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THE GENUS OF CURVES ON THE THREE DIMENSIONAL QUADRIC

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Abstract. By means of an *ad hoc* modification of the so-called "Castelnuovo-Harris analysis" we derive an upper bound for the genus of integral curves on the three dimensional nonsingular quadric which lie on an integral surface of degree $2k$, as a function of k and the degree d of the curve. In order to obtain this we revisit the Uniform Position Principle to make its use computation-free. The curves which achieve this bound can be conveniently characterized.

Introduction

The objects of investigation of this paper are the following two connected problems. *What are the possible geometric genera of integral curves C of degree d lying on a nonsingular three dimensional quadric Q_3 in $\mathbb{P}^4$ and on an integral surface S of degree $2k$ contained in Q_3?* As it is shown in this paper the above genera are bounded above by a function of d and k. *What is the structure of the curves for which the genus is maximum with respect to k and d?*

The above problems are natural questions stemming from the analogue problems that one can state by replacing, in what above, Q_3 by $\mathbb{P}^3$ and $2k$ by k. These were answered completely in the paper [JH]. The paper [G-P] (and its refinement contained in [E-P]) deals with the very similar questions of (i) determining the biggest possible genus for curves of degree d in $\mathbb{P}^3$ which do not lie on a surface of degree less than k, or lie on a surface of degree k and of (ii) understanding the curves for which the genus is the maximum possible. Going back to the quadric body ([A-S], §6) gives an answer to the problem of determining the maximum possible genus for curves which lie on a surface of degree $2k$ under the assumption $d > 2k(k-1)$. To do so they use the technique of [G-P], coupled with the idea of considering only hyperplane sections which are tangent to the quadric Q_3.

In this work an upper bound for the above genera is worked out with no assumptions on the degree d. The bound is obtained pursuing some

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numerical properties of embedded curves; a certain maximization process is involved (cf. §2). In analyzing the curves that should achieve that bound, the unpleasant answer is that some systems of invariants are inconsistent with each other so that, except for some special cases in which the bound is sharp and the curves of maximal possible genus are characterized, the bound turns out to be *not* sharp: the biggest possible genus is strictly smaller than the derived upper bound. It is appropriate to say that some geometric information gets lost in the process. At present the author is unable to bridge the gap between the bound obtained in this paper and "the real bound." He conjectures that the extremal curves should be special curves (cf. Definition 1.3) so that the right bound should then be (3.5.1) (see also §4, Question A).

The Paper [C-C-D] deals with questions (i) and (ii) above in the context of the arithmetic genus for curves in $\mathbb{P}^4$ of degree "sufficiently big."

The paper is organized as follows. §1 contains the statement of Theorem 1.4, which is the main result of this paper: it gives the upper bound, it says exactly when it is sharp and characterizes the curves of maximum genus in the cases in which the bound is sharp; the proof of (1.4.1) is in §2 and the one of (1.4.2) is in §3. This section also contains some preparatory material. The reader acquainted with the paper [JH] will realize how big is the debt of the present work towards it. However certain subtleties associated with the possibility of having to deal with "unbalanced" curves on nonsingular quadric surfaces had to be circumvented by means of a systematic use of the Uniform Position Principle; this section contains some alternative formulations of it. §2 is a Castelnuovo-Harris type approach to the determination of the wanted upper bound. It is an *ad hoc* modification of the above mentioned paper of Harris. It contains also a bound for the genus of curves for which the general hyperplane section is not contained in any curve "of type k" (see Theorem 2.10). §3 discusses the bound obtained in §2. Moreover it deals with a special class of curves (see Definition 1.3) which arise naturally in the contest of curves with the biggest possible genus. §4 is speculative in nature: it raises two questions that the author could not answer.

Note added in proof. Theorem 1.4 has been used by the author in his study of codimension two subvarieties of quadrics (cf. [D1]). In particular, it was instrumental **1)** in proving that, for every $n \geq 4$ (the cases $n = 4$ and $n = 6$ are due to Arrondo-Pedreira-Sols and Fania-Ottaviani, respectively), there are only finitely many components of the Hilbert scheme of Q_n corresponding to codimension two nonsingular subvarieties of Q_n which are not of general type (see [D2], which generalizes [E-P]), **2)** in studying the adjunction-theoretic structure of codimension two nonsingular subvarieties of quadrics (see [D3]) and **3)** in classifying subvarieties of degree at most 10 and scrolls (see [D4]).

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§1. Preliminaries

The basic notation is the one of [Ha].
The ground field is the field of complex numbers $\mathbb{C}$.
Q_i denotes a smooth i-dimensional quadric in a projective space $\mathbb{P}^{i+1}$.
When there is no danger of confusion, little distinction is made between Cartier divisors and associated rank one locally free sheaves and the additive and tensor product notation are sometimes used at the same time.
The topological space will be sometimes dropped when one is dealing with cohomology groups and their dimensions.
In this paper the use of the adjective *general* in connection with an element H of $\check{\mathbb{P}}$ is a quantifier; it means that there exists a Zariski dense open subset W of $\check{\mathbb{P}}$, such that for every $H \in W, \ldots$
$\lfloor t \rfloor$ denotes the biggest integer smaller than or equal to t.

The following two sets of data are fixed throughout the sequel of the paper:

1.1. *C is an integral curve lying on a smooth three-dimensional quadric Q_3, k is a positive integer, S_k is an integral surface in $|\mathcal{O}_{Q_3}(k)|$ containing C, d and g are the degree and the geometric genus of C, respectively.*

Definition 1.2. Define n_0 and ϵ when $d > 2k(k-1)$ and θ_0 and ϵ'

when $d \leq 2k(k-1)$ as follows:

$$\begin{aligned} n_0 &:= \left\lfloor \frac{d-1}{2k} \right\rfloor + 1; \\ d &\equiv -\epsilon \pmod{2k}, \qquad 0 \leq \epsilon \leq 2k-1; \\ \theta_0 &:= \left\lfloor \frac{d-1}{2k} \right\rfloor + 1; \\ d &\equiv -\epsilon' \pmod{2\theta_0}, \qquad 0 \leq \epsilon' \leq 2\theta_0 - 1. \end{aligned}$$

The following class of curves plays a central role in the understanding of the curves whose genus is the maximum possible. Arithmetically Cohen-Macauley is denoted by a.C.M..

