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ON PLAIN LATTICE POINTS WHOSE COORDINATES ARE RECIPROCALS MODULO A PRIME

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Abstract. We consider, for a given large prime p, the problem of covering a square $[0,p]\times[0,p]$ with discs center at the lattice point (x,y), x and y subject to condition $xy\equiv 1 \pmod p$ and with radius r. We are concerned with the size of r.

§1. Introduction

In this paper we consider, for each given large prime P, the problem of covering a 2-dimensional box $[0,P]\times[0,P]$ with discs $C_{(x,y)}(r)$ center at the lattice point (x,y), x and y subject to the condition $xy\equiv 1 \pmod P$ and with the least possible radius r. In other words, we wish to determine the infimum $r(P)$ of r satisfying

$$\bigcup_{\substack{x=1\\ xy\equiv 1 \ (\mathrm{mod}\ P)}}^{P-1} C_{(x,y)}(r) \supset [0,P]\times[0,P].$$

When $r=\sqrt{P}$, the area of the left-hand side member is roughly P^2, even if discs do not overlap. Thus it may be too optimistic to expect to have

$$r(P) = \text{constant times } \sqrt{P},$$

and actually for $P=5$, $r=\sqrt{5}$ is not large enough, but it would be reasonable to conjecture that

$$r(P) = P^{\frac{1}{2}+\varepsilon}$$

for every $\varepsilon>0$. If this is the case, we may claim that the lattice points (x,y) with $xy\equiv 1 \pmod P$ are "uniformly distributed." Towards this conjecture, we shall prove the following theorem.

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THEOREM. *For $P \gg 1$, we have*

$$r(P) \ll P^{\frac{3}{4}} \log P.$$

We can generalize the above problem to cover the higher dimensional box $[0, M] \times [0, M] \times \cdots \times [0, M]$ with the spheres

$$C_{(x_1, \cdots, x_N)}(r)$$

center at the lattice point $(x_1, \cdots, x_N)$ which satisfies $x_1 \cdots x_N \equiv 1 \pmod M$, and with the radius r, where M is a natural number. We shall give only a short notice for the generalization in the last section.

We may remark here that a related problem has been treated by Dinaburg and Sinai [1] and Fujii [2].

§2. Proof of Theorem

We shall give the following more general theorem.

THEOREM. *Suppose that M, K_1, K_2, H_1 and H_2 are integers satisfying $M \geq 3$, $0 \leq K_1 < K_2 < M$, $0 \leq H_1 < H_2 < M$, $K_2 - K_1 \geq 2$ and $H_2 - H_1 \geq 2$. Then we have*

$$\begin{aligned}\sharp\{(x,y) \mid\ & x, y \in \mathbf{Z},\ xy \equiv 1 \pmod M, K_1 < x \leq K_2, H_1 < y \leq H_2\} \\ &= \frac{\varphi(M)}{M^2}(K_2 - K_1)(H_2 - H_1) \\ &\quad + O(\sqrt{M}\sigma_0(M)\sigma_{-\frac{1}{2}}(M) \log(K_2 - K_1) \cdot \log(H_2 - H_1)),\end{aligned}$$

where O does not depend on K_1, K_2, H_1, H_2 and M , $\varphi(M)$ is the Euler function and we put $\sigma_a(M) = \sum_{d|M} d^a$.

This implies the following corollary immediately.

COROLLARY. *Suppose that M is a sufficiently large integer M and*

$$\frac{r}{\log r} \gg M^{\frac{5}{4}} \left(\frac{\sigma_0(M)\sigma_{-\frac{1}{2}}(M)}{\varphi(M)}\right)^{\frac{1}{2}},$$

then we have

$$\bigcup_{\substack{x=1 \\ xy\equiv 1 \pmod M}}^{M} C_{(x,y)}(r) \supset [0, M] \times [0, M].$$

Clearly, we get our theorem in the introduction.

