

ON DIFFERENTIAL POLYNOMIALS, II

HISASI MORIKAWA

Abstract. In Part II, we shall be concerned with applications of classical invariant theory, to statistic physics and to theta functions. Main theorem in Chapter 2 is stated as follows:

For a partition function

$$\xi(s) = \sum_{l=1}^{\infty} \gamma_l s^{dl}$$

satisfying $\gamma_l \geq 0$ ($l \geq 1$) and $\alpha > 0$, the $2n$ -apolar of $\xi(s)$

$$A_{2n}(\xi(s), \xi(s)) = s^{2n} \sum_{l=0}^{2n} (-1)^k \binom{2n}{k} \left(\frac{d}{ds}\right)^{2n-k} \xi(s) \left(\frac{d}{ds}\right)^k \xi(s)$$

has the expansion

$$A_{2n}(\xi(s), \xi(s)) = \sum_{l=2}^{\infty} \beta_{n,l} s^{-\alpha l}$$

such that $\beta_{n,l} \geq 0$ ($l \geq 2$). This means, for a given partition function $\xi(s)$ with nonnegative relative probabilities, we construct a sequence of partition functions $A_{2n}(\xi(s), \xi(s))_{n \geq 1}$ with the same properties, which may be considered a sequence of symbolical higher derivative of $\xi(s)$. The main theorem in Chapter 3 is stated as follows: For given theta functions $\varphi_1(z)$ and $\varphi_2(z)$ of level n_1 and n_2 respectively, in g variables $z = (z_1, z_2, \dots, z_g)$, then $r = (r_1, r_2, \dots, r_g)$ -apolar

$$A_r(\varphi_1(z), \varphi_2(z)) = \sum_{0 \leq j \leq r-j} \frac{(-1)^{|j|}}{n_1^{|j|} n_2^{|r-j|}} \binom{r}{j} \left(\frac{\partial}{\partial z}\right)^j \varphi_1(z) \left(\frac{\partial}{\partial z}\right)^{r-j} \varphi_2(z)$$

is a theta function of level $n_1 + n_2$, and

$$\begin{aligned} \left(\frac{\partial}{\partial z}\right)^h \varphi_1(z) \left(\frac{\partial}{\partial z}\right)^k \varphi_2(z) &= \sum_{0 \leq j \leq h+k} \sum_l (-1)^{|l|} n_1^{|l|} n_2^{|j-l|} \binom{h}{l} \binom{k}{j-l} \\ &\quad \frac{(-1)^{|h|} (n_1 n_2)^{|h+k-j|}}{(n_1 + n_2)^{|h+k|}} \left(\frac{\partial}{\partial z}\right)^j A_{h+k-j}(\varphi_1(z), \varphi_2(z)). \end{aligned}$$

§2. Apolars of partition functions

For a partition function $\zeta(s)$, we define its entropy $S(\zeta)$ and the central moments $M_n(\zeta)$ of $S(\zeta)$ as follows,

$$S(\zeta) = -\frac{s\zeta^{(1)}(s)}{\zeta(s)} + \log \zeta(s),$$

$$M_n(\zeta) = (-s)^n \sum_{l=0}^n (-1)^l \binom{n}{l} \frac{\zeta^{(n-l)}(s)}{\zeta(s)} \left(\frac{\zeta^{(1)}(s)}{\zeta(s)} \right)^l \quad (n \geq 2).$$

On the other hand, in the invariant theory on formal power series

$$f(\xi | t) = \sum_{l=0}^{\infty} \binom{\omega}{l} \xi^{(l)} t^l,$$

the generators of semi-invariants are given by the same polynomial sequence,

$$\phi_n(\xi) = \sum_{l=0}^n (-1)^l \binom{n}{l} \xi^{(n-l)} \xi^{(1)l} \xi^{(0)n-l-1} \quad (n \geq 2),$$

i.e.,

$$M_n(\xi) = \left(\frac{-s}{\xi(s)} \right)^n \phi_n(\xi) \quad (n \geq 2).$$

So we may define semi-invariants of partition functions. Most interesting semi-invariants of a partition function $\xi(s)$ are its apolars,

$$A_{2n}(\xi, \xi) = s^{2n} \sum_{l=0}^{2n} (-1)^l \binom{2n}{l} \xi^{(2n-l)}(s) \xi^{(l)}(s) \quad (1 \leq n < \infty),$$

For a partition function $\xi(s) = \sum_{l=1}^{\infty} \gamma_l s^{-\alpha l}$ with $\alpha > 0$ and $\gamma_l \geq 0$ ($l \geq 1$), the $2n$ -apolar has the expansion

$$A_{2n}(\xi(s), \xi(s)) = \sum_{l=0}^{\infty} \beta_{2n,l} s^{-\alpha l}$$

with non-negative coefficients

$$\beta_{2n,l} = \sum_{h=1}^{l-1} \gamma_{l-h} \gamma_h \sum_{j=0}^{2n} (-1)^j \binom{2n}{j} [\alpha h]^{(j)} [\alpha(l-h)]^{(2n-j)},$$

where

$$[\alpha h]^{(j)}[\alpha(l-h)]^{(2n-j)} = \alpha(\alpha h + 1) \cdots (\alpha h + j - 1)(\alpha(l-h))(\alpha(l-h) + 1) \cdots (\alpha(l-h) + 2n - j + 1).$$

MAIN THEOREM. Let $\xi(s) = \sum_{l=1}^{\infty} \gamma_l^{l=0} s^{-\alpha l}$ be a partition function, such that $\gamma_l \geq 0$ ($l \geq 1$) and $\alpha > 0$. Then the $2n$ -apolar $A_{2n}(\xi, \xi)$ has the expansion,

$$\sum_{l=2}^{\infty} \beta_{2n,l} s^{-\alpha l},$$

$$\beta_{2n,l} = \sum_{h=1}^{l-1} \left\{ \alpha^{2n} (l-2h)^{2n} + c_{2n,l,1}(\alpha)(l-2h)^{2(n-1)} + \cdots + c_{2n,l,n-1}(\alpha)(l-2h)^2 + c_{2n,l,n}(\alpha) \right\} \gamma_{l-h} \gamma_h \quad (l \geq 2),$$

with polynomials in α , $c_{2n,l,1}(\alpha), \dots, c_{2n,l,n}(\alpha)$ whose coefficients are non-negative integers. Consequently all the coefficients $\beta_{2n,l}$ of the expansion of $A_{2n}(\xi, \xi)$ are non-negative.

This suggests that from a partition functions $\zeta(s)$ with non-negative coefficients we may construct relating partition functions with non-negative coefficients, $A_2(\xi, \xi)$, $A_4(\xi, \xi)$, $A_6(\xi, \xi), \dots$

2.1. Outline of invariant theory on formal power series

First of all let us recollect the outline of the invariant theory on formal power series

$$(2.1) \quad f_j(\xi_j | t) = \sum_{l=0}^{\infty} \binom{\omega_j}{l} \xi_j^{(l)} t^l \quad (1 \leq j \leq r),$$

where $\omega_1, \omega_2, \dots, \omega_r$ are complex numbers and

$$\xi_j = \left(\xi_j^{(0)}, \xi_j^{(1)}, \xi_j^{(2)}, \xi_j^{(3)}, \dots \right) \quad (1 \leq j \leq r)$$

are independent variable vectors of infinite length. The details of the theory are found in [3].

$sl(2, C)$ -action on $\xi_j^{(0)}, \xi_j^{(1)}, \xi_j^{(2)}, \dots$ is given by Cayley-Aronholdt operators,

$$(2.2) \quad \begin{cases} D\xi_j^{(l)} = l\xi_j^{(l-1)} \\ \Delta\xi_j^{(l)} = (\omega_j - l)\xi_j^{(l+1)} \\ H\xi_j^{(l)} = (\omega_j - 2l)\xi_j^{(l)} \end{cases} \quad (0 \leq l < \infty; 1 \leq j \leq r),$$

with the relations,

$$(2.3) \quad \begin{cases} [D, \Delta] = H \\ [H, D] = 2D \\ [H, \Delta] = -2\Delta \end{cases}$$

In $C[\xi_1, \xi_2, \dots, \xi_r]$, degree, weight and index are defined as follows,

$$(2.4) \quad \deg \xi_j^{(l)} = 1, \quad \text{weight } \xi_j^{(l)} = l, \quad \text{index } \xi_j^{(l)} = \omega_j - 2l \\ (0 \leq l < \infty; 1 \leq j \leq r).$$

A polynomial $\varphi(\xi)$ in $C[\xi_1, \xi_2, \dots, \xi_r]$ is called a semi-invariant of index ν , if $\varphi(\xi)$ satisfies

$$(2.5) \quad D\varphi(\xi) = 0, \quad H\varphi(\xi) = \nu\varphi(\xi).$$

Semi-invariants form a graded algebra

$$(2.6) \quad S = \bigoplus_{\nu} S^{[\nu]}$$

where $S^{[\nu]}$ is the vector space of semi-invariants of index ν .

For a complex number ν we mean by t^ν the convergent power series

$$\sum_{l=0}^{\infty} \binom{\nu}{l} (t-1)^l$$

in the disc $\{t \mid |t-1| < 1\}$, which is a branch of the many valued function t^ν .

The germ of $GL(2, C)$ acts on $f_j(\xi_j \mid t)$ ($1 \leq j \leq r$) as follows;

$$(2.7) \quad f_j \left(\begin{pmatrix} \delta & \beta \\ \gamma & \alpha \end{pmatrix} \xi_j \mid t \right) = (\gamma t + \delta)^{\omega_j} f_j \left(\xi_j \mid \frac{\alpha t + \beta}{\gamma t + \delta} \right) \quad (1 \leq j \leq r),$$

i.e.,

$$(2.8) \quad \left(\begin{pmatrix} \delta & \beta \\ \gamma & \alpha \end{pmatrix} \xi_j \right) = \sum_{p=0}^l \sum_{q=0}^{\infty} \binom{l}{p} \binom{\omega_j - l}{q} \xi_j^{(l-p+q)} \alpha^{l-p} \beta^q \gamma^p \delta^{\omega_j - l - q},$$

$$\begin{aligned}
& \left(\begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix} \xi_j \right)^{(l)} = \sum_{p=0}^l \binom{l}{p} \xi^{(l-p)} \gamma^p \\
(2.9) \quad & \left(\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \xi_j \right)^{(l)} = \sum_{q=0}^{\infty} \binom{\omega_j - l}{q} \xi_j^{(l-q)} \beta^q \\
& \left(\begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \alpha \end{pmatrix} \xi_j^{(l)} \right) = \alpha^{2l-\omega_j} \xi_j^{(l)} \quad (0 \leq l < \infty; 1 \leq j \leq r)
\end{aligned}$$

A covariant of index ν means a formal power series

$$(2.10) \quad F(\xi_1, \dots, \xi_r; t) = \sum_{l=0}^{\infty} \binom{\nu}{l} C_l(\xi) t^l$$

with coefficients $C_l(\xi)$ in $C[\xi_1, \xi_2, \dots, \xi_r]$ such that

$$(2.11) \quad F\left(\begin{pmatrix} \delta\beta \\ \gamma\alpha \end{pmatrix} \xi_1, \dots, \begin{pmatrix} \delta\beta \\ \gamma\alpha \end{pmatrix} \xi_r; t\right) = (\gamma t + \delta)^\nu F\left(\xi_1, \dots, \xi_r; \frac{\alpha t + \beta}{\gamma t + \delta}\right)$$

for the germ of $SL(2, C)$.

We denote by $C^{[\nu]}$ the vector space of covariants of index ν , and we call the direct sum

$$C = \bigoplus_{\nu} C^{[\nu]}$$

the graded algebra of covariants.

The next four are the fundamental theorems in the invariant theory.

THEOREM. (Robert's theorem) *The mapping $\exp[t\Delta]$ defined by*

$$(2.12) \quad \exp[t\Delta]\varphi(\xi) = \sum_{l=0}^{\infty} \Delta^l \varphi(\xi) \frac{t^l}{l!}$$

is a graded algebra isomorphism of S onto C such that

$$(2.13) \quad \exp[t\Delta]S^{[\nu]} = C^{[\nu]}.$$

THEOREM. (Structure theorem on the graded algebra of semi-invariants) *Denoting polynomials*

$$(2.14) \quad \phi_n(\xi_j) = \sum_{l=0}^{\infty} (-1)^l \binom{n}{l} \xi_j^{(n-l)} \xi_j^{(1)l} \xi_j^{(0)} \quad (2 \leq n < \infty; 1 \leq j \leq r),$$

we obtain a system of generators

$$\begin{aligned} \xi_1^{(0)}, \xi_2^{(0)}, \dots, \xi_r^{(0)}; \xi_j^{(0)} \xi_k^{(1)} - \xi_k^{(0)} \xi_j^{(1)} & \quad (1 \leq i < k \leq r), \\ \phi_n(\xi_j) & \quad (2 \leq n < \infty; 1 \leq j \leq r) \end{aligned}$$

of the algebra $S[\xi_1^{(0)^{-1}}, \xi_2^{(0)^{-1}}, \dots, \xi_r^{(0)^{-1}}]$.

THEOREM. (Semi-simplicity theorem) *Assume that*

- i) $\omega_1, \omega_2, \dots, \omega_r$ are positive integers, or
- ii) $\sum_{j=1}^r d_j \omega_j \neq 0, 1, 2, 3, \dots$ for all non-negative integral vector $(d_1, d_2, \dots, d_r) \neq (0, 0, \dots, 0)$.

Then $C[\xi_1, \dots, \xi_r]$ is a semi-simple $sl(2, C)$ -module with decomposition

$$(2.15) \quad C \oplus \left(\bigoplus_{\nu} (\dim S^{[\nu]}) W^{[\nu]} \right),$$

where $W^{[\nu]}$ is the simple $sl(2, C)$ -module with the basis $[e_1, e_2, e_3, \dots]$ such that

$$(2.16) \quad \begin{aligned} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} e_l &= (\nu - l + 1) e_{l-1}, & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} e_l &= (l + 1) e_{l+1}, \\ \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} e_l &= (\nu - 2l) e_l. & & (0 \leq l \leq \infty) \end{aligned}$$

THEOREM. (Gram's theorem) *Under the same condition on $\omega_1, \omega_2, \dots, \omega_r$, the following two statements on an ideal I in $C[\xi_1, \xi_2, \dots, \xi_r]$ are equivalent,*

- i) I is an $sl(2, C)$ -admissible ideal,
- ii) there exists a set of index homogeneous semi-invariants $\{\varphi_\lambda\}_{\lambda \in \Lambda}$ such that I is generated by $\{\Delta^l \varphi_\lambda \mid 0 \leq l < \infty, \lambda \in \Lambda\}$.