DEFINITION 1.3. A curve C as in (1.1) is said to be in the class $\mathfrak{S}(d,k)$, if it is nonsingular, projectively normal and linked, in a complete intersection on Q_3 of type (k, n_0) if $d > 2k(k-1)$ ((θ_0, k) if $d \leq 2k(k-1)$), to an (*a fortiori*) a.C.M. curve D_ϵ ($D_{\epsilon'}$, respectively) of degree ϵ (ϵ', respectively) lying on a quadric surface hyperplane section of Q_3.

The following is the main result of this paper: it is a bound for the geometric genus of curves as in (1.1) in terms of d and k.

THEOREM 1.4. *Notation as in* (1.1) *and* (1.2). *Assume* $d > 2k(k-1)$. *Then*

$$g - 1 \leq \pi(d,k) - \Xi, \tag{1.4.1}$$

where

$$\pi(d,k) = \begin{cases} \dfrac{d^2}{4k} + \dfrac{1}{2}(k-3)d - \dfrac{\epsilon^2}{4k} - \epsilon(\dfrac{k-\epsilon}{2}), & 0 \leq \epsilon \leq k, \\ \dfrac{d^2}{4k} + \dfrac{1}{2}(k-3)d - (k-\tilde{\epsilon})(\dfrac{\tilde{\epsilon}}{2} - \dfrac{\tilde{\epsilon}}{4k} + \dfrac{1}{4}), & k+1 \leq \epsilon \leq 2k-1, \\ & \tilde{\epsilon} := \epsilon - k; \end{cases}$$

and

$$\Xi = \Xi(d,k) = \begin{cases} 0 & \textit{if } \epsilon = 0,\ 1,\ 2,\ 2k-1, \\ 1 & \textit{if else.} \end{cases}$$

(1.4.2) *The bound is sharp for* $\epsilon = 0,\ 1,\ 2,\ 3,\ 2k-2,\ 2k-1$. *A curve achieves such a maximum possible genus if and only if it is in the class*

$\mathfrak{S}(d,k)$, except, possibly, the cases $\epsilon = 3,\ 2k-2$.
Assume $d \leq 2k(k-1)$. Then the analogous statements with $\pi'(d,k) = \pi(d, \lfloor (d-1)/2k \rfloor + 1) = \pi(d,\theta_0)$ and with Ξ', ϵ', (θ_0, k) and $D_{\epsilon'}$ replacing Ξ, ϵ, (k, n_0) and D_ϵ respectively, hold.

The following, which is proven in [JH], page 194, is stated for the reader's convenience; it is one of the two main ingredients of the analysis:

LEMMA 1.5. (Gieseker) *Let $E \subseteq H^0(\mathbb{P}^1, \mathcal{O}(l-1))$, $\{0\} \neq F \subseteq H^0(\mathbb{P}^1, \mathcal{O}(l))$ be two vector spaces of dimensions e and f respectively, such that: $E \times H^0(\mathbb{P}^1, \mathcal{O}(1)) \subseteq F$. Then either $f \geq e+2$, or $|F|$ equals the complete linear system $|\mathcal{O}_{\mathbb{P}^1}(f-1)|$ plus $(l-f+1)$ fixed points.*

The following two lemmata are nothing else but a reformulation of the *Uniform Position Principle* (U.P.P.) (cf. [A-C-G-H], pages 111–113) in terms of subvarieties and of coherent sheaves respectively, rather than in terms of linear systems. The use of this principle is the second main ingredient. First some notation.

Let $\mathcal{C}$ be an integral curve of degree d in a projective space $\mathbb{P}$ of any dimension, H a hyperplane, Γ the corresponding hyperplane section of $\mathcal{C}$.

Let $\mathfrak{J}$ be the incidence correspondence in $\mathbb{P} \times \check{\mathbb{P}}$ defined by $\{(p; H) \mid p \in H\}$ with first and second second projections p and q respectively, $\mathcal{F}$ a coherent sheaf on $\mathfrak{J}$. By abuse of notation H can and will denote the hyperplane and the corresponding point of $\check{\mathbb{P}}$.
Let $\mathfrak{I}(\delta)$, $1 \leq \delta \leq d$ be the incidence correspondences in $\mathcal{C}^\delta \times \check{\mathbb{P}}$, defined by $\{(p_1, \ldots, p_\delta; H) \mid p_i \in H,\ \forall i\}$, where $\mathcal{C}^\delta$ denotes the δ-fold product of $\mathcal{C}$. The essence of the U.P.P. is that *the spaces $\mathfrak{I}(\delta)$ are irreducible.* This principle should be regarded as a fundamental property of curves in projective space. Finally define $\hat{\mathfrak{I}}(\delta)$ to be the quotient of $\mathfrak{I}(\delta)$ by the action of the symmetric group S_δ: $\hat{\mathfrak{I}}(\delta) = \mathfrak{I}(\delta)/S_\delta$. The spaces $\hat{\mathfrak{I}}(\delta)$ are irreducible as well.

LEMMA 1.6. (U.P.P. 1) *Notation as above. Let $\mathfrak{B}$ be a Zariski closed subset of $\mathfrak{J}$ and $\mathfrak{B}_H \subseteq \mathfrak{B}$ be the Zariski closed subset cut by a general hyperplane H, i.e. $\mathfrak{B} \cap q^{-1}(H)$. Then, either $\Gamma \subseteq \mathfrak{B}_H$, or $\Gamma \cap \mathfrak{B}_H = \emptyset$.*

Proof. Define $\Gamma' := \Gamma \cap \mathfrak{B}_H$, and let δ be the cardinality of Γ'; δ is constant on a Zariski dense open subset of $\check{\mathbb{P}}$. Without loss of generality assume $\delta > 0$. By shrinking the above set to another Zariski dense open set W, if necessary, one can assume that the incidence correspondence $\mathfrak{I}(\delta)$, restricted over W, is a connected étale covering of degree $\binom{d}{\delta}\delta!$. Clearly the corresponding covering associated with $\hat{\mathfrak{I}}(\delta)$ has degree $\binom{d}{\delta}$. The assignment $\{W \ni H\} \longrightarrow \{\Gamma' \in \hat{\mathfrak{I}}(\delta)\}$, defines a holomorphic section over W of the latter covering; this is a contradiction unless $\delta = d$. $\square$

Let B be any algebraic scheme of dimension b. Consider the following decomposition of sets: $\text{Support}(B_{red}) = B_b \cup B_{b-1} \cup \ldots B_1 \cup B_0$, where B_i denotes the union of all the supports of the components of B of dimension i.