To prove the theorem at the beginning of this section, we introduce the following functions $\chi_K(x)$ and $\chi_H(x)$ on the set of integers. They are periodic functions with period M and satisfying, for $0 \leq x < M$

$$\chi_K(x) = \begin{cases} 1 & \text{if } K_1 < x \leq K_2 \\ 0 & \text{otherwise,} \end{cases}$$

$$\chi_H(x) = \begin{cases} 1 & \text{if } H_1 < x \leq H_2 \\ 0 & \text{otherwise,} \end{cases}$$

and we put

$$\tilde{\chi}_K(x) = \sum_{y=1}^{M} \chi_K(y) e(\frac{yx}{M})$$

and

$$\tilde{\chi}_H(x) = \sum_{y=1}^{M} \chi_H(y) e(\frac{yx}{M}),$$

where $e(x)$ denotes $e^{2\pi i x}$. Then we have

$$\chi_K(x) = \frac{1}{M} \sum_{z=1}^{M} \tilde{\chi}_K(z) e(-\frac{xz}{M})$$

and

$$\tilde{\chi}_K(0) = \sum_{y=1}^{M} \chi_K(y) = K_2 - K_1.$$

We shall use the following lemma.

LEMMA. *Let d be a natural number satisfying $d \mid M$. Then we have*

$$\sum_{\substack{x=1 \\ (x,M)=d}}^{M-1} |\tilde{\chi}_K(x)| \ll \frac{M}{d} \log(K_2 - K_1)$$

and

$$\sum_{x=1}^{M-1} |\tilde{\chi}_K(x)| \ll M \log(K_2 - K_1).$$

Proof. For a real number a, we put

$$\|a\| = \min([a]+1-a, a-[a]),$$

where $[a]$ is the largest integer not exceeding a.

We shall prove the first inequality. Denoting by U the left hand side of the first inequality, we have

$$\begin{aligned}
U &= \sum_{\substack{x=1 \\ (x,M)=d}}^{M-1} |\tilde{\chi}_K(x)| = \sum_{\substack{x=1 \\ (x,M)=d}}^{M-1} |\sum_{y=1}^{M} \chi_K(y) e(\frac{yx}{M})| \\
&= \sum_{\substack{1 \le x \le \frac{M-1}{d} \\ (x, \frac{M}{d})=1}} |\sum_{y=1}^{M} \chi_K(y) e(\frac{yx}{\frac{M}{d}})| = \sum_{\substack{1 \le x \le \frac{M-1}{d} \\ (x, \frac{M}{d})=1}} | \sum_{K_1 < y \le K_2} e(\frac{yx}{\frac{M}{d}})| \\
&\ll \sum_{\substack{1 \le x \le \frac{M-1}{d} \\ (x, \frac{M}{d})=1}} \min\left(\frac{1}{\|\frac{x}{M/d}\|}, K_2 - K_1\right) \\
&\ll \sum_{\substack{1 \le x \le \frac{M-1}{d} \\ (x, \frac{M}{d})=1, \frac{x}{M/d} \le \frac{1}{2}}} \min\left(\frac{M}{xd}, K_2 - K_1\right) \\
&\quad + \sum_{\substack{1 \le x \le \frac{M-1}{d} \\ (x, \frac{M}{d})=1, \frac{x}{M/d} > \frac{1}{2}}} \min\left(\frac{1}{1-\frac{x}{M/d}}, K_2 - K_1\right) \\
&\ll \sum_{\substack{1 \le x \le \frac{M-1}{d}, (x, \frac{M}{d})=1 \\ \frac{x}{M/d} \le \frac{1}{2}, \frac{M}{xd} \ge K_2 - K_1}} (K_2 - K_1) + \sum_{\substack{1 \le x \le \frac{M-1}{d}, (x, \frac{M}{d})=1 \\ \frac{x}{M/d} \le \frac{1}{2}, \frac{M}{xd} < K_2 - K_1}} \frac{M}{xd} \\
&\quad + \sum_{\substack{1 \le \frac{M}{d} - y \le \frac{M-1}{d}, (\frac{M}{d} - y, \frac{M}{d})=1 \\ \frac{\frac{M}{d} - y}{M/d} > \frac{1}{2}, \frac{M}{dy} \ge K_2 - K_1}} (K_2 - K_1) + \sum_{\substack{1 \le \frac{M}{d} - y \le \frac{M-1}{d}, (\frac{M}{d} - y, \frac{M}{d})=1 \\ \frac{\frac{M}{d} - y}{M/d} > \frac{1}{2}, \frac{M}{dy} < K_2 - K_1}} \frac{M}{dy} \\
&\ll \sum_{1 \le x \le \frac{M}{d(K_2 - K_1)}} (K_2 - K_1) + \sum_{\frac{M}{d(K_2 - K_1)} < x \le \frac{M}{2d}} \frac{M}{xd}.
\end{aligned}$$

