon) = 0$, we get Theorem 3.2 (4) by (3.97). □

For our later purpose (cf. §6), we need the following.

LEMMA 3.5. *Let (X, g) be an irreducible algebraic variety in $\mathbb{P}^N(\mathbb{C})$ with the Bergmann metric. Let $\pi : X' \to X - \Sigma_X$ be a m-sheeted covering of $X - \Sigma_X$. Then*

$$W^{1,2}_0(X', \pi^*g) = W^{1,2}(X', \pi^*g)$$

and Theorem 3.2 holds for (X', π^*g).

Proof. By [L-T], it is sufficient to show Theorem 3.2 for (X', π^*g). Considering the pull back of the family of cut-off functions of Theorem 3.2 for (X, g), it satisfies the conditions of Theorem 3.2, since π is a finite covering. □

§4. A comparison theorem for the heat kernel of projective varieties

In this section, we recall the results of [L-T] concerning the upper bound of the heat kernel of projective algebraic varieties with the Bergmann metric.

Let M be a projective algebraic variety of pure dimension n in the projective space $\mathbb{P}^N(\mathbb{C})$. Let g be the Bergmann metric of M; i.e., the restriction of the Fubini-Study metric of $\mathbb{P}^N(\mathbb{C})$. Let S be the singular set of M. Consider the Dirichlet Laplacian of $M - S$; i.e., the Friedrichs extension of the Laplacian on $C_0^\infty(M - S)$ which is defined by:

$$\Delta := \delta_{\max} d_{\min}. \tag{4.1}$$

Since $d_{\max} = d_{\min}$ on the space of functions (cf. [L-T, Theorem 4.1] and Corollary 3.1), we can ignore the Dirichlet boundary condition. As the semi-group generated by the Dirichlet Laplacian has a smooth kernel function denoted by $K_M(t,x,y)$. In particular, we denote by $K_{\mathbb{P}^n}(t,x,y)$ the heat kernel of complex projective space of dimension n with the Fubini-Study metric.

Let $r(x,y)$ be the distance of two points x and y in $\mathbb{P}^n(\mathbb{C})$ relative to the Fubini-Study metric. By the symmetry on $\mathbb{P}^n(\mathbb{C})$,

$$K_{\mathbb{P}^n}(t,x,y) = K_{\mathbb{P}^n}(t, r(x,y)) \tag{4.2}$$

where $K_{\mathbb{P}^n}(t,s)$ is a function of t and s. Now we can state the result of Li-Tian.

THEOREM 4.1. ([L-T]) *Let (M,g) be a projective algebraic variety of pure dimension n with the Bergmann metric. Then the following comparison theorem of the heat kernels holds for all $(t,x,y) \in [0,\infty) \times (M-S) \times (M-S)$:*

$$K_M(t,x,y) \leq K_{\mathbb{P}^n}(t, r_{\mathbb{P}^N}(x,y))$$

where $r_{\mathbb{P}^N}(x,y)$ is the distance from x to y in $\mathbb{P}^N(\mathbb{C})$.

Combining Theorem 4.1 and Theorem 2.1, we have the following corollary.

COROLLARY 4.1. *Let (M, g) be the same as in Theorem 4.1. Then there is a constant $C(n)$ which depends only on n such that for every $f \in C_0^\infty(M - S)$,*

$$\|f\|_{\frac{2n}{n-2}} \leq C(n)(\|df\|_2 + \|f\|_2) \qquad (n > 1),$$

$$\|f\|_4 \leq C(1)(\|df\|_2 + \|f\|_2) \qquad (n = 1).$$

COROLLARY 4.2. *Let $\pi : \mathfrak{X} \to \Delta(1)$ be a one parameter degenerating family of projective algebraic manifolds of dimension n in a fixed projective space $\mathbb{P}^N(\mathbb{C})$ as in the introduction. Let G be an arbitrary Riemannian metric of $\mathfrak{X}$. Set $X_t := \pi^{-1}(t)$ and $g_t := G|_{X_t}$ for $t \in \Delta$. Then there exists a constant $C > 0$ independent of $t \in \Delta(1/2) - \{0\}$ such that for every $f \in C^\infty(X_t)$, the following inequality holds:*

$$\|f\|_{\frac{2n}{n-2},t} \leq C(\|df\|_{2,t} + \|f\|_{2,t}) \qquad (n > 1),$$
$$\|f\|_{4,t} \leq C(\|df\|_{2,t} + \|f\|_{2,t}) \qquad (n = 1)$$

where $\|\cdot\|_{p,t}$ is the L^p-norm with respect to g_t.

Proof. Since $\mathfrak{X}|_{\Delta(1/2)} \Subset \mathbb{P}^N(\mathbb{C}) \times \Delta(1)$, there is a constant C_0 such that

$$C_0^{-1}(g_{\mathbb{P}^N(\mathbb{C})} + |dt|^2) \leq g \leq C_0(g_{\mathbb{P}^N(\mathbb{C})} + |dt|^2) \tag{4.3}$$

on $\pi^{-1}(\Delta(1/2))$ where $g_{\mathbb{P}^N(\mathbb{C})}$ is the Fubini-Study metric of $\mathbb{P}^N(\mathbb{C})$. which implies

$$C_0^{-1} g_{\mathbb{P}^N(\mathbb{C})}|_{X_t} \leq g_t \leq C_0 g_{\mathbb{P}^N(\mathbb{C})}|_{X_t}. \tag{4.4}$$

By Corollary 4.1 and (4.4), we obtain the desired estimates. □

§5. Continuity of the spectrum in the parameter

Let (M, g) be a Riemannian manifold of dimension n with finite volume, and let $\{(M_i, g_i)\}_{i \geq 1}$ be a sequence of compact Riemannian manifolds of dimension n. In this section, we shall show that the spectrum of the Laplacian of (M_i, g_i), converges to that of the Dirichlet Laplacian of (M, g) as $i \to \infty$ under certain conditions, which is an abstract version of our Main Theorem. Introduce the following condition for (M, g).