2.2. Central moments $M_n(Z)$ of entropy

The scale-change invariant derivation

$$(2.17) \quad \theta = -s \frac{d}{ds}$$

is a nice tool for the study of partition functions.

We choose a generic partition function

$$(2.18) \quad Z(s) = \exp[u(s)]$$

and define the entropy of $Z(s)$ by

$$(2.19) \quad S(Z) = \frac{\theta Z(s)}{Z(s)} + \log Z(s) = \theta u(s) + u(s) = (\theta + 1)u(s).$$

The central n -th moment $M_n(Z)$ is defined by

$$(2.20) \quad M_n(Z) = (-s)^n \sum_{l=0}^n \binom{n}{l} \frac{Z^{(n-l)}(s)}{Z(s)} \left(\frac{Z^{(1)}(s)}{Z(s)} \right)^l.$$

Let us show the justification of the definitions in Boltzmann's sense. Let $\xi(s)$ be a partition function given by an integral

$$\xi(s) = \int_{-\infty}^{\infty} \exp[-sf(x)] dx,$$

where we assume that d/dx and the integral are freely commutative. Then we have

$$\begin{aligned} & - \int_{-\infty}^{\infty} \log \left\{ \frac{\exp[-sf(x)]}{\xi(s)} \right\} \frac{\exp[-sf(x)]}{\xi(s)} dx \\ &= \int_{-\infty}^{\infty} (sf(x) + \log \xi(s)) \frac{\exp[-sf(x)]}{\xi(s)} dx \\ &= \frac{-s \int_{-\infty}^{\infty} \frac{d}{ds} \exp[-sf(x)] dx}{\xi(s)} + \log \xi(s) \\ &= \frac{-s \frac{d}{ds} \int_{-\infty}^{\infty} \exp[-sf(x)] dx}{\xi(s)} + \log \xi(s) \\ &= -\frac{\frac{d}{ds} \xi(s)}{\xi(s)} + \log \xi(s) = \frac{\theta \xi(s)}{\xi(s)} + \log \xi(s) = (\theta + 1)u(s), \end{aligned}$$

$$\begin{aligned} & \int_{-\infty}^{\infty} \left\{ \log \left\{ \frac{\exp[-sf(x)]}{\xi(s)} \right\} - S(Z) \right\}^n \frac{\exp[-sf(x)]}{\xi(s)} dx \\ &= \int_X \left(-sf(x) + s \frac{\xi^{(1)}(s)}{\xi(s)} \right)^n \frac{\exp[-sf(x)]}{\xi(s)} dx \end{aligned}$$

$$\begin{aligned}
&= (-s)^n \sum_{l=0}^n (-1)^l \binom{n}{l} \left(\frac{\xi^{(1)}(s)}{\xi(s)} \right)^l \frac{\int_{-\infty}^{\infty} \left(\frac{d}{ds} \right)^{n-l} \exp[-sf(x)] dx}{\xi(s)} \\
&= (-s)^n \sum_{l=0}^n (-1)^l \binom{n}{l} \left(\frac{\xi^{(1)}(s)}{\xi(s)} \right)^l \frac{\left(\frac{d}{ds} \right)^{n-l} \exp[-sf(x)] dx}{\xi(s)} \\
&= (-s)^n \sum_{l=0}^n (-1)^l \binom{n}{l} \frac{\xi^{(n-l)}(s)}{\xi(s)} \left(\frac{\xi^{(1)}(s)}{\xi(s)} \right)^l.
\end{aligned}$$

LEMMA 2.1.

$$(2.21) \quad (-s)^n \left(\frac{d}{ds} \right)^n = [\theta]^{(n)} = \theta(\theta+1) \cdots (\theta+n-1),$$

$$(2.22) \quad M_2(Z) = \theta(\theta+1)u(s) = [\theta]^{(2)}u(s),$$

$$(2.23) \quad (-s)^n \left(\frac{d}{ds} \right)^n = (\theta+2)(\theta+3) \cdots (\theta+n-1)M_2(Z) \\ = [\theta+2]^{(n-2)}M_2(Z).$$

Proof. If we assume (2.19) for $n-1$,

$$\begin{aligned}
(-s)^n \left(\frac{d}{ds} \right)^n &= \left(-s \frac{d}{ds} \right) (-s)^{n-1} \left(\frac{d}{ds} \right)^{n-1} + (n-1)(-s)^{n-1} \left(\frac{d}{ds} \right)^{n-1} \\
&= \theta[\theta]^{(n-1)} + (n-1)[\theta]^{(n-1)}(\theta+n-1) = [\theta]^{(n)}.
\end{aligned}$$

From the definition of $M_2(Z)$,

$$\begin{aligned}
M_2(s) &= (-s)^2 \left\{ \frac{Z^{(1)}(s)}{Z(s)} - \left(\frac{Z^{(1)}(s)}{Z(s)} \right)^2 \right\} = (-s)^2 \frac{d}{ds} \left(\frac{\frac{d}{ds} Z(s)}{Z(s)} \right) \\
&= (-s)^2 \left(\frac{d}{ds} \right)^2 u(s) = \theta(\theta+1)u(s) = [\theta]^{(2)}u(s).
\end{aligned}$$

(2.21) is a direct consequence of (2.19) and (2.20).

Let us express the generating function

$$(2.24) \quad M(Z, t) = 1 + \sum_{n=2}^{\infty} M_n(Z) \frac{t^n}{n!},$$

where $M_1(Z) \equiv 0$.

PROPOSITION 2.1.

$$(2.25) \quad M(Z, t) = \frac{Z(s(1-t))}{Z(s)} \exp \left[st \frac{Z^{(1)}(s)}{Z(s)} \right]$$

$$(2.26) \quad \begin{aligned} M(Z, t) &= \exp \left[\sum_{j=2}^{\infty} (-s)^j \left(\frac{d}{ds} \right)^j u(s) \frac{t^j}{j!} \right] \\ &= \exp \left[\sum_{j=2}^{\infty} [\theta + 2]^{(j-2)} M_2(Z) \frac{t^j}{j!} \right]. \end{aligned}$$

Proof. Using Taylor expansion, we have

$$\begin{aligned} & \frac{Z(s(1-t))}{Z(s)} \exp \left[st \frac{Z^{(1)}(s)}{Z(s)} \right] \\ &= \left(\sum_{h=0}^{\infty} \frac{Z^{(h)}(s)}{Z(s)} \frac{(-st)^h}{h!} \right) \left(\sum_{l=0}^{\infty} \left(\frac{Z^{(1)}(s)}{Z(s)} \right)^l \frac{l! (lt)^l}{l!} \right) \\ &= 1 + \sum_{n=2}^{\infty} \left\{ \sum_{l=0}^n \binom{n}{l} \frac{Z^{(n-l)}(s)}{Z(s)} (-st)^{n-l} \left(\frac{Z^{(1)}(s)}{Z(s)} \right)^l (st)^l \right\} \\ &= 1 + \sum_{n=2}^{\infty} \left\{ (-s)^n \sum_{l=0}^n (-1)^l \binom{n}{l} \frac{Z^{(n-l)}(s)}{Z(s)} \left(\frac{Z^{(1)}(s)}{Z(s)} \right)^l \right\} \frac{t^n}{n!} \\ &= 1 + \sum_{n=2}^{\infty} M_n(Z) \frac{t^n}{n!} = M(Z, t), \end{aligned}$$

$$\begin{aligned} M(Z, t) &= \frac{Z(s(1-t))}{Z(s)} \exp \left[st \frac{Z^{(1)}(s)}{Z(s)} \right] \\ &= \exp \left[u(s-st) - u(s) + st \frac{d}{ds} u(s) \right] \\ &= \exp \left[\sum_{l=0}^{\infty} (-s)^l \left(\frac{d}{ds} \right)^l u(s) \frac{t^l}{l!} - u(s) - s \frac{d}{ds} u(s) t \right] \\ &= \exp \left[\sum_{j=2}^{\infty} (-s)^j \left(\frac{d}{ds} \right)^j u(s) \frac{t^j}{j!} \right] = \exp \left[\sum_{j=2}^{\infty} [\theta + 2]^{(j-2)} M_2(Z) \frac{t^j}{j!} \right] \end{aligned}$$

We obtain the differential polynomial expressions $M_n(Z)$ in $M_2(Z)$ with respect to the derivation θ .

THEOREM 2.1.

$$(2.27) \quad M_n(Z) = n! \sum_{\substack{\sum_j l_j = n, \\ j \geq 2}} \prod_j \frac{1}{l_j!} \left(\frac{(-s)^j \left(\frac{d}{ds}\right)^j u(s)}{j!} \right)^{l_j}$$

$$= n! \sum_{\substack{\sum_j l_j = n, \\ j \geq 2}} \prod_j \frac{1}{l_j!} \left(\frac{[\theta]^{(j-2)} M_2(Z)}{j!} \right)^{l_j}.$$

Proof. From Proposition 2.1 it follows

$$\begin{aligned} 1 + \sum_{n=2}^{\infty} M_n(Z) \frac{t^n}{n!} &= \exp \left[\sum_{j=2}^{\infty} (-s)^j \left(\frac{d}{ds}\right)^j u(s) \frac{t^j}{j!} \right] \\ &= \prod_{j=2}^{\infty} \exp \left[(-s)^j \left(\frac{d}{ds}\right)^j u(s) \frac{t^j}{j!} \right] \\ &= \prod_{j=2}^{\infty} \left\{ \sum_{l_j=2}^{\infty} \left(\frac{(-s)^j \left(\frac{d}{ds}\right)^j u(s)}{j!} \right)^{l_j} \frac{t^{j l_j}}{l_j!} \right\} \\ &= 1 + \sum_{n=2}^{\infty} \left\{ \sum_{\substack{\sum_j l_j = n, \\ j \geq 2}} \prod_j \frac{1}{l_j!} \left(\frac{(-s)^j \left(\frac{d}{ds}\right)^j u(s)}{j!} \right)^{l_j} \frac{n!}{l_j!} t^{j l_j} \right\} \\ &= 1 + \sum_{n=2}^{\infty} \left\{ n! \sum_{\substack{\sum_j l_j = n, \\ j \geq 2}} \prod_j \frac{1}{l_j!} \left(\frac{(-s)^j \left(\frac{d}{ds}\right)^j u(s)}{j!} \right)^{l_j} \right\} \frac{t_n}{n!} \\ &= 1 + \sum_{n=2}^{\infty} \left\{ n! \sum_{\substack{\sum_j l_j = n, \\ j \geq 2}} \prod_j \frac{1}{l_j!} \left(\frac{[\theta + 2]^{(j-2)} M_2(Z)}{j!} \right)^{l_j} \right\} \frac{t_n}{n!} \end{aligned}$$

$M(Z, s)$ satisfies a partial differential equation, from which we obtain a recurrence relation between $(M_n(Z))_{n \geq 2}$.

PROPOSITION 2.2.

$$(2.28) \quad s \frac{\partial}{\partial s} M(Z, t) + (1-t) \frac{\partial}{\partial t} M(Z, t) = t M_2(Z) M(Z, t).$$

$$(2.29) \quad \begin{aligned} \theta M_n(Z) &= -s \frac{d}{ds} M_n(Z) \\ &= M_{n+1}(Z) - nM_n(Z) - nM_2(Z)M_{n-1}(Z) \quad (n \geq 2). \end{aligned}$$

$$(2.30) \quad (\theta + n)M_n(Z) = M_{n+1}(Z) - nM_2(Z)M_{n-1}(Z) \quad (n \geq 2).$$

Proof. From Proposition 2.1 it follows

$$\begin{aligned} M(Z, t) &= \frac{Z(s(1-t))}{Z(s)} \exp \left[st \frac{Z^{(1)}(s)}{Z(s)} \right] \\ &= \frac{Z(s(1-t))}{Z(s)} \exp \left[st \frac{d}{ds} u(s) \right], \end{aligned}$$

$$\begin{aligned} s \frac{\partial}{\partial s} M(Z, t) + (1-t) \frac{\partial}{\partial t} M(Z, t) &= \left(s \frac{\partial}{\partial s} + (1-t) \frac{\partial}{\partial t} \right) Z(s(1-t)) \frac{1}{Z(s)} \exp \left[st \frac{d}{ds} u(s) \right] \\ &\quad + \left(s \frac{\partial}{\partial s} + (1-t) \frac{\partial}{\partial t} \right) \left(\frac{1}{Z(s)} \right) Z(s(1-t)) \exp \left[st \frac{d}{ds} u(s) \right] \\ &\quad + \frac{Z(s(1-t))}{Z(s)} \left(s \frac{\partial}{\partial s} + (1-t) \frac{\partial}{\partial t} \right) \exp \left[st \frac{d}{ds} u(s) \right] \\ &= -s \frac{d}{ds} u(s) Z(s(1-t)) \exp \left[st \frac{d}{ds} u(s) \right] \\ &\quad + (st + s(1-t)) \frac{d}{ds} u(s) \frac{Z(s(1-t))}{Z(s)} \exp \left[st \frac{d}{ds} u(s) \right] \\ &\quad + \frac{Z(s(1-t))}{Z(s)} t s^2 \left(\frac{d}{ds} \right)^2 u(s) \exp \left[st \frac{d}{ds} u(s) \right] \\ &= t(-s)^2 \left(\frac{d}{ds} \right)^2 \frac{Z(s(1-t))}{Z(s)} \exp \left[st \frac{d}{ds} u(s) \right] = tM_2(Z)M(t) \end{aligned}$$

$$\begin{aligned} &\left(s \frac{\partial}{\partial s} + (1-t) \frac{\partial}{\partial t} \right) M(Z, t) - tM_2(Z)M(Z, t) \\ &= \sum_{n=2}^{\infty} \frac{d}{ds} M_n(Z) \frac{t^n}{n!} + (1-t) \sum_{n=2}^{\infty} M_n(Z) \frac{t^{n-1}}{(n-1)!} \\ &\quad - tM_2(Z) - \sum_{n=2}^{\infty} M_2(Z) \frac{t^{n+1}}{n!}. \end{aligned}$$

Comparing the coefficients of t^n of the both sides, we obtain (2.28) and (2.29).

