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EXTENSION OF CR STRUCTURES ON THREE DIMENSIONAL PSEUDOCONVEX CR MANIFOLDS

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Abstract. Let $\overline{M}$ be a smoothly bounded orientable pseudoconvex *CR* manifold of finite type and $dim_{\mathbb{R}} M = 3$. Then we extend the given *CR* structure on M to an integrable almost complex structure on S_g^+ which is the concave side of M and $M \subset bS_g^+$.

§1. Introduction

Let $\widetilde{M}$ be a smooth orientable manifold of dimension $2n-1$ and let $\overline{M} \subset \widetilde{M}$ be a smoothly bounded *CR* manifold with a given *CR* structure $\mathcal{S}$ of dimension $n-1$. Since $\widetilde{M}$ is orientable, there are smooth real nonvanishing 1-form η and smooth real vector field X_0 on $\widetilde{M}$ so that $\eta(X) = 0$ for all $X \in \mathcal{S}$ and $\eta(X_0) = 1$. We define the Levi form of $\mathcal{S}$ on $\overline{M}$ by $i\eta([X', \overline{X}''])$.

In [4], Catlin has considered an extension problem of a given *CR* structure on M to an integrable almost complex structure on a $2n$-dimensional manifold Ω with boundary so that the extension is smooth up to the boundary and so M lies in $b\Omega$. Under certain conditions on the Levi form (cf., [4, Theorem 1.1, Theorem 1.3]), this leads to a solution of the Kuranishi problem [1, 9, 13], which is to show that an abstract *CR* manifold can be locally embedded in $\mathbb{C}^n$.

In this paper, we consider an extension problem of a given *CR* structure on M when M is a pseudoconvex *CR* manifold of finite type and $\dim_{\mathbb{R}} M = 3$. For a given positive continuous function g on M, where $g = 0$ on bM, we define

$$S_g^+ = \{(x,t) \in M \times [0,\infty);\ 0 \le t \le g(x)\}.$$

Then our main result is the following theorem:

THEOREM 1.1. *Let $\overline{M} \subset \widetilde{M}$ be a smoothly bounded orientable pseudoconvex CR manifold of finite type with given CR structure $\mathcal{S}$ on M and*

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$\dim_{\mathbb{R}} M = 3$. *Then there exists a positive continuous function* g *on* M *and a smooth integrable almost complex structure* $\mathcal{L}$ *on* S_g^+ *such that for all* $x \in M$, $\mathcal{L}_{(x,0)} \cap \mathbb{C}TM = S_x$. *Furthermore, if* $\mathcal{J}_{\mathcal{L}}: TS_g^+ \to TS_g^+$ *is the map associated with the complex structure* $\mathcal{L}$, *then* $dt(\mathcal{J}_{\mathcal{L}}(X_0)) < 0$ *at all points of* $M_0 = \{(x, 0);\ x \in M\}$.

Note that we extend the given *CR* structure on M to the concave side (instead of convex side) of M. We also note that if M is strongly pseudoconvex, this case was handled in [4, Theorem 1.1]. Theorem 1.1, in general, would not imply the local embedding of M into $\mathbb{C}^2$ (cf., [2, 6]). But we have the following theorem as an application of Theorem 1.1.

THEOREM 1.2. *Let* D *be a complex manifold with* C^∞ *boundary and* $\dim_{\mathbb{C}} D = 2$. *Suppose that the almost complex structure on* D *extends smoothly to a manifold* $\overline{M} \subset bD$ *where* $\overline{M}$ *is compact pseudoconvex CR manifold of finite type with smooth boundary and* $\dim_{\mathbb{R}} M = 3$. *Then* D *can be embedded in a larger complex manifold* Ω *so that* M *lies in the interior of* Ω *as a real hypersurface.*

Remark 1.3. In [5], the author showed that any smooth compact pseudoconvex complex manifold $\overline{D}$ of finite type with $\dim_{\mathbb{C}} D = n$, $n \geq 2$, can be embeded into a larger complex manifold Ω. Theorem 1.2 is a generalization of this result to non-compact complex manifolds of complex dimension 2.

In [4], Catlin has introduced certain nonlinear equations which come from deformation theory of an almost complex structure. The linearized forms of these equations are simply the $\overline{\partial}$–operator from $\Lambda^{0,1} \otimes T^{1,0}$ to $\Lambda^{0,2} \otimes T^{1,0}$. The solutions of these equations represent sucessive corrections that must be made in the iterative process of solving the nonlinear equation. To overcome difficulties in subelliptic estimates for $\overline{\partial}$ near bM, we choose a Hermitian metric on S_g^+ so that S_g^+ takes on the form $S_\varepsilon = M \times [0, \varepsilon]$, where M is a complete noncompact manifold. To this end, we choose, for each $x_0 \in M$, a noneuclidean ball that is of size $\delta = g(x_0)$ in the transverse holomorphic direction and of size $\tau(x_0, \delta)$ in the tangential holomorphic direction. Some technical difficulties in constructing the quantity $\tau(x_0, \delta)$ is handled in Section 3. Here we introduce special coordinate changes (Proposition 3.1) so that the tangential vector field L_1 can be written in a suitable form. These change of coordinates will have an independent interest in studying the *CR* manifolds of finite type. To study the behavior of $\tau(x_0, \delta)$, we introduce a smoothly varying function $\mu(x, \delta)$ which is defined invariantly. Then it follows that $\tau(x, \delta) \approx \mu(x, \delta)$ (Proposition 3.2), and hence $\tau(x, \delta)$ is defined invariantly. Also $\tau(x, \delta)$ satisfies "doubling property" (Corollary 3.3), which is one of a crucial property of $\tau(x, \delta)$. Equipped with all of these necessary properties of $\tau(x, \delta)$, we perform some careful subelliptic estimates of the $\overline{\partial}$ type equation on each of these noneuclidean balls (Section 4). Then this will give us the estimates so called "tame estimates", which are required in the Nash-Moser method for the approximate solution to the linearized equation. Then the rest of the procedure is similar to those of Catlin's, which uses the simplified version of Nash-Moser theorem [12].