LEMMA 1.7. (U.P.P. 2) *Notation as above. Let H be a general hyperplane. Consider the natural map: $H^0(\mathcal{F}_{|q^{-1}(H)}) \otimes_{\mathcal{O}_H} \check{\mathcal{F}}_{|q^{-1}(H)} \xrightarrow{\eta} \mathcal{O}_H$, and the coherent ideal sheaf $\mathcal{I}_{\mathfrak{B}_H,H} := \text{Im}(\eta)$. Then for each positive integer i we have that either $\Gamma \subseteq \mathfrak{B}_{Hi}$, or $\Gamma \cap \mathfrak{B}_{Hi} = \emptyset$.*

Proof. Generic flatness (cf. [Mu], Lecture 8) and semicontinuity give a Zariski dense open subset $W \subseteq \check{\mathbb{P}}$ over which $q_*\mathcal{F}$ is a locally free coherent sheaf and the natural maps $q_*\mathcal{F} \otimes_{\mathcal{O}_W} k(H') \to h^0(\mathcal{F}_{|q^{-1}(H')})$ are isomorphisms $\forall\, H' \in W$. Pick $\varsigma \gg 0$ such that $q_*\mathcal{F} \otimes \mathcal{O}_{\check{\mathbb{P}}}(\varsigma)$ is spanned by global sections on $\check{\mathbb{P}}$; then the following diagram commutes and has surjective vertical arrows, $\forall\, H' \in W$:

$$\begin{tikzcd}
H^0(\mathcal{F} \otimes q^*\mathcal{O}_{\check{\mathbb{P}}}(\varsigma)) \otimes \check{\mathcal{F}} \otimes q^*\mathcal{O}_{\check{\mathbb{P}}}(-\varsigma) \arrow[r] \arrow[d] & \mathcal{O}_{\mathfrak{I}} \arrow[d] \\
H^0(\mathcal{F}_{|q^{-1}(H')}) \otimes_{\mathcal{O}_{q^{-1}(H')}} \check{\mathcal{F}}_{|q^{-1}(H')} \arrow[r] & \mathcal{O}_{q^{-1}(H')}.
\end{tikzcd}$$

Hence, for a general H, the set $\mathfrak{B}_{Hi}$ is the restriction to H of a Zariski closed subset $\mathfrak{B}'$ of $\mathfrak{I}$. Let $\Gamma'_H := \mathfrak{B}_{Hi} \cap \Gamma$ and δ be its cardinality; shrink W, if necessary, in order for δ to be constant over W. One can now conclude as in the previous lemma. $\square$

Remark 1.8. The above proposition is still valid, after obvious changes, if one replaces $\mathbb{P}$ by some closed subscheme $\mathcal{C} \subseteq \mathfrak{T} \subseteq \mathbb{P}$. In this paper $\mathfrak{T} = Q_3$.

Remark 1.9. It is maybe worthwhile to observe that (1.6) and (1.7) are both equivalent to the irreducibility of the varieties $\hat{\mathfrak{I}}(\delta)$, $1 \leq \delta \leq d$.

§2. Deriving the upper bound

The following is a presentation of the relevant invariants and of how to use them to give an upper bound on g as a function of d and k (cf. [JH]).

Consider the following natural morphisms: $\widehat{C} \xrightarrow{\nu} C \xhookrightarrow{\iota} Q_3 \hookrightarrow \mathbb{P}^4$, where $\widehat{C} \xrightarrow{\nu} C$ denotes the normalization of C, the other two arrows the given embeddings. All sheaves of the form $\mathcal{O}(h)$ are pull-backs from $\mathbb{P}^4$; the sheaves on $\hat{C}$ are pull-backs via ν. Let $\rho := \iota \circ \nu$ and ρ_l be the map induced in cohomology by ρ; define:

$$\alpha_l := \dim_{\mathbb{C}}[\mathrm{Im} H^0(Q_3, \mathcal{O}_{Q_3}(l)) \xrightarrow{\rho_l} H^0(\widehat{C}, \mathcal{O}_{\widehat{C}}(l)].$$

Let H be a general hyperplane of $\mathbb{P}^4$, $\Gamma := C \cap H$, $Q_2 := Q_3 \cap H$; then for every l there is the map: $H^0(Q_3, \mathcal{I}_{\Gamma,Q_3}(l)) \xrightarrow{\sigma_l} H^0(\widehat{C}, \mathcal{I}_{\Gamma,\widehat{C}}(l)) \simeq H^0(\widehat{C}, \mathcal{O}_{\widehat{C}}(l-1))$.
Since $\mathrm{Im}(\rho_{l-1}) \subseteq \mathrm{Im}(\sigma_l)$, $\mathrm{Ker}(\rho_l) = \mathrm{Ker}(\sigma_l) = H^0(\mathcal{I}_{C,Q_3}(l))$, $H^1_*(\mathcal{O}_{Q_3}) = 0$, one gets the following chain of relations at the end of which the quantities β_l are defined:

$$\begin{aligned}\alpha_l - \alpha_{l-1} &= \dim \mathrm{Im}(\rho_l) - \dim \mathrm{Im}(\rho_{l-1}) \geq \dim \mathrm{Im}(\rho_l) - \dim \mathrm{Im}(\sigma_l)\\ &= h^0(\mathcal{O}_{Q_3}(l)) - h^0(\mathcal{I}_{\Gamma,Q_3}(l) = h^0(\mathcal{O}_{Q_2}(l)) - h^0(\mathcal{I}_{\Gamma,Q_2}(l))\\ &=: \beta_l.\end{aligned}$$

One may think of β_l as the number of independent conditions that Γ imposes on $|\mathcal{O}_{Q_2}(l)|$.
Define:

$$\begin{aligned}\gamma_l &:= \beta_l - \beta_{l-1}\\ &= [h^0(\mathcal{O}_{Q_2}(l)) - h^0(\mathcal{I}_{\Gamma,Q_2}(l))] - [h^0(\mathcal{O}_{Q_2}(l-1)) - h^0(\mathcal{I}_{\Gamma,Q_2}(l-1))].\end{aligned}$$

These "second differences" are quantities that can be realized geometrically as follows: consider the following exact sequence defining a general conic Q_1 (to be chosen so that it is smooth and it does not meet Γ) in Q_2:

$$0 \to \mathcal{I}_{\Gamma,Q_2}(-1+l) \to \mathcal{I}_{\Gamma,Q_2}(l) \to \mathcal{O}_{Q_1}(l) \to 0.$$