2.3. Inhomogeneous semi-invariants of partition functions

For a given system of generic partition functions $(Z_1(s), Z_2(s), \dots, Z_r(s))$,

$$(2.31) \quad \xi_j^{(l)} \longrightarrow (-s)^l Z_j^{(l)}(s) = (-s)^l \left(\frac{d}{ds} \right)^l Z_j(s) \\ (0 \leq l < \infty; 1 \leq j \leq r),$$

so that

$$(2.32) \quad D \left((-s)^l Z_j^{(l)}(s) \right) = l(-s)^{l-1} Z_j^{(l-1)}(s) \\ \Delta \left((-s)^l Z_j^{(l)}(s) \right) = (\omega_j - l) Z_j^{(l+1)}(s) \quad (0 \leq l < \infty; 1 \leq j \leq r). \\ H \left((-s)^l Z_j^{(l)}(s) \right) = (\omega_j - 2l) Z_j^{(l)}(s)$$

The images of semi-invariants by the specialization are called semi-invariants of the system of partition functions $(Z_1(s), Z_2(s), \dots, Z_r(s))$. We denote by the same notations as 2, Λ ,

$$S = \bigoplus_{\nu} S^{[\nu]}$$

the graded algebra of semi-invariants of $(Z_1(s), Z_2(s), \dots, Z_r(s))$.

There are many standard ways to construct semi-invariants from given semi-invariants. One of most important one is making apolars.

Let $\varphi(s)$ and $\phi(s)$ be semi-invariants of index μ and ν , respectively. Then n -th apolar of $\varphi(s)$ and $\phi(s)$ is defined by

$$(2.33) \quad A_n(\varphi, \phi)(s) = \sum_{l=0}^n (-1)^l \binom{n}{l} \frac{\Delta^{n-l} \varphi(s)}{(\mu)_{n-l}} \frac{\Delta^l \phi(s)}{(\nu)_l}.$$

The n -th apolar $A_n(\varphi, \phi)(s)$ is a semi-invariant of index $M - \nu - 2n$, which is independent on the choice of $(\omega_1, \omega_2, \dots, \omega_r)$.

EXAMPLE.

$$A_n(Z_j, Z_k) = \sum_{l=0}^n (-1)^l \binom{n}{l} (-s)^{n-l} Z_j^{(n-l)}(s) (-s)^l Z_k^{(l)}(s) \\ A_1(Z_j, Z_k) = (-s) Z_j^{(1)}(s) Z_k(s) - Z_j(s) (-s) Z_k^{(1)}(s)$$

$$A_2(Z_j, Z_j) = 2 \left\{ (-s)^2 Z_j^{(2)}(s) Z_j(s) - \left((-s) Z_j^{(1)}(s) \right)^2 \right\}$$

$$A_{2m+1}(Z_j, Z_j) = 0$$

$$A_{2m}(Z_j, Z_j) = \sum_{l=0}^{2m} (-1)^l \binom{2m}{l} (-s)^{2m-l} Z_j^{(2m-l)}(s) (-s)^l Z_j(s)$$

Another standard way of construction is making Hankel determinants,

$$\text{Hank}_{2q}(Z_j) = \begin{vmatrix} Z_j(s), & (-s)Z_j^{(1)}(s), & (-s)^2 Z_j^{(2)}(s), & \dots, & (-s)^q Z_j^{(q)}(s) \\ (-s)Z_j^{(1)}(s), & (-s)^2 Z_j^{(2)}(s), & (-s)^3 Z_j^{(3)}(s), & \dots, & (-s)^{q+1} Z_j^{(q+1)}(s) \\ \vdots & \vdots & \vdots & & \vdots \\ (-s)^q Z_j^{(q)}(s), & (-s)^{q+1} Z_j^{(q+1)}(s), & (-s)^{q+2} Z_j^{(q+2)}(s), & \dots, & (-s)^{2q} Z_j^{(2q)}(s) \end{vmatrix}.$$

EXAMPLE.

$$\begin{aligned} \text{Hank}_2(Z_j) &= \begin{vmatrix} Z_j(s) & (-s)Z_j^{(1)}(s) \\ (-s)Z_j^{(1)}(s) & (-s)^2 Z_j^{(2)}(s) \end{vmatrix} \\ &= (-s)^2 Z_j^{(2)}(s) Z_j(s) - \left((-s) Z_j^{(1)}(s) \right)^2, \\ \text{Hank}_4(Z_j) &= \begin{vmatrix} Z_j(s) & (-s)Z_j^{(1)}(s) & (-s)^2 Z_j^{(2)}(s) \\ (-s)Z_j^{(1)}(s) & (-s)^2 Z_j^{(2)}(s) & (-s)^3 Z_j^{(3)}(s) \\ (-s)^2 Z_j^{(2)}(s) & (-s)^3 Z_j^{(3)}(s) & (-s)^4 Z_j^{(4)}(s) \end{vmatrix} \\ &= (-s)^4 Z_j^{(4)}(s) (-s)^2 Z_j^{(2)}(s) Z_j(s) \\ &\quad + 2(-s)^3 Z_j^{(3)}(s) (-s)^2 Z_j^{(2)}(s) (-s) Z_j^{(1)}(s) \\ &\quad - \left((-s)^3 Z_j^{(3)}(s) \right)^2 Z_j(s) - (-s)^4 Z_j^{(4)}(s) \left((-s)^2 Z_j^{(2)}(s) \right)^2 \\ &\quad - \left((-s)^2 Z_j^{(2)}(s) \right)^3 \end{aligned}$$

One more important construction is making Wronskians. Let $\varphi_1(Z), \varphi_2(Z), \dots, \varphi_n(Z)$ be semi-invariants of index $\nu_1, \nu_2, \dots, \nu_n$, respectively. Wronskian of $\varphi_1(Z), \varphi_2(Z), \dots, \varphi_n(Z)$ is defined by

$$W_n(\varphi_1, \dots, \varphi_n)(Z)$$

$$= \begin{vmatrix} \varphi_1(Z) & \varphi_2(Z) & \cdots & \varphi_n(Z) \\ \frac{1}{(\nu_1)_1} \Delta \varphi_1(Z) & \frac{1}{(\nu_2)_1} \Delta \varphi_2(Z) & \cdots & \frac{1}{(\nu_n)_1} \Delta \varphi_n(Z) \\ \frac{1}{(\nu_1)_2} \Delta^2 \varphi_1(Z) & \frac{1}{(\nu_2)_2} \Delta^2 \varphi_2(Z) & \cdots & \frac{1}{(\nu_n)_2} \Delta^2 \varphi_n(Z) \\ \vdots & \vdots & \cdots & \vdots \\ \frac{1}{(\nu_1)_{n-1}} \Delta^{n-1} \varphi_1(Z) & \frac{1}{(\nu_2)_{n-1}} \Delta^{n-1} \varphi_2(Z) & \cdots & \frac{1}{(\nu_n)_{n-1}} \Delta^{n-1} \varphi_{n-1}(Z) \end{vmatrix}.$$

which is a semi-invariant of index $\sum_{j=1}^n \nu - n(n-1)$. Wronskians are independent on the choice of $\omega_1, \omega_2, \dots, \omega_n$.

Since degree, weight and index are defined by

$$\begin{aligned} \deg((-s)^l Z^{(l)}(s)) &= 1, & \text{weight}((-s)^l Z_j^{(l)}(s)) &= l, \\ \text{index}((-s)^l Z^{(l)}(s)) &= \omega_j & (0 \leq l < \infty; 1 \leq j \leq r), \end{aligned}$$

from the index formula for semi-invariants, we observe

$$(2.34) \quad S^{[\nu]} = \bigoplus_{\nu = \sum_j d_j \omega_j - 2p} S[d_1, d_2, \dots, d_r; p].$$

where $S[d_1, \dots, d_r; p]$ is the vector space of semi-invariants separately homogeneous of degree (d_1, d_2, \dots, d_r) an isobaric of weight p . The decomposition

$$(2.35) \quad S = \bigoplus_{d_1, \dots, d_r, p} S[d_1, d_2, \dots, d_r; p]$$

is independent on the choice of $\omega_1, \omega_2, \dots, \omega_n$.

DEFINITION 2.1. An inhomogeneous semi-invariant of homogeneous degree (d_1, \dots, d_r) and isobaric of weight p is a polynomial $\varphi(\dots, (-s)^l Z^{(l)}(s) / Z_j(s), \dots)$ in $(-s)^l Z_j^{(l)}(s)$ ($0 \leq l < \infty; 1 \leq j \leq r$) such that $\varphi(\dots, (-s)^l Z^{(l)}(s), \dots)$ belongs to $S[d_1, d_2, \dots, d_r; p]$.

THEOREM 2.2. *The algebra of inhomogeneous semi-invariants S is a polynomial algebra in*

$$\frac{(-s)Z_j^{(1)}(s)}{Z_j(s)} - \frac{(-s)Z_1^{(1)}(s)}{Z_1(s)} \quad (2 \leq j \leq r)$$

and the central moments of entropies

$$M_l(Z_j) \quad (2 \leq l < \infty; 1 \leq j \leq r),$$

i.e.

$$S = C \left[\left(\frac{(-s)Z_j^{(1)}(s)}{Z_j(s)} - \frac{(-s)Z_1^{(1)}(s)}{Z(s)} \right)_{2 \leq j \leq r}, (M_l(Z_j))_{\substack{2 \leq l < \infty \\ 1 \leq j \leq r}} \right].$$

This is a direct consequence of the structure theorem on semi-invariants, because

$$\begin{aligned} M_l(Z_j) &= \sum_{l=0}^n (-1) \binom{n}{l} \frac{(-s)^{n-l} Z_j^{n-l}(s)}{Z_j(s)} \left(\frac{(-s)Z_j^{(1)}(s)}{Z_j(s)} \right)^l \\ &= \frac{1}{Z_j(s)^n} \sum_{l=0}^n \binom{n}{l} (-s)^{n-l} Z^{(n-l)}(s) \left((-s)Z_j^{(1)}(s) \right)^l Z_j(s)^{n-l-1} \\ &= \frac{1}{Z_j(s)^n} \phi_n(\xi) \Big|_{\xi_j^{(l)} = (-s)^l Z_j^{(l)}(s)} \end{aligned}$$

2.4. Apolars of realistic partition functions

In the former paragraphs, we have treat only abstract formal aspects of partition functions and their semi-invariants, however the most essential points are reality and positivities of coefficients of expansions of partition functions and certain their semi-invariants.

In the present paragraph we mean by a realistic partion function, a partition function

$$(2.36) \quad \xi(s) = \sum_{l=1}^{\infty} \gamma_l s^{-\alpha l}$$

such that $\gamma_l \geq 0$ ($l \geq 1$) and $\alpha > 0$.

In order to calculate the coefficients of $A_{2n}(\xi, \xi)$, we need Stirling numbers $S_l^{(n)}$, which are defined by the coefficients in $[x]^{(n)}$,

$$(2.37) \quad [x]^{(n)} = x(x+1) \cdots (x+n-1) = \sum_{l=1}^n S_l^{(n)} x^l$$

LIST OF $S_l^{(n)}$.

$$\begin{aligned} [x]^{(1)} &= x \\ [x]^{(2)} &= x^2 + x \\ [x]^{(3)} &= x^3 + 3x^2 + 2x \end{aligned}$$

$$\begin{aligned}
[x]^{(4)} &= x^4 + 6x^3 + 11x^2 + 6x \\
[x]^{(5)} &= x^5 + 10x^4 + 35x^3 + 50x^2 + 24x \\
[x]^{(6)} &= x^6 + 15x^5 + 85x^4 + 225x^3 + 274x^2 + 120x \\
[x]^{(7)} &= x^7 + 21x^6 + 175x^5 + 735x^4 + 1624x^3 + 1764x^2 + 720x
\end{aligned}$$

$S_l^{(n)}$ ($1 \leq l \leq n$) are calculated by the recurrence relation:

$$(2.38) \quad S_n^{(n)} = 1, \quad S_1^{(n)} = (n-1)!, \quad S_l^{(n)} = S_{l-1}^{(n-1)} + (n-1)S_l^{(n-1)}.$$

The coefficients of $2n$ -apolar of $\xi(s) = \sum_{l=1}^{\infty} \gamma_l s^{-\alpha l}$

$$\begin{aligned}
A_{2n}(\xi(s), \xi(s)) &= s^{2n} \sum_{j=0}^{2n} (-1)^j \binom{2n}{j} \left(\frac{\partial}{\partial s} \right)^{(j)} \xi(s) \left(\frac{\partial}{\partial s} \right)^{2n-j} \xi(s) \\
&= \sum_{l=2}^{\infty} \beta_{2n,l} s^{-\alpha l}
\end{aligned}$$

are given by

$$(2.39) \quad \beta_{2n,l} = \sum_{h=1}^{l-1} \gamma_{l-h} \gamma_h \sum_{j=0}^{2n} (-1)^j \binom{2n}{j} [\alpha h]^{(j)} [\alpha(l-h)]^{(2n-j)}.$$

The next inequalities are the key for the estimation of $\beta_{2n,l}$.