CONDITION. **(C1)** (1) *There exists a sequence of cut-off functions* $\{\rho_i\}_{i=0}^{\infty} \subset C_0^0(M) \cap W^{1,2}(M)$ *which satisfies the following conditions:*

$$\lim_{i \to \infty} \|\rho_i - 1\|_2 + \|d\rho_i\|_2 = 0, \quad 0 \leq \rho_i \leq 1.$$

(2) $S_i := \{x \in M; \rho_i(x) = 0\}$ *has finitely many smooth boundaries.*
(3)

$$\lim_{i \to \infty} \mathrm{vol}(S_i) = 0.$$

Set $\gamma(i) := \|\rho_i - 1\|_2 + \|d\rho_i\|_2$. Then $\gamma(i) \to 0$ as $i \to \infty$.

Remark 5.1. From the argument of [L-T, Theorem 4.1], under (C1), it follows that $W^{1,2}(M) = W_0^{1,2}(M)$ with respect to g, or equivalently $d_{\max} = d_{\min}$ for (M, g).

Introduce the following condition for the family $\{(M_i, g_i)\}$:

CONDITION. **(C2)** *There exists a sequence of open subsets* $\{S_i\}$ *of* M *and a sequence of into-diffeomorphisms* $\{f_i\}_{i \geq 1}$ $f_i : M - S_i \hookrightarrow M_i$ *which satisfies the following conditions:*
(1) *For every* $i \geq 1$,

$$\frac{1}{2} g \leq f_i^* g_i \leq 2g$$

on $M - S_i$.
(2) *If we trivially extend* $f_i^* g_i$ *to* M *by setting* $f_i^* g_i := 0$ *on* S_i, *then*

$$\lim_{i \to \infty} f_i^* g_i = g$$

almost everywhere on M.
(3) *Setting* $K_i := M_i - f_i(M - S_i)$,

$$\lim_{i \to \infty} \mathrm{vol}(K_i) = 0.$$

If $\{S_i\}$ *is the same one as in* (C1), *then* (C2) *is said to be subject to* (C1).

Let $\{\lambda_1(i) \leq \lambda_2(i) \leq \cdots\}$ be the spectrum of Δ_i, the Laplacian of (M_i, g_i), and $\{\lambda_1 \leq \lambda_2 \leq \cdots\}$ the spectrum of $\Delta := \delta_{\max} d_{\min}$, the Dirichlet Laplacian of (M, g), counted with multiplicities. Our goal in this section is to prove the following theorem.

THEOREM 5.1. *Let $\{(M_i, g_i)\}$ be a sequence of compact Riemannian manifolds and (M, g) a Riemannian manifold with finite volume satisfying* (C1). *If* (C2) *subject to* (C1) *is satisfied for $\{(M_i, g_i)\}$ and (M, g), and if the Sobolev inequality is uniform; i.e., there exists a constant $C > 0$ independent of i such that*

$$\|\phi\|_{\frac{2n}{n-2},i} \leq C(\|d\phi\|_{2,i} + \|\phi\|_{2,i})$$

for every $\phi \in C^\infty(M_i)$ where $\|\cdot\|_{p,i}$ is the L^p-norm of (M_i, g_i), then

$$\lim_{i\to\infty} \lambda_k(i) = \lambda_k.$$

As mentioned in the introduction, we prove the theorem only when $n > 2$. To show that the spectrum of the Dirichlet Laplacian of M consists of discrete eigenvalues, we need the Rellich lemma for M.

LEMMA 5.1. *Under the assumption of Theorem* 5.1, *the Sovolev inequality holds on M. Namely there exists a constant $C > 0$ such that for any $\psi \in C_0^\infty(M)$, the following inequality holds:*

$$\|\psi\|_{\frac{2n}{n-2}} \leq C(\|d\psi\|_2 + \|\psi\|_2).$$

Proof. Let $\psi \in C_0^\infty(M)$. Since $(f_i^{-1})^*(\rho_i\psi)$ is a smooth function on M_i, we get by the Sobolev inequality on M_i and (C2),

$$\|\rho_i\psi\|_{\frac{2n}{n-2}} \leq 4^n C(\|d\psi\|_2 + \|\psi\|_2 + \|\psi\|_\infty \|d\rho_i\|_2).$$

Since $\|d\rho_i\|_2 \leq \gamma(i)$ by (C1), we obtain the desired inequality, taking the limit as $i \to \infty$. □

PROPOSITION 5.1. (Rellich lemma) *Inclusion $W^{1,2}(M) \hookrightarrow L^2(M)$ is compact.*

Proof. Let $\{f_n\}$ be a bounded sequence in $W^{1,2}(M)$; i.e.,

$$\|f_n\|_2 + \|df_n\|_2 \leq C_0 < \infty$$

for all n. Since $M - S_i$ has a smooth compact boundary, Rellich lemma holds for any $M - S_i$ (cf. [G-T, Theorem 7.26]). By the diagonalization argument, we can choose a subsequence $\{f_{nm}\}$ which converges in every $L^2(M - S_i)$. By the Hölder inequality, Lemma 5.1 and Remark 5.1, we get

$$
\begin{aligned}
\|f_{nm}\|_{L^2(S_i)} &\leq \operatorname{vol}(S_i)^{\frac{1}{n}} \|f_{nm}\|_{L^{\frac{2n}{n-2}}(S_i)} \\
&\leq C \operatorname{vol}(S_i)^{\frac{1}{n}} (\|f_{nm}\|_2 + \|df_{nm}\|_2) \\
&\leq CC_0 \operatorname{vol}(S_i)^{\frac{1}{n}}.
\end{aligned} \tag{5.1}
$$