Let

$$E_l := \mathrm{Im}[H^0(\mathcal{I}_{\Gamma,Q_2}(l)) \to H^0(\mathcal{O}_{Q_1}(l))],$$

and

$$e_l := \dim_{\mathbb{C}}(E_l).$$

Then: $\gamma_l = h^0(\mathcal{O}_{Q_1}(l)) - [h^0(\mathcal{I}_{\Gamma,Q_2}(l)) - h^0(\mathcal{I}_{\Gamma,Q_2}(l-1))]$.
It is now clear that γ_l measures the incompleteness of the linear systems induced on Q_1 by $|\mathcal{I}_{\Gamma,Q_2}(l)|$:

$$\gamma_l = 2l + 1 - e_l.$$

Let:

$$\theta := \min\{t \in \mathbb{N} \mid h^0(\mathcal{I}_{\Gamma,Q_2}(t)) > 0\}.$$

By the existence of S_k, one infers that $\theta \leq k$. Let:

$$n := \min\{\nu \in \mathbb{N} \mid |\mathcal{I}_{\Gamma,Q_2}(\nu)| \text{ is not empty and does not have fixed components}\}.$$

Since $\gamma_l = \beta_l - \beta_{l-1}$, and $\beta_l = d$, $\forall l \gg 0$ ($h^1(\mathcal{I}_{\Gamma,Q_2}(l) = 0$, $\forall l \gg 0$), one sees that $\gamma_l = 0$, $\forall l \gg 0$. Define

$$m := \min\{\mu \in \mathbb{N} \mid \gamma_\mu = 0\}.$$

Clearly $\gamma_l = 0$, $\forall l \geq m$.

Following Halphen, Castelnuovo, and more recently Gruson-Peskine and Harris, by choosing $\lambda \gg 0$, one gets:

$$\begin{aligned}
g - 1 &= (\text{Riemann-Roch}) \\
d\lambda - h^0(\mathcal{O}_{\widehat{C}}(\lambda)) &\leq (\alpha_l \leq h^0(\mathcal{O}_{\widehat{C}}(\lambda))) \\
d\lambda - \alpha_\lambda &\leq (\beta_t \leq \alpha_t - \alpha_{t-1}) \\
d\lambda - \sum_{t=0}^{\lambda} \beta_t &= (\beta_t = \sum_{l=0}^{t} \gamma_l) \\
d\lambda - \sum_{l=0}^{\lambda} (\lambda - l + 1)\gamma_l &= (\sum_{l=0}^{\lambda} \gamma_l = d) \\
&= \sum_{l=0}^{\lambda} (l-1)\gamma_l.
\end{aligned} \tag{2.1}$$

The next step is to maximize the above sum with respect to some constraints on the numbers γ_l. For the sake of clarity the analysis of these quantities is divided into three cases: $d > 2k(k-1)$, $d \leq 2k(k-1)$ and $\theta =\leq k-1$, $d \leq 2k(k-1)$ and $\theta = k$. It is not necessary to distinguish between the last two cases; however if one assumes $\theta = k$ then one gets the smaller upper bound (2.10), and does so without assuming the existence of the surface S_k.

The Case $d > 2k(k-1)$.

By (1.2): $d = 2n_0k - \epsilon$.

If $d > 2k(k-1)$, then equality holds in the inequality $\theta \leq k$; for if one chooses H general then $D_k := S_k \cap Q_2$ will be an integral curve which will not contain any of the components of $D_\theta \in |\mathcal{I}_{\Gamma,Q_2}(\theta)|$ so that, by computing intersections on Q_2, one gets: $D_k \cdot D_\theta = 2\theta k \geq d > 2k(k-1)$, that is $\theta > (k-1)$.
It follows that the linear systems E_l are empty in the range $[0, k-1]$:

$$\gamma_l = 2l + 1, \qquad \forall l \in [0, k-1].$$

Since D_k is an integral curve the linear systems $|\mathcal{I}_{\Gamma,Q_2}(l)| = D_k + |\mathcal{O}_{Q_2}(l-k)|$, in the range $[k, n-1]$, so that, on the general Q_1, $E_l = D_k \cap Q_1 + |\mathcal{O}_{Q_1}(l-k)|$; it follows that:

$$\gamma_i = 2k, \qquad \forall i \in [k, n-1].$$

The above interval is empty if and only if $|\mathcal{I}_{\Gamma,Q_2}(k)|$ is free of fixed components, which in turn is equivalent to the statement that D_k moves; this last condition implies of course $h^0(\mathcal{I}_{\Gamma,Q_2}(k)) \geq 2$ so that, if $n = k$ then $\gamma_k \leq 2k-1$.

As in [JH] it is now time to use Gieseker's Lemma; it allows to understand better the behavior of the quantities γ_l in the third remaining interval $[n, m]$.

LEMMA 2.2. *If $k < n$ then one has the following information as to the behavior of the quantities γ_l: $\gamma_{n-1} - \gamma_n \geq 1$; $\gamma_{l-1} - \gamma_l \geq 2$, $\forall l \in [n, m-1]$; $\gamma_{m-1} - \gamma_m = \gamma_{m-1} \geq 1$. If $k = n$ then the same conditions hold except, possibly, the first one; in any case $\gamma_k \leq 2k-1$.*

Proof. The only difference between the two possibilities $k < n$ and $k = n$ lies, possibly, in $(\gamma_{n-1} - \gamma_n)$. By what has been shown above, the second statement for the case $k = n$ is clear.
Assume therefore that $k < n$. One has $E_j \subseteq H^0(\mathcal{O}_{Q_1}(j)) \simeq H^0(\mathcal{O}_{\mathbb{P}^1}(2j))$ and $H^0(\mathcal{O}_{\mathbb{P}^1}(1)) \times H^0(\mathcal{O}_{\mathbb{P}^1}(1)) \times E_{j-1} \subseteq E_j$. One applies Lemma (1.5) twice for every index j in the range considered, keeping in mind that, since Q_1 does not meet Γ, the lack of fixed components for $|\mathcal{I}_{\Gamma,Q_2}(j)|$ implies the base-point-freeness of the corresponding E_j. It follows that $e_l - e_{l-1} \geq 4$, except possibly $l = n$, m where $e_l - e_{l-1} \geq 3$. $\square$

Since $|\mathcal{I}_{\Gamma,Q_2}(n)|$ does not have fixed components, any curve in that linear system cuts on D_k a set of $2nk$ points (counted with multiplicities) that contains Γ, so that $2nk \geq d$:

$$n \geq n_0 = \lfloor \frac{d-1}{2k} \rfloor + 1.$$

One can summarize the information on $\gamma : [0, m] \to \mathbb{N}$ as follows:

$$\begin{aligned} \gamma_l &= 2l+1, && l \in [0, k-1]; \\ \gamma_l &= 2k, && l \in [k, n-1]; \\ \gamma_n &\leq 2k-1; && \\ \gamma_l - \gamma_{l+1} &\geq 2, && l \in [n, m-2]; \\ \gamma_l &= 0, && l \geq m; \\ \sum_{l=0}^{m} \gamma_l &= d. && \end{aligned}$$

After (2.1), the goal is to maximize $\sum_{l=0}^{m}(l-1)\gamma_l$, subject to the above constraints. One can start by reducing the process to the case in which $n = n_0$.

