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# INTEGRATION OF LOCAL ACTIONS ON HOLOMORPHIC FIBER SPACES

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**Abstract.** It is proved that every holomorphically convex complex space endowed with an action of a compact Lie group $K$ can be realized as an open $K$-stable subspace of a holomorphically convex space endowed with a holomorphic action of the complexified group $K^{\mathbb{C}}$. Similar results are obtained for holomorphic $K$-bundles over such spaces.

Let $G$ be a real Lie group which acts by holomorphic transformations on a (reduced) complex space $X$. Suppose that the Lie algebra of the complexification $G^{\mathbb{C}}$ of $G$ (see [Ho, p. 204]) is the complexification of the Lie algebra of $G$. This holds for example in the case where $G$ is simply connected. Then, by integrating the holomorphic vector fields given by the $G$-action, the complexification $G^{\mathbb{C}}$ acts locally and holomorphically on $X$ (see [K]).

Adapting the terminology of Palais (see [P]), we say that a complex space $X^*$ which contains $X$ as an open subset is a globalization of the complex $G$-space $X$ whenever the local $G^{\mathbb{C}}$-action on $X$ extends to a global holomorphic action on $X^*$ and $G^{\mathbb{C}} \cdot X = X^*$.

The following results are proved in this paper.

THEOREM 1. *Let $G$ be a compact Lie group and $X$ a holomorphically convex complex $G$-space $X$. Then there exists a globalization $X^*$ of $X$ satisfying the following conditions*

(i) *$X^*$ is holomorphically convex*

(ii) *Every $G$-equivariant holomorphic map $\psi$ from $X$ into a complex space*

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*$Y$ where $G^{\mathbb{C}}$ acts holomorphically extends to a $G^{\mathbb{C}}$-equivariant holomorphic map $\psi^*: X^* \to Y$.*

Part (ii) of Theorem 1 implies that the globalization of the Remmert reduction of $X$ (see [GR, p. 221]) is the Remmert reduction of $X^*$, since, in the special case where $X$ is assumed to be Stein, the globalization $X^*$ is also a Stein space ([H1]) (see §7 for the precise statement).

Theorem 1 is a special case of

THEOREM 2. *Let $G$ be a compact Lie group, $X$ a holomorphically convex complex $G$-space and $P \xrightarrow{\pi} X$ a holomorphic principal $G$-bundle over $X$ with complex structure group $S$. Then the bundle $P \xrightarrow{\pi} X$ extends to a holomorphic principal $G^{\mathbb{C}}$-bundle $P^* \xrightarrow{\pi^*} X^*$ which also has $S$ as structure group.*

Theorem 2 has an important application when $X$ is a Stein space. Under the assumption that a certain bundle of Lie groups over $X$ extends to $X^*$, there is an equivariant version of Grauert's Oka Principle ([HK]), i.e.,

$$\mathrm{Bund}_{\mathcal{O}}(X)^G \cong \mathrm{Bund}_{\mathcal{C}^0}(X)^G$$

where $\mathrm{Bund}_{\mathcal{O}}(X)^G$ (resp. $\mathrm{Bund}_{\mathcal{C}^0}(X)^G$) denotes the isomorphism classes of holomorphic (resp. topological) $G$-bundles over $X$ with fixed complex structure group $S$ and fiber. Theorem 2 implies that this assumption in [HK] is superfluous.

Local differentiable actions and their globalizations have been extensively studied by Palais in [P]. In the category of possibly non Hausdorff manifolds he has given necessary and sufficient conditions for a local action to admit a globalization. Although Palais' globalization is in general non Hausdorff, it gives important insight into the behavior of a local action. In order to verify the Hausdorff property in our setting, it is necessary to use tools of complex geometry. The proof of Theorem 1 is carried out by combining Palais' method with complex analytic techniques, e.g., properties of the Remmert reduction and Stein theory. It should also be noted that the proof of Theorem 1 for Stein spaces (see [H1]) can be substatially simplified by incorporation of Palais' techniques (§7 Remark).

We would like to thank W. Kaup for calling Palais' results to our attention. For the convenience of the reader, we recall in the first three paragraphs those parts of this theory which are used here. Of course we formulate them in the context of complex spaces.

As an application we show in §5 that a holomorphically separable complex $G$-space $X$ admits a possibly non Haudorff globalization if $G$ is a compact group or vector group.

In §6 we discuss "globalizations" of $G$-equivariant holomorphic maps. These results are used to prove Theorem 1 in §7 and Theorem 2 in §8.

A consequence to Hamiltonian actions of a compact group $G$ on a holomorphically convex Kählerian space is given in §9.

## §1. Local actions

Let $G$ be a real connected Lie group and $X$ a complex space which is assumed to be reduced. A *local action of G on X* is given by a real analytic map $\Phi: \Omega \to X$ where $\Omega$ is an open neighborhood of the neutral section $\{e\} \times X$ in $G \times X$ such that

(i) for all $x \in X$ the open subset $\Omega(x) := \{g \in G;\ (g, x) \in \Omega\}$ of $G$ is connected,

(ii) for fixed $g \in G$ the map $x \to \Phi(g, x) =: g \cdot x$ is holomorphic when defined, $e \cdot x = x$ and

(iii) if $(gh, x) \in \Omega$, $(h, x) \in \Omega$ and $(g, h \cdot x) \in \Omega$, then $(gh) \cdot x = g \cdot (h \cdot x)$

hold.

For a complex Lie group $G$ the local action is said to be holomorphic if $\Phi$ is holomorphic.

There is a natural notion of equivalence of local $G$-actions on $X$ which is given by restricting $\Omega$ to a smaller neighborhood of $\{e\} \times X$ in $G \times X$. To an equivalence class of a local action one can assign a linear map from the Lie algebra $\mathfrak{g}$ of $G$ into the Lie algebra $\mathrm{Vec}(X)$ of holomorphic vector fields on $X$. The map $\lambda: \mathfrak{g} \to \mathrm{Vec}(X)$ is closed by $\xi \to \xi_X$, where the holomorphic vector field $\xi_X$ on $X$ is defined by

$$\xi_X(f) = \left(\frac{d}{dt}\right)_{t=0} f \circ \exp t\xi.$$

The map $\lambda$ is a homomorphism of Lie algebras if we view $\mathfrak{g}$ as the Lie algebra of right invariant vector fields on $G$.

Conversely, if there is a homomorphism $\lambda: \mathfrak{g} \to \mathrm{Vec}(X)$ of Lie algebras, then up to equivalence there exists a unique local $G$-action on $X$ which induces the given $\lambda$ (see [K]).

If $G$ is a complex Lie group, then local holomorphic $G$-actions correspond to $\mathbb{C}$-linear homomorphisms.

For a Lie group $G$ which is not connected let $G_e$ be the connected component of the neutral element $e \in G$. A local $G$-action will always mean a local $G_e$-action.

## §2. Foliations associated to local actions

Let $G$ be a Lie group and $X$ a space. For a $G$-action on $X$ we have an induced diagonal action of $G$ on $G \times X$ given by $g \cdot (h, x) = (gh, g \cdot x)$. The quotient map

$$G \times X \longrightarrow X^* := (G \times X)/G, \qquad (h, x) \longrightarrow [h, x]$$

is equivariant with respect to the "right" $G$-action on $G \times X$ which is given by $r(g) \cdot (h, x) = (hg^{-1}, x)$ and the induced action on $X^*$ satisfies

$$[hg^{-1}, x] = [e, (gh^{-1} \cdot x].$$

In particular there is a tautological $G$-equivariant isomorphism $X \rightarrow X^*$, $x \rightarrow [e, x]$.

Under certain assumptions there exists a reasonable quotient $X^*$ of $G \times X$ also when one consider a local $G$-action on $X$. In this case the above map leads to an open locally $G$-equivariant embedding of $X$ into $X^*$ which is in general not surjective. Thus after adding to $X$ some points one obtains a set $X^*$ containing $X$ such that the local action on $X$ is induced by a global $G$-action on $X^*$.