Let $\epsilon > 0$ be an arbitrary given number. By (C1), there exists $i(\epsilon)$ such that

$$
CC_0 \operatorname{vol}(S_i)^{\frac{1}{n}} < \frac{\epsilon}{2} \quad for \quad i \geq i(\epsilon). \tag{5.2}
$$

Therefore for $i \geq i(\epsilon)$, we get

$$
\|f_{nm} - f_{nm'}\|_2 \leq \|f_{nm} - f_{nm'}\|_{L^2(M-S_i)} + \frac{\epsilon}{2}. \tag{5.3}
$$

Since $\{f_{nm}\}$ converges on $M - S_i$, there exists $m(\epsilon)$ such that if $m, m' \geq m(\epsilon)$,

$$
\|f_{nm} - f_{nm'}\|_{L^2(M-S_i)} < \frac{\epsilon}{2}, \tag{5.4}
$$

which combined with (5.3) yields $\|f_{nm} - f_{nm'}\|_2 < \epsilon$ for $m, m' \geq m(\epsilon)$. □

COROLLARY 5.1. *The spectrum of the Dirichlet Laplacian of (M, g) consists of discrete eigenvalues.*

Let $\{\phi_k(i)\}$ be a complete orthonormal system of $L^2(M_i)$ which consists of eigenfunctions of Δ_i; i.e.,

$$
\Delta_i \phi_k(i) = \lambda_k(i)\phi_k(i), \quad (\phi_k(i), \phi_l(i))_i = \delta_{kl} \tag{5.5}
$$

where $(\cdot, \cdot)_i$ stands for the inner product of $L^2(M_i)$. Set

$$
\psi_k(i) := \rho_i f_i^* \phi_k(i) \in C_0^\infty(M - S_i). \tag{5.6}
$$

From the definition, Proposition 2.1 and (C), it follows that for all i and k

$$
\|\psi_k(i)\|_2 \leq C' \lambda_k(i)^n \tag{5.7}
$$

where C' is a constant depending only on C, the constant in Theorem 5.1.

PROPOSITION 5.2. *For every $N \geq 0$, there exists a subsequence $\{i(\nu)\}$ such that the following formulae hold for $0 \leq k \leq N$:*

(1)
$$\lim_{\nu \to \infty} \lambda_k(i(\nu)) = \lambda_k.$$

(2)
$$s - \lim_{\nu \to \infty} \psi_k(i(\nu)) = \phi_k \quad \text{in } L^2(M)$$
$$w - \lim_{\nu \to \infty} \psi_k(i(\nu)) = \phi_k \quad \text{in } W^{1,2}(M)$$

where ϕ_k is the eigenfunction of Δ such that $(\phi_k, \phi_l) = \delta_{kl}$ and $\Delta\phi_k = \lambda_k \phi_k$.

For the proof, we need the following.

LEMMA 5.2. *Suppose that Proposition 5.2 is true for N. Then,*

$$\limsup_{i \to \infty} \lambda_{N+1}(i) \leq \lambda_{N+1}.$$

Proof. By Proposition 5.2 for N, we may assume

$$s - \lim_{i \to \infty} \psi_k(i) = \phi_k \quad \text{in } L^2(M) \tag{5.8}$$

$$w - \lim_{i \to \infty} \psi_k(i) = \phi_k \quad \text{in } W^{1,2}(M) \tag{5.9}$$

for $0 \leq i \leq N$. Let ϕ_{N+1} be the eigenfunction of Δ_0 such that

$$(\phi_i, \phi_{N+1}) = \delta_{i,N+1} \quad \text{and} \quad \Delta_0 \phi_{N+1} = \lambda_{N+1} \phi_{N+1} \tag{5.10}$$

for $i \leq N+1$. Since $W^{1,2}(M) = W_0^{1,2}(M)$, there is a sequence $\{\phi_{N+1,i}\} \subset C_0^\infty(M)$ such that

$$\|\phi_{N+1,i} - \phi_{N+1}\|_{L_1^2} \leq \frac{1}{i}, \quad \operatorname{supp} \phi_{N+1,\nu} \subset M - S_i. \tag{5.11}$$

Set

$$\chi_{N+1}(i) := (f_i^{-1})^* \phi_{N+1,i} \in C_0^\infty(M_i - K_i), \tag{5.12}$$

$$\xi_{N+1}(i) := \chi_{N+1}(i) - \sum_{k \leq N} (\chi_{N+1}(i), \phi_k(i))_i \phi_k(i), \tag{5.13}$$

$$P(\phi_{N+1,i}) := \xi_{N+1}(i)/\|\xi_{N+i}(i)\|_{2,i} \in C^\infty(M_i). \tag{5.14}$$

It follows from the definition

$$\|P(\phi_{N+1,i})\|_{2,i} = 1, \quad P(\phi_{N+1,i}) \perp \{\phi_0(i), \cdots, \phi_N(i)\}, \tag{5.15}$$

$$\|dP(\phi_{N+1,i})\|_{2,i}^2 \geq \inf_{f \perp \{\phi_0(i),\cdots,\phi_N(i)\}} \|df\|_2^2/\|f\|_2^2 = \lambda_{N+1}(i), \tag{5.16}$$

$$\|dP(\phi_{N+1,i})\|_{2,i}^2 = \frac{\|d\chi_{N+1}(i) - \sum_{k\leq N}(\chi_{N+1}(i), \phi_k(i))_i d\phi_k(i)\|_{2,i}^2}{\|\chi_{N+1}(i) - \sum_{k\leq N}(\chi_{N+1}(i), \phi_k(i))_i \phi_k(i)\|_{2,i}^2}. \tag{5.17}$$