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§2. Deformation of almost complex structures

Let M be a *CR* manifold as in section 1 and set $\Omega = M \times (-1, 1)$. In this section we extend a given *CR* structure on M to an almost complex manifold Ω, and we consider a deformation problem of an almost complex structure on Ω so that the new (deformed) amost complex structure is integrable (or close to be integrable).

Since Ω is an almost complex manifold of $\dim_{\mathbb{R}} \Omega = 4$, there is a subbundle $\mathcal{L}$ of $\mathbb{C}T\Omega$ of dimension 2 (over $\mathbb{C}$) such that $\mathcal{L} \cap \overline{\mathcal{L}} = \{0\}$. Let A be a smooth section of $\Gamma^1(\mathcal{L}) = \Lambda^{0,1}(\mathcal{L}) \otimes \mathcal{L}$, where $\Lambda^{0,1}(\mathcal{L})$ denotes the set of $(0, 1)$ forms with respect to $\mathcal{L}$. Observe that if A is sufficiently small, then the bundle $\mathcal{L}^A = \{L + \overline{A}(L);\ L \in \mathcal{L}\}$ defines a new almost complex structure and if $\overline{L}'$ and $\overline{L}''$ are sections of $\overline{\mathcal{L}}$, then $\overline{L}' + A(\overline{L}')$ and $\overline{L}'' + A(\overline{L}'')$ are sections of $\overline{\mathcal{L}^A}$. Similarly, if ω is a section of $\Lambda^{1,0}(\mathcal{L})$, then $\omega - A^*\omega$ is a section of $\Lambda^{1,0}(\mathcal{L}^A)$ where the adjoint A^* maps from $\Lambda^{1,0}(\mathcal{L})$ to $\Lambda^{0,1}(\mathcal{L})$ and is defined by

$$(A^*\omega)(\overline{L}) = \omega(A(\overline{L})), \tag{2.1}$$

for all $\overline{L} \in \overline{\mathcal{L}}$ and $\omega \in \Lambda^{1,0}$. We want to choose A so that

$$(\omega - A^*\omega)([L' + A(L'), L'' + A(L'')]) = 0.$$

By linearizing, i.e., by ignoring terms where A or A^* appear more than once, we obtain

$$\omega([L', A(L'')]) + \omega([A(L'), L'']) - A^*\omega([L', L'']) = -\omega([L', L'']). \tag{2.2}$$

Let $L = L' + L''$ denote the decomposition of a vector $L \in \mathbb{C}T_z$ where $L' \in \mathcal{L}_z$ and $L'' \in \overline{\mathcal{L}}_z$. For sections $\overline{L}_1$, $\overline{L}_2$ of $\overline{\mathcal{L}}$, we define

$$(2.3) \qquad (D_2A)(\overline{L}_1, \overline{L}_2) = [\overline{L}_1, A(\overline{L}_2)]' - [\overline{L}_2, A(\overline{L}_1)]' - A([\overline{L}_1, \overline{L}_2]'').$$

Note that this definition is linear in $\overline{L}_1$ and $\overline{L}_2$ so D_2A is a section of $\Gamma^2 = \Lambda^{0,2}(\mathcal{L}) \otimes \mathcal{L}$. It follows from (2.1) and (2.3) that (2.2) is equivalent to the equation

$$(2.4) \qquad D_2A = -F,$$

where F is a section of Γ^2 defined by

$$(2.5) \qquad F(\overline{L}_1, \overline{L}_2) = [\overline{L}_1, \overline{L}_2]'.$$

Note that F measures the extend to which $\mathcal{L}$ fails to be integrable. If $\mathcal{L}$ defines a CR structure on $M \subset b\Omega$ and if we want $\mathcal{L}_A$ to define the same CR structure on M, then this means that A must satisfy $A(\overline{L}') = 0$ on M whenever $\overline{L}'$ is a section of $\overline{\mathcal{L}}$ that is tangent to M. This is a Dirichlet condition on some of the components of the solution of (2.4).

Since $\dim_{\mathbb{C}} \Omega = 2$, it follows that $D_3B = 0$ for all $B \in \Gamma^2$, where $D_3: \Gamma^2 \longrightarrow \Gamma^3$ is defined by

$$\begin{aligned} &D_3B(\overline{L}_1, \overline{L}_2, \overline{L}_3) \\ &\quad = [\overline{L}_1, B(\overline{L}_2, \overline{L}_3)]' - [\overline{L}_2, B(\overline{L}_1, \overline{L}_3)]' + [\overline{L}_3, B(\overline{L}_1, \overline{L}_2)]' \\ &\qquad - B([\overline{L}_1, \overline{L}_2]'', \overline{L}_3) + B([\overline{L}_1, \overline{L}_3]'', \overline{L}_2) - B([\overline{L}_2, \overline{L}_3]'', \overline{L}_1). \end{aligned}$$

Now set $\Omega = M \times (-1, 1)$. Then we have the following formal solution of the extension problem [4, Theorem 4.1].