THEOREM 2.3. *For real-non-negative x and y*

$$(2.40) \quad \sum_j (-1)^j \binom{2n}{j} [x]^{(j)} [y]^{(2n-j)} \geq 0,$$

$$(2.41) \quad (y-x) \sum_j (-1)^j \binom{2n+1}{j} [x]^{(j)} [y]^{(2n+1-j)} \geq 0 \quad (n = 0, 1, 2, \dots).$$

Proof. For $n = 0, 1$, we have

$$(y-x)(-[x]^{(1)} + [y]^{(1)}) = (y-x)^2 > 0$$

$$[x]^{(2)} - 2[x]^{(1)}[y]^{(1)} + [y]^{(2)} = (x-y)^2 + (x+y) \geq 0.$$

Assuming (2.40) and (2.41) for $2n$ and $2n + 1$, we have

$$\begin{aligned}
& \sum_j (-1)^j \binom{2n+2}{j} [x]^{(j)} [y]^{(2n+2-j)} \\
&= \sum_j (-1)^j \binom{2n+1}{j} [x]^{(j)} [y]^{(2n+1-j+1)} \\
&\quad + \sum_j (-1)^j \binom{2n+1}{j-1} [x]^{((j-1)+1)} [y]^{(2n+1-(j-1))} \\
&= \sum_j (-1)^j \binom{2n+1}{j} [x]^{(j)} [y]^{(2n+1-j)} (y + 2n + 1 - j) \\
&\quad - \sum_j (-1)^{j-1} \binom{2n+1}{j-1} [x]^{(j-1)} [y]^{(2n+1-j)} (x + j - 1) \\
&= \sum_j (-1)^j \binom{2n+1}{j} [x]^{(j)} [y]^{(2n+1-j)} (y + 2n + 1 - j - x - j) \\
&= \sum_j (-1)^j \binom{2n+1}{j} [x]^{(j)} [y]^{(2n+1-j)} (y - x + 2n + 1 - j - j) \\
&= (y - x) \sum_j (-1)^j \binom{2n+1}{j} [x]^{(j)} [y]^{(2n+1-j)} \\
&\quad + \sum_j (-1)^j \binom{2n+1}{j} (2n + 1 - j) [x]^{(j)} [y]^{(2n+1-j)} \\
&\quad - \sum_j (-1)^j \binom{2n+1}{j} j [x]^{(j)} [y]^{(2n+j-1)} \\
&\geq (2n + 1) \sum_j (-1)^j \binom{2n}{j} [x]^{(j)} [y]^{(2n-j+1)} \\
&\quad + (2n + 1) \sum_j (-1)^{j-1} \binom{2n}{j-1} [x]^{(j-1+1)} [y]^{(2n-(j-1))} \\
&= (2n + 1) \sum_j (-1)^j \binom{2n}{j} [x]^{(j)} [y]^{(2n-j)} \{y + 2n - j + x + j\} \\
&= (2n + 1)(x + y + 2n) \sum_j (-1)^j \binom{2n}{j} [x]^{(j)} [y]^{(2n-j)} \geq 0,
\end{aligned}$$

$$\begin{aligned}
& (y-x) \sum_j (-1)^j \binom{2n+2+1}{j} [x]^{(j)} [y]^{(2n+2-j+1)} \\
&= (y-x) \sum_j (-1)^j \binom{2n+2}{j} [x]^j [y]^{(2n+2-j)} (y+2n+2-j) \\
&\quad + (y-x) \sum_j (-1)^j \binom{2n+2}{j-1} [x]^{(j-1)} (x+j-1) [y]^{(2n+2-(j-1))} \\
&= (y-x) \sum_j (-1)^j \binom{2n+2}{j} [x]^{(j)} [y]^{(2n+2-j)} (y+2n+2-j-x-j) \\
&= (y-x) \sum_j (-1)^j \binom{2n+2}{j} [x]^{(j)} [y]^{(2n+2-j)} (y-x+2n+2-j-j) \\
&= (y-x)^2 \sum_j (-1)^j \binom{2n+2}{j} [x]^{(j)} [y]^{(2n+2-j)} \\
&\quad + (y-x) \sum_j (-1)^j \binom{2n+2}{j} (2n+2-j) [x]^{(j)} [y]^{(2n+1-j)} (y+2n+1-j) \\
&\quad - (y-x) \sum_j (-1)^j \binom{2n+2}{j} j [x]^{(j-1)} (x+j-1) [y]^{(2n+1-(j-1))} \\
&\geq (2n+2)(y-x) \sum_j (-1)^j \binom{2n+1}{j} [x]^{(j)} [y]^{(2n+1-j)} (y+2n+1-j) \\
&\quad + (2n+2)(y-x) \sum_j (-1)^{j-1} \binom{2n+1}{j-1} [x]^{(j-1)} [y]^{(2n+1-(j-1))} (x+j-1) \\
&= (2n+2)(y-x) \sum_j (-1)^j \binom{2n+1}{j} [x]^{(j)} [y]^{(2n+1-j)} (y+2n+1-j+x+j) \\
&= (2n+2)(x+y+2n+1)(y-x) \sum_j (-1)^j \binom{2n+1}{j} [x]^{(j)} [y]^{(2n+1-j)} \geq 0.
\end{aligned}$$

The main theorem is a direct consequence of the inequalities (2.40) and (2.41).

THEOREM 2.4. *If $\alpha > 0$ and $\gamma_l \geq 0$ ($l \geq 0$), the $2n$ -apolar $A_{2n}(\xi(s)$,*

$\xi(s)$ of the realistic partition function

$$\xi(s) = \sum_{l=1}^{\infty} \gamma_l s^{-\alpha l},$$

has the expansion

$$A_{2n}(\xi(s), \xi(s)) = \sum_{l=2}^{\infty} \beta_{2n,l} s^{-\alpha l}$$

with non-negative coefficients

$$(2.42) \quad \beta_{2n,l} = \sum_{h=1}^{l-1} \gamma_{l-h} \gamma_h \sum_{j=0}^{2n} (-1)^j \binom{2n}{j} [\alpha h]^{(j)} [\alpha(l-h)]^{(2n-j)} \geq 0$$

($l \geq 2$).

Proof. From the definition of $2n$ -apolar and from Theorem 2.3 it follows

$$\begin{aligned} A_{2n}(\xi(s), \xi(s)) &= \sum_{h,k=1}^{\infty} \gamma_h \gamma_k A_{2n}(s^{-\alpha h}, s^{-\alpha k}) \\ &= \sum_{k,h=1}^{\infty} \sum_{j=0}^{2n} (-1)^j \binom{2n}{j} (-\alpha h)(-\alpha h_1) \\ &\quad \cdots (-\alpha h - j + 1)(-\alpha k)(-\alpha k - 1) \\ &\quad \cdots (-\alpha k - 2n + j + 1) s^{-\alpha(h+k)} \\ &= \sum_{l=2}^{\infty} \sum_{h=1}^{l-1} \gamma_{l-h} \gamma_h \left(\sum_{j=0}^{2n} (-1)^j \binom{2n}{j} [\alpha h]^{(j)} [\alpha(l-h)]^{(2n-j)} \right) s^{-\alpha l}, \\ \beta_{2n,l} &= \sum_{h=1}^{l-1} \gamma_{l-h} \gamma_h \left(\sum_{j=0}^{2n} (-1)^j \binom{2n}{j} [\alpha h]^{(j)} [\alpha(l-h)]^{(2n-j)} \right) \geq 0. \end{aligned}$$

The next three are the concrete calculations of the coefficients of $A_2(\xi(s), \xi(s))$, $A_4(\xi(s), \xi(s))$ and $A_6(\xi(s), \xi(s))$.

EXAMPLE 2.1. Let $\xi(s) = \sum_{l=1}^{\infty} \gamma_l s^{-\alpha l}$ be a realistic partition function. Then we have the expansion,

$$(2.43) \quad \begin{aligned} A_2(\xi, \xi) &= s^2(2\xi^{(2)}(s)\xi(s) - 2\xi^{(l)}(s)^2) \\ &= \sum_{l=2}^{\infty} \left[\sum_{h=1}^{l-1} \{\alpha^2(l-2h)^2 + \alpha l\} \gamma_{l-h} \gamma_h \right] s^{-\alpha l} \end{aligned}$$

Consequently $A_2(\xi, \xi)$ is also a realistic partition function.

Proof. Watching the list of Stirling numbers, we observe,

$$\begin{aligned}
A_2(\xi, \xi) &= \sum_{l=2}^{\infty} \left[\sum_{h=1}^{l-1} \left\{ [\alpha h]^{(2)} + [\alpha(l-h)]^{(2)} \right. \right. \\
&\quad \left. \left. - 2[\alpha h]^{(1)}[\alpha(l-h)]^{(1)} \right\} \gamma_{l-h} \gamma_h \right] s^{-\alpha l} \\
&= \sum_{l=2}^{\infty} \left[\sum_{h=1}^{l-1} \left\{ \alpha^2(l^2 - 4lh + h^2) + \alpha l \right\} \gamma_{l-h} \gamma_h \right] s^{-\alpha l} \\
&= \sum_{h=1}^{\infty} \left[\sum_{l=h+1}^{\infty} \left\{ \alpha^2(l-2h)^2 + \alpha l \right\} \gamma_{l-h} \gamma_h \right] s^{-\alpha l}.
\end{aligned}$$

EXAMPLE 2.2. For a realistic $\xi(s) = \sum_{l=1}^{\infty} \gamma_l s^{-\alpha l}$, we have

$$\begin{aligned}
(2.44) \quad A_4(\xi, \xi) &= s^4 \left(2\xi^{(4)}(s)\xi(s) - 8\xi^{(3)}(s)\xi^{(1)}(s) + 6\xi^{(2)}(s)^2 \right) \\
&= \sum_{l=2}^{\infty} \left[\sum_{h=1}^{l-1} \left\{ \alpha^4(l-2h)^4 + (6\alpha^3 l + 8\alpha^2)(l-2h)^2 \right. \right. \\
&\quad \left. \left. + 3\alpha^2 l^2 + 6\alpha l \right\} \gamma_{l-h} \gamma_h \right] s^{-\alpha l}.
\end{aligned}$$

Consequently $A_4(\xi, \xi)$ is also a realistic partition function.

Proof. Watching the list of Stirling numbers, we observe,

$$\begin{aligned}
A_4(\xi, \xi) &= \sum_{l=2}^{\infty} \left[\sum_{h=1}^{l-1} \left\{ [\alpha h]^{(4)} + [\alpha(l-h)]^{(4)} - 4[\alpha h]^{(3)}[\alpha(l-h)]^{(1)} \right. \right. \\
&\quad \left. \left. - 4[\alpha(l-h)]^{(3)}[\alpha h]^{(1)} + 6[\alpha h]^{(2)}[\alpha(l-h)]^{(2)} \right\} \gamma_{l-h} \gamma_h \right] s^{-\alpha l} \\
&= \sum_{l=2}^{\infty} \left[\sum_{h=1}^{l-1} \left\{ \alpha^4 h^4 + 6\alpha^3 h^3 + 11\alpha^2 h^2 + 6\alpha h + \alpha^4(l-h)^4 \right. \right. \\
&\quad + 6\alpha^3(l-h)^3 + 11\alpha^2(l-h)^2 + 6\alpha(l-h) \\
&\quad - 4(\alpha^3 h^3 + 3\alpha^2 h^2 + 2\alpha h)(\alpha(l-h)) \\
&\quad - 4(\alpha^3(l-h)^3 + 3\alpha^2(l-h) + 2\alpha(l-h))\alpha h \\
&\quad \left. \left. + 6(\alpha^2 h^2 + \alpha h)(\alpha^2(l-h)^2 + \alpha(l-h)) \right\} \gamma_{l-h} \gamma_h \right] s^{-\alpha l}
\end{aligned}$$

$$\begin{aligned}
&= \sum_{l=2}^{\infty} \left[\sum_{h=1}^{l-1} \left\{ \alpha^4 (h^4 - 4h^3(l-h) + 6h^2(l-h) \right. \right. \\
&\quad \left. \left. - 4h(l-h)^3 + (l-h)^4) + 6\alpha^3 (l^3 - 4l^2h + 4lh^2) \right. \right. \\
&\quad \left. \left. + \alpha^2 (11h^2 - 11(l-h)^2 - 10h(l-h)) + 6\alpha l \right\} \gamma_{l-h} \gamma_h \right] s^{-\alpha l} \\
&= \sum_{l=2}^{\infty} \left[\sum_{h=1}^{l-1} \left\{ \alpha^4 (l-2h)^4 + 6\alpha^3 l (l-2h)^2 \right. \right. \\
&\quad \left. \left. + 8\alpha^2 (l-2h)^2 + 3\alpha^2 l^2 + 6\alpha l \right\} \gamma_{l-h} \gamma_h \right] s^{-\alpha l} \\
&= \sum_{l=2}^{\infty} \left[\sum_{h=1}^{l-1} \left\{ \alpha^4 (l-2h)^4 + (6\alpha^3 l + 8\alpha^2) (l-2h)^2 \right. \right. \\
&\quad \left. \left. + 3\alpha^2 l^2 + 6\alpha l \right\} \gamma_{l-h} \gamma_h \right] s^{-\alpha l}.
\end{aligned}$$

EXAMPLE 2.3. For a realistic $\xi(s) = \sum_{l=1}^{\infty} \gamma_l s^{-\alpha l}$, we have

$$\begin{aligned}
(2.45) \quad A_6(\xi, \xi) &= s^6 \left(2\xi^{(6)}(s)\xi(s) - 12\xi^{(5)}(s)\xi^{(1)}(s) \right. \\
&\quad \left. + 30\xi^{(4)}(s)\xi^{(2)}(s) - 20\xi^{(3)}(s)\xi^{(2)}(s) \right) \\
&= \sum_{l=2}^{\infty} \left[\sum_{h=1}^{l-1} \left\{ \alpha^6 (l-2h)^6 + (15\alpha^5 l + 40\alpha^4) (l-2h)^4 \right. \right. \\
&\quad \left. \left. + (45\alpha^4 l^2 + 210\alpha^3 l + 184\alpha^2) (l-2h)^2 \right. \right. \\
&\quad \left. \left. + 15\alpha^3 l^3 + 90\alpha^2 l^2 + 120\alpha l \right\} \gamma_{l-h} \gamma_h \right] s^{-\alpha l}
\end{aligned}$$