By computation,

$$\begin{aligned}(\chi_{N+1}(i), \phi_k(i))_i &= ((f_i^{-1})^* \phi_{N+1,i}, \phi_k(i))_i \\ &= \int_{M_i - K_i} (f_i^{-1})^* \phi_{N+1,i} \phi_k(i) dv_i \\ &= \int_{M - S_i} \phi_{N+1,i} f_i^* \phi_k(i) f_i^* dv_i.\end{aligned} \tag{5.18}$$

Set

$$G_i := \begin{cases} f_i^* dv_i/dv & (M - S_i) \\ 0 & (S_i). \end{cases} \tag{5.19}$$

By the definition and (C2), it follows that

$$\|G_i\|_\infty \leq 2^n. \tag{5.20}$$

From (5.18) and (5.19), it follows that

$$\begin{aligned}(\chi_{N+1}(i), \phi_k(i))_i &= (\phi_{N+1,i}, f_i^* \phi_k(i) G_i) \\ &= (\phi_{N+1,i}, \psi_k(i)) + (\phi_{N+1,i}, (1-\rho_i) f_i^* \phi_k(i)) \\ &\qquad + (\phi_{N+1,i}, f_i^* \phi_k(i)(G_i - 1)) \\ &= (\phi_{N+1}, \psi_k(i)) + (\phi_{N+1,i} - \phi_{N+1}, \psi_k(i)) \\ &\qquad + (\phi_{N+1,i}, (1-\rho_i) f_i^* \phi_k(i)) \\ &\qquad + (\phi_{N+1,i}, f_i^* \phi_k(i)(G_i - 1)).\end{aligned} \tag{5.21}$$

By the hypothesis,

$$\lim_{i\to\infty}(\phi_{N+1}, \psi_k(i)) = (\phi_{N+1}, \phi_k) = 0. \tag{5.22}$$

By (5.7) and (5.11),

$$\begin{aligned} |(\phi_{N+1,i} - \phi_{N+1}, \psi_k(i))| &\le \|\phi_{N+1,i} - \phi_{N+1}\|_2 \|\psi_k(i)\|_2 \\ &\le \frac{2C\lambda_k^n}{i}. \end{aligned} \tag{5.23}$$

From (C2), Proposition 2.1 and the hypothesis, it follows that for $k \le N$

$$\|\phi_k(i)\|_\infty \le C\lambda_k(i)^n \le C'(\lambda_k(i)^n + 1) \tag{5.24}$$

where $C' > 0$ is a constant independent of i and k. By (C1) and (5.24),

$$\begin{aligned} |(\phi_{N+1,i}, (1-\rho_i)f_i^*\phi_k(i))| &\le \|\phi_{N+1,i}\|_2 \|1-\rho_i\|_2 \|\phi_k(i)\|_\infty \\ &\le 2C'(\lambda_k^n + 1)\|1-\rho_i\|_2 \\ &\le 2C'(\lambda_k^n+1)\gamma(i) \end{aligned} \tag{5.25}$$

In the same manner,

$$|(\phi_{N+1,i}, f_i^*\phi_k(i)(G_i - 1))| \le 2C'(\lambda_k^n + 1)\|G_i - 1\|_2. \tag{5.26}$$

Since $\|G_i - 1\|_\infty \le 2^n + 1$ and $\lim_{i\to\infty}(G_i(x) - 1) = 0$ for almost every $x \in M$ by (C2) and (5.19), the Lebesgue convergence theorem implies

$$\lim_{i\to\infty} \|G_i - 1\|_2 = 0, \tag{5.27}$$

which combined with (5.26) yields

$$\lim_{i\to\infty}(\phi_{N+1,i}, f_i^*\phi_k(i)(G_i - 1)) = 0. \tag{5.28}$$

By (5.21), (5.22), (5.23), (5.35) and (5.28), we get

$$\lim_{i\to\infty}(\chi_{N+1}(i), \phi_k(i))_i = 0, \tag{5.29}$$

which yields

$$\begin{aligned} \lim_{i\to\infty} \|\chi_{N+1}(i) - \sum_{k\le N}(\chi_{N+1}(i), \phi_k(i))_i\phi_k(i)\|_{2,i}^2 &= \lim_{i\to\infty} \|\chi_{N+1}(i)\|_{2,i}^2 \\ &= \lim_{i\to\infty} \|(f_i^{-1})^*\phi_{N+1}\|_{2,i}^2 \\ &= 1, \end{aligned} \tag{5.30}$$

$$\begin{aligned}\lim_{i\to\infty}\|d\chi_{N+1}(i)-\sum_{k\le N}(\chi_{N+1}(i),\phi_k(i))_i d\phi_k(i)\|_{2,i}^2 \\ &= \lim_{i\to\infty}\|d\chi_{N+1}(i)\|_{2,i}^2 \\ &= \|d\phi_{N+1}\|_2^2+\lim_{i\to\infty}\int_{M-S_i}|d\phi_{N+1}|^2(G_i^2-1)dv.\end{aligned} \tag{5.31}$$

By the Lebesgue convergence theorem again, we get

$$\lim_{i\to\infty}\int_{M-S_i}|d\phi_{N+1}|^2(G_i^2-1)dv=0, \tag{5.32}$$

which combined with (5.30) and (5.31) yields

$$\lim_{i\to\infty}\|dP(\phi_{N+1,i})\|_{2,i}=\lambda_{N+1}. \tag{5.33}$$

By (5.16) and (5.33) we get

$$\limsup_{i\to\infty}\lambda_{N+1}(i)\le\lambda_{N+1}.$$

□

Proof of Proposition 5.2. We prove the proposition by induction. It is clear by Theorem 3.1 that the proposition holds for $N=0$. Therefore we may assume the proposition for $0\le i\le N$, and prove it for $N+1$.