THEOREM 2.1. *Suppose that M is an orientable CR manifold of dimension $2n - 1$ such that the CR dimension equals $n - 1$. Then there exists an almost complex structure $\mathcal{L}^*$ on $\Omega = M \times (-1, 1)$ such that $\mathcal{L}^*$ is an extension of the CR structure on M, and such that it is integrable to infinite order at M in the sense that if ω is a section of $\Lambda^{1,0}(\mathcal{L}^*)$ and $\overline{L}_1$, $\overline{L}_2$ are sections of $\overline{\mathcal{L}}^*$, then $\omega([\overline{L}_1, \overline{L}_2])$ vanishes to infinite order along M.*

The next theorem shows that the above formal extension is essentially unique.

THEOREM 2.2. ([4, Theorem 4.2]) *Let M and Ω be as in Theorem 2.1. Suppose that $\mathcal{L}$ and $\mathcal{X}$ are almost complex structures on Ω that extend the CR structure on $M_0 = \{(x,0); x \in M\}$, and that are integrable to infinite order on M_0 as in Theorem 2.1. Then, there exists a diffeomorphism G of Ω onto itself that is the identity when $t = 0$ and such that $G_*\mathcal{X}$ approximates $\mathcal{L}$ to infinite order near M_0 in the sense that if X is a section of $\mathcal{L}$, then G_*X differs from a section of $\mathcal{L}$ by a vector field which vanishes to infinite order on M_0.*

Now assume that $\dim_{\mathbb{R}} M = 3$ and let $\Omega = M \times (-1,1)$. By Theorem 2.1, we have an almost complex structure $\mathcal{L}^*$ that is integrable to infinite order along $M_0 = \{(x,0);\ x \in M\}$. Let η be a smooth non-vanishing one form on M that satisfies $\eta(L) = 0$ for all $L \in \mathcal{S}_x$ $x \in M$, and that defines the Levi form of M as in Section 1. We can clearly extend η to all of Ω so that it still annihilates $\mathcal{S}_{(x,t)}$ for all $(x,t) \in \Omega$, where $\mathcal{S}_{(x,t)}$ now denotes the space of vectors in $\mathcal{L}^*_{(x,t)}$ that are tangent to the level set of the auxiliary coordinate t.

Choose a smooth real vector field X_0 on Ω that satisfies $X_0 t \equiv 0$ and $\eta(X_0) \equiv 1$ in Ω. Set $Y_0 = -\mathcal{J}_{\mathcal{L}^*}(X_0)$ so that $X_0 + iY_0$ is a section of $\mathcal{L}^*$ that is transverse to the level set of t. Let $G\colon \Omega \longrightarrow \Omega$ be a diffeomorphism such that G fixes M_0 and

$$G_*Y_{0|_{(x,0)}} = \frac{\partial}{\partial t|_{(x,0)}},\ x \in M.$$

Since M is orientable, we may assume that $dt(\mathcal{J}_{\mathcal{L}^*}(X_0))$ is always negative. Thus $dt(Y_0) > 0$ along M_0, which shows that G preserves the sides of M_0; i.e., G maps $\Omega^+ = \{(x,t); 0 \le t < 1\}$ into itself. If we set $\mathcal{L}^0 = G_*\mathcal{L}^*$, then clearly $\widetilde{Z} = -iG_*(X_0 + iY_0)$ is a section of $\mathcal{L}^0$ such that along M_0,

$$\widetilde{Z} = -iX_0 + \frac{\partial}{\partial t}.$$

If we write $\widetilde{Z} = \widetilde{X} + g(x,t)\frac{\partial}{\partial t}$, where $\widetilde{X}t \equiv 0$, then we set $L_2 = g^{-1}\widetilde{Z}$. Then $L_2 = \frac{\partial}{\partial t} + X$ where $Xt \equiv 0$. We fix a smooth metric $\langle\ ,\ \rangle_0$ that is Hermitian with respect to the structure $\mathcal{L}^0$ on Ω, and let $\{L_1, L_2\}$ be an orthonormal frame defined in a neighborhood of $p \in M$. Note that along M, we have $L_2 = \frac{\partial}{\partial t} - iX_0$ and $dt = \frac{1}{2}(dt + i\eta) + \frac{1}{2}(dt - i\eta)$, which implies that $\partial t = \frac{1}{2}(dt + i\eta)$. Hence $\partial t(L) = \frac{1}{2}dt(L) + \frac{i}{2}\eta(L)$ and

$$\partial t([X_1, \overline{X}_2]) = \frac{i}{2}\eta([X_1, \overline{X}_2]) \tag{2.6}$$

for all $X_1, X_2 \in \mathcal{S}_{(x,t)}$, along M.

DEFINITION 2.3. We say $p \in \overline{M}$ is of finite type if there exist a list of vector fields $L^1, \dots, L^m$, with $L^i = L_1$ or $\overline{L}_1$, $i = 1, 2, \dots m$, so that $\partial t([L^m, [L^{m-1}, \dots [L^2, L^1] \dots]) \neq 0$ at p. The smallest integer m satisfying $\partial t([L^m, [L^{m-1}, \dots [L^2, L^1] \dots]) \neq 0$ is called the type at $p \in \overline{M}$.

It is obvious that this definition is an open condition. Observe also that, if $p \in M$ is of type m, then $L_1, \overline{L}_1, [L^m, \dots, [L^2, L^1] \dots]$ span all local vector fields tangent to M because $\partial t(L_1) \equiv 0$.

§3. Special Frames for Almost–Complex Structures

Let M, Ω, X_0, L_1, L_2 and $\mathcal{L}^0$ be as in Section 2. In this section, we will construct special coordinate functions defined in a neighborhood of $z_0 \in M$.