Remark. It should be noted that $m \leq n_0 + k$. This is a straightforward consequence of the constraints on γ. In particular one could already find an a priori upper bound for $\sum(l-1)\gamma_l$ by adding up setting, for example, $\gamma_l = 2k$.

LEMMA 2.3. *Given any function γ subject to the above constraints there exists a function $\tilde{\gamma}$, subject to the same constraints, for which the corresponding $n = n_0$ (here n is the first number greater or equal to k for which $\gamma_n < 2k$) and for which $\sum(l-1)\gamma_l \leq \sum(l-1)\tilde{\gamma}_l$.*

Proof. Assume $n - n_0 =: \xi > 0$, otherwise there is nothing to show. One has:

$$d = \sum_0^{n-1} \gamma_l + \sum_n^m \gamma_l = -k^2 + 2n_0 k + 2\xi k + \sum_n^m \gamma_l = -k^2 + d + \epsilon + 2\xi k + \sum_n^m \gamma_l;$$

it follows that

$$k^2 = \epsilon + 2\xi k + \sum_n^m \gamma_l.$$

By the above, $k \geq 2$, so that $2\xi k \geq 4$ and $\sum_n^m \gamma_i \leq k^2 - 4$. It follows that one of the following conditions must hold:

a) there is an index $n \leq j \leq m-1$ for which $\gamma_{j-1} - \gamma_j \geq 5$;

b) there are two distinct indices $j_1 < j_2$ as in a) for which $\sum_1^2(\gamma_{j_t-1} - \gamma_{j_t}) \geq 6$;

c) there are three distinct indices $j_1 < j_2 < j_3$, as in a) such that $\sum_1^3(\gamma_{j_t-1} - \gamma_{j_t}) \geq 8$;

d) there are four distinct indices $j_1 < j_2 < j_3 < j_4$, as in a) such that $\sum_1^4(\gamma_{j_t-1} - \gamma_{j_t}) \geq 10$.

In case a) one decreases (increases) γ_{j-1} (γ_j) by one. In case b) either one is also in case a) or one can decrease (increase) γ_{j_1-1} (γ_{j_2}) by one. Similarly in the remaining cases. As a consequence of this process, the constraints are respected but $\sum_0^m(l-1)\gamma_l$ increases. Since this sum is bounded from above by the above remark the process must come to an end, *i.e.* one can modify any γ to a $\tilde{\gamma}$ for which the corresponding $\xi = 0$. ∎

COROLLARY 2.4. *The following function $\tilde{\gamma}$ satisfies the constraints and maximizes $\sum_0^m(l-1)\gamma_l$:*

if $0 \leq \epsilon \leq k$:

$$\begin{aligned}
\tilde{\gamma}_l &= 2l+1, && 0 \leq l \leq k-1,\\
\tilde{\gamma}_l &= 2k, && k \leq l < n_0,\\
\tilde{\gamma}_l &= 2(k+n_0-l)-1, && n_0 \leq l \leq n_0+k-\epsilon-1,\\
\tilde{\gamma}_l &= [2(k+n_0-l)-1]-1, && n_0+k-\epsilon \leq l \leq n_0+k-1,\\
\tilde{\gamma}_l &= 0, && n_0+k \leq l,
\end{aligned}$$

if $k+1 \leq \epsilon \leq 2k-1$, let $\tau := \epsilon - k$, then:

$$\begin{aligned}
\tilde{\gamma}_l &= 2l+1, && 0 \leq l \leq k-1,\\
\tilde{\gamma}_l &= 2k, && k \leq l < n_0,\\
\tilde{\gamma}_l &= [2(k+n_0-l)-1]-1, && n_0 \leq l \leq n_0+k-\tau-1,\\
\tilde{\gamma}_l &= [2(k+n_0-l)-1]-2, && n_0+k-\tau \leq l \leq n_0+k-1,\\
\tilde{\gamma}_l &= 0, && n_0+k \leq l,
\end{aligned}$$

Proof. By the previous lemma one can assume $n = n_0$; it remains to define $\tilde{\gamma}_l$ in such a way that $\sum_{n_0}^m \gamma_l = k^2 - \epsilon$ and $\sum(l-1)\gamma_l$ is maximized. First define $\breve{\gamma}_l = 2(k+n_0-l)-1$ for $l \in [n_0, n_0+k-1]$. Now one has to delete from the graph of $\breve{\gamma}\epsilon$ points; $\tilde{\gamma}$ is the way to delete those points while maintaining the constraints and meeting the above maximization requirements. ∎

2.5. If one adds up $\sum_{l=0}^{n_0+k}(l-1)\tilde{\gamma}_l$, one gets the desired function $\pi = \pi(d,k)$ for which $(g-1) \leq \pi$. For its explicit form see Theorem 1.4.

Remark. The above is the bound obtained in [A-S], §6 for curves C of degree $d > 2k(k-1)$ contained in an integral surface of degree $2k$. As it will be shown in §3, the bound (2.5) is not quite sharp.

The Case $d \leq 2k(k-1)$, $\theta \leq k-1$.

In this case the analysis of the behavior of the function γ associated with C is analogous to the first case. The twist is the behavior of γ in the interval $[\theta, n-1]$. The following takes care of that interval.