Palais made this precise in [P]. In order to present and to show how to carry over his results to local actions on complex spaces we need some preparation.

Let $\Phi: \Omega \rightarrow X$ be a local action of a connected Lie group $G$ on the complex space $X$. A product $N \times U$, where $N$ is an open connected neighborhood of the neutral element $e \in G$ and $U$ is an open subset of $X$ satisfying

(a) $N = N^{-1}$,

(b) $N^2 \times U \subset \Omega$ and

(c) $N^2 \times N^2 \cdot U \subset \Omega$

is called an *elementary slice pair.*

The union of all elementary slice pairs covers a neighborhood of $\{e\} \times X$ and the image $W$ of an elementary slice pair $N \times U$ with respect to the map

$$\Psi: N \times U \longrightarrow G \times X, \qquad (g, x) \longrightarrow (g, g \cdot x)$$

is open. Moreover $\Psi$ is an isomorphism onto $W$. The images of all possible $W$ translated by the right action of $G$ on $G \times X$, which is given by right multiplication on the first component, cover $G \times X$. For $h \in G$ and an elementary slice pair $N \times U$ the isomorphism

$$\Psi_h : Nh \times U \longrightarrow Wh, \qquad (g, x) \longrightarrow (g, (gh^{-1}) \cdot x),$$

where $Wh := \{(gh, y) \in G \times X;\ (g, y) \in W\}$, is called a *slice chart with respect to $h$ and $N \times U$ for the local diagonal $G$-action on $G \times X$.*

PROPOSITION 1. *Let $\Psi_j : N_j g_j \times U_j \to W_j$ be slice charts, $j = 1, 2$. For any $s \in N_1 g_1 \cap N_2 g_2$ there are isomorphisms*

$$\Psi_j(s) : N_j g_j \times (s g_j^{-1}) \cdot U_j \longrightarrow W_j \qquad (g, x) \longrightarrow (g, (gs^{-1}) \cdot x)$$

*such that*

$$\Psi_j(s)^{-1}(W_1 \cap W_2) = (N_1 g_1 \cap N_2 g_2) \times ((s g_1^{-1}) \cdot U_1 \cap (s g_2^{-1}) \cdot U_2).$$

*In particular, $\Psi_1(s) = \Psi_2(s)$ on $\Psi_1(s)^{-1}(W_1 \cap W_2) = \Psi_2(s)^{-1}(W_1 \cap W_2)$.*

*Proof.* Since $s g_j^{-1} \in N_j$, $\{s g_j^{-1}\} \times (s g_j^{-1}) \cdot U_j \subset \Omega$ and $\Psi_j$ is an isomorphism, the map

$$\Psi_j(s) : N_j g_j \times (s g_j^{-1}) \cdot U_j \to W_j, \quad (g, x) \to (g, (g g_j^{-1}) \cdot ((s g_j^{-1})^{-1} \cdot x))$$

is also an isomorphism. For $(g, x) \in N_j g_j \times (s g_j^{-1}) \cdot U_j$ it follows that $g g_j^{-1} g_j s^{-1} = g s^{-1} \in N_j^2$, $x \in N_j \cdot U_j$. Thus $N_j^2 \times N_j \cdot U_j \subset \Omega$ implies

$$(g g_j^{-1}) \cdot ((s g_j^{-1})^{-1} \cdot x) = (g s^{-1}) \cdot x.$$

Now $\Psi_j(s)(N_j g_j \times (s g_j^{-1}) \cdot U_j) = \Psi_j(N_j g_j \times U_j)$. Thus $g \in N_j g_j$ and $y \in N_j g_j$ and $y \in N_j \cdot U_j$ for $(g, y) \in W_j$. Moreover $s g^{-1} \in N_j^2$ and $N_j^2 \times N_j \cdot U_j \subset \Omega$. Therefore $\Psi_j(s)^{-1} : W_j \to N_j g_j \times (s g_j^{-1}) \cdot U_j$ is given by $(g, y) \to (g, (s g^{-1}) \cdot y)$. □

COROLLARY 1. *For slice charts $\Psi_j$: $N_j g_j \times U_j \to W_j$, $(g, x) \to (g, (g g_j^{-1}) \cdot x)$, $j = 1, 2$ there exist open subsets $U_{12}$ resp. $U_{21}$ of $U_1$ resp. $U_2$ such that*

$$\Psi_1^{-1}(W_1 \cap W_2) = (N_1 g_1 \cap N_2 g_2) \times U_{12}$$

*resp.*

$$\Psi_2^{-1}(W_1 \cap W_2) = (N_1 g_1 \cap N_2 g_2) \times U_{21}.$$

*For any* $s \in N_1 g_1 \cap N_2 g_2$ *the isomorphism*

$$\Psi_{21}: \Psi_1^{-1}(W_1 \cap W_2) \longrightarrow \Psi_2^{-1}(W_1 \cap W_2), \qquad \Psi_{21} := \Psi_2^{-1} \circ \Psi_1$$

*is given by* $(g, x) \to (g, (g_2 s^{-1}) \cdot ((s g_1^{-1} \cdot x)).$ □

Using slice charts one can define a *leaf topology* on $G \times X$. A subset $V$ of $G \times X$ is a neighborhood of $(h_0, y_0)$ in this topology if and only if there exists a slice chart $\Psi: Nh \times U \to W$ such that

$$(h_0, y_0) \in \Psi(Nh \times \{x_0\}) \subset V$$

for some $x_0 \in U$.

A connected component of $G \times X$ in the leaf topology is called a *leaf of the local G-action on* $G \times X$. The leaf topology depends only on the equivalence class of the local $G$-action on $X$. Since $G \times X$ is a disjoint union of leaves, one has a quotient space $X^*$ of $G \times X$ whose points are the leaves. Endow $X^*$ with the quotient topology and denote by $\Pi: G \times X \to X^*$ the quotient map.

Corollary 1 has the following interpretation. A slice chart $\Psi_j: N_j g_j \times U_j \to W_j$ is a homeomorphism if we view $N_j g_j \times U_j$ as the disjoint union of manifolds $N_j g_j \times \{x\}$, $x \in U_j$, and $W_j$ as an open subset of $G \times X$ with respect to the leaf topology. The manifold structure on $N_j g_j$ induces a manifold structure on $\Psi_j(N_j g_j \times \{x\})$ for each $x \in U_j$ and Corollary 1 says that the slice charts $\Psi_j$ are analytically compatible charts $G \times X$ with respect to the leaf structure. Each connected component, i.e., each leaf, is a regulary embedded submanifold of the analytic space $G \times X$. If the given local $G$-action on $X$ is holomorphic, then each leaf is a complex submanifold of $G \times X$.

PROPOSITION 2. (c.f. [P, p. 10]) *Let* $\Sigma_0$ *be a leaf,* $z_0 \in \Sigma_0$ *and let* $\Psi_0: N_0 g_0 \times U_0 \to W_0$ *be a slice chart such that* $z_0 \in \Sigma_0 \cap W_0$. *For every* $z_1 \in \Sigma_0$ *there exists a slice chart* $\Psi_1: N_1 g_1 \times U_1 \to W_1$ *with* $z_1 \in W_1$, $N_1 \subset N_0$, *an open analytic embedding* $\beta: W_1 \to W_0$ *and an open holomorphic embedding* $f: U_1 \to U_0$ *such that*

(i) $\Psi_0^{-1} \circ \beta \circ \Psi_1(g, x) = (g g_1^{-1} g_0, f(x))$ *for* $(g, x) \in N_1 g_1 \times U_1$ *and*

(ii) *if a leaf* $\Sigma$ *intersects* $W_1$ *non trivially, then* $\Psi_1(N_1 g_1 \times \{x\}) \subset \Sigma$ *and* $\Psi_0(N_0 g_0 \times \{f(x)\}) \subset \Sigma$ *for some* $x \in U_1$.