First, we note that $X_0 t \equiv 0$ on Ω and hence there is a neighborhood V_{z_0} of z_0 such that there exist coordinates (u_1, u_2, u_3, u_4) with the property that $u_4 = t$ and $u_k(u', t) = u_k(u', 0)$, $k < 4$ for $(u', t) \in V_{z_0}$, and that $\partial/\partial u_3 = -X_0$ at all points of $M \cap V_{z_0}$. For any point $x_0 \in V_{z_0} \cap M$, we define an affine transformation $C_{x_0}: \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ so that if $(x'_0, 0) \in \mathbb{R}^4$ are the coordinates of x_0, then

$$C_{x_0}(u', t) = (P_{x_0}(u' - x'_0), t),$$

where the 3×3 constant matrix P_{x_0} is chosen so that if new coordinates $x = (x_1, \dots, x_4)$ are defined by $x = C_{x_0}(u)$, then

$$L_{1|_{x_0}} = \frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}, \text{ and } X_{0|_{x_0}} = -\frac{\partial}{\partial x_3}. \tag{3.1}$$

Note that the second equality actually implies that $X_0 = -\frac{\partial}{\partial x_3}$ at all points of $V_{z_0} \cap M$ and that $L_2 = \frac{\partial}{\partial t} - i\frac{\partial}{\partial x_3}$ along $M \cap V_{z_0}$. We also note that the matrix P_{x_0} is uniquely determined by (3.1) and depends smoothly on $x_0 \in V_{z_0} \cap M$.

PROPOSITION 3.1. *For each $x_0 \in V_{z_0} \cap M$ and positive integer m, there are smooth coordinates $x = (x_1, x_2, x_3, x_4)$, $x(x_0) = 0$, defined near x_0 such that in x coordinates the vector field L_1 can be written as*

$$L_1 = \left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right) + \sum_{l=1}^{2} b_l(x)\frac{\partial}{\partial x_l} + (e(x) + ia(x))\frac{\partial}{\partial x_3}, \tag{3.2}$$

where $b_1(0) = b_2(0) = 0$, *and* $e(x)$, $a(x)$ *are real functions satisfying*

$$\frac{\partial^{j+k} e(x_0)}{\partial x_1^j \partial x_2^k} = 0,\ j+k \leq m,\ \textit{and}\ \frac{\partial^k a(x_0)}{\partial x_2^k} = 0,\ k \leq m. \tag{3.3}$$

Proof. Let us write the vector field L_1 in terms of the coordinate functions (x_1, x_2, x_3, t) satisfying (3.1):

$$\begin{aligned} L_1 = &\left(\frac{\partial}{\partial x_1} + \sum_{l=1}^{2} b_l^1(x)\frac{\partial}{\partial x_l} + e(x)\frac{\partial}{\partial x_3}\right) \\ &- i\left(\frac{\partial}{\partial x_2} + \sum_{l=1}^{2} b_l^2(x)\frac{\partial}{\partial x_l} + a(x)\frac{\partial}{\partial x_3}\right), \end{aligned} \tag{3.4}$$

where $e(x)$, $a(x)$ and b_l^i, $1 \leq i,\ l \leq 2$ are smooth real valued functions satisfying $e(0) = a(0) = b_l^i(0) = 0$. Therefore (3.3) holds for $j + k \leq 0$. By induction, assume that we have coordinate functions x_1, x_2, x_3 and t such that L_1 can be written as (3.4), where the coefficient functions $e(x)$ and $a(x)$ satisfy:

$$\frac{\partial^{j+k} e}{\partial x_1^j \partial x_2^k}(0) = 0,\ j+k \leq l-1,\ \text{and},\ \frac{\partial^k a}{\partial x_2^k}(0) = 0,\ k \leq l-1. \tag{3.5}$$

Set

$$\begin{aligned} &\widetilde{x}_1 = x_1,\ \widetilde{x}_2 = x_2,\ \text{and} \\ &\widetilde{x}_3 = x_3 - \sum_{j+k=l} \frac{1}{(j+1)!k!}\frac{\partial^l e(0)}{\partial x_1^j \partial x_2^k} x_1^{j+1} x_2^k. \end{aligned}$$

Then, in terms of $\widetilde{x}$-coordinates, L_1 can be written as:

$$\begin{aligned} L_1 = &\left(\frac{\partial}{\partial \widetilde{x}_1} + \sum_{l=1}^{2} \widetilde{b}_l^1(\widetilde{x})\frac{\partial}{\partial \widetilde{x}_l} + \widetilde{e}(\widetilde{x})\frac{\partial}{\partial \widetilde{x}_3}\right) \\ &+ i\left(\frac{\partial}{\partial \widetilde{x}_2} + \sum_{l=1}^{2} \widetilde{b}_l^2(\widetilde{x})\frac{\partial}{\partial \widetilde{x}_l} + \widetilde{a}(\widetilde{x})\frac{\partial}{\partial \widetilde{x}_3}\right), \end{aligned}$$

where

$$\frac{\partial^{j+k} \widetilde{e}}{\partial \widetilde{x}_1^j \partial \widetilde{x}_2^k}(0) = 0,\ 1 \leq j+k \leq l,\ \text{and}\ \frac{\partial^k \widetilde{a}}{\partial \widetilde{x}_2^k}(0) = 0,\ k \leq l-1.$$

We also perform another change of coordinates:

$$x_1 = \widetilde{x}_1,\ x_2 = \widetilde{x}_2,\ x_3 = \widetilde{x}_3 - \frac{1}{(l+1)!}\frac{\partial^l \widetilde{a}(0)}{\partial \widetilde{x}_2^l}\widetilde{x}_2^{l+1}.$$