PROPOSITION 2.6. *$\gamma_l = 2\theta$, $\forall l \in [\theta, n-1]$.*

Proof. Let l be in the above range. Using the notation of (1.7) define $\mathfrak{J}_{Q_3} := p^{-1}Q_3$, $\mathfrak{C} := p^{-1}C$ and define $\mathcal{F}(l) := \mathcal{I}_{\mathfrak{C},\mathfrak{J}_{Q_3}} \otimes p^*\mathcal{O}_{Q_3}(l)$. The proof of Lemma 1.7 and Remark 1.8 imply that for every l the fixed component F_l of $|\mathcal{I}_{\Gamma,Q_2}(l)|$ contains all of Γ. Clearly $F_\theta \supseteq F_{\theta+1} \supseteq \ldots \supseteq F_{n-1}$.
If $F_\theta \supsetneq F_l$, for some l, then the curve $F_\theta - F_l$ would be free to move in $|\mathcal{I}_{\Gamma,Q_2}(\theta)|$, a contradiction. It follows that $F_\theta = \ldots = F_{n-1}$.
To conclude one has to show that F_θ is actually a member of $|\mathcal{I}_{\Gamma,Q_2}(\theta)|$.
One can choose a line $\ell \subseteq \check{\mathbb{P}}^4$ such that:

i) it defines a pencil of hyperplane sections of Q_3 based on a smooth conic $\bar{Q}_1$ that does not meet C,

ii) it meets the open set W of (1.7) and

iii) it meets the open set of $\check{\mathbb{P}}^4$ for which Γ has cardinality d.

Using the same method as in the quoted lemma one constructs a surface $\tilde{S}$ on $q^{-1}(\ell)$ (which is the blowing up of Q_3 along $\bar{Q}_1$), that cuts on the general element of the pencil the corresponding curve F_θ. This surface descends to Q_3 as a surface S that cuts on, the general element of the pencil, a curve of the form $F_\theta + \mu\bar{Q}_1$, where μ is some integer. Since $Pic(Q_3) \simeq Pic(\mathbb{P}^4)$, one sees that $S \in |\mathcal{O}_{Q_3}(\zeta)|$, for some integer ζ; it follows that $F_\theta \in |\mathcal{I}_{\Gamma,Q_2}(\chi)|$, for some integer χ. By the minimality of θ one concludes $\theta = \chi$. □

Remark. The same method as above offers an alternative way to prove, less elementarily but in an unifying way, that $\gamma_l = 2k$, $\forall l \in [k, n-1]$ in the case $d > 2k(k-1)$.

Now one can repeat the analysis of the case $d > 2k(k-1)$ and obtain an analogous function $\tilde{\gamma}$ as follows: substitute k and n_0 by θ_0 and k respectively, for if one does so, then m will be maximized.

2.7. Adding up one gets, as in (2.5), a function $\pi' = \pi'(d,k)$ that bounds $g-1$ from above. By construction $\pi'(d,k) = \pi(d, \lfloor \frac{d-1}{2k} \rfloor + 1) = \pi(d, \theta_0)$.

Remark. The bound π' is not quite sharp as well (see §3.).

The Case $d \leq 2k(k-1)$, $\theta = k$.

In what follows the surface S_k will play no role. Hence the only assumptions needed are:

2.8. $C \subseteq Q_3$ *is an integral curve of degree* $d \leq 2k(k-1)$, *for which the general hyperplane section* $\Gamma \subseteq Q_2$ *is not contained in any curve belonging to the linear system* $|\mathcal{O}_{Q_2}(k-1)|$.

Clearly $\theta \geq k$; as in the previous case $\gamma_l = 2\theta$, if $l \in [\theta, n-1]$; also Lemma 2.2 holds with k replaced by θ.

Now one starts modifying γ, if necessary, to maximize $\sum(l-1)\gamma_l$. First of all, since $|\mathcal{O}_{Q_2}(k-1)| \simeq \mathbb{P}^{k^2}$, one has $d > k^2 = \sum_{l=0}^{k-1} \gamma_l$. Next, since the numbers γ_l must add up to $d \leq 2k(k-1)$, after reducing oneself, as in Lemma 2.3, to the case $k = \theta = n$, it is easy to see which function $\tilde{\gamma}$ maximizes m, and thus $\sum(l-1)\gamma_l$:

let ν, ϵ be the unique non-negative integers such that

$$d = k^2 + \nu^2 + \epsilon, \qquad 0 \leq \epsilon \leq 2\nu, \tag{2.9}$$

then define $\tilde{\gamma}$ as follows:

if $0 \leq \epsilon \leq \nu$, then

$$\begin{aligned}
\tilde{\gamma}_l &= 2l+1, & 0 \leq l \leq k-1;\\
\tilde{\gamma}_l &= [2(k+\nu-l)-1]+1, & k \leq l \leq k+\epsilon-1;\\
\tilde{\gamma}_l &= [2(k+\nu-l)-1], & k+\epsilon \leq l \leq k+\nu-1;\\
\tilde{\gamma}_l &= 0, & k+\nu \leq l;
\end{aligned}$$

if $\nu+1 \leq \epsilon \leq 2\nu$, let first $\tau := \epsilon - \nu$ and

$$\begin{aligned}
\tilde{\gamma}_l &= 2l+1, & 0 \leq l \leq k-1;\\
\tilde{\gamma}_l &= [2(k+\nu-l)-1]+2, & k \leq l \leq k+\tau-1;\\
\tilde{\gamma}_l &= [2(k+\nu-l)-1]+1, & k+\tau \leq l \leq k+\nu-1;\\
\tilde{\gamma}_l &= 0, & k+\nu \leq l.
\end{aligned}$$

Remark. Even without adding up, at this point one already knows, since $\theta_0 < k$, that the result will be strictly smaller than the corresponding π' of (2.7).

The proof of (1.4.1) is now complete. By adding up what above one gets the following:

THEOREM 2.10. *Assumptions and notation as in* (2.8) *and* (2.9). *The geometric genus of* C *satisfies the following bound:*

$$g-1 \leq \begin{cases} (k-\frac{3}{2})d - \frac{1}{3}(k^3-\nu^3) - \frac{1}{6}(k-\nu) + \frac{1}{2}\epsilon^2, & \text{if } 0 \leq \epsilon \leq \nu; \\ (k-\frac{3}{2})d - \frac{1}{3}(k^3-\nu^3) - \frac{1}{6}(k-\nu) & \\ \qquad + \frac{1}{2}\nu^2 + \frac{1}{2}(\epsilon-\nu)(\epsilon-\nu+k-3), & \text{if } \nu+1 \leq \epsilon \leq 2\nu. \end{cases}$$

§3. Discussion: When is the bound sharp? When is it not?

Assume the curve C has geometric genus maximum with respect to the upper bounds π, π' of (2.5) and (2.7). In particular $\gamma = \tilde{\gamma}$ and the inequalities in (2.1) are all equalities. By the following elementary claim, *if such a curve exists* then it will be smooth and projectively normal.