If $\frac{M}{d(K_2-K_1)} \geq 1$, then

$$U \ll \frac{M}{d(K_2-K_1)}(K_2-K_1) + \frac{M}{d}\left(\log(\frac{M}{2d}) - \log(\frac{M}{d(K_2-K_1)}) + 1\right)$$
$$\ll \frac{M}{d} + \frac{M}{d}\log(\frac{e}{2}(K_2-K_1)) \ll \frac{M}{d}\log(K_2-K_1).$$

If $\frac{M}{d(K_2-K_1)} < 1$, then

$$U \ll \sum_{1 \leq x \leq \frac{M}{2d}} \frac{M}{xd} \ll \frac{M}{d}(\log(\frac{M}{2d}) + 1) \ll \frac{M}{d}\log(K_2-K_1).$$

Thus we get the first inequality.

We can prove the second inequality by modifying the above argument as follows.

$$\begin{aligned}\sum_{x=1}^{M-1} |\tilde{\chi}_K(x)| &= \sum_{1 \leq x \leq M-1} |\sum_{K_1 < y \leq K_2} e(\frac{yx}{M})| \\ &\ll \sum_{1 \leq x \leq M-1} \min(\frac{1}{\|\frac{x}{M}\|}, K_2-K_1) \\ &\ll \sum_{1 \leq x \leq \frac{M}{(K_2-K_1)}} (K_2-K_1) + \sum_{\frac{M}{(K_2-K_1)} < x \leq \frac{M}{2}} \frac{M}{x} \\ &\ll M\log(K_2-K_1).\end{aligned}$$

Thus we have completed the proof of the lemma.

We now proceed to the proof of the theorem.

Putting

$$S = \sharp\{(x,y) \mid xy \equiv 1 \ (\mathrm{mod}\, M), K_1 < x \leq K_2, H_1 < y \leq H_2\},$$

we have

$$S = \sum_{\substack{x=1 \\ xy \equiv 1 \ (\mathrm{mod}\, M)}}^{M} \chi_K(x)\chi_H(y)$$

$$= \frac{1}{M^2} \sum_{\substack{x=1 \\ xy\equiv 1 \pmod M}}^{M} \sum_{z_1=1}^{M} \sum_{z_2=1}^{M} \tilde{\chi}_K(z_1) e(-\frac{xz_1}{M}) \tilde{\chi}_H(z_2) e(-\frac{yz_2}{M})$$

$$= \frac{1}{M^2} \sum_{\substack{x=1 \\ xy\equiv 1 \pmod M}}^{M} \tilde{\chi}_K(0)\tilde{\chi}_H(0)$$

$$+\frac{1}{M^2}\tilde{\chi}_K(0) \sum_{\substack{x=1 \\ xy\equiv 1 \pmod M}}^{M} \sum_{z_2=1}^{M-1} \tilde{\chi}_H(z_2) e(-\frac{yz_2}{M})$$

$$+\frac{1}{M^2}\tilde{\chi}_H(0) \sum_{\substack{x=1 \\ xy\equiv 1 \pmod M}}^{M} \sum_{z_1=1}^{M-1} \tilde{\chi}_K(z_1) e(-\frac{xz_1}{M})$$

$$+\frac{1}{M^2} \sum_{\substack{x=1 \\ xy\equiv 1 \pmod M}}^{M} \sum_{z_1=1}^{M-1} \sum_{z_2=1}^{M-1} \tilde{\chi}_K(z_1)\tilde{\chi}_H(z_2) e(-\frac{xz_1+yz_2}{M})$$

$$= S_1 + S_2 + S_3 + S_4, \quad \text{say.}$$

It is easy to see

$$S_1 = \frac{1}{M^2}\varphi(M)(K_2-K_1)(H_2-H_1).$$

S_2 is clearly equal to

$$\frac{1}{M^2}(K_2-K_1)\sum_{z_2=1}^{M-1}\tilde{\chi}_H(z_2) \sum_{\substack{x=1 \\ xy\equiv 1 \pmod M}}^{M} e(-\frac{yz_2}{M}).$$