By Lemma 5.2, we may assume $0<\lambda_{N+1}(i)\le 2\lambda_{N+1}$. By the definition, Proposition 2.1, (C1) and (C2), we have

$$\begin{aligned}\|\psi_{N+1}(i)\|_2+\|d\psi_{N+1}(i)\|_2 \\ &= \|\rho_i f_i^*\phi_{N+1}(i)\|_2+\|d(\rho_i f_i^*\phi_{N+1}(i))\|_2 \\ &\le \|\rho_i\|_2\|\phi_{N+1}(i)\|_\infty+\|d\rho_i\|_2\|\phi_{N+1}(i)\|_\infty \\ &\qquad +\|\rho_i\|_\infty\|f_i^*d\phi_{N+1}(i)\|_2 \\ &\le 2C\lambda_{N+1}^n+2C\lambda_{N+1}^n\gamma(i)+2C\lambda_{N+1},\end{aligned} \tag{5.34}$$

which combined with Proposition 5.1 implies that there is a subsequence $\{i(\nu)\}$ and $\psi\in W^{1,2}(M)$ such that

$$\lim_{\nu\to\infty}\lambda_{N+1}(i(\nu))=\liminf_{i\to\infty}\lambda_{N+1}(i) \tag{5.35}$$

and $\{\psi_{N+1}(i(\nu))\}$ converges to ψ weakly in $W^{1,2}(M)$ and strongly in $L^2(M)$. Since

$$\begin{aligned}
\|\psi_{N+1}(i)\|_2 &= \|\rho_i f_i^*\phi_{N+1}(i)\|_2 \\
&\geq \|f_i^*\phi_{N+1}(i)\|_2 - \|(1-\rho_i)f_i^*\phi_{N+1}(i)\|_2 \\
&\geq \frac{1}{2^n}\|\phi_{N+1}(i)\|_{L^2(M_i-K_i)} - C\lambda_{N+1}^n\|1-\rho_i\|_2 \\
&\geq \frac{1}{2^n}(1-\|\phi_{N+1}(i)\|_{L^2(K_i)}) - C\lambda_{N+1}^n\gamma(i) \\
&\geq \frac{1}{2^n}(1-2C\lambda_{N+1}^n \operatorname{vol}(K_i)) - C\lambda_{N+1}^n\gamma(i),
\end{aligned} \tag{5.36}$$

we have

$$\|\psi\|_2 = \lim_{\nu\to\infty}\|\psi_{N+1}(i_\nu)\|_2 \geq \frac{1}{2^n}, \tag{5.37}$$

which shows $\psi \neq 0$. Let us show that $(\psi, \phi_k) = 0$ for $k \leq N$ and $\|\psi\|_2 = 1$. By computation,

$$\begin{aligned}
&|(\psi_{N+1}(i), \psi_k(i))| \\
&\quad = |(\rho_i f_i^*\phi_{N+1}(i), \rho_i f_i^*\phi_k)| \\
&\quad \leq \|1-\rho_i^2\|_2\|\phi_{N+1}(i)\|_\infty\|\phi_k(i)\|_\infty + |(f_i^*\phi_{N+1}(i), f_i^*\phi_k(i))| \\
&\quad \leq C\lambda_{N+1}^{2n}\gamma(i) + |\int_{M_i-K_i}\phi_{N+1}(i)\phi_k(i)G_i^{-1}dv_i| \\
&\quad \leq C\lambda_{N+1}^{2n}\gamma(i) + |(\phi_{N+1}(i), \phi_k(i))_i| \\
&\qquad\qquad + \int_{M_i}|\phi_{N+1}(i)\phi_k(i)(G_i^{-1}-1)dv_i| \\
&\qquad\qquad + \int_{K_i}|\phi_{N+1}(i)\phi_k(i)G_i^{-1}dv_i| \\
&\quad \leq C\lambda_{N+1}^{2n}\gamma(i) + C2^n\lambda_{N+1}^{2n}\operatorname{vol}(K_i) + C\lambda_{N+1}^{2n}\int_{M_i}|G_i^{-1}-1|dv_i \\
&\quad \leq C\lambda_{N+1}^{2n}(\gamma(i) + 2^n\operatorname{vol}(K_i) + \int_M |G_i^{-1}-1|dv)
\end{aligned} \tag{5.38}$$

where $G_i^{-1} \in L^\infty(M_i)$ is defined by

$$G_i^{-1} := \begin{cases} (dv_i - (f_i^{-1})^*dv)/dv_i & (M_i - K_i) \\ 0 & (K_i). \end{cases} \tag{5.39}$$

It is clear, by the definition, that $|G_i^{-1}| = |(f_i^{-1})^* G_i| \leq 2^n$. By (C2) and the Lebesgue convergence theorem, we get

$$(\psi, \phi_k) = \lim_{\nu \to \infty} (\psi_{N+1}(i(\nu)), \psi_k(i(\nu)) = 0. \tag{5.40}$$

In the same way, we can show

$$\|\psi\|_2 = 1. \tag{5.41}$$

Next let us show that $\Delta\psi = a\psi$ where $a = \liminf_{\nu \to \infty} \lambda_{N+1}(i(\nu))$.