*Proof.* Since a leaf is connected, it is sufficient to show that the set of $z_1 \in \Sigma_0$ such that the statement of Proposition 2 holds is open and closed in the leaf topology. But this is a consequence of Proposition 1. □

COROLLARY 2. *The quotient map* $\Pi: G \times X \to X^*$ *is open.*

## §3. Univalent actions

For a local $G$-action of a connected Lie group $G$ on a complex space $X$ the projection $p_G: G \times X \to G$ is an analytic map with respect to the leaf structure on $G \times X$. Moreover, the restriction $p_\Sigma$ of $p_G$ to a leaf $\Sigma$ is a local isomorphism. In particular $p_G(\Sigma)$ is open in $G$.

The local $G$-action on $X$ is said to be *univalent* if for each leaf $\Sigma$ the map $p_\Sigma$ is injective (see [P, p. 62]). In this case there is an analytic isomorphism $q_\Sigma: p_G(\Sigma) \to \Sigma$ which is the inverse of $p_\Sigma$.

A univalent local $G$-action on $X$ is regular, i.e., each point $z_0$ of $G \times X$ is in the image of a slice chart $\Psi: N g_0 \times U \to W$ such that

(*) a leaf $\Sigma$ intersects $W$ non trivially if and only if $\Sigma \cap W = \Psi(N g_0 \times \{x\})$ for some $x \in U$.

For a regular local action the leaves are closed in $G \times X$.

A slice chart $\Psi: N g_0 \times U \to W$ which has the property (*) is called a *regular slice chart.* For a univalent local action every slice chart is regular.

Although the quotient $X^*$ is in general non Hausdorff one has the following

THEOREM 1. (c.f. [P, p. 63]) *For a univalent* $G$*-action on* $X$ *the quotient* $X^*$ *is a possibly non Hausdorff complex space such that every slice chart* $\Psi: N g_0 \times U \to W$ *inudeces an open holomorphic embedding* $\Psi^*: U \to X^*$ *which makes the diagram*

$$\begin{array}{ccc} N g_0 \times U & \xrightarrow{\Psi} & G \times X \\ p_U \downarrow & & \downarrow \Pi \\ U & \xrightarrow[\Psi^*]{} & X^* \end{array}$$

*commutative.*

*Proof.* Each point $z \in G \times X$ lies in the image of a slice chart $\Psi: Ng_0 \times U \to W$. Since $\Pi$ is open and $\Psi$ is a regular slice chart, the induced map $\Psi^*: U \to \Pi(W)$ is a homeomorphism. From §2 Propostion 2 it follows that two regular charts $\Psi_j$ induce holomorphically compatible maps $\Psi_j^*$. $\square$

The group $G$ acts by multiplication from the right on the first component of $G \times X$. Since for every $g \in G$ the map $r_g: G \times X \to G \times X$, $r_g(h, x) = (hg^{-1}, x)$ maps leaves onto leaves, there is an induced $G$-action on the quotient $X^*$. For a univalent local $G$-action on $X$ this action is compatible with the complex structure of $X^*$.

THEOREM 2. (c.f. [P, p. 71]) *For a univalent local $G$-action on $X$ the map $\iota: X \to X^*$, $\iota(x) = \Pi(e, x)$, is a locally $G$-equivariant open holomorphic embedding which has a the following universality property.*

*If $G$ acts locally and univalently on a complex space $Y$, then for every locally $G$-equivariant holomorphic map $\psi: X \to Y$ there exists a unique $G$-equivariant holomorphic map $\psi^*: X^* \to Y^*$ such that the diagram*

$$\begin{array}{ccc} X & \xrightarrow{\iota} & X^* \\ \psi \downarrow & & \downarrow \psi^* \\ Y & \xrightarrow[\iota]{} & Y^* \end{array}$$

*commutes.*

*Proof.* The map $\iota: X \to X^*$ is locally biholomorphic and by univalency it is injective. Thus $\iota$ is an open locally equivariant holomorphic embedding.

Since $G \cdot \iota(X) = X^*$, the map $\psi^*: X^* \to Y^*$ is unique. In order to prove existence we set $\hat{\psi}: G \times X \to G \times Y$, $\hat{\psi}(g, x) = (g, \psi(x))$. Then $\hat{\psi}$ maps leaves into leaves and is $G$-equivariant with respect to the right $G$-action. Thus we obtain an equivariant map between the quotients $\psi^*: X^* \to Y^*$. Using regular charts on $X$ and $Y$ one sees that $\psi^*$ is holomorphic. $\square$

For a complex space $X$ with a local $G$-action a not necessarily Hausdorff complex space $X^*$ with a global $G$-action and a locally $G$-equivariant open holomorphic embedding $\iota: X \to X^*$ is called a *universal globalization of the local $G$-action on $X$* if the universality property of Theorem 2 is satisfied. Thus a universal globalization of a univalent local $G$-action always exists and it is unique up to biholomorphisms. In this case we have $G \cdot \iota(X) = X^*$. If $X^*$ exists, then we will always identify $X$ with its image in $X^*$.

PROPOSITION. (c.f. [P, p. 70]) *Let $X^*$ be a universal globalization of a local $G$-action on $X$. For a leaf $\Sigma$ of $G \times X$ and $(g_0, x_0) \in \Sigma$ we have $p_G(\Sigma) = \Omega_0 g_0$ where $\Omega_0$ is the connected component of $e$ in $\{g \in G;\ g \cdot x_0 \in X\}$. The map*

$$q_\Sigma : \Omega_0 g_0 \longrightarrow \Sigma, \qquad q_\Sigma(g) := (g, (g g_0^{-1}) \cdot x_0)$$

*is the inverse of $p_\Sigma : \Sigma \to \Omega_0 g_0$, $p_\Sigma := p_G|\Sigma$.*

*Proof.* Since the map $q_\Sigma$ is continuous with respect to the leaf topology, the image $\Sigma_0 := q_\Sigma(\Omega_0 g_0)$ is an open subset of $\Sigma$. In fact it is also closed. For this let $(\bar{g}, \bar{x})$ be in the closure of $\Sigma_0$ and let $\Psi: N\bar{g} \times U \to W$, $(g, x) \to (g, (g\bar{g}^{-1}) \cdot x)$ be a slice chart such that $\bar{x} \in U$. Since $\Psi(N\bar{g} \times \{\bar{x}\})$ is open in $\Sigma$, $N\bar{g} \cap \Omega_0 g_0$ is non empty and $N\bar{g}g_0^{-1} \cup \Omega_0$ is a connected neighborhood of $e \in G$. Moreover, there exists an $s \in N\bar{g} \cap \Omega_0 g_0$ such that $(s, (s\bar{g}^{-1}) \cdot \bar{x}) = (s, (s g_0^{-1}) \cdot x_0)$. This implies $N\bar{g}g_0^{-1} = N \cdot \bar{x} \subset X$, i.e., $N\bar{g}g_0^{-1} \subset \Omega_0$ and $(\bar{g}, \bar{x}) \in \Sigma_0$. □

*Remark.* It follows that a local $G$-action is univalent if and only if it admits a universal globalization. Moreover, for a univalent local $G$-action the map $q_\Sigma : \Omega_0 g_0 \to \Sigma$ is an isomorphism. Note that the universality property of $X^*$ was not used in the above proof. Hence, if $X$ admits some globalization, then it has also a universal one.

EXAMPLE. Let $G$ be a complex Lie group and $H$ a closed complex subgroup. The local holomorphic $G$-action on a domain $X \subset G/H$ is univalent and the universal globalization $X^*$ is $G$-homogeneous. Thus $X^* = G/\tilde{H}$ where $\tilde{H}$ is an open subgroup of $H$.