Then, in terms of x-coordinates, L_1 can be written as in (3.4) satisfying (3.5) with $l-1$ replaced by l. If we proceed up to m steps, we will have coordinate functions (x_1, x_2, x_3, t) defined near $x_0 \in M \cap V_{z_0}$ satisfying (3.2) and (3.3). □

We first construct continuously varying non-isotrophic balls that are defined invariantly. Let $\{\chi_\nu\}_{\nu\in I}$ be a partition of unity subordinated to the coordinate neighborhoods $\{U_\nu\}_{\nu\in I}$ of Ω. Let m be a given positive integer. Let us fix $\delta > 0$ for a moment. For any j, k with $j > 0$, define

$$\begin{aligned}
\mathcal{L}^\nu_{j,k}\partial\overline{\partial}\eta(x) &= \frac{i}{2}L_1^{j-1}\overline{L}_1^k\eta([L_1, \overline{L}_1])(x),\ x \in U_\nu, \\
C_l^\nu(x) &= \sum_{j+k=l} |\mathcal{L}^\nu_{j,k}\partial\overline{\partial}\eta(x)|^2,\ l = 1, \ldots, m,\ \text{and}, \\
C_l(x) &= \sum_{\nu\in I}\chi_\nu C_l^\mu(x).
\end{aligned}$$

Set $M = (m+1)!$ and define

$$\mu(x,\delta) = \left(\sum_{l=1}^m C_l^{M/l+1}(x)\delta^{-2M/l+1}\right)^{-1/2M}. \tag{3.6}$$

By (2.6) and Proposition 2.4 it follows that $\sum_{l=1}^m C_l(x) > 0$ if the type at x is less than or equal to m. Therefore $\mu(x,\delta)$ is defined intrinsically and it is a smooth function of $\delta > 0$ and x for x satisfying $\sum_{l=1}^m C_l(x) > 0$.

We want to define another quantity, $\tau(x_0, \delta)$, related to the coordinate functions defined in Proposition 3.1. Let $x_0 \in M$ be a point whose type is less than or equal to m. Let us take the coordinate functions $x = (x_1, x_2, x_3, t)$ defined near x_0 where the vector field L_1 has the representation as in (3.2), where the coefficient functions $e(x)$ and $a(x)$ of $\partial/\partial x_3$ satisfy the estimates in (3.3).

Set

$$\widetilde{a}(x) := \frac{\partial}{\partial x_1}a(x) = \text{Re}\left[\frac{\partial}{\partial z_1}a(x)\right],$$

and set $z_1 = \frac{1}{2}(x_1 - ix_2)$ and $z_2 = \frac{1}{2}(t - ix_3)$. Since $a(x_0) = 0$, the Taylor expansion of $\widetilde{a}(x)$ at x_0 has the expression (in terms of (z_1, z_2)-coordinates) as:

$$\widetilde{a}(x) = \sum_{0 \le j+k \le m-1} \widetilde{a}_{jk}(x_0) z_1^j \overline{z}_1^k + \mathcal{O}(|z_1|^m + |z_2|\,|z|),\ z = (z_1, z_2).$$

Now set

$$A_l(x_0) = \max\{|\widetilde{a}_{jk}(x_0)|;\ j + k = l\},\ l = 0, 1, \ldots, m-1,$$

and set

$$\tau(x_0, \delta) = \min_{0 \le l \le m-1} \{(\delta / A_l(x_0))^{1/l+2}\}. \tag{3.7}$$

Assuming that the type at x_0 is less than or equal to m, it follows that $a_{jk}(x_0) \neq 0$ for some $j + k = l \le m - 1$ and hence $\tau(x_0, \delta)$ is well defined. It also satisfies the estimate:

$$\delta^{1/2} \lesssim \tau(x_0, \delta) \lesssim \delta^{1/m+1}.$$

Let us consider the following balls defined in terms of $\tau(x_0, \delta)$:

$$Q_\delta(x_0) = \{(x_1, x_2, x_3, t) \colon |x_1|, |x_2| < \tau(x_0, \delta), |x_3|, |t| < \delta\}.$$

We want to study the relations between $\tau(x_0, \delta)$ and $\mu(x, \delta)$ for $x \in Q_\delta(x_0)$, where $\mu(x, \delta)$ is defined as in (3.6). Set $D_1 = \partial / \partial z_1$ for a convenience. If we combine the definition of $\tau(x_0, \delta)$ and the fact that $\eta(L_1) \equiv 0$, we obtain by induction that

$$|D_1^j \overline{D}_1^k \eta(\frac{\partial}{\partial x_i})(x_0)| \lesssim \delta \tau(x_0, \delta)^{-(j+k+1)},\ \text{for} j + k \le m,\ i = 1, 2. \tag{3.8}$$

Note that $\eta([L_1, \overline{L}_1])$ can be written as

$$\eta([L_1, \overline{L}_1]) = (-2i \operatorname{Re}[\frac{\partial}{\partial z_1} a]) \eta(\frac{\partial}{\partial x_3}) + R_0, \tag{3.9}$$

where R_0 satisfies, from the estimates in (3.3) and (3.8) that,

$$|D_1^j \overline{D}_1^k R_0(x_0)| \lesssim \delta \tau(x_0, \delta)^{-(j+k+1)},\ j + k + 1 \le m. \tag{3.10}$$

Combining (3.7)–(3.10), we get:

$$|D_1^j \overline{D}_1^k \eta([L_1, \overline{L}_1])(x_0)| \lesssim \delta \tau(x_0, \delta)^{-(j+k+2)},\ j + k + 2 \le m.$$