Choose an arbitrary $\chi \in C_0^\infty(M)$. Since $\Delta = \delta_{\max} d_{\min}$ for (M, g), it is sufficient to show

$$(d\psi, d\chi) = a(\psi, \chi) \tag{5.42}$$

to prove

$$\Delta\psi = a\psi. \tag{5.43}$$

By the definition, we get

$$\begin{aligned} (d\psi, d\chi) &= \lim_{\nu \to \infty} (d\psi_{N+1}(i(\nu)), d\chi) \\ &= \lim_{\nu \to \infty} (d\phi_{N+1}(i(\nu)), d((f_{i(\nu)}^{-1})^* \chi))_{i(\nu)} \\ &= \lim_{\nu \to \infty} (\Delta_{i(\nu)} \phi_{N+1}(i(\nu)), (f_{i(\nu)}^{-1})^* \chi)_{i(\nu)} \\ &= \lim_{\nu \to \infty} \lambda_{N+1}(i(\nu))(\phi_{N+1}(i(\nu)), (f_{i(\nu)}^{-1})^* \chi)_{i(\nu)} \\ &= a(\psi, \chi), \end{aligned} \tag{5.44}$$

which proves (5.43). Since $\psi \perp \{\phi_0, \cdots, \phi_N\}$ by (5.40), we get

$$a \geq \lambda_{N+1} = \inf_{f \perp \{\phi_0, \cdots, \phi_N\}} \|df\|^2 / \|f\|^2. \tag{5.45}$$

Lemma 5.2 and (5.45) imply $\lambda_{N+1} = \limsup \lambda_{N+1}(i) = \liminf \lambda_{N+1}(i)$. This prove the proposition for N+1. $\square$

Proof of Theorem 5.1. We prove the theorem by induction. Since $\lambda_0(i)$ 0 for all i, it holds for $k = 0$. We assume the theorem for $k \leq n$ and prove it for $k = n + 1$. For the proof, it is sufficient to show

$$\limsup_{i \to \infty} \lambda_{n+1}(i) \leq \lambda_{n+1} \leq \liminf_{i \to \infty} \lambda_{n+1}(i). \tag{5.46}$$

We can choose sequences $\{i_\nu\}$ and $\{i_\mu\}$ such that

$$\lim_{\nu\to\infty}\lambda_{n+1}(i_\nu)=\limsup_{i\to\infty}\lambda_{n+1}(i),\qquad \lim_{\mu\to\infty}\lambda_{n+1}(i_\mu)=\liminf_{i\to\infty}\lambda_{n+1}(i).$$

By Propostioin 5.2, choosing subsequence $\{i_{\nu,k}\}$ and $\{i_{\mu,l}\}$ if needed, we have

$$\lim_{k\to\infty}\lambda_{n+1}(i_{\nu,k})=\lambda_{n+1},\qquad \lim_{l\to\infty}\lambda_{n+1}(i_{\mu,l})=\lambda_{n+1}.$$

This implies (5.46) and completes the proof. □

§6. Proof of Main Theorem

In this section, we use the same notations as in §0, §1, §3 and §5.

In view of the proof of Theorem 5.1, it is sufficient to show the following proposition **(P)** to prove Main Theorem:

(P) For every sequence $\{t_n\}$ with $\lim_{n\to\infty} t_n = 0$, there exists a subsequence $\{t_{n,i}\}$ such that

$$\lim_{\nu\to\infty}\sigma(\Delta_{X_{t_{n,i}}})=\sigma(\Delta_Z). \tag{6.1}$$

Let $\{t_n\}$ be given. For every i, we can find $t_{n,i}$ with $|t_{n,i}|<\gamma(1/i)$ where $\gamma(\epsilon)$ is the same one as in Theorem 1.1, and obtain a subsequence $\{t_{n,i}\}$. Set $(M,g):=(Z-\Sigma_Z,g_Z)$, $M_i:=X_{t_{n,i}}$ and $g_i:=g_{t_{n,i}}$.

It is clear that (M,g) is a Riemannian manifold with finite volume. To apply Theorem 5.1 to (M,g) and (M_i,g_i), we shall verify (C1) and (C2) for them.

At first we must find a family of cut-off functions verifying (C1) for $(Z-\Sigma_Z,g_Z)$. By Theorem 3.2 and Lemma 3.5, there is a family of cut-off functions $\{\rho_i\}\subset C_0^0(Z-\Sigma_Z)\cap W^{1,2}(Z)$ which satisfies (C1) for (Z,g) where $g:=\iota^*(g_{\mathbb{P}^N(\mathbb{C})}|X_0)$ is the pull back of the Bergmann metric of X_0. Since g_Z is a restriction of some Riemannian metric of $\mathfrak{X}$, it is quasi-isometric to g. Therefore, $\{\rho_i\}$ also satisfies (C1) for $(Z-\Sigma_Z,g_Z)$.

Next we must construct maps $f_i : M-S_i\to M_i$ verifying (C2) subject to (C1). Set $f_i:=f_{t_{n,i}}^{(1/i)} : Z-S_i\to X_{t_{n,i}}$ where $f_t^\epsilon : Z-\Sigma_\epsilon\to X_t$ is the same map constructed in Theorem 1.1. We remark that f_i is well-defined on $Z-S_i$, since $f_{t_{n,i}}^{(1/i)}$ is defined on $Z-\Sigma_{Z,\frac{1}{i}}$ by Theorem 1.1 and $Z-S_i\Subset Z-\Sigma_{Z,\frac{2}{i}}$ from Theorem 3.2. By Theorem 1.1 and (4.4), $\{f_i\}$ satisfies (C2) subject to (C1).

Finally, we must verify the uniformity of the Sobolev inequality for $\{(M_i, g_i)\}$. But this follows from Corollary 4.2. Therefore, we can apply Theorem 5.1 to (Z, g_Z) and $\{(X_{t_{n,i}}, g_{t_{n,i}})\}$, and obtain (6.1). $\square$

Appendix

Let X be an irreducible algebraic variety of dimension n in $\mathbb{P}^N(\mathbb{C})$, and Y an irreducible subvariety of dimension $d(< n)$ in X. For $y \in Y_{\mathrm{reg}}$, let N_y be a linear subspace of dimension $N - d$ in $\mathbb{P}^N(\mathbb{C})$ which contains y such that Y and N_y intersects transversally at y; i.e., $T_y\mathbb{P}^N(\mathbb{C}) = T_yY \oplus T_yN_y$.