## §4. Local holomorphic actions induced by global real actions

Let $G$ be a connected Lie group which is a subgroup of its complexification $G^{\mathbb{C}}$ and let $X$ be a complex $G$-space. The $G$-action induces a local holomorphic $G^{\mathbb{C}}$-action on $X$. Every leaf $\Sigma \subset G^{\mathbb{C}} \times X$ is a complex $G$-manifold and the projection $p_\Sigma : \Sigma \to G^{\mathbb{C}}$ is a $G$-equivariant locally biholomorphic map. Hence $\Sigma$ is a Riemann domain over $G^{\mathbb{C}}$ with the additional property that $p_\Sigma : \Sigma \to G^{\mathbb{C}}$ is $G$-equivariant, i.e., $\Sigma$ is a Riemann $G$-domain over $G^{\mathbb{C}}$. The complex manifold $G^{\mathbb{C}}$ is a Stein manifold which contains $G$ as a closed totally real submanifold (see e.g. [H2]).

A Riemann domain over a Stein manifold has an envelope of holomorphy (see [R]). The envelope of holomorphy $\hat{\Sigma}$ of $\Sigma$ is again a Riemann

$G$-domain over $G^{\mathbb{C}}$. The diagram

$$\begin{array}{ccc} \Sigma & \xrightarrow{q} & \hat{\Sigma} \\ p_\Sigma \searrow & & \swarrow p_{\hat{\Sigma}} \\ & G^{\mathbb{C}} & \end{array}$$

commutes and all maps are locally biholomorphic and $G$-equivariant. In particular if $q$ and $p_{\hat{\Sigma}}$ are injective, then $p_\Sigma$ is injective. Thus, if $G^{\mathbb{C}}$ is such that any Stein Riemann $G$-domain over $G^{\mathbb{C}}$ is schlicht, then every local $G^{\mathbb{C}}$-action on holomorphically separable $G$-space is univalent (see proof of §5 Proposition 2). Note that a local $G^{\mathbb{C}}$-action on $X$ can be univalent only if all leaves $\Sigma \subset G^{\mathbb{C}} \times X$ are holomorphically separable.

In order to describe the geometry of the leaves $\Sigma \subset G^{\mathbb{C}} \times X$, let $K$ be a maximal compact subgroup of $G$. There is a $G$-equivariant real analytic isomorphism $G \times_K M \to G^{\mathbb{C}}$, $[g, x] \to g \cdot x$, where $M$ is a closed $K$-invariant real analytic submanifold of $G^{\mathbb{C}}$ (see [A] and [HHK]) and $G \times_K M$ denotes the quotient of $G \times M$ with respect to the $K$-action which is given by $k \cdot (g, x) = (gk^{-1}, k \cdot x)$. This implies that $\Sigma$ is $G$-equivariantly and real analytically isomorphic to $G \times_K N$ where $N := (p_\Sigma)^{-1}(M)$. The diagram

$$\begin{array}{ccccc} N & \longrightarrow & G \times_K N & \xrightarrow{\sim} & \Sigma \\ p \downarrow & & \downarrow & & \downarrow p_\Sigma \\ M & \longrightarrow & G \times_K M & \xrightarrow{\sim} & G^{\mathbb{C}} \end{array}$$

commutes, where $p$ is given by restricting $p_\Sigma$ to $N$. Since the $G$-action on $G^{\mathbb{C}}$ and $\Sigma$ is free, $K$ acts freely on $M$ and on $N$. Thus $p_\Sigma$ is injective if and only if $p$ induces an injection $\bar{p}\colon N/K \to M/K$. In general, $\bar{p}$ is only a local isomorphism.

If $\dim G = \dim M/K \geq 2$, then it is possible to construct a Riemann $G$-domain $\Sigma$ over $G^{\mathbb{C}}$ which is not schlicht. For this one has to choose a connected manifold $N_0$ and a local isomorphism $\bar{p}\colon N_0 \to M/K$ which is not injective. Pulling back the $K$-principal bundle $M \to M/K$ with respect to $\bar{p}$, we obtain a $K$-principal bundle $p\colon N \to N_0$. After identifying $G \times_K M$ with $G^{\mathbb{C}}$, we see that there is a local isomorphism $p_\Sigma\colon \Sigma \to G^{\mathbb{C}}$, where $\Sigma = G \times_K N$ is a real analytic manifold.

## §5. Basic examples

It is a simple matter to give concrete example of local actions which are not univalent.

EXAMPLES. (c.f. [P, p. 88]) (a) $\mathbb{C}^*$ acts on $\mathbb{C}^*$ by multiplication and therefore $\mathbb{C}^*$ acts locally on the universal covering $\mathbb{C}$ of $\mathbb{C}^*$. The local $\mathbb{C}^*$-action on $\mathbb{C}$ is not univalent.
(b) Every complex Lie group $G$ of positive dimension contains an open connected subset $Y$ whose fundamental group is isomorphic to $\mathbb{Z}$. The group $G$ acts locally on $Y$ by left multiplication. The local action on the universal covering $X$ of $Y$ is not univalent, since otherwise there would be a locally biholomorphic $G$-equivariant map from $X^*$ into $G$ which is not injective.

For later use we note an important

PROPOSITION 1. (c.f. [P, p. 84]) *A local $\mathbb{R}$-action on a complex space $X$ is univalent.*

*Proof.* A leaf $\Sigma \subset \mathbb{R} \times X$ is a connected real one dimensional manifold. Since a local diffeomorphism from $\Sigma$ into $\mathbb{R}$ is automatically injective, the statement follows. □

*Remark* 1. There exist local $\mathbb{R}$-actions which do not admit any globalization which is a Hausdorff topological space (c.f. [P, p. 86]).

For the next example of a univalent local action we consider a complex space $X$ with an action of a vector group $G \cong \mathbb{R}^m$.

PROPOSITION 2. *A local holomorphic $\mathbb{C}^m$-action on a holomorphically separable complex space $X$ which is induced by a global $\mathbb{R}^m$-action is univalent.*

*Proof.* A leaf $\Sigma \subset \mathbb{C}^m \times X$ is a connected holomorphically separable manifold with an $\mathbb{R}^m$-action and an $\mathbb{R}^m$-equivariant locally biholomorphic map $p_\Sigma \colon \Sigma \to \mathbb{C}^m$, where $\mathbb{R}^m$ acts by addition on $\mathbb{C}^m$. Thus $\Sigma \stackrel{p_\Sigma}{\to} \mathbb{C}^m$ is a holomorphically separable tube domain over $\mathbb{C}^m$. It is sufficient to prove the following

CLAIM. *A holomorphically separable tube domain over $\mathbb{C}^m$ is schlicht, i.e., $p_\Sigma$ is injective.*

In order to prove the claim we first note that $\Sigma$ embeds equivariantly into its envelop of holomorphy which is agian a tube domain over $\mathbb{C}^m$. Thus we may assume that $\Sigma$ is a pseudoconvex tube domain over $\mathbb{C}^m$. By a result of K. Stein (see [St]) the universal covering of $\Sigma$ is also pseudoconvex.

Hence, in addition, we may assume $\Sigma$ to be simply connected. Now an argument of Yang (see [Y, p. 278, proof of Theorem 1]) shows that $p_\Sigma$ is injective. $\square$

A similar result holds for local actions of $G^{\mathbb{C}}$ which are induced by a global action of a compact group $G$.

PROPOSITION 3. *Let $G$ be a compact connected Lie group which acts globally on a holomorphically separable complex space $X$. Then the induced local $G^{\mathbb{C}}$-action on $X$ is univalent.*

*Proof.* In this case a leaf $\Sigma$ is a holomorphically separable complex manifold where $G$ acts globally. Moreover, since $\Sigma \overset{p_\Sigma}{\to} G^{\mathbb{C}}$ is $G$-equivariant and locally biholomorphic, $\Sigma$ does not contain any proper $G$-invariant closed complex submanifold. From §4.1 Proposition in [H1] it follows that $p_\Sigma$ is injective. $\square$

*Remark* 2. It is likely that Propositions 2 and 3 are special cases of the same statement for Lie groups $G$ with a bi-invariant Riemannian metric, i.e., for Lie groups such that the universal covering is isomorphic to a product of a vector group and a compact semisimple Lie group.