Similarly, by applying L_1 or $\overline{L}_1$ to $\eta([L_1, \overline{L}_1])$ sucessively, we obtain by induction that

$$\mathcal{L}_{j,k}\eta(x) = D_1^{j-1}\overline{D}_1^k \left[\mathrm{Re}(D_1 a)\eta(\frac{\partial}{\partial x_3})\right] + E_{j+k-1}, \tag{3.11}$$

where E_{j+k-1} satisfies

$$|D_1^s \overline{D}_1^t E_{j+k-1}(x_0)| \lesssim \delta\tau(x_0,\delta)^{-(j+k+s+t)}, \; j+k+s+t \leq m. \tag{3.12}$$

Therefore for any j, k, s, t with $j+k+s+t \leq m$, it follows from (3.11) that

$$|D_1^s \overline{D}_1^t \mathcal{L}_{j,k}\eta(x_0)| \lesssim \delta\tau(x_0,\delta)^{-(s+t+j+k+1)}. \tag{3.13}$$

If we use the Taylor series method and the estimates in (3.13), we obtain that

$$|\mathcal{L}_{j,k}\eta(x)| \lesssim \delta\tau(x_0,\delta)^{-(j+k+1)}, \; x \in Q_\delta(x_0).$$

Since this implies that

$$C_l(x) \lesssim \delta^2\tau(x_0,\delta)^{-2(l+1)}, \; x \in Q_\delta(x_0), \; l \leq m,$$

we conclude from the definition of $\mu(x,\delta)$ in (3.6) that

$$\tau(x_0,\delta) \lesssim \mu(x,\delta) \text{ when } x \in Q_\delta(x_0). \tag{3.14}$$

Conversely, let us prove that $\mu(x,\delta) \lesssim \tau(x_0,\delta)$. Define

$$T(x_0,\delta) = \min\{l : (\delta/A_l(x_0))^{1/l+2} = \tau(x_0,\delta)\}. \tag{3.15}$$

By the definition of $\tau(x_0,\delta)$ and $T(x_0,\delta)$, there must exist integers j, k with $(j-1)+k = T(x_0,\delta)$, $j \geq 1$, so that

$$|\widetilde{a}_{j-1,k}(x_0)| = |\frac{1}{(j-1)!k!}D_1^{j-1}\overline{D}_1^k \left[\mathrm{Re}\, D_1 a\right](x_0)| = \delta\tau(x_0,\delta)^{-j-k-1}.$$

If we apply the estimates in (3.12) and (3.13) with $s+t=0$ and the fact that $\tau(x_0,\delta) \ll 1$ if δ is small, it follows that

$$|\mathcal{L}_{j,k}\eta(x_0)| \geq \frac{1}{2} j!k!\delta\tau(x_0,\delta)^{-j-k-1}.$$

Then, again by using the estimates in (3.13) and the Taylor series method, we obtain that

$$|\mathcal{L}_{j,k}\eta(x)| \approx \delta\tau(x_0,\delta)^{-j-k-1},$$

which implies that

$$\mu(x,\delta) \lesssim \tau(x_0,\delta), \ x \in Q_\delta(x_0). \tag{3.16}$$

If we combine (3.14) and (3.16), we have proved the following proposition.

PROPOSITION 3.2. *If* $x \in Q_\delta(x_0)$, *then*

$$\tau(x_0,\delta) \approx \mu(x,\delta). \tag{3.17}$$

COROLLARY 3.3. *Suppose* $x \in Q_\delta(x_0)$. *Then*

$$\tau(x_0,\delta) \approx \tau(x,\delta).$$

Proof. If we set $x = x_0$ in (3.17), then we see that $\mu(x_0,\delta) \approx \tau(x_0,\delta)$. Since this holds for $x_0 = x$, it follows that $\mu(x,\delta) \approx \tau(x,\delta)$. Hence we have $\tau(x_0,\delta) \approx \tau(x,\delta)$. □

Remark 3.4. $\mu(x,\delta)$ is defined intrinsically, that is, independent of coordinate functions. Therefore, Proposition 3.2 shows that the quantity $\tau(x_0,\delta)$ is defined invariantly, up to a universal constant, with respect to coordinate functions.

Assume $\overline{M} \subset \widetilde{M}$ and let $\varphi \in C^\infty(\overline{M})$ be a smooth real-valued function such that $\varphi(x) > 0$ for $x \in M$, and $\varphi(x) = 0$, $d\varphi(x) \neq 0$ for $x \in bM$. We can extend φ to Ω by requiring that it be independent of t. Let us denote by T_p the type at a point $p \in \overline{M}$ and define

$$T(\overline{M}) = \max\{T_p;\ p \in \overline{M}\}.$$

Since type condition is an open condition, we see that $T(\overline{M})$ is well defined and is finite. In the sequal, we assume that $T(\overline{M}) = m < \infty$. We define $r \in C^\infty(\Omega)$ by $r(x,t) = t(\phi(x))^{-2m}$ and for any ε, σ, $0 < \varepsilon \le \sigma \le 1$, we define

$$S_{\varepsilon,\sigma} = \{(x,t) \in \emptyset;\ \varphi(x) > 0 \text{ and } 0 \le r(x,t) \le \varepsilon\sigma^{3\cdot 2^{m-1}}\}.$$