Here, by a linear subspace, we mean a subvariety of the form $H_1 \cap \cdots \cap H_r$ where H_i is a hyperplane in $\mathbb{P}^N(\mathbb{C})$.

PROPOSITION A.1. *There exists a nonempty Zariski open subset U_Y $(\subset Y - \Sigma_Y)$ of Y such that if $y \in U_Y$, then*

$$\dim_y N_y \cap X = n - d.$$

For the proof, we need several lemmas. In what follows, rings in consideration are noetherian commutative with 1, and all schemes are assumed to be Noetherian.

Let A be a local ring and $\mathfrak{M}$ be the maximal ideal of A. Let I and J be proper ideals of A. Let $\bar{A} := A/I$ be the residue ring and $\pi : A \to \bar{A}$ be the natural projection. Let $\overline{\mathfrak{M}} := \pi(\mathfrak{M})$ be the maximal ideal of $\bar{A}$. We denote by $\mathrm{Gr}_I A$ the graded $\bar{A}$-algebra defined by

$$\mathrm{Gr}_I A := \oplus_{i=0}^{\infty} I^i/I^{i+1} \quad (I^0 := A). \tag{A.1}$$

For $x \in A$, we attach an integer $\nu_I(x)$ defined by

$$\nu_I(x) := \sup\{j \in \mathbb{Z}_{\geq 0}; x \in I^j\} \quad (x \neq 0), \quad \nu_I(0) = \infty. \tag{A.2}$$

For $x \in A$, we attach an element $\mathrm{in}_I(x)$ of $\mathrm{Gr}_I A$ defined by

$$\mathrm{in}_I(x) := x \mod I^{\nu_I(x)+1} \in I^{\nu_I(x)}/I^{\nu_I(x)+1} \quad (x \neq 0), \quad \mathrm{in}_I(0) = 0. \tag{A.3}$$

Finally, we define an ideal $\mathrm{Gr}_I(J, A)$ of $\mathrm{Gr}_I A$ by

$$\mathrm{Gr}_I(J, A) := \{\mathrm{in}_I(x); x \in J\}. \tag{A.4}$$

LEMMA A.1. ([M, Theorem 15.7]) *Let A be a local ring, and $I \subsetneq A$ be an ideal. Then,*

$$\dim A = \dim \mathrm{Gr}_I A.$$

LEMMA A.2. ([H, Chap. 2, §2, Lemma 5]) *There exists an isomorphism between graded algebras as follows:*

$$\mathrm{Gr}_{I+J/J} A/J \cong \mathrm{Gr}_I A/\mathrm{Gr}_I(J, A).$$

LEMMA A.3. ([M, Theorem 15.1]) *Let $\phi : A \to B$ be a flat local homomorphism of local rings. Then,*

$$\dim B = \dim A + \dim B/\mathfrak{M}B.$$

LEMMA A.4. *Let A be a local ring with the maximal ideal $\mathfrak{M}$, and B be a flat A-algebra with $B \neq \mathfrak{M}B$. Then,*

$$\dim B \geq \dim A + \dim B/\mathfrak{M}B.$$

Proof. Take a maximal ideal $\bar{\mathfrak{n}}$ of $B/\mathfrak{M}B$ such that $\dim B/\mathfrak{M}B = \dim(B/\mathfrak{M}B)_{\bar{\mathfrak{n}}}$ where $(B/\mathfrak{M}B)_{\bar{\mathfrak{n}}}$ is the localization of $B/\mathfrak{M}B$ by the maximal ideal $\bar{\mathfrak{n}}$.

Let $\pi : B \to B/\mathfrak{M}B$ be the projection. We set $\mathfrak{n} := \pi^{-1}(\bar{\mathfrak{n}})$. Then, by Lemma A.3, we have

$$\dim B \geq \dim B_{\mathfrak{n}} = \dim A + \dim(B/\mathfrak{M}B)_{\bar{\mathfrak{n}}} = \dim A + \dim B/\mathfrak{M}B.$$

□

Finally, we need the following theorem:

THEOREM A.1. ([H, Chap. 2, Sect. 1, Theorem 1]) *Let X be a scheme and $Y \subset X$ be a reduced closed subscheme. Then, there exists a nonempty Zariski open subset $U \subset X$ such that*
(1) $U \cap Y \neq \emptyset$.
(2) U is normally flat along $U \cap Y$. Or equivalently, $\mathrm{Gr}_{\mathcal{I}_{Y,y}} \mathcal{O}_{X,y}$ is $\mathcal{O}_{Y,y}$-flat for every $y \in U \cap Y$ where $\mathcal{O}_X$ (resp. $\mathcal{O}_Y$) is the structure sheaf of X (resp. Y), $\mathcal{I}_Y (\subset \mathcal{O}_X)$ is the defining ideal sheaf of Y, and $\mathcal{O}_{X,y}$ (resp. $\mathcal{O}_{Y,y}$, $\mathcal{I}_{Y,y}$) denotes the stalk at y.

Proof of Proposition A.1. By virtue of Theorem A.1, it suffices to prove that $\dim_y N_y \cap X = n-d$ if $y \in Y_{\mathrm{reg}}$ and if X is normally flat along Y at y; i.e., $\mathrm{Gr}_{\mathcal{I}^X_{Y,y}} \mathcal{O}_{X,y}$ is $\mathcal{O}_{Y,y}$-flat.

We can take a minimal set of generators $z = (z_1, \cdots, z_d)$ (consisting of d-elements) of $\mathcal{I}^{\mathbb{P}^N}_{N_y,y}$. Set $A = \mathcal{O}_{X,y}$, $\mathfrak{M} = \mathfrak{M}_{X,y}$ and $I = \mathcal{I}_{Y,y}$. We also set $\bar{A} = A/I$ and $\bar{\mathfrak{M}} = \mathfrak{M}/I$.