*Remark* 3. One can construct a holomorphically separable complex $G$-manifold $X$ with an induced univalent local $G^{\mathbb{C}}$-action which admits some Hausdorff globalization such that the universal globalization $X^*$ is not a Haudorff topological space as follows.

If $\Delta$ denotes the unit disc in $\mathbb{C}$, then $Z := \Delta \times G^{\mathbb{C}}$ is a holomorphically separable manifold. Let $B$ be a closed $G$-invariant subset of $G^{\mathbb{C}}$ such that $G^{\mathbb{C}} \setminus B$ is not connected and set $A := \{0\} \times B$. Then $X := Z \setminus A$ is a holomorphically separable complex $G$-manifold with an induced univalent local $G^{\mathbb{C}}$-action. Moreover $Z$ is a Hausdorff globalization of $X$ and the universal globalization $X^*$ is not a Hausdorff topological space.

Note that it is note possible to construct such that an example for a compact group $G$ and a Stein manifold $X$ (see §7).

## §6. Relative globalizations

Let $X$ and $Y$ be complex spaces and let the connected Lie group $G$ act locally on $X$ and $Y$. If $\phi: X \to Y$ is a locally equivariant holomorphic map, then in general univalency of the local action on $Y$ (resp. $X$) does not imply univalency of the local action on $X$ (resp. $Y$).

EXAMPLES. (a) If $X$ is the product $Y \times Z$ and $H$ is some Lie group which acts locally and not univalently on $Z$, then $X$ considered as a space with a local $G \times H$-action is not univalent.
(b) The group $\mathbb{R}^2$ acts on $Y = \mathbb{C}^2 \setminus \mathbb{R}^2$ by addition. Thus $\mathbb{C}^2$-acts locally on $Y$ and $\mathbb{C}^2$ is the universal globalization of $Y$. Every non trivial covering $X$ of $Y$ is an example of a local $\mathbb{C}^2$-action on a complex manifold which is not univalent (c.f. [P, p. 88]). Note that $X$ is a tube domain over $\mathbb{C}^2$ which is not holomorphically separable (c.f. §5 Proposition 2).

In this section we are interested in conditions for which univalency of the local action on $Y$ implies univalency of the local action on $X$. The examples show that one has to make restrictions on the group as well as on the map $\phi: X \to Y$.

As a first step observe that the map $\hat{\phi}: G \times X \to G \times Y$, $\hat{\phi} = \mathrm{id}_G \times \phi$ is continuous with respect to the leaf topology and maps a leaf $\Sigma$ in $G \times X$ into a leaf $\Lambda$ in $G \times Y$. Moreover we have the following

PROPOSITION. *For univalent local $G$-actions on $X$ and $Y$ the map $\hat{\phi}$ maps a leaf $\Sigma \subset G \times X$ isomorphically onto an open subset of a leaf $\Lambda \subset G \times Y$. If $\phi$ is proper, then $\hat{\phi}$ maps $\Sigma$ isomorphically onto $\Lambda$.*

*Proof.* For $(g_0, x_0) \in \Sigma$ let $\Lambda$ be the leaf through $\hat{\phi}(g_0, x_0) := (g_0, y_0)$ where $y_0 = \phi(x_0)$. The maps

$$q_\Sigma: p_G(\Sigma) \longrightarrow \Sigma, \qquad g \longrightarrow (g, (gg_0^{-1}) \cdot x_0)$$

and

$$q_\Lambda: p_G(\Lambda) \longrightarrow \Lambda, \qquad g \longrightarrow (g, (gg_0^{-1} \cdot y_0)$$

are isomorphisms. Since $q_\Lambda^{-1} \circ \hat{\phi} \circ q_\Sigma: p_G(\Sigma) \to p_G(\Lambda)$ is the inclusion map, the first part of Proposition 1 follows.

For a proper map $\phi$ the induced map $\hat{\phi}$ is also proper. Since $\Sigma$ is closed in $G \times X$ (see §3), this implies that $\hat{\phi}(\Sigma)$ is closed in $\Lambda$. Consequently we have $\hat{\phi}(\Sigma) = \Lambda$. ∎

Let $G$ act locally on $X$ and $Y$ and let $\phi: X \to Y$ be locally $G$-equivariant holomorphic map. The map $\phi$ is called *univalent* if $\hat{\phi} := \mathrm{id}_G \times \phi$ maps each leaf in $G \times X$ injectively into a leaf in $G \times Y$. We say that $\phi$ has the *lifting property with respect to the local action of $G$* if $\hat{\phi}$ maps each leaf in $G \times X$ isomorphically onto a leaf in $G \times Y$ (this terminology is explained by Lemma 1 below).

COROLLARY. *For local $\mathbb{R}$-actions on $X$ and $Y$ a locally equivariant proper holomorphic map $\phi: X \to Y$ has the lifting property.*

□

LEMMA 1. *Let $G$ be a Lie group which acts locally on $X$ and $Y$. If the $G$-action on $Y$ is univalent and the locally equivariant holomorphic map $\phi$ has the lifting property, then the $G$-action on $X$ is univalent.*

*Proof.* Let $\Sigma$ be a leaf in $G \times X$ and $(g_0, x_0) \in \Sigma$. For $y_0 := \phi(x_0)$ denote by $\Lambda$ the leaf containing $(g_0, y_0)$. The statement follows from $p_\Sigma = p_\Lambda \circ \hat{\phi}$ where $p_\Sigma := p_G|\Sigma$ and $p_\Lambda := p_G|\Lambda$. □

*Remark* 1. The conclusion of Lemma 1 also holds for a univalent map.

Let $G$ be a Lie group such that the universal covering $\tilde{G}_e$ of the connected component of the neutral element $e \in G$ is a product of a compact group and a vector group. In this case the universal complexification $G^{\mathbb{C}}$ of $G$ is *polar*, i.e., the map

$$G \times i\mathfrak{g} \longrightarrow G^{\mathbb{C}}, \qquad (g, \xi) \longrightarrow g \exp \xi$$

is a real analytic $G$-equivariant isomorphism, where $\mathfrak{g}$ denotes the Lie algebra of $G$.

Let $Z$ be a not necessarily Hausdorff complex space with a holomorphic $G^{\mathbb{C}}$-action. A $G$-invariant subset $A$ of $Z$ is said to be *orbit convex* if for all $a \in A$ and $\xi \in i\mathfrak{g}$ the set $\{t \in \mathbb{R};\ (\exp t\xi) \cdot a \in A\}$ is connected.

If $G$ acts on a complex space $Y$ such that the local $G^{\mathbb{C}}$-action on $Y$ is univalent, then the universal globalization $Y^*$ of $Y$ is called *orbit convex* if $Y$ considered as a $G_e$-invariant subset of $Y^*$ is orbit convex.

LEMMA 2. *Let $G$ be a connected Lie group such that $G^{\mathbb{C}}$ is polar and let $X$ and $Y$ be complex $G$-spaces such that $Y$ admits an orbit convex globalization $Y^*$ with respect to the induced local $G^{\mathbb{C}}$-action.*

*If $\phi: X \to Y$ is a $G$-equivariant holomorphic map which has the lifting property with respect to every local $\mathbb{R}$-action given by the homomorphisms $\gamma_\xi: \mathbb{R} \to G^{\mathbb{C}}$, $t \to \exp t\xi$, $\xi \in i\mathfrak{g}$, then $\phi$ has the lifting property with respect to the local $G^{\mathbb{C}}$-action.*

*In particular the universal globalization $X^*$ exists and it is orbit convex. Moreover if $Y^*$ is Hausdorff then $X^*$ is also Hausdorff.*

*Proof.* For any $x_0 \in X$ and $y_0 := \phi(x_0) \in Y$ we have to show that $\hat{\phi} = \mathrm{id}_{G^{\mathbb{C}}} \times \phi$ maps the leaf $\Sigma \subset G \times X$ through the point $(e, x_0)$ isomorphically onto the leaf $\Lambda \subset G \times Y$ which contains $(e, y_0)$. Denote by $\tilde{\phi}: \Sigma \to \Lambda$ the map $\hat{\phi}$ restricted to the leaf $\Sigma$. Since $\tilde{\phi}$ is a local isomorphism, it is sufficient to prove the existence of a continuous section $\sigma: \Lambda \to \Sigma$, i.e., a map $\sigma$ such that $\tilde{\phi} \circ \sigma = \mathrm{id}_\Lambda$. By univalency of the local $G^{\mathbb{C}}$-action on $Y$, this is equivalent to the construction of a lifting $q_\Sigma: p_{G^{\mathbb{C}}}(\Lambda) \to \Sigma$ of the isomorphism $q_\Lambda: p_{G^{\mathbb{C}}}(\Lambda) \to \Lambda$, $h \to (h, h \cdot y_0)$ with $q_\Sigma(e) = (e, x_0)$.