Proof. Watching the list of Stirling numbers, we observe,

$$\begin{aligned}
A_6(\xi, \xi) &= s^6 \left(2\xi^{(6)}(s)\xi(s) - 12\xi^{(5)}(s)\xi^{(1)}(s) 30\xi^{(4)}(s)\xi^{(2)}(s) - 20\xi^{(3)}(s)\xi^{(2)}(s) \right) \\
&= \sum_{l=2}^{\infty} \left[\sum_{h=1}^{l-1} \left\{ [\alpha h]^{(6)} + [\alpha(l-h)]^{(6)} - 6[\alpha h]^{(5)}[\alpha(l-h)]^{(1)} \right. \right. \\
&\quad \left. \left. - 6[\alpha(l-h)]^{(5)}[\alpha h]^{(1)} + 15[\alpha h]^{(4)}[\alpha(l-h)]^{(2)} \right. \right. \\
&\quad \left. \left. + 15[\alpha(l-h)]^{(4)}[\alpha h]^{(2)} - 20[\alpha h]^{(3)}[\alpha(l-h)]^{(3)} \right\} \gamma_{l-h} \gamma_h \right] s^{-\alpha l}
\end{aligned}$$

$$\begin{aligned}
&= \sum_{l=2}^{\infty} \left[\sum_{h=1}^{l-1} \left\{ \alpha^6 h^6 + 15\alpha^5 h^5 + 85\alpha^4 h^4 + 225\alpha^3 h^3 + 274\alpha^2 h^2 \right. \right. \\
&\quad + 120\alpha h + \alpha^6 (l-h)^6 + 15\alpha^5 (l-h)^5 + 85\alpha^4 (l-h)^4 \\
&\quad + 225\alpha^3 (l-h)^3 + 274\alpha^2 (l-h)^2 + 120\alpha (l-h) \\
&\quad - 6 \left(\alpha^5 h^5 + 10\alpha^4 h^4 + 35\alpha^3 h^3 + 50\alpha^2 h^2 + 24\alpha h \right) \alpha (l-h) \\
&\quad - 6 \left(\alpha^5 (l-h)^5 + 10\alpha^4 (l-h)^4 + 35\alpha^3 (l-h)^3 + 50\alpha^2 (l-h)^2 \right. \\
&\quad \left. + 24\alpha (l-h) \right) \alpha h + 15(\alpha^4 h^4 + 6\alpha^3 h^3 + 11\alpha^2 h^2 \\
&\quad + 6\alpha h)(\alpha^2 (l-h)^2 + \alpha (l-h)) + 15(\alpha^4 (l-h)^4 + 6\alpha^3 (l-h)^3 \\
&\quad + 11\alpha^2 (l-h)^2 + 6\alpha (l-h))(\alpha^2 h^2 + \alpha h) \\
&\quad \left. - 20(\alpha^3 h^3 + 3\alpha^2 h^2 + 2\alpha h)(\alpha^3 (l-h)^3 \right. \\
&\quad \left. + 3\alpha^2 (l-h)^2 + 2\alpha (l-h)) \right\} \Bigg] \\
&= \sum_{l=2}^{\infty} \left[\sum_{h=1}^{l-1} \left\{ \alpha^6 (l^6 - 12l^5 h + 60l^4 h^2 - 160l^3 h^3 + 240l^2 h^4 \right. \right. \\
&\quad - 192lh^5 + 64h^6) + 15\alpha^5 (l^5 - 8l^4 h + 24l^3 h^2 - 32l^2 h^3 + 16lh^4) \\
&\quad + 5\alpha^4 (17l^4 - 100l^3 h + 228l^2 h^2 - 256lh^3 + 128h^4) \\
&\quad + \alpha^3 (225l^3 - 840l^2 h + 840lh^2) \\
&\quad \left. + \alpha^2 (274l^2 - 736lh + 736h^2) + 120\alpha l \right\} \gamma_{l-h} \gamma_h \Bigg] s^{-\alpha l} \\
&= \sum_{l=2}^{\infty} \left[\sum_{h=1}^{l-1} \left\{ \alpha^6 (l-2h)^6 + 15\alpha^5 l (l-2h)^4 + 5\alpha^4 (8(l-2h)^4 \right. \right. \\
&\quad + 9l^2 (l-2h)^2) + 210\alpha^3 l (l-2h)^2 + 15\alpha^3 l^3 + 184\alpha^2 (l-2h)^2 \\
&\quad \left. + 90\alpha^2 l^2 + 120\alpha l \right\} \gamma_{l-h} \gamma_h \Bigg] s^{-\alpha l} \\
&= \sum_{l=2}^{\infty} \left[\sum_{h=1}^{l-1} \left\{ \alpha^6 (l-2h)^6 + (15\alpha^5 l + 40\alpha^4) (l-2h)^4 \right. \right. \\
&\quad + (45\alpha^4 l^2 + 210\alpha^3 l + 184\alpha^2) (l-2h)^2 \\
&\quad \left. + 15\alpha^3 l^3 + 90\alpha^2 l^2 + 120\alpha l \right\} \gamma_{l-h} \gamma_h \Bigg] s^{-\alpha l}.
\end{aligned}$$

PROPOSITION 2.3. *If two realistic partition functions $\xi(s) = \sum_{l=1}^{\infty}$*

$\gamma_l s^{-\alpha l}$ and $\tilde{\xi}(s) = \sum_{l=1}^{\infty} \tilde{\gamma}_l s^{-\alpha l}$ satisfy $A_2(\xi, \xi) = A_2(\tilde{\xi}, \tilde{\xi})$, then $\xi(s) = \tilde{\xi}(s)$.

Proof. Putting $A_2(\xi, \xi) = \sum_{l=2}^{\infty} \beta_l s^{-\alpha l}$, we observe

$$\beta_l = \sum_{h=1}^{l-1} \{\alpha^2(l-2h)^2 + \alpha l\} \gamma_{l-h} \gamma_h,$$

$$2\alpha\gamma_1^2 = 2\alpha\tilde{\gamma}_1^2,$$

$$\begin{aligned} & 2 \sum_{r=1}^m \{\alpha^2(2m+1-2r)^2 + (2m+1)\alpha\} \gamma_{2m+1-r} \gamma_r \\ &= 2 \sum_{r=1}^m \{\alpha(2m+1-2r) + (2m+1)\alpha\} \tilde{\gamma}_{2m+1-r} \tilde{\gamma}_r \quad (m = 0, 1, 2, 3, \dots), \end{aligned}$$

$$\begin{aligned} & 2 \sum_{r=1}^{m-1} \{\alpha^2(2m-2r)^2 + 2m\alpha\} \gamma_{2m-r} \gamma_r + 2m\alpha\gamma_m^2 \\ &= 2 \sum_{r=1}^{m-1} \{\alpha^2(2m-2r)^2 + 2m\alpha\} \tilde{\gamma}_{2m-r} \tilde{\gamma}_r + 2m\alpha\tilde{\gamma}_m^2 \quad (m = 1, 2, 3, \dots). \end{aligned}$$

Since $\alpha > 0$ and $\tilde{\gamma}_l, \gamma_l \geq 0$ ($l \geq 1$), we conclude $\tilde{\gamma}_l = \gamma_l$ ($l \geq 1$).

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§3. Differential polynomials in theta functions

In the present chapter we shall be concerned with differential polynomials in theta functions. On the algebra of differential polynomials in theta functions, Heisenberg Lie algebra acts so nicely that the subalgebra of theta functions coincides with the subalgebra consisting of so called semi-invariants. This means the invariant theoretic point of view is a helpful method. By analogy of Hilbert operator we are able to calculate concrete expressions, and again apolars take important parts.

Notations.

$$\mathbf{Z}_{\geq 0} = \{\text{non-negative integer}\},$$

$$\mathbf{Z}_{\geq 0}^g = \{j = (j_1, j_2, \dots, j_g) \mid j_i \in \mathbf{Z}_{\geq 0} \ (1 \leq j \leq g)\},$$

$$|j| = j_1 + j_2 + \dots + j_g$$

$$j + \varepsilon_i = (j_1, \dots, j_{i-1}, j_i + 1, j_{i+1}, \dots, j_g)$$

$$j - \varepsilon_i = \begin{cases} (j_1, \dots, j_{i-1}, j_i - 1, j_{i+1}, \dots, j_g) & (j_i \geq 1) \\ 0 & (j_i = 0), \end{cases}$$

$$j! = j_1! j_2! \dots j_g!, \binom{j}{h} = \frac{j!}{(j-h)!h!} = \binom{j_1}{h_1} \binom{j_2}{h_2} \dots \binom{j_g}{h_g},$$

$z = (z_1, z_2, \dots, z_g)$, $u = (u_1, u_2, \dots, u_g)$: systems of complex variables independent each other,

τ : a complex symmetric $g \times g$ -matrix whose imaginary part is positive definite,

$$\begin{aligned} \left(u + l + \frac{a}{n}\right) &= \left(u_1 + l_1 + \frac{a_1}{n}\right)^{j_1} \left(u_2 + l_2 + \frac{a_2}{n}\right)^{j_2} \\ &\quad \dots \left(u_g + l_g + \frac{a_g}{n}\right)^{j_g}, \\ \left(2\pi\eta\sqrt{-1}u + \frac{\partial}{\partial Z}\right)^j &= \left(2\pi\eta\sqrt{-1}u_1 + \frac{\partial}{\partial Z_1}\right)^{j_1} \left(2\pi\eta\sqrt{-1}u_2 + \frac{\partial}{\partial Z_2}\right)^{j_2} \\ &\quad \dots \left(2\pi\eta\sqrt{-1}u_g + \frac{\partial}{\partial Z_g}\right)^{j_g} \end{aligned}$$

3.1. Theta functions and auxiliary theta functions

First of all we shall recollect some basic definitions and results in the previous article [3].

DEFINITION 3.1. A theta function of level n ($n \geq 1$) means an entire function $f(z) = (z_1, z_2, \dots, z_g)$ satisfying the difference equations.

$$(3.1) \quad \begin{aligned} f(z + \hat{b} + b\tau) \\ = \exp[-\pi n \sqrt{-1} \{b\tau^t b + 2z^t b\}] f(z) \quad ((\hat{b}, b) \in \mathbf{Z}^g \times \mathbf{Z}^g). \end{aligned}$$

Denoting of $\theta_0^{(n)}$ the vector space of theta functions of level n , we obtain the graded algebra of theta functions.

$$(3.2) \quad \theta_0 = \bigoplus_{n \geq 1} \theta_0^{(n)}$$

Theta series

$$(3.3) \quad \begin{aligned} \mathcal{Y}^{(n)} \left[\begin{array}{c} \frac{a}{n} \\ 0 \end{array} \right] (\tau | z) \\ = \sum_{l \in \mathbf{Z}^g} \exp \left[\pi n \sqrt{-1} \left\{ \left(l + \frac{a}{n} \right) \tau^t \left(l + \frac{a}{n} \right) + 2z^t \left(l + \frac{a}{n} \right) \right\} \right] \\ (a \in \mathbf{Z}^g / n\mathbf{Z}^g), \end{aligned}$$

form a canonical basis of $\theta_0^{(n)}$, and consequently

$$(3.4) \quad \dim_{\mathbf{C}} \theta_0^{(n)} = n^g.$$

DEFINITION 3.2. An auxiliary theta of level n means a function $\varphi(u, z) = (u_1, u_2, \dots, u_g, z_1, z_2, \dots, z_g)$ such that

1. $\varphi(u, z)$ is a polynomial in $u = (u_1, u_2, \dots, u_g)$ whose coefficients are entire functions in $z = (z_1, z_2, \dots, z_g)$,
2. $\varphi(u + b, z + \hat{b} + b\tau) = \exp[-\pi n \sqrt{-1} \{b\tau^t b + 2z^t b\}] \varphi(u, z) \quad ((\hat{b}, b) \in \mathbf{Z}^g \times \mathbf{Z}^g)$.

Denoting of $\theta^{(n)}$ the vector space of auxiliary theta functions of level n , we obtain the graded algebra of auxiliary theta functions,

$$(3.5) \quad \theta^{(n)} \cap \theta_0 = \theta_0^{(n)} \quad (n \geq 1).$$

Auxiliary theta series

$$\begin{aligned}
(3.6) \quad \vartheta_j^{(n)} \begin{bmatrix} \frac{a}{n} \\ 0 \end{bmatrix} (\tau \mid u, Z) &= \left(2\pi n \sqrt{-1} u + \frac{\partial}{\partial z} \right)^j \vartheta_j^{(n)} \begin{bmatrix} \frac{a}{n} \\ 0 \end{bmatrix} (\tau \mid Z) \\
&= (2\pi n \sqrt{-1})^{|j|} \sum_{l \in \mathbf{Z}^g} \left(u + l + \frac{a}{n} \right)^j \\
&\quad \exp \left[\pi n \sqrt{-1} \left\{ \left(l + \frac{a}{n} \right) \tau^t \left(l + \frac{a}{n} \right) + 2z^t \left(l + \frac{a}{n} \right) \right\} \right] \\
&\quad (j \in \mathbf{Z}_{\geq 0}^g, a \in \mathbf{Z}^g / n\mathbf{Z}^g)
\end{aligned}$$

from a canonical basis of $\theta^{(n)}$. Denoting by $\theta_j^{(n)}$ the vector space spanned by

$$(3.7) \quad \vartheta_j^{(n)} \begin{bmatrix} \frac{a}{n} \\ 0 \end{bmatrix} (\tau \mid u, z) \quad (a \in \mathbf{Z}^g / n\mathbf{Z}^g),$$

we obtain a fine decomposition of θ ,

$$\begin{aligned}
(3.8) \quad \theta &= \bigoplus_{j \in \mathbf{Z}_{\geq 0}^g} \bigoplus_{n \geq 1} \theta_j^{(n)}, \quad \theta^{(n)} = \bigoplus_{j \in \mathbf{Z}_{\geq 0}^g} \theta_j^{(n)}. \\
\theta_j &= \Delta^j \theta_0, \quad \theta_j^{(n)} = \Delta^j \theta_0^{(n)}, \quad (j \in \mathbf{Z}_{\geq 0}^g, n \geq 1)
\end{aligned}$$

and consequently

$$(3.9) \quad \dim_{\mathbf{C}} \theta_j^{(n)} = n^g \quad (n \geq 1, j \in \mathbf{Z}_{\geq 0}^g).$$

3.2. Action of Heisenberg Lie algebra

We denote the projection operators

$$\begin{aligned}
(3.10) \quad \sigma_j &: \theta \longrightarrow \theta_j \\
\sigma^{(n)} &: \theta \longrightarrow \theta^{(n)} \quad (j \in \mathbf{Z}_{\neq 0}^g, n \neq 1) \\
\sigma_j^{(n)} &: \theta \longrightarrow \theta_j^{(n)}
\end{aligned}$$