CLAIM 3.1. *C is smooth if and only if ρ_l is surjective $\forall l \gg 0$. Moreover if C is smooth it is projectively normal (i.e. ρ_l is surjective $\forall l$) if and only if $\beta_l = \alpha_l - \alpha_{l-1}$, $\forall l$.*

Proof. (Cf. [JH], page 193). The first part is clear since the normalization map $\nu^* : \mathcal{O}_C \to \mathcal{O}_{\hat{C}}$ has zero cokernel if and only if C is smooth. As to the second part one argues as follows. If ρ_l is surjective for every l, then $\sigma_l = \rho_{l-1}$ for every l as well. Conversely assume C is not projectively normal and let l_0 be any index such that ρ_{l_0+1} is surjective but ρ_{l_0} is not. Since $h^1(\mathcal{I}_{C,Q_3}(l_0+1)) = 0$, σ_{l_0+1} is surjective. It follows that $\alpha_{l_0+1} - \alpha_{l_0} > \beta_{l_0}$. $\square$

CLAIM 3.2. *If $d > 2k(k-1)$, then there exists an integral surface $S_{n_0} \in |\mathcal{I}_{C,Q_3}(n_0)|$ such that $S_k \not\subseteq S_{n_0}$. If $d \leq 2k(k-1)$, then there exists an integral surface $S_{\theta_0} \in |\mathcal{I}_{C,Q_3}(\theta_0)|$.*

Proof. Assume first that $d > 2d(k-1)$. Then since $|\mathcal{I}_{\Gamma,Q_2}(n_0)|$ is free of fixed components, one finds in it an element F_{n_0} that does not contain the irreducible curve D_k. The projective normality of C translates into the surjection $H^0(\mathcal{O}_{Q_3}(l)) \twoheadrightarrow H^0(\mathcal{O}_C(l))$, $\forall l$. This, in turn, is equivalent to $H^1(\mathcal{I}_{C,Q_3}(l)) = 0$, $\forall l$. Applying this to the case $l = n_0 - 1$ one gets the surjection $H^0(\mathcal{I}_{\Gamma,Q_3}(n_0)) \twoheadrightarrow H^0(\mathcal{I}_{\Gamma,Q_2}(n_0))$.
Therefore F_{n_0} can be lifted to a surface $S_{n_0} \in |\mathcal{I}_{C,Q_3}(n_0)|$. Since $Pic(Q_3) = \mathbb{Z}$ it follows that this surface is integral otherwise one would find $n_1 < n_0$ for which there is an element $F_{n_1} \in |\mathcal{I}_{\Gamma,Q_2}(n_1)|$ not containing D_k, a contradiction, since then $|\mathcal{I}_{\Gamma,Q_2}(n_1)|$ would be free of fixed components.
If $d \leq 2k(k-1)$ then there is a unique element $F_{\theta_0} \in |\mathcal{I}_{\Gamma,Q_2}(\theta_0)|$; one can lift it to a surface $S_{\theta_0} \in |\mathcal{I}_{C,Q_3}(\theta_0)|$ which is integral by the minimality property of θ_0. $\square$

By what has just been shown, C is residual to a curve D_ϵ of degree ϵ if $d > 2k(k-1)$ ($D_{\epsilon'}$ of degree ϵ' if $d \leq 2k(k-1)$) in a complete intersection on Q_3 of type (k, n_0) ((θ_0, k), respectively).
The following lemma is the technical device needed to relate C and D_ϵ (D'_ϵ). The proof is a mere generalization of [JH], page 199, where the case $S \simeq \mathbb{P}^2$ was dealt with. It will be used here only in the case $S \simeq Q_2$; proving it in a more general form is not more costly.

LEMMA 3.3. *Let S be a normal and projective surface, $\mathcal{O}_S(1)$ a nef and big line bundle on it, F and G two curves in $|\mathcal{O}_S(n)|$ and $|\mathcal{O}_S(m)|$ respectively without any common component. Denote by $\tilde{\Gamma}$ their scheme-theoretic intersection. Assume $\tilde{\Gamma} = \Gamma + \Gamma'$, where Γ is reduced and disjoint from Γ' and $\Gamma \subseteq S_{reg}$. Then:*

$$h^1(S, \mathcal{I}_{\Gamma,S}(n+m-l) \otimes \omega_S) = h^0(S, \mathcal{I}_{\Gamma',S}(l)), \qquad \forall l < m, n.$$

Proof. Let $\pi : S' \to S$ be the blowing up of S along Γ, E the exceptional divisor. Since F and G meet transversally at Γ one gets the following relations concerning strict transforms: $F' = \pi^* F - E$, $G' = \pi^* G - E$. Denote by Γ'' the scheme on S' isomorphic to Γ' via π, and by $\mathcal{O}_{S'}(v)$ the pull back $\pi^*\mathcal{O}_{S'}(v)$. By taking the cohomology of the following resolution:

$$\begin{aligned} 0 \to \pi^*\mathcal{O}_S(-n-m+l) + 2E \\ \to \pi^*\mathcal{O}_S(-n+l) + E \oplus \pi^*\mathcal{O}_S(-m+l) + E \to \mathcal{I}_{\Gamma''}(l) \to 0, \end{aligned}$$

one gets, for $l < n$, m:

$$0 \to H^0(\mathcal{I}_{\Gamma''}(l)) \xrightarrow{b} H^1(\mathcal{O}_{S'}(-n-m+l)+2E)$$
$$\to H^1(\mathcal{O}_{S'}(-n+l)+E) \oplus H^1(\mathcal{O}_{S'}(-m+l)+E) \to \dots$$

The above vector space is zero, for $l < n$, m, as it is now shown. Leray spectral sequence gives $H^1(\mathcal{O}_{S'}(-t+l)+E) = H^1(\mathcal{O}_S(-t+l))$, $\forall t$. The latter group is zero (this is a well-known argument): take a desingularization $\mathcal{S} \to S$, pull back $\mathcal{O}_S(-t+l)$ to a nef and big $\mathcal{O}_{\mathcal{S}}(-t+l)$; Kawamata-Viewheg vanishing (cf. C-K-M, Lecture 8) descends, again by Leray spectral sequence, to S.
Next, S' being normal it is Cohen-Macauley. Using Serre Duality:

$$H^1(\mathcal{O}_{S'}(-n-m+l)\otimes\mathcal{O}_{S'}(2E)) \simeq H^1(\pi^*(\omega_S\otimes\mathcal{O}_S(n+m-l))\otimes\mathcal{O}_{S'}(-E))^\vee.$$