For $h \in p_{G^{\mathbb{C}}}(\Lambda)$ let $g \in G$ and $\xi \in i\mathfrak{g}$ be such that $h = g \exp \xi$. Since $Y^*$ is an orbit convex globalization, we have $g \exp t\xi \in p_{G^{\mathbb{C}}}(\Lambda)$ for all $t \in [0,1]$. Therefore $\exp t\xi \in p_{G^{\mathbb{C}}}(\Lambda)$ for $t \in [0,1]$. Using the lifting property of $\phi$ with respect to the local $\mathbb{R}$-action given by $\xi \in i\mathfrak{g}$, we see that $\beta_\xi: [0,1] \to X$, $\beta_\xi(t) := (\exp t\xi) \cdot x_0$ make sense. Thus we can define the analytic map $q_\Sigma: p_{G^{\mathbb{C}}}(\Lambda) \to \Sigma$ by $g_\Sigma(h) = (h, g \cdot \beta_\xi(1))$, where $h = g \exp \xi$. The equivariance of the map $\phi$ shows that $q_\Sigma$ is a lifting of $q_\Lambda$.

Thus the universal globalization $X^*$ exists (Lemma 1).

CLAIM. *For the induced map $\phi^*: X^* \to Y^*$ one has $X = (\phi^*)^{-1}(Y)$.*

*Proof of Claim.* For $x \in X^*$ and $\phi^*(x) = y \in Y$ let $g \in G$, $\xi \in i\mathfrak{g}$ and $\bar{x} \in X$ be such that $g^{-1} \cdot x = (\exp \xi) \cdot \bar{x}$, and $\bar{y} := \phi^*(\bar{x}) \in Y$. Then by orbit convexity $(\exp t\xi) \cdot \bar{y} \in Y$ for all $t \in [0,1]$. The lifting property implies $(\exp t\xi) \cdot \bar{x} \in X$ for all $t \in [0,1]$. In particular $x \in X$ and the claim follows.

Since $X = (\phi^*)^{-1}(Y)$ and $\phi^*$ is $G^{\mathbb{C}}$-equivariant, it follows that $X$ is orbit convex in $X^*$. Finally to prove Hausdorffness we have to separate points of $X^*$ which lie on a fibre of $\phi^*$. But $X^* = G^{\mathbb{C}} \cdot X$ and the equivariance of $\phi^*$ implies that these points belong to $g \cdot X$ for some $g \in G^{\mathbb{C}}$. □

THEOREM. *Let $G$ be a Lie group with polar complexification $G^{\mathbb{C}}$ and $Y$ a complex space with global $G$-action which admits an orbit convex globalization $Y^*$ with respect to the induced local $G^{\mathbb{C}}$-action.*

*Then every $G$-equivariant proper holomorphic map $\phi: X \to Y$ has the lifting property with respect to $G^{\mathbb{C}}$. In particular the universal globalization $X^*$ exists, it is orbit convex and if $Y^*$ is Hausdorff then $X^*$ is also Hausdorff. The induced map $\phi^*: X^* \to Y^*$ is proper.*

*Proof.* By the above Corollary a proper map has the lifting property with respect to all local $\mathbb{R}$-action. Thus Lemma 2 applies. The properness of $\phi^*$ follows from the properness of the restrictions $\phi^*|g \cdot X: g \cdot X \to g \cdot Y$ for every $g \in G^{\mathbb{C}}$. □

*Remark* 2. Let $G$ be a Lie group with polar complexifcation $G^{\mathbb{C}}$ and $X$ a complex $G$-space with an orbit convex and Hausdorff globalization $X^*$ with respect to the local $G^{\mathbb{C}}$-action. Then $\Omega(x) = \{g \in G_e^{\mathbb{C}};\ g \cdot x \in X\}$ is connected and contains $G_e$ for all $x \in X$. For every $\gamma \in G$ the holomorphic maps

$$\begin{aligned} \Omega(x) &\longrightarrow X^*, \qquad & h &\longrightarrow \gamma \cdot (h \cdot x) \quad \text{and} \\ \Omega(x) &\longrightarrow X^*, \qquad & h &\longrightarrow (\gamma h \gamma^{-1}) \cdot (\gamma \cdot x) \end{aligned}$$

agree on $G_e$. Therefore they are equal. This implies that the $G$-action on $X$ extends to a $G$-action on $X^*$. From this it follows that the $G$-action extends to a holomorphic $G^{\mathbb{C}}$-action on $X^*$.

## §7. Holomorphically convex spaces

Let $X$ be a holomorphically convex space, i.e., for every compact subset $C$ of $X$ the holomorphically convex hull $\hat{C} = \{x \in X;\ |f(x)| \leq \sup_{y \in C} |f(y)| \text{ for all } f \in \mathcal{O}(X)\}$ is also compact. We recall the following result of Remmert (see [GR, p. 221]).

> *There exist a Stein space $Y$ and a proper surjective holomorphic map $\phi: X \to Y$ such that*
>
> (i) *all fibers of $\phi$ are connected and*
>
> (ii) *the induced sheaf homomorphism from $\mathcal{O}_Y$ into the direct image sheaf $\phi_* \mathcal{O}_X$ is an isomorphism.*

The map $\phi: X \to Y$ is called the *Remmert reduction* of $X$. From a set theoretical point of view $Y$ is obtained by identifying those point of $X$ which cannot be separated by holomorphic functions. On the level of holomorphic functions one can identify $\mathcal{O}(X)$ with $\mathcal{O}(Y)$.

The Remmert reduction $\phi: X \to Y$ is equivariantly with respect to the group $\mathrm{Aut}_{\mathcal{O}}(X)$ of biholomorphic maps on $X$, i.e., to every biholomorphic map $g \in \mathrm{Aut}_{\mathcal{O}}(X)$ there exists a biholomorphic map on $Y$ which we also denote by $g$, such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{g} & X \\ \phi \downarrow & & \downarrow \phi \\ Y & \xrightarrow[g]{} & Y \end{array}$$

commutes. Moreover, the corresponding homomorphism $\mathrm{Aut}_{\mathcal{O}}(X) \to \mathrm{Aut}_{\mathcal{O}}(Y)$ is continuous with respect to the compact open topology. It follows that an action of a Lie group $G$ on $X$ pushes down to a $G$-action on $Y$ such that $\phi\colon X \to Y$ is equivariant. The base $Y$ of the Remmert reduction is a Stein $G$-space. In [H1] the following is proved:

> *Let $G$ be a compact Lie group and $Y$ a Stein $G$-space. Denote by $G^{\mathbb{C}}$ the universal complexification of $G$. Then there exists a universal globalization $Y^*$ of the local $G^{\mathbb{C}}$-action on $Y$ such that*
>
> (i) *$Y^*$ is Hausdorff*
>
> (ii) *$Y^*$ is a Stein space and*
>
> (iii) *$Y$ considered as an open subset of $Y^*$ is orbit convex.*

*Remark.* As a consequence of the result in §5, $Y^*$ exists as a possibly not Hausdorff complex space. Since the leaves of the local diagonal $G^{\mathbb{C}}$-action on $G^{\mathbb{C}} \times Y$ are closed (see §3) the $G$-invariant holomorphic functions on $G^{\mathbb{C}} \times Y$ separate the fibers of the quotient map $\Pi\colon G^{\mathbb{C}} \times Y \to Y^*$ (see [H1, §2.3 Corollary 2]). It follows that $Y^*$ is Hausdorff.