The last partial sum on x is

$$\sum_{\substack{x=1 \\ (x,M)=1}}^{M} e(-\frac{xz_2}{M}) = \sum_{d|M}\mu(d) \sum_{x=1,d|x}^{M} e(-\frac{xz_2}{M})$$

$$= \sum_{d|M}\mu(d) \sum_{1\le x\le \frac{M}{d}} e(-\frac{xz_2}{M/d})$$

$$= \begin{cases} M\sum_{d|M}\frac{\mu(d)}{d} & \text{if } M/d \mid z_2 \\ 0 & \text{otherwise,} \end{cases}$$

where $\mu(d)$ is the Möbius function. Hence, we get

$$S_2 = \frac{1}{M^2}(K_2 - K_1)M \sum_{d|M} \frac{\mu(d)}{d} \sum_{\substack{z_2=1 \\ M/d|z_2}}^{M-1} \tilde{\chi}_H(z_2).$$

The last partial sum on z_2 is equal to

$$\begin{aligned}
&\sum_{1\le \frac{Mz}{d} \le M} \tilde{\chi}_H(\frac{Mz}{d}) - \tilde{\chi}_H(0) \\
&= \sum_{1\le z\le d} \sum_{y=1}^{M} \chi_H(y) e(\frac{yMz}{dM}) - (H_2 - H_1) \\
&= \sum_{y=1}^{M} \chi_H(y) \sum_{1\le z\le d} e(\frac{yz}{d}) - (H_2 - H_1) \\
&= d \sum_{\substack{y=1 \\ d|y}}^{M} \chi_H(y) - (H_2 - H_1) \\
&= d([\frac{H_2}{d}] - [\frac{H_1}{d}] + O(1)) - (H_2 - H_1) \\
&= d(\frac{H_2}{d} - \frac{H_1}{d} + O(1)) - (H_2 - H_1) \\
&= O(d).
\end{aligned}$$

Consequently, we get

$$\begin{aligned}
S_2 &= \frac{1}{M^2}(K_2 - K_1)M \sum_{d|M} \frac{\mu(d)}{d} O(d) \\
&\ll \frac{1}{M^2}(K_2 - K_1)M \sum_{d|M} |\mu(d)| \\
&\ll \sum_{d|M} |\mu(d)| \ll \sigma_0(M).
\end{aligned}$$

Similarly, we get

$$S_3 \ll \sigma_0(M).$$

Finally, using the estimate on the Kloosterman sum (cf. p.35 of Hooley

[3], for example), we get

$$
\begin{aligned}
S_4 &= \frac{1}{M^2}\sum_{z_1=1}^{M-1}\sum_{z_2=1}^{M-1}\tilde{\chi}_K(z_1)\tilde{\chi}_H(z_2) \sum_{\substack{x=1\\ xy\equiv 1 \pmod M}}^{M} e(-\frac{xz_1+yz_2}{M})\\
&\ll \frac{1}{M^2}\sum_{z_1=1}^{M-1}\sum_{z_2=1}^{M-1}|\tilde{\chi}_K(z_1)||\tilde{\chi}_H(z_2)|\sqrt{M}\sigma_0(M)(M,z_1)^{\frac{1}{2}}\\
&\ll \frac{1}{M^2}\sqrt{M}\sigma_0(M)\sum_{d|M}\sqrt{d}(\sum_{\substack{z_1=1,\\ (z_1,M)=d}}^{M-1}|\tilde{\chi}_K(z_1)|)(\sum_{z_2=1}^{M-1}|\tilde{\chi}_H(z_2)|)\\
&\ll \frac{1}{M^2}\sqrt{M}\sigma_0(M)\sum_{d|M}\sqrt{d}\frac{M}{d}\log(K_2-K_1)\cdot M\log(H_2-H_1)\\
&\ll \sqrt{M}\sigma_0(M)\sigma_{-\frac{1}{2}}(M)\log(K_2-K_1)\cdot\log(H_2-H_1).
\end{aligned}
$$

All of these estimates lead to the theorem at the beginning of this section.