The quantities ϵ and σ will be fixed later. If we set $g(x) = \epsilon \cdot \sigma^{3\cdot 2^{m-1}} \cdot \varphi(x)^{2m}$, then $S_{\varepsilon,\sigma}$ will be the required manifold S_g^+ of Section 1. We define a subbundle of $\mathcal{L}^0$ on $S_{\varepsilon,\sigma}$ by letting $\mathcal{R}_{(x,t)} = \{L \in \mathcal{L}^0_{(x,t)};\ Lr = 0\}$. Clearly the map H defined by $H(L) = L - (Lr)(L_2 r)^{-1} L_2$ defines an isomorphism of $\mathcal{S}$ onto $\mathcal{R}$ (at all points of $S_{\varepsilon,\sigma}$). We define a weighted metric $\langle\ ,\ \rangle$ on $\mathcal{L}^0$ by the relations

$$\begin{aligned}&\langle H(L_1), H(L_1)\rangle = \mu(z, \varepsilon\phi(z)^{2m})^{-2}\langle L_1, L_1\rangle_0,\\&\langle L_2, L_2\rangle = \varepsilon^{-2}\phi(z)^{-4m}, \text{ and}\\&\langle L_2, H(L_1)\rangle = 0,\end{aligned}$$

where $L_1 \in \mathcal{S}$. Since $\mu(x,\delta)$ is a smooth function of x and δ, it follows that $\langle\ ,\ \rangle$ is a smooth Hermitian metric on $\mathcal{L}^0$.

We now show how $S_{\varepsilon,\sigma}$ can be covered by special coordinate neighborhoods such that on each such neighborhood there is a frame $\mathcal{L}$ that satisfies good estimates:

PROPOSITION 3.5. *There exist constants ε_0 and σ_0 such that if $0 < \varepsilon < \varepsilon_0$ and $0 < \sigma < \sigma_0$, then on $S_{\varepsilon,\sigma}$ there exist for all $x_0 \in M$ with $\varphi(x_0) > 0$ a neighborhood $W(x_0) \subset S_{\varepsilon,\sigma}$ with the following properties:*

(i) *On $W(x_0)$ there are smooth coordinates $y_1, \ldots, y_4$ so that $W(x_0) = \{y;\ |y'| < \sigma, 0 \le y_4 \le \sigma^{3\cdot 2^{m-1}}\}$, where $y' = (y_1, y_2, y_3)$ is independent of t and where the function y_4 is defined by $y_4 = \varepsilon^{-1}\varphi(x)^{-2m}t$. Thus, $M_0 \cap W(x_0)$ and $M_\sigma \cap W(x_0)$ correspond to the points in $W(x_0)$ where $y_4 = 0$ and $\sigma^{3\cdot 2^{m-1}}$, respectively. Moreover, the point $(x_0, 0) \in \Omega$ (which we identify with x_0) corresponds to the origin.*

(ii) *The above coordinate charts are uniformly smoothly related in the sense that if $W(\widetilde{p}_0)$ and $W(x_0)$ intersect, and if $\widetilde{y}$ and y_0 are the associated coordinates, then*

$$|D^\alpha(\widetilde{y} \circ (y_0)^{-1})| \le C_{|\alpha|} \tag{3.18}$$

holds on that portion of $\mathbb{R}^4$ where $\widetilde{y} \circ (y_0)^{-1}$ is defined. The constant $C_{|\alpha|}$ is independent of $\widetilde{p}_0$ and x_0.

(iii) *On $W(x_0)$, there exists a smooth frame L_1, L_2 for $\mathcal{L}$ such that if ω^1, ω^2 is the dual frame, and if L_k and ω^k are written as $\sum_{j=1}^4 b_{kj}\frac{\partial}{\partial y_j}$ and $\sum_{j=1}^4 d_{kj}dy_j$, then*

$$\sup_{y\in W(x_0)} \left\{|D_y^\alpha b_{kj}(y)| + |D_y^\alpha d_{kj}(y)|\right\} \lesssim C_{|\alpha|},$$

where $C_{|\alpha|}$ is independent of x_0, j, k.

(iv) *There are independent constants $c > 0$ and $C > 0$ such that if $B_b(x)$ denotes the ball of radius b about $x \in S_{\varepsilon,\sigma}$ with respect to the metric $\langle\ ,\ \rangle$, then*

$$B_{c\sigma}(x_0) \subset W(x_0) \subset B_{C\sigma}(x_0), \tag{3.19}$$

and if $\operatorname{Vol} B_b(x_0)$ *denotes the volume of* $B_b(x_0)$ *with respect to* $\langle\ ,\ \rangle$, *then*

$$cb^3\sigma^{3\cdot 2^{m-1}} \leq \operatorname{Vol} B_b(x_0) \leq Cb^3\sigma^{3\cdot 2^{m-1}}. \tag{3.20}$$

Proof. We first cover $\overline{M}$ by a finite number of neighborhoods V_ν, $\nu = 1, \ldots, N$, in Ω such that in each V_ν there exist coordinates $(u_1, \ldots, u_4)$ with the property that $u_4 = t$ and that $u_k(u', t) = u_k(u', 0)$, $k < 4$, for $(u', t) \in V_\nu$, and that $\frac{\partial}{\partial u_3} = -X_0$ at all points of $M \cap V_\nu$.

For any point $x_0 \in M \cap V_\nu$, we take coordinate functions $x = (x_1, \ldots, x_4)$ constructed as in Proposition 3.1. In terms of x-coordinates, L_1^ν and L_2^ν can be written as:

$$L_1^\nu = \frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2} + \sum_{l=1}^{3} a_l(x)\frac{\partial}{\partial x_l}, \text{ and} \tag{3.21}$$

$$L_2^\nu = \frac{\partial}{\partial t} - i\frac{\partial}{\partial x_3} + \sum_{l=1}^{3} b_l(x)\frac{\partial}{\partial x_l},$$

where $a_3(x) = e(x) + ia(x)$, and where $e(x)$, $a(x)$ satisfy estimates in (3.3). Set $z_1 = 1/2(x_1 - ix_2)$ and $z_2 = 1/2(t - ix_3)$. Since $a_3(x_0) = 0$, the Taylor expansion of $a_3(x)$ at x_0 has the expression:

$$a_3(x) = \sum_{1 \leq j+k \leq m} a_{jk}(x_0) z_1^j \bar{z}_1^k + \mathcal{O}(|z_1|^{m+1} + |z_2|\,|z|). \tag{3.22}$$