We introduce differential operators, derivations, acting on θ as follows,

$$\begin{aligned}
(3.11) \quad \varepsilon &= \sum_{n \geq 1} n \sigma^{(n)} \\
D_i &= \sum_{n \geq 1} \frac{1}{2\pi \sqrt{-1}} \frac{\partial}{\partial u_i} \sigma^{(n)} \quad (1 \leq i \leq g), \\
\Delta_i &= \sum_{n \geq 1} \left(2\pi n \sqrt{-1} u_i + \frac{\partial}{\partial z_i} \right) \sigma^{(n)}
\end{aligned}$$

then the action on the canonical basis are given by

$$\begin{aligned}
& \varepsilon \vartheta_j^{(n)} \begin{bmatrix} \frac{a}{n} \\ 0 \end{bmatrix} (\tau \mid u, z) = n \vartheta_j^{(n)} \begin{bmatrix} \frac{a}{n} \\ 0 \end{bmatrix} (\tau \mid u, z) \\
(3.12) \quad & D_i \vartheta_j^{(n)} \begin{bmatrix} \frac{a}{n} \\ 0 \end{bmatrix} (\tau \mid u, z) = n j_i \vartheta_{j-\varepsilon_i}^{(n)} \begin{bmatrix} \frac{a}{n} \\ 0 \end{bmatrix} (\tau \mid u, z) \\
& \Delta_i \vartheta_j^{(n)} \begin{bmatrix} \frac{a}{n} \\ 0 \end{bmatrix} (\tau \mid u, z) = \vartheta_{j+\varepsilon_i}^{(n)} \begin{bmatrix} \frac{a}{n} \\ 0 \end{bmatrix} (\tau \mid u, z) \\
& \qquad (1 \leq i \leq g, j \in \mathbf{Z}_{\geq 0}^g, a \in \mathbf{Z}^g/n\mathbf{Z}^g, n \geq 1).
\end{aligned}$$

The operators $\varepsilon, D_1, D_2, \dots, D_g, \Delta_1, \Delta_2, \dots, \Delta_g$ satisfy the Heisenberg Lie algebra relations;

$$\begin{aligned}
(3.13) \quad & [\varepsilon, D_i] = [\varepsilon, \Delta_i] = [D_i, D_k] = [\Delta_i, \Delta_k] = 0 \\
& [D_i, \Delta_k] = \begin{cases} \varepsilon & (i = k) \\ 0 & (i \neq k), \end{cases}
\end{aligned}$$

Denoting

$$D^l = D_1^{l_1} D_2^{l_2} \dots D_g^{l_g}, \quad \Delta^l = \Delta_1^{l_1} \Delta_2^{l_2} \dots \Delta_g^{l_g} \quad (l \in \mathbf{Z}_{\geq 0}^g),$$

we have

$$\begin{aligned}
& \frac{1}{p!n^{|p|}} D^p \vartheta_j^{(n)} \begin{bmatrix} \frac{a}{n} \\ 0 \end{bmatrix} (\tau \mid u, z) = \binom{j}{p} \vartheta_{j-p}^{(n)} \begin{bmatrix} \frac{a}{n} \\ 0 \end{bmatrix} (\tau \mid u, z) \\
(3.14) \quad & \frac{1}{j!n^{|j|}} D^j \vartheta_j^{(n)} \begin{bmatrix} \frac{a}{n} \\ 0 \end{bmatrix} (\tau \mid u, z) = \vartheta_j^{(n)} \begin{bmatrix} \frac{a}{n} \\ 0 \end{bmatrix} (\tau \mid u, z) \\
& \qquad (j, p \in \mathbf{Z}_{\geq 0}^g, a \in \mathbf{Z}^g/n\mathbf{Z}^g, n \geq 1).
\end{aligned}$$

The next lemma is the key for our purpose, which is the analogy of Hilbert operators in the classical invariant theory.

LEMMA 3.1. *Projection operators $\sigma_j^{(n)}$; $\theta \rightarrow \theta_j^{(n)}$ are given concretely as follows;*

$$(3.15) \quad \sigma_j^{(n)} = \sum_{p \in \mathbf{Z}_{\geq 0}^g} \frac{(-1)^{|p|}}{p!j!n^{|p|}n^{|j|}} \Delta^{p+j} D^{p+j} \sigma_j^{(n)} \quad (j \in \mathbf{Z}_{\geq 0}^g, n \geq 1).$$

Proof. It is sufficient to prove

$$\left(\sum_p \frac{(-1)^{|p|}}{p!j!n^{|p+j|}} \Delta^{p+j} D^{p+j} \right) \vartheta_k^{(n)} \begin{bmatrix} \frac{a}{n} \\ 0 \end{bmatrix} (\tau | u, z) = \begin{cases} \vartheta_n^{(n)} \begin{bmatrix} \frac{a}{n} \\ 0 \end{bmatrix} (\tau | u, z) & (k = j) \\ 0 & (k \neq j). \end{cases}$$

From (3.12) and (3.14) it follows

$$\begin{aligned} & \left(\sum_p \frac{(-1)^{|p|}}{p!j!n^{|p+j|}} \Delta^{p+j} D^{p+j} \right) \vartheta_k^{(n)} \begin{bmatrix} \frac{a}{n} \\ 0 \end{bmatrix} (\tau | u, z) \\ &= \left(\sum_p \frac{(-1)^{|p|}}{p!n^{|p|}} \Delta^{p+j} D^p \right) \left(\frac{1}{j!n^{|j|}} D^j \vartheta_k^{(n)} \begin{bmatrix} \frac{a}{n} \\ 0 \end{bmatrix} (\tau | u, z) \right) \\ &= \left(\sum_p (-1)^{|p|} \Delta^{p+j} \frac{1}{p!n^{|p|}} D^p \right) \binom{k}{j} \vartheta_{k-j}^{(n)} \begin{bmatrix} \frac{a}{n} \\ 0 \end{bmatrix} (\tau | u, z) \\ &= \sum_p (-1)^{|p|} \Delta^{p+j} \binom{k}{j} \binom{k-j}{p} \vartheta_{k-j-p}^{(n)} \begin{bmatrix} \frac{a}{n} \\ 0 \end{bmatrix} (\tau | u, z) \\ &= \left(\sum_p (-1)^{|p|} \binom{k-j}{p} \binom{k}{j} \right) \vartheta_k^{(n)} \begin{bmatrix} \frac{a}{n} \\ 0 \end{bmatrix} (\tau | u, z) \\ &= \begin{cases} \vartheta_k^{(n)} \begin{bmatrix} \frac{a}{n} \\ 0 \end{bmatrix} (\tau | u, z) & (k = j) \\ 0 & (k \neq j) \end{cases} \end{aligned}$$

Since $\theta = \bigoplus_{j \in \mathbf{Z}_{\geq 0}^g} \Delta^j \theta_0$, each auxiliary theta $\varphi(u, z)$ can be expressed uniquely

$$\varphi(u, z) = \sum_{j \in \mathbf{Z}_{\geq 0}^g} \Delta^j \Psi_j(z)$$

with theta functions $\Psi_j(z)$. We denote the linear operators by

$$(3.16) \quad L_j \varphi(u, z) = \Psi_j(z) \quad (j \in \mathbf{Z}_{\geq 0}^g)$$

i.e.,

$$\varphi(u, z) = \sum_j \Delta_j^j \Psi_j(z)$$

From Lemma 3.1 we observe that

LEMMA 3.2.

$$(3.17) \quad \begin{cases} L_0 = \sum_{n \geq 1} \sum_{p \in \mathbf{Z}_{\geq 0}^g} \frac{(-1)^{|p|}}{p!n^{|p|}} \Delta^p D^p \sigma^{(n)} \\ L_j = \sum_{n \geq 1} \sum_{p \in \mathbf{Z}_{\geq 0}^g} \frac{(-1)^{|p|}}{p!j!n^{|p+j|}} \Delta^p D^{p+j} \sigma^{(n)} \end{cases} \quad (j \in \mathbf{Z}_{\geq 0}^g)$$

Proof. Since Δ^j induces a vector space isomorphisms of $\theta_0^{(n)}$ onto $\Delta\theta_0^{(n)}$ and $\sigma_j + \Delta^j L_j$, from Lemma 3.1 we conclude

$$L_j = \sum_{n \geq 1} \sum_{p \in \mathbf{Z}_{\geq 0}^g} \frac{(-1)^{|p|}}{p!j!n^{|p+j|}} \Delta^p D^{p+j} \sigma^{(n)}.$$

LEMMA 3.3.

$$(3.18) \quad L_0 \Delta_i = 0 \quad (1 \leq i \leq g).$$

Proof. For any theta functions $\Psi_l(z)$

$$L_0 \left(\sum_l \Delta^l \Psi_l(z) \right) = \Psi_l(z),$$

hence $L_0 \Delta^l \Psi_l(z) = 0$ ($l \neq 0 \in \mathbf{Z}_{\geq 0}^g$).

LEMMA 3.4. For theta functions $\varphi_1(z)$ and $\varphi_2(z)$,

$$(3.19) \quad L_0 \left(\Delta^j \varphi_1(z) \Delta^k \varphi_2(z) \right) = (-1)^{|j|} L_0 \left(\varphi_1(z) \Delta^{j+k} \varphi_2(z) \right) \quad (j, k \in \mathbf{Z}_{\geq 0}^g).$$

Proof. First we shall show

$$(*) \quad \Delta^j \varphi_1(z) \Delta^k \varphi_2(z) - (-1)^{|j|} \varphi_1(z) \Delta^{j+k} \varphi_2(z) \in \bigcup_{1 \leq j \leq g} \Delta_i \theta$$

by induction on $|j|$. Assuming $(*)$ for j ($|j| \leq m$), then

$$\begin{aligned} \Delta^{j+\epsilon_i} \varphi_1(z) \Delta^k \varphi_2(z) &= \Delta_i \left(\Delta^j \varphi_1(z) \Delta^k \varphi_2(z) \right) - \Delta^j \varphi_1(z) \Delta^{k+\epsilon_i} \varphi_2(z) \\ &= (-1)^{|j+\epsilon_i|} \varphi_1(z) \Delta^{j+k+\epsilon_i} + \text{an element in } \bigcup_{1 \leq p \leq g} \Delta_p \theta. \end{aligned}$$

Hence from Lemma 3.3 we obtain

$$L_0 \left(\Delta^j \varphi_1(z) \Delta^k \varphi_2(z) \right) = (-1)^{|j|} L_0 \left(\varphi_1(z) \Delta^{j+k} \varphi_2(z) \right).$$

3.3. Apolars of theta functions

We shall define apolars of theta functions and give the concrete expression of the linear operators

$$L_j : \theta \longrightarrow \theta_0 \quad (j \in \mathbf{Z}_{\geq 0}^g).$$

DEFINITION 3.3. For each r in $\mathbf{Z}_{\geq 0}^g$ the r -th apolar of theta functions $\varphi_1(z)$ in $\theta_0^{(n_1)}$ and $\varphi_2(z)$ in $\theta_0^{(n_2)}$ is defined by

$$(3.20) \quad A_r(\varphi_1(z), \varphi_2(z)) = \sum_{0 \leq j \leq r} \frac{(-1)^{|j|}}{n_1^{|j|} n_2^{|r-j|}} \binom{r}{j} \Delta^j \varphi_1(z) \Delta^{r-j} \varphi_2(z).$$

LEMMA 3.5. For $\varphi_1(z)$ in $\theta_1^{(n)}$ and $\varphi_2(z)$ in $\theta_0^{(n_2)}$,

$$(3.21) \quad A_r(\varphi_1(z), \varphi_2(z)) \in \theta_0^{(n_1+n_2)}.$$

Proof. Obviously $A_r(\varphi_1(z), \varphi_2(z))$ belongs to $\theta^{(n_1+n_2)}$. It is sufficient to show $D_i A_r(\varphi_1(z), \varphi_2(z)) = 0$ ($1 \leq i \leq g$). By induction on $|j|$ we can prove

$$D_i \Delta^j \sigma^n = (n j_i \Delta^{j-\epsilon_i} + \Delta^n D_i) \sigma^n,$$

hence

$$\begin{aligned} & D_i A_r(\varphi_1(z), \varphi_2(z)) \\ &= \sum_j \frac{(-1)^{|j|}}{n_1^{|j|} n_2^{|r-j|}} \binom{r}{j} \left\{ (D_i \Delta^j \varphi_1(z)) \Delta^{r-j} \varphi_2(z) \right. \\ &\quad \left. + \Delta^j \varphi_1(z) (D_i \Delta^{r-j} \varphi_2(z)) \right\} \\ &= \sum_j \frac{(-1)^{|j|}}{n_1^{|j|} n_2^{|r-j|}} \binom{r}{j} \left\{ n_1 j_i \Delta^{j-\epsilon_i} \varphi_1(z) \Delta^{r-j} \varphi_2(z) \right. \\ &\quad \left. + n_2 (r_i - j_i) \Delta^j \varphi_1(z) \Delta^{r-j-\epsilon_i} \varphi_2(z) \right\} \\ &= \sum_j \frac{(-1)^{\sum_{k \neq i} j_k}}{n_1^{\sum_{k \neq i} j_k} n_2^{\sum_{k \neq k} (r_k - j_k)}} \\ &\quad \prod_{k \neq i} \binom{r_k}{j_k} \left\{ \sum_{j_i} (-1)^{j_i} \binom{r_i}{j_i} \frac{j_i}{n_1^{j_i-1} n_2^{r_i-j_i}} \Delta^{j-\epsilon_i} \varphi_1(z) \Delta^{r-j} \varphi_2(z) \right. \\ &\quad \left. + \sum_{j_i} (-1)^{j_i} \binom{r_i}{j_i} \frac{r_i - j_i}{n_1^{j_i} n_2^{r_i-j_i-1}} \Delta^j \varphi_1(z) \Delta^{r-j-\epsilon_i} \varphi_2(z) \right\} \end{aligned}$$

From

$$\sum_j (-1)^{j_i} \binom{r_i}{j_i} + \sum_j (-1)^{j_i} \binom{r_i}{j_i} (r - (j_i - 1)) = 0$$

we observe $D_i A_r(\varphi_1(z), \varphi_2(z)) = 0$ ($1 \leq i \leq g$).