By Leray spectral sequence one concludes using the isomorphism b. □

CLAIM 3.4. *D_ϵ lies on some quadric surface $\Sigma \subseteq Q_3$.*

Proof. If $\epsilon = 0$ there is nothing to prove. Let $\epsilon > 0$, and denote by Γ' the general hyperplane section of D_ϵ. Then $\gamma_{n_0+k-1} = 0$ and $\beta_{n_0+k-2+t} = d$, $\forall t \geq 0$. One has: $0 < \gamma_{n_0+k-2} = d - \beta_{n_0+k-3} = d - [d - h^1(\mathcal{I}_{\Gamma,Q_2}(n_0 + k-3))] = h^0(\mathcal{I}_{\Gamma'}(1))$; the last equality follows from (3.3). C being projectively normal, D_ϵ is a.C.M. (cf., for example, [Mi], Th 1.1). It follows that $h^0(\mathcal{I}_{D_\epsilon,\mathbb{P}^4}(1) > 0$, i.e. D_ϵ is contained in a hyperplane. □

EXAMPLE 3.5. Here the function γ and the genus of the curves in the class $\mathfrak{S}(d,k)$ are computed. The curves in these classes are the natural candidates to be the curves of maxima genera with respect to d and k. Since the two cases $d > 2k(k-1)$ and $d \leq 2k(k-1)$ are treated in the same way, the example is worked out only in the former case.

Let Γ' denote the general hyperplane section of D_ϵ. Assume first that ϵ is even: $\epsilon = 2\alpha$. Since D_ϵ is a.C.M., one sees that $D_\epsilon \in |\mathcal{O}_\Sigma(\alpha)|$. One takes the cohomology of the following projective resolutions of the twists of the ideal sheaf of Γ':

$$0 \to \mathcal{O}_{Q_2}(-1-\alpha+l) \to \mathcal{O}_{Q_2}(-1+l) \oplus \mathcal{O}_{Q_2}(-\alpha+l) \to \mathcal{I}_{\Gamma',Q_2}(l) \to 0.$$

To compute the quantities γ_l one argues as in (3.4) using Lemma 3.3: $\gamma_{n_0+k-2-l} = h^0(\mathcal{I}_{\Gamma',Q_2}(l+1)) - h^0(\mathcal{I}_{\Gamma',Q_2}(l))$, $\forall l \leq k-2$.

Now it is assumed that ϵ is odd: $\epsilon = 2\alpha - 1$. One can pick a line $\mathcal{L}$ on Σ so that $\mathcal{M} := D_\epsilon \cup \mathcal{L}$ is a curve in $|\mathcal{O}_\Sigma(\alpha)|$ (cf. [A-C-G-H], Ex. III D7). The general hyperplane section of $\mathcal{M}$ is $\Gamma'' = \Gamma' \cup p$, where p is the point hyperplane section of the line $\mathcal{L}$. In addition to the projective resolution for Γ'', which is the same as above, one also has the following exact sequences:

$$0 \to \mathcal{I}_{\Gamma'',Q_2}(l) \to \mathcal{I}_{\Gamma',Q_2}(l) \to \mathcal{O}_p \to 0.$$

Keeping in mind that $h^1(\mathcal{I}_{\Gamma'',Q_2}(l)) = 0$, $\forall l \geq \alpha$, Lemma 2.2 and the usual constraints a straightforward computation, analogous to the one of the case ϵ even, gives the desired quantities γ.

From what above one concludes that the function $\hat{\gamma}$ for these special curves is the following:

first let

$$\Delta := \begin{cases} 0 & \text{if } \epsilon = 0 \text{ or } \epsilon \text{ is odd,} \\ 1 & \text{if } \epsilon \text{ is even and } \epsilon \geq 2; \end{cases}$$

and let α be as above, then

$$\begin{aligned} &\hat{\gamma}_l = [2(n_0 + k - l) - 1], && n_0 \leq l \leq n_0 + k - \alpha - 2 \\ &\hat{\gamma}_l = [2(n_0 + k - l) - 1)] - \Delta, && l = n_0 + k - \alpha - 1, \\ &\hat{\gamma}_l = [2(n_0 + k - l) - 1] - 2, && n_0 + k - \alpha \leq l \leq n_0 + k - 2 \\ &\hat{\gamma}_l = 0, && n + k - 1 \leq l. \end{aligned}$$

Moreover by adding up one gets that the genera of these curves are:

(3.5.1)

$$g - 1 = \Pi := \begin{cases} \dfrac{1}{4k}d^2 + \dfrac{1}{2}(k-3)d - \dfrac{\epsilon}{2}[(k-1)(1 - \dfrac{\epsilon}{2k})] - \dfrac{1}{4}, & \text{if } \epsilon \text{ is odd,} \\ \dfrac{1}{4k}d^2 + \dfrac{1}{2}(k-3)d - \dfrac{\epsilon}{2}[(k-1)(1 - \dfrac{\epsilon}{2k})], & \text{if } \epsilon \text{ is even.} \end{cases}$$

The two functions $\tilde{\gamma}$ of (2.4) and $\hat{\gamma}$ coincide if and only if $\epsilon = 0,\ 1,\ 2,\ 2k-1$. This proves that the geometric genus of C achieves the bound π if and only if $\epsilon = 0,\ 1,\ 2,\ 2k-1$ and $C \in \mathfrak{S}(d,k)$. Conversely if $C \in \mathfrak{S}(d,k)$ then its associated function γ and its genus are as in (3.5.1).

As to the cases $\epsilon = 3,\ 2k-2$. By what above $g-1 < \pi$, and $\sum(l-1)\hat{\gamma}_l = \pi - 1$; it follows that this latter value is the sharp bound.

This proves (1.4.2) so that the proof of Theorem 1.4 is now complete.

§4. Two open questions

The author would like to pose the following two questions. The first one is the consequence of the incompleteness of Theorem 1.4. The answer to the second one would constitute a natural property of curves on quadrics.

QUESTION A. *Is it true that the curves of maximal genus with respect to* (d,k) *are the ones of the class* $\mathfrak{S}(d,k)$ *?*

A positive answer would give the sharp bound (3.5.1) and the complete characterization of the curves of maximal genus.

QUESTION B. *The* U.P.P. *is expressed in terms of the space of hyperplane sections of* Q_3*, i.e.* $\check{\mathbb{P}}^4$. *Does the analogue statement hold if one considers only hyperplanes which are tangent to* Q_3*, i.e. replacing* $\check{\mathbb{P}}^4$ *by* $\check{Q}_3$ *? If such a statement fails to be true, what are the implications for the embedded curve* C*?*

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