§3. Concluding remark

3-1. It is clear that our theorem could be refined slightly, if we take care of the condition

$$(x, \frac{M}{d}) = 1$$

in the process of the proof of our lemma, or by replacing $(M,z_1)^{\frac{1}{2}}$ by $(M,z_1,z_2)^{\frac{1}{2}}$ in the estimate of S_4.

3-2. To get a higher dimensional analogue of our theorem, it is enough to apply the esimate on the higher dimensional Kloosterman sum due to Deligne.

3-3. As a final remark, we mention a slightly different approach.

It is to reduce our problem to the following estimate on the incomplete Kloosterman sum:

$$\sum_{\substack{K_1<x\le K_2\\ (x,M)=1}} e(-\frac{\bar{x}z}{M}) \ll \sqrt{M}\sigma_0(M)\sqrt{(z,M)}\cdot\log(K_2-K_1),$$

where $2 \le K_2 - K_1 \le M$, z is an integer in $1 \le z \le M-1$ and $\bar{x}$ satisfies $\bar{x}x \equiv 1 \pmod{M}$. The above estimate can be obtained by modifying the proof of Lemma 4 on p.36 of Hooley [3].

Now

$$\begin{aligned}
&\sharp\{(x,y) \mid xy \equiv 1 \pmod{M}, K_1 < x \le K_2, H_1 < y \le H_2\} \\
&\quad = \sum_{\substack{K_1 < x \le K_2 \\ (x,M)=1}} \chi_H(\bar{x}) = \frac{1}{M} \sum_{\substack{K_1 < x \le K_2 \\ (x,M)=1}} \sum_{z=1}^{M} \tilde{\chi}_H(z) e(-\frac{\bar{x}z}{M}) \\
&\quad = \frac{1}{M} \sum_{\substack{K_1 < x \le K_2 \\ (x,M)=1}} \tilde{\chi}_H(0) + \frac{1}{M} \sum_{z=1}^{M-1} \tilde{\chi}_H(z) \sum_{\substack{K_1 < x \le K_2 \\ (x,M)=1}} e(-\frac{\bar{x}z}{M}) \\
&\quad = W_1 + W_2, \quad \text{say.}
\end{aligned}$$

It is easy to see

$$W_1 = \frac{1}{M^2}\varphi(M)(K_2 - K_1)(H_2 - H_1) + O(\frac{H_2 - H_1}{M}).$$

Using the above estimate on the incomplete Kloosterman sum and Lemma, we get

$$\begin{aligned}
W_2 &\ll \frac{1}{M} \sum_{z=1}^{M-1} |\tilde{\chi}_H(z)| \, | \sum_{\substack{K_1 < x \le K_2 \\ (x,M)=1}} e(-\frac{\bar{x}z}{M})| \\
&\ll \frac{\sqrt{M}\sigma_0(M)\log(K_2 - K_1)}{M} \sum_{z=1}^{M-1} |\tilde{\chi}_H(z)| \sqrt{(z,M)} \\
&\ll \frac{\sqrt{M}\sigma_0(M)\log(K_2 - K_1)}{M} \sum_{d|M} \sqrt{d} \sum_{\substack{z=1 \\ (M,z)=d}}^{M-1} |\tilde{\chi}_H(z)| \\
&\ll \sqrt{M}\sigma_0(M)\sigma_{-\frac{1}{2}}(M)\log(K_2 - K_1) \cdot \log(H_2 - H_1).
\end{aligned}$$

Thus we get

$$\begin{aligned}
&\sharp\{(x,y) \mid xy \equiv 1 \pmod{M}, K_1 < x \le K_2, H_1 < y \le H_2\} \\
&\qquad = \frac{1}{M^2}\varphi(M)(K_2 - K_1)(H_2 - H_1) \\
&\qquad\qquad + O(\sqrt{M}\sigma_0(M)\sigma_{-\frac{1}{2}}(M)\log(K_2 - K_1) \cdot \log(H_2 - H_1)),
\end{aligned}$$

which is the same as the assertion in the section 2.

If we assume a strong conjecture on the above incomplete Kloosterman sum, then we can certainly replace $\frac{3}{4}$ in the theorem in the introduction by a better constant.

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