THEOREM 3.1. *For theta functions $\varphi_1(z)$ in $\theta_0^{(n_1)}$ and $\varphi_2(z)$ in $\theta_0^{(n_2)}$*

$$(3.22) \quad \begin{aligned} L_0(\Delta^j \varphi_1(z) \Delta^k \varphi_2(z)) \\ = (-1)^{|j|} \left(\frac{n_1 n_2}{n_1 + n_2} \right)^{|j+k|} A_{j+k}(\varphi_1(z), \varphi_2(z)) \\ (j, k \in \mathbf{Z}_{\geq 0}^g). \end{aligned}$$

Proof. From (3.19)

$$L_0(\varphi_1(z) \Delta^k \varphi_2(z)) = (-1)^{|j|} L_0(\varphi_1(z) \Delta^{j+k} \varphi_2(z)),$$

it is sufficient to prove

$$L_0(\varphi_1(z) \Delta^k \varphi_2(z)) = \left(\frac{n_1 n_2}{n_1 + n_2} \right)^k A_k(\varphi_1(z), \varphi_2(z)).$$

since $D^p \varphi_1(z) = 0$ for $p \neq 0$ and (3.14) it follows,

$$\begin{aligned} L_0(\varphi_1(z) \Delta^k \varphi_2(z)) &= \sum_p \frac{(-1)^{|p|}}{p!(n_1 + n_2)^{|p|}} \Delta^p D^p \varphi_1(z) \Delta^k \varphi_2(z) \\ &= \sum_p \frac{(-1)^{|p|} n_2^{|p|}}{(n_1 + n_2)^{|p|}} \Delta^p \varphi_1(z) \frac{D^k \Delta^k \varphi_2(z)}{p! n_2^{|p|}} \\ &= \sum_p \frac{(-1)^{|p|} n_2^{|p|}}{(n_1 + n_2)^{|p|}} \Delta^p \varphi_1(z) \binom{k}{p} \Delta^{k-p} \varphi_2(z) \\ &= \sum_{q \leq p} \frac{(-1)^{|p|} n_2^{|p|}}{(n_1 + n_2)^{|p|}} \binom{k}{p} \binom{p}{q} \Delta^q \varphi_1(z) \Delta^{k-q} \varphi_2(z) \\ &= \sum_{q \leq p} (-1)^{|p-q|} \binom{k-q}{p-q} \left(\frac{n_2}{n_1 + n_2} \right)^{|p-q|} \\ &\quad \times \frac{n_1^{|q|} n_2^{|k|}}{(n_1 + n_2)^{|q|}} \frac{(-1)^{|q|}}{n_1^{|q|} n_2^{|k-q|}} \binom{k}{q} \Delta^q \varphi_1(z) \Delta^{k-q} \varphi_2(z) \end{aligned}$$

$$\begin{aligned}
&= \sum_q \left(\sum_l (-1)^{|l|} \binom{k-q}{l} \left(\frac{n_2}{n_1+n_2} \right)^{|l|} \right) \\
&\quad \times \frac{n_1^{|q|} n_2^{|k|}}{(n_1+n_2)^{|q|}} \frac{(-1)^{|q|}}{n_1^{|q|} n_2^{|k-q|}} \binom{k}{q} \Delta^q \varphi_1(z) \Delta^{k-q} \varphi_2(z) \\
&= \sum_q \left(1 - \frac{n_2}{n_1+n_2} \right)^{|k-q|} \frac{n_1^{|q|} n_2^{|k|}}{(n_1+n_2)^{|q|}} \frac{(-1)^{|q|}}{n_1^{|q|} n_2^{|l-q|}} \binom{k}{q} \Delta^q \varphi_1(z) \Delta^{k-q} \varphi_2(z) \\
&= \sum_q \left(\frac{n_1}{n_1+n_2} \right)^{|k-q|} \left(\frac{n_1}{n_1+n_2} \right)^{|q|} n_2^{|k|} \frac{(-1)^{|q|}}{n_1^{|q|} n_2^{|k-q|}} \binom{k}{q} \Delta^q \varphi_1(z) \Delta^{k-q} \varphi_2(z) \\
&= \left(\frac{n_1 n_2}{n_1+n_2} \right)^{|k|} A_k(\varphi_1(z), \varphi_2(z)),
\end{aligned}$$

THEOREM 3.2. For theta functions $\varphi_1(z)$ in $\theta_0^{(n_1)}$ and $\varphi_2(z)$ in $\theta_0^{(n_2)}$

$$\begin{aligned}
(3.23) \quad L_j \left(\Delta^h \varphi_1(z) \Delta^k \varphi_2(z) \right) &= \sum_l (-1)^{|l|} n_1^{|l|} n_2^{|j-l|} \binom{h}{l} \binom{k}{j-l} \\
&\quad \frac{(-1)^{|h|} (n_1 n_2)^{|h+k-j|}}{(n_1+n_2)^{|h+k|}} A_{h+k-j}(\varphi_1(z), \varphi_2(z)) \\
&\quad (j, h, k \in \mathbf{Z}_{\geq 0}^g; n_1, n_2 \geq 1).
\end{aligned}$$

Proof. From Lemma 3.2 the operator $L_j \sigma^{(n)}$ can be written as follows

$$L_j \sigma^{(n)} = L_0 \left(\frac{1}{j! n^{|j|}} D^j \sigma^{(n)} \right),$$

hence, using (3.14), we have

$$\begin{aligned}
&L_j(\Delta^h \varphi_1(z) \Delta^k \varphi_2(z)) \\
&= \frac{1}{(n_1+n_2)^{|j|}} L_0 \left(\frac{1}{j!} D^j (\Delta^h \varphi_1(z) \Delta^k \varphi_2(z)) \right) \\
&= \frac{1}{(n_1+n_2)^{|j|}} L_0 \left(\sum_l \frac{n_1^{|l|}}{l! n_1^{|l|}} D^l \Delta^h \varphi_1 \frac{n_2^{|j-l|}}{(j-l)! n_2^{|j-l|}} D^{j-l} \Delta^k \varphi_2(z) \right) \\
&= \frac{1}{(n_1+n_2)^{|j|}} L_0 \left(\sum_l n_1^{|l|} n_2^{|j-l|} \binom{h}{l} \binom{k}{j-l} \Delta^{h-l} \varphi_1(z) \Delta^{k-j+l} \varphi_2(z) \right) \\
&= \frac{1}{(n_1+n_2)^{|j|}} \sum_l n_1^{|l|} n_2^{|j-l|} \binom{h}{l} \binom{k}{j-l} L_0 \left(\Delta^{h-l} \varphi_1(z) \Delta^{k-j+l} \varphi_2(z) \right)
\end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{(n_1 + n_2)^{|j|}} \sum_l n_1^{|l|} n_2^{|j-l|} \binom{h}{l} \binom{k}{j-l} (-1)^{|h-l|} \\
 &\quad \times \left(\frac{n_1 n_2}{n_1 + n_2} \right)^{|h+k-j|} A_{h+k-j}(\varphi_1(z), \varphi_2(z)) \\
 &= \left(\sum_l (-1)^{|l|} n_1^{|l|} n_2^{|j-l|} \binom{h}{l} \binom{k}{j-l} \right) \\
 &\quad \times \frac{(-1)^h (n_1 n_2)^{|h+k-j|}}{(n_1 + n_2)^{|h+k|}} A_{h+k-j}(\varphi_1(z), \varphi_2(z)).
 \end{aligned}$$

Finally we obtain the next decomposition formula.

THEOREM 3.3.

$$\begin{aligned}
 (3.24) \quad &\Delta^h \varphi_1(z) \Delta^k \varphi_2(z) \\
 &= \sum_{0 \leq j \leq h+k} \sum_l (-1)^{|l|} n_1^{|l|} n_2^{|j-l|} \binom{h}{l} \binom{k}{j-l} \\
 &\quad \times \frac{(-1)^{|h|} (n_1 n_2)^{|h+k-j|}}{(n_1 + n_2)^{|h+k|}} \Delta^j A_{h+k-j}(\varphi_1(z), \varphi_2(z)) \\
 &\quad (\varphi_1(z) \in \theta_0^{(n_1)}, \varphi_2(z) \in \theta_0^{(n_2)}; h, k \in \mathbf{Z}_{\geq 0}^g).
 \end{aligned}$$

Specialization $u = (u_1, \dots, u_g) \rightarrow 0 = (0, \dots, 0)$ induces a graded algebra isomorphism of $\theta = \bigoplus_{j \in \mathbf{Z}_{\neq 0}^g} \theta_j$ onto the graded algebra $D(\theta_0)$ of differential polynomials in theta functions such that

$$(3.25) \quad \Delta^j \varphi(z) \longrightarrow \left(\frac{\partial}{\partial z} \right)^j \varphi(z) \quad (\varphi(z) \in \theta_0).$$

Since the apolars $A_r(\varphi_1(z), \varphi_2(z))$ of $\varphi_1(z)$ in $\theta_0^{(n_1)}$ and $\varphi_2(z)$ in $\theta_0^{(n_2)}$ belong to $\theta_0^{(n_1+n_2)}$, we may write

$$\begin{aligned}
 (3.26) \quad &A_r(\varphi_1(z), \varphi_2(z)) \\
 &= \sum_{0 \leq j \leq r} \frac{(-1)^{|j|}}{n_1^{|j|} n_2^{|r-j|}} \binom{r}{j} \left(\frac{\partial}{\partial z} \right)^j \varphi_1(z) \left(\frac{\partial}{\partial z} \right)^{r-j} \varphi_2(z).
 \end{aligned}$$

The next is the translation of Theorem 3.3 by means of the specialization $\Delta^j \rightarrow (\partial/\partial z)^j$ ($j \in \mathbf{Z}_{\geq 0}^g$).

THEOREM 3.4.

$$\begin{aligned}
 (3.27) \quad & \left(\frac{\partial}{\partial z}\right)^h \varphi_1(z) \left(\frac{\partial}{\partial z}\right)^k \varphi_2(z) \\
 &= \sum_{0 \leq j \leq h+k} \sum_l (-1)^l n_1^{|l|} n_2^{|j-l|} \binom{h}{l} \binom{k}{j-l} \\
 &\quad \times \frac{(-1)^{|h|} (n_1 n_2)^{|h+k-j|}}{(n_1 + n_2)^{|h+k|}} \left(\frac{\partial}{\partial z}\right)^j A_{h+k-j}(\varphi_1(z), \varphi_2(z)) \\
 &\quad (\varphi_1(z) \in \theta_0^{(n_1)}, \varphi_2(z) \in \theta_0^{(n_2)}; h, k \in \mathbf{Z}_{\geq 0}^g).
 \end{aligned}$$

3.4. A differential algebra expansion of the field of abelian functions

We mean by **A** the abelian variety defined by the period matrix (I, τ) , and mean by \mathbf{V}_0 the affine g -bundle over **A** defined by the quotient

$$\mathbf{V}_0 = \mathbf{C}^g \times \mathbf{C}^g / \sim,$$

where $(u, Z) \sim (u + b, z + \hat{b} + b\tau)$ ($(\hat{b}, b) \in \mathbf{Z}^g \times \mathbf{Z}^g$).

We denote by Ψ_0 the field of abelian functions on **A**, i.e., the quotients $\varphi_1(z)/\varphi_2(z)$ of theta functions of the same level.

DEFINITION 3.4. $F_j\Psi$ means the vector space over Ψ_0 spanned by

$$\frac{\Delta^k \varphi_1(z)}{\varphi_2(z)} \quad (0 \leq k \leq j),$$

where $\varphi_1(z)$ and $\varphi_2(z)$ run over theta functions of the same level. Then $F_0\Psi = \Psi_0$ and

$$(3.28) \quad \Psi = \bigcup_{j \in \mathbf{Z}_{\geq 0}^g} F_j\Psi$$

is a filtration of an algebra. We call the elements of Ψ auxiliary abelian functions.

THEOREM 3.5. *The algebra Ψ of auxiliary abelian functions in a polynomial algebra over Ψ_0 ,*

$$(3.29) \quad \Psi = \Psi_0 \left[\Delta_1 \vartheta \begin{bmatrix} o \\ o \end{bmatrix} (\tau | z), \dots, \Delta_g \vartheta \begin{bmatrix} o \\ o \end{bmatrix} (\tau | z) \right],$$

where $\vartheta \begin{bmatrix} o \\ o \end{bmatrix} (\tau | z)$ is the theta series of level one.