Since $\dim_y N_y \cap X \geq n-d$ from the intersection inequality of dimension (cf. [G-R, p. 102]), it suffices to show $\dim_y N_y \cap X \leq n-d$. By Lemma A.1 and A.2,

$$\begin{aligned}(\mathrm{A.5}) \quad \dim_y N_y \cap X &= \dim A/zA \\ &= \dim \mathrm{Gr}_{(I+zA)/zA} A/zA = \dim \mathrm{Gr}_I A/\mathrm{Gr}_I(zA, A).\end{aligned}$$

As Y and N_y intersects transversally at y, we have $\mathcal{I}^{\mathbb{P}^N}_{N_y,y} + \mathcal{I}^{\mathbb{P}^N}_{Y,y} = \mathfrak{M}_{\mathbb{P}^N,y}$. Hence, z is mapped to the regular system of parameter $\bar{z} = (\bar{z}_1, \cdots, \bar{z}_d)$ of $\bar{A} = A/I = \mathcal{O}_{Y,y} = \mathcal{O}_{\mathbb{P}^N,y}/\mathcal{I}^{\mathbb{P}^N}_{Y,y}$ by the natural projection map. In other words, we have $\bar{z}\bar{A} = \bar{\mathfrak{M}}$. This also shows that $\nu_I(z_i) = 0$ for each i; i.e., $z_i \notin I$ for each i. Here we consider z_i as an element of $A = \mathcal{O}_{X,y} = \mathcal{O}_{\mathbb{P}^N,y}/\mathcal{I}^{\mathbb{P}^N}_{X,y}$. This shows that $\mathrm{in}_I(z_i) = z_i \bmod I = \bar{z}_i$ is an element of the degree zero component $\bar{A} = A/I$ of $\mathrm{Gr}_I A$.

As $\mathrm{Gr}_I(zA, A)$ contains $\bar{z}$, it suffices to show that $\dim \mathrm{Gr}_I A/\bar{z}\mathrm{Gr}_I A \leq n-d$ by (A.5). Note that we have $\mathrm{Gr}_I A/\bar{z}\mathrm{Gr}_I A \cong \mathrm{Gr}_I A/\bar{\mathfrak{M}}\mathrm{Gr}_I A$. Hence, by Lemma A.1 and A.4, we have

$$(\mathrm{A.6}) \ \dim \mathrm{Gr}_I A/\bar{z}\mathrm{Gr}_I A \leq \dim \mathrm{Gr}_I A - \dim A/I = \dim A - \dim A/I = n-d.$$

This completes the proof. □

PROPOSITION A.2. *Let X be an irreducible algebraic variety of dimension n in $\mathbb{P}^N(\mathbb{C})$.*

Let $H_1, \cdots, H_r$ $(r \leq n)$ be hyperplanes in $\mathbb{P}^N(\mathbb{C})$. Let

$$X \cap H_1 \cap \cdots \cap H_r = \bigcup_{i=0}^{r} (X \cap H_1 \cap \cdots \cap H_r)^{(n-i)}$$
$$(\dim(X \cap H_1 \cap \cdots \cap H_r)^{(n-i)} = n-i)$$

be the decomposition into the pure dimensional components. Then,

$$\deg(X) \geq \deg(X \cap H_1 \cap \cdots \cap H_r)_{\mathrm{prop}}$$

where $(X \cap H_1 \cap \cdots \cap H_r)_{\text{prop}} := (X \cap H_1 \cap \cdots \cap H_r)^{(n-r)}$ *is the proper component of the intersection* $X \cap H_1 \cap \cdots \cap H_r$.

For the proof, we prepare a lemma.

LEMMA A.5. *Let* Y *be an irreducible algebraic variety in* $\mathbb{P}^N(\mathbb{C})$ *and* H *be a hyperplane in the same projective space. Then,*

$$\deg(Y) \geq \deg(Y \cap H)_{\text{prop}}.$$

Proof. By the intersection inequality of dimension (cf. [G-R, p. 102]), $\dim Y \cap H$ is equal to either $\dim Y - 1$ or $\dim Y$. When $\dim Y \cap H = \dim Y - 1$, then $\deg(Y) = \deg(Y \cap H)$ by the Bézout theorem. When $\dim Y \cap H = \dim Y$, $Y \cap H = Y$ and therefore $(Y \cap H)_{\text{prop}} = \emptyset$. Therefore, $\deg(Y \cap H)_{\text{prop}} = 0$ by the definition. This completes the proof. $\square$

Proof of Proposition A.2. We prove the propositon by induction. When $r = 0$, there is nothing to prove. We assume the proposition for $r \leq k$ and prove it for $r = k + 1$.

Let

$$(X \cap H_1 \cap \cdots \cap H_r)_{\text{prop}} = \bigcup_{\alpha} A_\alpha \tag{A.7}$$

be the irreducible decomposition. By the definition and the intersection inequality of dimension,

$$(X \cap H_1 \cap \cdots \cap H_r \cap H_{k+1})_{\text{prop}} \subset \bigcup_{\alpha} (A_\alpha \cap H_{k+1})_{\text{prop}}. \tag{A.8}$$

By Lemma A.5 and the hypothesis of induction, we have

$$\begin{aligned} \deg(X) \geq \sum_{\alpha} \deg(A_\alpha) &\geq \sum_{\alpha} \deg(A_\alpha \cap H_{k+1})_{\text{prop}} \\ &\geq \deg(X \cap H_1 \cap \cdots \cap H_{k+1})_{\text{prop}} \end{aligned} \tag{A.9}$$

This completes the proof. $\square$

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