THEOREM. *Let $G$ be a compact Lie group which acts on a holomorphically convex complex space $X$. Then there exists a universal globalization $X^*$ of the local $G^{\mathbb{C}}$-action on $X$ such that*

(i) *$X^*$ is Hausdorff*

(ii) *$X^*$ is holomorphically convex and the diagram*

$$\begin{array}{ccc} X & \longrightarrow & X^* \\ \phi \downarrow & & \downarrow \phi^* \\ Y & \longrightarrow & Y^* \end{array}$$

*commutes, where the horizontal maps are given by globalizations and the vertical maps are Remmert reductions.*

(iii) *$X$ considered as an open subset of $X^*$ is orbit convex.*

*Proof.* Since the Remmert reduction maps $X$ equivariant onto the Stein space $Y$, there exists a universal Hausdorff globalization $Y^*$ of the local $G^{\mathbb{C}}$-action on $Y$. Moreover $Y$ considered as an open subset of $Y^*$ is orbit convex. By §6 Theorem, this implies the existence of a Hausdorff orbit convex globalization $X^*$ of $X$ and a holomorphic $G^{\mathbb{C}}$-equivariant proper map $\phi^*: X^* \to Y^*$ such that the diagram

$$\begin{array}{ccc} X & \longrightarrow & X^* \\ \phi \downarrow & & \downarrow \phi^* \\ Y & \longrightarrow & Y^* \end{array}$$

commutes. Since $\phi^*$ is proper and $Y^*$ is Stein, it follows that $X^*$ is holomorphically convex. The equivariance of $\phi^*$ implies $\mathcal{O}_{Y^*} \cong (\phi^*)_* \mathcal{O}_{X^*}$. Therefore $\phi^*: X^* \to Y^*$ is the Remmert reduction of $X^*$. □

## §8. Globalization of bundles

In this section we study the action of a Lie group $G$ which has a polar complexification $G^{\mathbb{C}}$ on bundles. Let $P \xrightarrow{\pi} X$ be a holomorphic principal bundle over a complex space $X$ with complex structure group $S$. We assume that $S$ acts from the right on $P$ and for $s \in S$ denote by the same symbol $r_s$ the maps $P \to P$, $p \to p \cdot s$ and $S \to S$, $x \to x \cdot s$. An $S$-invariant vector field $\xi_P$ on $P$ induces a vector field $\xi_X$ on $X$ such that $\xi_X(\pi(p)) = \pi_* \xi_P(p)$ for every $p \in P$. In general, if $\xi$ is a holomorphic vector field on a complex space $Z$, then the corresponding local $\mathbb{R}$-action on $Z$ is univalent (see §5 Proposition 1). Thus with respect to the universal globalization $Z^*$ of $Z$ we can associate to $\xi$ and $z \in Z$ the connected component $I(z, \xi)$ containing zero in $\{t \in \mathbb{R};\ t \cdot z \in Z\}$.

LEMMA. *Let $P \xrightarrow{\pi} X$ be a holomorphic principal bundle over $X$ with structure group $S$. Then $\pi$ has the lifting property with respect to any local $\mathbb{R}$-action on $P$ which is given by an $S$-invariant vector field $\xi_P$ on $P$.*

*Proof.* Let $\xi_X$ be the vector field on $X$ which is induced by $\xi_P$. It is sufficient to show the following

CLAIM. (c.f. [GH]) *For every $x_0 \in X$ and $p \in \pi^{-1}(x_0)$ we have $I(x_0, \xi_X) = I(p, \xi_P)$, i.e., $\xi_P$ can be integrated as far as $\xi_X$.*

We have $I(p, \xi_P) \subset I(x_0, \xi_X)$ for every $p \in \pi^{-1}(x_0)$ (§6 Proposition).

For $t_1 \in I(x_0, \xi_X)$ let $U$ be an open neighborhood of $x_1 = t_1 \cdot x_0$ such that $P|U$ is trivial, i.e., $P|U \cong U \times S$. In this trivialization the vector field $\xi_P$ is of the form

$$\xi_P(u, h) = (\xi_X(u), (r_h)_* \xi(u)),$$

where $\xi$ is a real analytic map from $U$ into the Lie algebra $\mathfrak{s}$ of $S$. Let $J$ denote a compact neighborhood of $t_1 \in I(x_0, \xi_X)$ such that $t \cdot x_0 \in U$ for $t \in J$. Then there exists a solution $t \to \phi(t) \in S$ of equation

$$(r_{\phi(t)^{-1}})_* \dot{\phi}(t) = A(t),$$

where $A(t) = \xi(t \cdot x_0)$, with $\phi(t_1) = e$ and which is defined on an open neighborhood of $J$ (see [KN, p. 69]). Using the trivialization $P|U \cong U \times S$ we see that for $h \in S$

$$\gamma_h(t) = (t \cdot x_0, \phi(t) \cdot h)$$

defines an integral curve $\gamma_h$ of the vector field $\xi_P$ with $\gamma_h(t_1) = (x_1, h)$. Using this, one sees that $I(p, \xi_P)$. This proves the claim $\square$

By an action of a Lie group $G$ on the principal bundle $P$ we mean a group homomorphism $\rho$ from $G$ into the group $\mathrm{Aut}_S(P)$ of $S$-equivariant biholomorphisms of $P$ such that the map $G \times P \to P$, $(g, p) \to \rho(g)(p) := g \cdot p$ is real analytic. If $\rho$ is given, then we say that $P$ is a holomorphic principal $G$-bundle. For a complex Lie group $G$ this will automatically mean that the map $G \times P \to P$ is holomorphic.

Let $G$ be a Lie group which admits a polar complexfication $G^{\mathbb{C}}$ and $X$ a complex $G$-space with an orbit convex globalization $X^*$.

THEOREM. *A holomorphic principal $G$-bundle $P \xrightarrow{\pi} X$ with complex structure group $S$ extends to a holomorphic principal $G^{\mathbb{C}}$-bundle $P^* \xrightarrow{\pi^*} X^*$. Moreover, if $X^*$ is Hausdorff, then $P^*$ is Hausdorff.*

*Proof.* Since $\pi: P \to X$ has the lifting property with respect to every local $\mathbb{R}$-action on $P$ which is induced by the homomorphisms $\mathbb{R} \to G^{\mathbb{C}}$, $t \to \exp t\xi$, $\xi \in i\mathfrak{g}$, there exists a universal globalization $P^*$ of $P$ with respect to $G^{\mathbb{C}}$ which is Hausdorff if $X^*$ is Hausdorff. Let $\pi^*: P^* \to X^*$ be the map induced by $\pi: P \to X$. For a subset $U$ of $X$ we set $P^*|U := (\pi^*)^{-1}(U)$. In order to prove that $P^*$ is a principal bundle one may proceed as follows. We have $(\pi^*)^{-1}(X) = P$ (§6 Lemma 2). Thus for each $g \in G^{\mathbb{C}}$ we obtain a biholomorphic map $\Phi_g: P^*|g \cdot X \to P$, $q \to g^{-1} \cdot q$. For $s \in S$ and $g \in G^{\mathbb{C}}$ let $s_g: P^*|g \cdot X \to P^*|g$ be defined by $s_g(q) = g \cdot ((g^{-1} \cdot q) \cdot s)$.