Proof. We shall prove

$$(*) \quad F_r \Psi \subset \Psi_0 \left[\Delta_1 \vartheta \begin{bmatrix} o \\ o \end{bmatrix} (\tau | z), \dots, \Delta_g \vartheta \begin{bmatrix} o \\ o \end{bmatrix} (\tau | Z) \right]$$

by induction on $|r|$. Assume $(*)$ for j satisfying $|j| \neq r$. Then taking h, k such that $h + k = r$, $|h + k| = r$, from 3.4 we have

$$\begin{aligned} & \frac{\Delta^h \vartheta \begin{bmatrix} o \\ o \end{bmatrix} (\tau | Z) \Delta^k \vartheta \begin{bmatrix} o \\ o \end{bmatrix} (\tau | Z)}{\vartheta \begin{bmatrix} o \\ o \end{bmatrix} (\tau | Z)^2} \\ &= \sum_{0 \leq j \leq j+k=r} \sum_l (-1)^{|l|} \binom{h}{l} \binom{k}{j-l} \\ & \quad \times \frac{(-1)^{|h|} \Delta^j A_{r-j} (\vartheta \begin{bmatrix} o \\ o \end{bmatrix} (\tau | Z), \vartheta \begin{bmatrix} o \\ o \end{bmatrix} (\tau | Z))}{2^{|r|} \vartheta \begin{bmatrix} o \\ o \end{bmatrix} (\tau | Z)^2} \\ &\equiv \frac{(-1)^{|h|} \Delta^r \vartheta \begin{bmatrix} o \\ o \end{bmatrix} (\tau | Z)^2}{2^{|h|} \vartheta \begin{bmatrix} o \\ o \end{bmatrix} (\tau | Z)^2} \equiv \frac{(-1)^{|h|} \Delta^r \vartheta \begin{bmatrix} o \\ o \end{bmatrix} (\tau | Z)}{2^{|r|-1} \vartheta \begin{bmatrix} o \\ o \end{bmatrix} (\tau | Z)} \pmod{\bigcup_{j \not\leq r} F_j \Psi}. \end{aligned}$$

This means

$$\frac{\Delta^r \vartheta \begin{bmatrix} o \\ o \end{bmatrix} (\tau | z)}{\vartheta \begin{bmatrix} o \\ o \end{bmatrix} (\tau | z)} \in \Psi_0 \left[\Delta_1 \vartheta \begin{bmatrix} o \\ o \end{bmatrix} (\tau | z), \dots, \Delta_g \vartheta \begin{bmatrix} o \\ o \end{bmatrix} (\tau | z) \right].$$

Again from (3.24), for theta functions $\varphi_1(z), \varphi_2(z)$ in $\theta_0^{(n)}$ we have

$$\frac{\Delta^h \varphi_1(z) \Delta^k \vartheta \begin{bmatrix} o \\ o \end{bmatrix} (\tau | z)}{\varphi_2(z) \vartheta \begin{bmatrix} o \\ o \end{bmatrix} (\tau | z)}$$

$$\begin{aligned}
&= \sum_{0 \leq j \leq h+k=r} \sum_l (-1)^{|l|} n^{|l|} \binom{h}{l} \binom{k}{j-l} \frac{(-1)^{|l|} n^{|r-j|}}{(n+1)^{|r|}} \\
&\quad \times \frac{\Delta^j A_{r-j}(\varphi_1(z)) \vartheta \begin{bmatrix} o \\ o \end{bmatrix}(\tau | z)}{\varphi_2(z) \vartheta \begin{bmatrix} o \\ o \end{bmatrix}(\tau | z)} \\
&\equiv \frac{n^{|h|}}{(n+1)^{|r|}} \frac{\Delta^r(\varphi(z)) \vartheta \begin{bmatrix} o \\ o \end{bmatrix}(\tau | z)}{\varphi_2(z) \vartheta \begin{bmatrix} o \\ o \end{bmatrix}(\tau | z)} \\
&\equiv \frac{n^{|h|}}{(n+1)^{|r|}} \frac{\Delta^r \varphi_1(z)}{\varphi_2(z)} + \frac{n^{|h|}}{(n+1)^{|r|}} \frac{\Delta^r \vartheta \begin{bmatrix} o \\ o \end{bmatrix}(\tau | z)}{\vartheta \begin{bmatrix} o \\ o \end{bmatrix}(\tau | z)} \pmod{\bigcup_{|j| \neq r} F_j \Psi}.
\end{aligned}$$

Choosing h, k such that $h+k=r$, $|h|, |k| \leq r$, we conclude

$$\frac{\Delta^r \varphi(z)}{\varphi_2(z)} \in \Psi_0 \left[\Delta_1 \vartheta \begin{bmatrix} o \\ o \end{bmatrix}(\tau | z), \dots, \Delta_g \vartheta \begin{bmatrix} o \\ o \end{bmatrix}(\tau | z) \right].$$

The next is the translation of Theorem 3.5.

THEOREM 3.6. *The polynomial algebra over Ψ_0*

$$(3.30) \quad \Psi_0 \left[\frac{\frac{\partial}{\partial z_1} \vartheta \begin{bmatrix} o \\ o \end{bmatrix}(\tau | z)}{\vartheta \begin{bmatrix} o \\ o \end{bmatrix}(\tau | z)}, \dots, \frac{\frac{\partial}{\partial z_g} \vartheta \begin{bmatrix} o \\ o \end{bmatrix}(\tau | z)}{\vartheta \begin{bmatrix} o \\ o \end{bmatrix}(\tau | z)} \right]$$

coincides with the algebra over Ψ_0 spanned by

$$\frac{\left(\frac{\partial}{\partial z} \right)^j \vartheta^{(n)} \begin{bmatrix} a \\ o \end{bmatrix}(\tau | z)}{\vartheta^{(n)} \begin{bmatrix} o \\ o \end{bmatrix}(\tau | z)} \quad (j \in \mathbf{Z}_{\geq 0}^g; a, b \in \mathbf{Z}^g/n\mathbf{Z}^g; n \geq 1).$$

Moreover derivations act on generators as follows,

$$\begin{aligned}
 (3.31) \quad & \frac{\partial}{\partial z_j} \left(\frac{\frac{\partial}{\partial z_j} \vartheta \begin{bmatrix} o \\ o \end{bmatrix} (\tau | z)}{\vartheta \begin{bmatrix} o \\ o \end{bmatrix} (\tau | z)} \right) \\
 &= \frac{\frac{\partial}{\partial z_j} \vartheta \begin{bmatrix} o \\ o \end{bmatrix} (\tau | z)}{\vartheta \begin{bmatrix} o \\ o \end{bmatrix} (\tau | z)} \frac{\frac{\partial}{\partial z_j} \vartheta \begin{bmatrix} o \\ o \end{bmatrix} (\tau | z)}{\vartheta \begin{bmatrix} o \\ o \end{bmatrix} (\tau | z)} - \frac{1}{2} \frac{A_{\epsilon_i + \epsilon_j} (\vartheta \begin{bmatrix} o \\ o \end{bmatrix} (\tau | z), \vartheta \begin{bmatrix} o \\ o \end{bmatrix} (\tau | z))}{\vartheta \begin{bmatrix} o \\ o \end{bmatrix} (\tau | z)^2},
 \end{aligned}$$

where

$$\begin{aligned}
 & A_{\epsilon_i + \epsilon_j} \left(\vartheta \begin{bmatrix} o \\ o \end{bmatrix} (\tau | z), \vartheta \begin{bmatrix} o \\ o \end{bmatrix} (\tau | z) \right) \\
 &= 2 \frac{\partial^2}{\partial z_i \partial z_j} \vartheta \begin{bmatrix} o \\ o \end{bmatrix} (\tau | z) - 2 \frac{\partial}{\partial z_i} \vartheta \begin{bmatrix} o \\ o \end{bmatrix} (\tau | z) \frac{\partial}{\partial z_j} \vartheta \begin{bmatrix} o \\ o \end{bmatrix} (\tau | z).
 \end{aligned}$$

The derivations $\partial/\partial z_1, \dots, \partial/\partial z_g$ map the field Ψ_0 into itself.

3.5. Decomposition of differential polynomials

We choose a system of independent variables

$$s = (s_1, s_2, \dots, s_g)$$

and a system of dependent variables

$$Z(s) = (Z_1(s), Z_2(s), \dots, Z_N(s))$$

with positive integral valued weight

$$\text{weight } Z_k(s) = n_k \quad (n \leq k \leq N)$$

we regard $(Z_k^{(j)}(s))$ also a system of dependent variables with respect to $s = (s_1, s_2, \dots, s_g)$, where $Z_k^{(j)} = (\partial/\partial s)^j Z_k(s)$ and $\text{weight } Z_k^{(j)}(s) = n_k$ ($j \in \mathbf{Z}_{\geq 0}^g, 1 \leq k \leq N$).

We introduce a system of derivatives of the polynomial algebra

$$\Omega = Q[\dots, Z_k^{(j)}, \dots]$$

as follows,

$$(3.32) \quad \begin{cases} \varepsilon = \sum_{n \geq 1} n \sigma(n) \\ D_i = \sum_{j \in \mathbf{Z}_{\geq 0}^g, 1 \leq k \leq N} n_k j_i Z_k^{(j-\epsilon_i)} \frac{\partial}{\partial Z_k^{(j)}} \\ \Delta_i = \frac{\partial}{\partial s_1} \end{cases} \quad (1 \leq i \leq g),$$

where

$$\sigma^{(n)} : \Omega \longrightarrow \Omega^{(n)}$$

means the projection of Ω onto the vector subspace $\Omega^{(n)}$ consisting of elements of weight n in Ω . Then the derivations $\varepsilon, D_1, \dots, D_g, \Delta_1, \dots, \Delta_1, \dots, \Delta_g$ satisfy Heisenberg Lie algebra relations:

$$(3.33) \quad \begin{cases} [\varepsilon, D_i] = [D_i, D_k] = [\Delta_i, \Delta_k] = 0 \\ [D_i, \Delta_k] = \begin{cases} (i = k) \\ (i \neq k). \end{cases} \end{cases}$$

DEFINITION 3.5. An element ϕ in $\Omega^{(n)}$ is called a semi-invariant of weight n , if $D_i \phi = 0$ ($1 \leq i \leq g$). Semi-invariants of weight n form a vector space $\Omega_0^{(n)}$ and the graded algebra

$$\Omega_0 = \bigoplus_{n \geq 1} \Omega_0^{(n)}$$

is called the graded algebra of semi-invariants in Ω .

LEMMA 3.6.

$$(3.34) \quad D_i \Delta^p = p_i \varepsilon \Delta^{p-\epsilon_i} + \Delta^p D_i \quad (p \in \mathbf{Z}_{\geq 0}^g),$$

$$(3.35) \quad \frac{D^p \Delta^j \varphi}{n! p!} = \binom{j}{p} \Delta^{j-p} \varphi \quad (j, p \in \mathbf{Z}_{\geq 0}^g, \varphi \in \Omega_0^{(n)}).$$

Proof. Assuming (3.34) for p_i , we have

$$\begin{aligned} D_i \Delta^{p+\epsilon_i} &= D_i \Delta^p \Delta_i = (p_i \varepsilon \Delta^{p-\epsilon_i} + \Delta^p D_i) \Delta_i \\ &= p_i \varepsilon \Delta^p + \Delta^p \varepsilon + \Delta^{p+\epsilon_i} D_i = (p_i + 1) \varepsilon \Delta^p + \Delta^{p+\epsilon_i} D_i. \end{aligned}$$

THEOREM 3.7. We denote

$$(3.36) \quad L_j = \sum_{n \geq 1} \sum_{p \in \mathbf{Z}_{\geq 0}^g} \frac{(-1)^{|p|}}{p!j!n^{|p+\epsilon|}} \Delta^p D^{p+j} \sigma^{(n)},$$

then

$$(3.37) \quad D_i L_i = 0$$

$$(3.38) \quad L_j \Delta^k | \Omega_0 = \begin{cases} \text{id}_{\Omega_0} & (j = k) \\ 0 & (j \neq k) \end{cases}$$

$$(3.39) \quad \sum_{j \in \mathbf{Z}_{\geq 0}^g} \Delta^j L_j = \text{id}_{\Omega}$$

Proof. From the relation

$$\sum_p (-1)^{|p|} \binom{m}{p} = \begin{cases} 1 & (m = 0) \\ 0 & (m \neq 0), \end{cases}$$

it follows

$$\begin{aligned} \sum_j \Delta^j L_j &= \sum_{n \geq 1} \sum_p \frac{(-1)^{|p|}}{n^{|j+p|} p! j!} \Delta^{i+p} D^{j+p} \sigma^{(n)} \\ &= \sum_{n \geq 1} \frac{1}{m! n^{|m|}} \left(\sum_p (-1)^{|p|} \binom{m}{p} \right) \Delta^m D^m \sigma^{(n)} - \sum_{n \geq 1} \sigma^{(n)} = \text{id}_{\Omega} \end{aligned}$$

From (3.35), for each φ in $\Omega_0^{(n)}$, we have

$$\begin{aligned} L_j \Delta^k \varphi &= \sum \frac{(-1)^{|p|} (p+j)! \Delta^p}{p! j! n^{|p+j|} (p+j)!} D^{p+j} \Delta^k \varphi \\ &= \sum (-1)^{|p|} \binom{p+j}{p} \binom{k}{p+j} \Delta^p \Delta^{k-j-p} \varphi \\ &= \sum_p (-1)^{|p|} \binom{k-j}{p} \binom{k}{j} \Delta^{k-j} \varphi = \varphi. \end{aligned}$$

This means $L_j \Delta^k | \Omega_0 = \text{id}_{\Omega_0}$. Since $\varepsilon \sigma^{(n)} = \eta \sigma^{(n)}$, from (3.34) we have

$$\begin{aligned} D_i L_i &= \sum_{n \geq 1} \sum_p \frac{(-1)^{|p|}}{n^{|p+j|} p! j!} D_i \Delta^p D^{j+p} \sigma^{(n)} \\ &= \sum_{n \geq 1} \sum_p \frac{(-1)^{|p|}}{n^{|p+j|} p! j!} (p + i \varepsilon \Delta^{p_i - \varepsilon_i} dj + p + \Delta^p D^{j+p+\varepsilon_i}) \sigma^{(n)} \\ &= \sum_{n \geq 1} \left(\sum_p \frac{(-1)^{|p-\varepsilon_i|}}{n^{|p-\varepsilon_i+j|} (p-\varepsilon_i)! j!} \Delta^{p-1} D^{j+p} \right. \\ &\quad \left. + \sum_p \frac{(-1)^{|p|}}{n^{|p+j|} p! j!} \Delta^p D^{j+p+\varepsilon_i} \right) \sigma^{(n)} = 0 \end{aligned}$$

Now we obtain the decomposition formula.

THEOREM 3.8. L_i is a vector space isomorphism of $\Delta^j \Omega_0$ onto Ω_0 , and $\Delta^j L_i$ is the projection of Ω onto $\Delta^j \Omega_0$ in the decomposition:

$$(3.40) \quad \Omega = \bigoplus_{j \in \mathbf{Z}_{\geq 0}^g} \Delta^j \Omega_0 = \bigoplus_{n \geq 1} \bigoplus_{j \in \mathbf{Z}_{\geq 0}^g} \Delta^j \Omega_0^{(n)}.$$

Theorem 3.8 states that each differential polynomial ϕ is uniquely written

$$\phi = \sum_h \left(\frac{\partial}{\partial s} \right)^h L_h \phi$$

with semi-invariant $L_h \phi$.

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*The University of Aizu
 Tsuruga, Ikki-machi, Aizu-Wakamatsu City
 Fukushima, 965 Japan*