CLAIM. *For $g_1$, $g_2 \in G^{\mathbb{C}}$ we have $s_{g_1} = s_{g_2}$ on $P^*|g_1 \cdot X \cap g_2 \cdot X$.*

*Proof of Claim.* For $x \in X$ let $\Omega(x) := \{h \in G^{\mathbb{C}};\ h \cdot x \in X\}$. The set $\Omega(x)$ contains the image of $G$ in $G^{\mathbb{C}}$ and it is invariant with respect to the

$G$-action on $G^{\mathbb{C}}$ which is given by left-multiplication. Since $X^*$ is an orbit convex globalization, it follows that $\Omega(x)/G$ is connected. The holomorphic maps from $\Omega(x)$ into $P$ given by

$$h \longrightarrow (h \cdot p) \cdot s$$

resp.

$$h \longrightarrow h \cdot (p \cdot s),$$

where $p \in \pi^{-1}(x)$ is fixed, agree on the image of $G$ in $G^{\mathbb{C}}$. The Identity Theorem implies that

$$(h \cdot p) \cdot s = h \cdot (p \cdot s)$$

for all $h \in \Omega(x)$.

For $g_1,\ g_2 \in G^{\mathbb{C}},\ s \in S$ and $q \in P^*|g_1 \cdot X \cap g_2 \cdot X$ set $p := g_2^{-1} \cdot q$, $x = \pi(p)$ and $h := g_1^{-1} g_2$. It follows from the above consideration that $s_{g_1}(q) = g_1 \cdot ((g_1^{-1} \cdot q) \cdot s) = g_1 \cdot ((h g_2^{-1} \cdot q) \cdot s) = (g_1 h) \cdot ((g_2^{-1} \cdot q) \cdot s) = s_{g_2}(q)$ holds. This proves the claim.

As a consequence we see that the $S$-action on $P$ extends to a fiberwise free and transitive action on $P^*$ which commutes with the $G^{\mathbb{C}}$-action. Using the maps $\Phi_g$ one sees that $P^* \xrightarrow{\pi^*} X^*$ is a holomorphic principal $G^{\mathbb{C}}$-bundle over $X^*$. □

COROLLARY. *Let $G$ be a compact Lie group, $X$ a holomorphically convex $G$-space and $P \xrightarrow{\pi} X$ a holomorphic principal $G$-bundle over $X$ with structure group $S$. Then $P \xrightarrow{\pi} X$ extends to a holomorphic principal $G^{\mathbb{C}}$-bundle $P^* \xrightarrow{\pi^*} X^*$.*

□

As we already mentioned in the introduction, the Corollary has an important application to the equivariant version of Grauert's Oka Principle. In order to explain this, let $G$ be a compact Lie group and $X$ a Stein space. We fix a complex Lie group $S$ and denote by $\mathrm{Bund}_{\mathcal{O}}(X)^G$ (resp. $\mathrm{Bund}_{\mathcal{C}^0}(X)^G$) the isomorphisms classes of holomorphic (resp. topological) principal $G$-bundle over $X$ with complex structure group $S$. Under the assumption that a certain $G$-bundle of Lie groups over $X$ extends to a $G^{\mathbb{C}}$-bundle over $X^*$ it has been shown in [HK] that the natural map

$$\mathrm{Bund}_{\mathcal{O}}(X)^G \longrightarrow \mathrm{Bund}_{\mathcal{C}^0}(X)^G$$

is an isomorphism. The Corollary implies that this assumption is superfluous.

## §9. Hamiltonian actions on Kählerian spaces

Let $X$ be a complex space. A Kählerian structure $\omega$ on $X$ is given by an open covering $\{U_\alpha\}$ of $X$ together with a family of smooth strictly plurisubharmonic functions $\rho_\alpha\colon U_\alpha \to \mathbb{R}$ such that $h_{\alpha\beta} = \rho_\alpha - \rho_\beta$ is pluriharmonic on $U_\alpha \cap U_\beta$. Note that for smooth $X$ one obtains the usual definition of a Kählerian manifold whose Kähler form is given locally by $\omega_\alpha = \frac{i}{2}\partial\bar{\partial}\rho_\alpha$. For smooth $X$ we will not distinguish between $\omega$ as defined above and the associated Kähler form $\omega := \frac{i}{2}\partial\bar{\partial}\rho_\alpha$.

For a complex $G$-space $X$ one has a natural notion of a $G$-invariant Kählerian structure $\omega$. A moment map on a complex $G$-space $X$ with respect to such an invariant Kählerian structure is a smooth map $\mu\colon X \to \mathfrak{g}^*$ such that

(i) $\mu$ is $G$-equivariant and

(ii) $d\mu_\xi = \iota_{\xi_X}\omega$ for every $\xi \in \mathfrak{g}^*$

holds on every smooth $G$-stable complex submanifold $Y$ of $X$. Here $\omega$ denotes the Kähler form induced on $Y$, $\xi_X$ the vector field on $X$ induced by $\xi$, $\mu_\xi$ the $\xi$th component of $\mu$ and $\iota$ is the usual contraction.

If the complexification $G^{\mathbb{C}}$ acts on $X$, then the following result is well know (see e.g. [GS], [HL], [Ki], [N], [S]).
*The reduction $\mu^{-1}(0)/G$ is in a natural way a Kählerian complex space.*

One can construct the complex structure on $\mu^{-1}(0)/G$ as it follows. Let $X(\mu)$ be the set of semistable points with respect to $\mu^{-1}(0)$, i.e., $X(\mu) = \{x \in X;\ \overline{G^{\mathbb{C}} \cdot x} \cap \mu^{-1}(0) \neq \emptyset\}$. The relation $x \sim y$ if and only if $\overline{G^{\mathbb{C}} \cdot x} \cap \overline{G^{\mathbb{C}} \cdot y} \cap X(\mu) \neq \emptyset$ is in fact an equivalence relation on $X(\mu)$, $X(\mu)$ is open in $X$ and $\mu^{-1}(0)/G$ is isomorphic to $X(\mu)//G^{\mathbb{C}}$. The sheaf of holomorphic functions can be "identified" with the sheaf of $G^{\mathbb{C}}$-invariant holomorphic functions on $X(\mu)$.

Now assume that $X$ is a Kählerian $G$-space with moment map $\omega$ and also assume that $X$ has an orbit convex globalization $X^*$, e.g., $X$ is holomorphically convex. Then we can set $X^*(\mu) = \{x \in X^*;\ \overline{G^{\mathbb{C}} \cdot x} \cap \mu^{-1}(0) \neq \emptyset\}$. In this case the above result can be extended as follows.

QUOTIENT THEOREM. *The set $X^*(\mu)$ of semistable points is open in $X^*$. The quotient $X^*(\mu)//G^{\mathbb{C}}$ has a structure of a complex space such that the quotient map $\pi\colon X^*(\mu) \to X^*(\mu)//G^{\mathbb{C}}$ is holomorphic and*

(i) *for any open subset $Q$ of $X^*(\mu)//G^{\mathbb{C}}$ we have $\mathcal{O}(Q) \cong \mathcal{O}(\pi^{-1}(Q))^{G^{\mathbb{C}}}$,*

(ii) *the embedding* $\mu^{-1}(0) \rightarrow X^*(\mu)$ *induces a homeomorphism* $\mu^{-1}(0)/G \rightarrow X^*(\mu)//G^{\mathbb{C}}$ *and*

(iii) $X^*(\mu)//G^{\mathbb{C}}$ *is Kählerian space whose Kählerian structure is compatible with the symplectic reduction* $\mu^{-1}(0) \rightarrow \mu^{-1}(0)/G$.

The main ingredient in the proof is a slice theorem of Luna type (see e.g. [H1], [HL], [L], [S]). In our setting this can be formulated as follows.

SLICE THEOREM. *For every* $y \in \mu^{-1}(0)$ *we have* $H := (G_y)^{\mathbb{C}} = (G^{\mathbb{C}})_y$ *and there exists a locally closed* $H$*-invariant complex subspace* $S$ *of* $X^*$ *which contains* $y$ *such that* $U := G^{\mathbb{C}} \cdot S$ *is open in* $X^*$ *and the natural map*

$$G^{\mathbb{C}} \times_H S \longrightarrow H, \quad [g,x] \longrightarrow g \cdot x$$

*is biholomorphic.*

The proofs of these theorems are very similar to those of the corresponding result in [HL] and they will therefore be omitted.

*Remark.* $X^*(\mu)$ is always Hausdorff.

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