Set $\delta = \varepsilon\phi(x_0)^{2m}$, and set

$$T_m(x) = \sum_{1 \leq j+k \leq m} a_{jk}(x_0) z_1^j \bar{z}_1^k = \widetilde{T}_m(x_1, x_2, 0, 0)$$

for a convenience. We take the quantity $\mu(x_0, \delta)$ and the corresponding quantity $\tau(x_0, \delta)$, for the function $a_3(x)$ (or $\widetilde{a}(x) = \partial/\partial x_1 a(x)$), as defined in (3.6) and (3.7). By virtue of Proposition 3.1, and by the definition of $\tau(x_0, \delta)$, it follows that $|a_{jk}(x_0)| \leq \delta\tau(x_0, \delta)^{-j-k-1}$, $j + k \leq m$, and hence Proposition 3.2 implies that

$$|a_{jk}(x_0)| \lesssim \delta\mu(x_0, \delta)^{-j-k-1}. \tag{3.23}$$

We define new coordinates $y = (y_1, \dots, y_4)$ by means of dilation map $D_{\varepsilon,x_0}: \mathbb{R}^4 \longrightarrow \mathbb{R}^4$:

$$\begin{aligned} y &= D_{\varepsilon,x_0}(x) \\ &= (\mu(x_0,\delta)^{-1}x_1, \mu(x_0,\delta)^{-1}x_2, \varepsilon^{-1}\varphi(x_0)^{-2m}x_3, \varepsilon^{-1}\varphi(x)^{-2m}x_4), \end{aligned}$$

where $\varphi(x)$ is the function φ expressed in the x-coordinates of x_0. In terms of the y-coordinates, we define an open set $W_b(x_0)$ by

$$W_b(x_0) = \{x \in V_\nu \cap S_{\varepsilon,\sigma};\ |y_k(x)| < b,\ k = 1,2,3,\ 0 \le y_4(x) \le \sigma^{3\cdot 2^{m-1}}\}.$$

Note that in $W_b(x_0)$, $y_4 = 0$ and $y_4 = \sigma^{3\cdot 2^{m-1}}$ coincide with $r = 0$ and $r = \varepsilon\sigma^{3\cdot 2^{m-1}}$, respectively, the boundaries of $S_{\varepsilon,\sigma}$. We define a frame L_1, L_2 in $W_b(x_0)$ by setting

$$\text{(3.24)} \qquad \begin{aligned} &L_1 = \mu(x,\delta)(L_1^\nu - dL_2^\nu) = \mu(x,\delta)H(L_1^\nu), \quad \text{and} \\ &L_2 = \varepsilon\varphi(x)^{2m}L_2^\nu, \end{aligned}$$

where $d = (L_1^\nu r)(L_2^\nu r)^{-1}$. Assuming that L_1^ν and L_2^ν have the expressions as in (3.21) in V_ν, we set $A_l(y) = a_l \circ D_{\varepsilon,x_0}^{-1}(y)$, $D(y) = d \circ D_{\varepsilon,x_0}^{-1}$, $\Phi = \phi \circ D_{\varepsilon,x_0}^{-1}$, $B_l(y) = b_l \circ D_{\varepsilon,x_0}^{-1}(y)$, and $\Phi_l = \frac{\partial\varphi}{\partial_l} \circ D_{\varepsilon,x_0}^{-1}$. Then we conclude that in the y-coordinate of $W_b(x_0)$,

$$\text{(3.25)} \qquad \begin{aligned} L_1 = {} & \frac{\mu(x,\delta)}{\mu(x_0,\delta)}\left[\frac{\partial}{\partial y_1} - i\frac{\partial}{\partial y_2} + \sum_{l=1}^{2}(A_l - DB_l)\frac{\partial}{\partial y_l}\right] \\ & + \mu(x,\delta)\delta^{-1}\left(A_3 - D(B_3 - i)\right)\frac{\partial}{\partial y_3}. \end{aligned}$$

Observe that since the diameter in the x-coordinates of $W_b(x_0)$ is $\mathcal{O}(b\mu(x_0,\delta)) \ll \varphi(x_0)$, it is clear that $\mu(x,\delta)\mu(x_0,\delta)^{-1}$ and $\Phi\varphi(x_0)^{-1}$ are very close to 1 in $W_b(x_0)$ if b is small. We set

$$|f|_{m,W_b(x_0)} = \sup\{|D_y^\alpha f(y)|;\ y \in W_b(x_0), |\alpha| \le m\},$$

and we extend this norm to vector fields and 1-forms by using the coefficients of $\frac{\partial}{\partial y_j}$ or dy_j. From the expression of $a_3(x)$ in (3.22) and by virtue of the estimates in (3.3) and (3.23), it follows that

$$\lim_{\sigma\to 0} |\delta^{-1}\mu(x,\delta)A_3(y) - T_m(y)|_{k,W_b(x_0)} = 0,$$

when $b \leq \sqrt{\sigma}$. Similarly, by direct calculation, one obtains that

$$D = \frac{-2\varepsilon m y_4 \Phi^{2m-1}(\Phi_1 + i\Phi_2 + \sum_{l=1}^{3} A_l \Phi_l)}{1 + 2i\varepsilon m \Phi^{2m-1}\Phi_3 y_4 - \sum_{l=1}^{3} 2\varepsilon m \Phi^{2m-1}\Phi_l y_4}. \tag{3.26}$$

Note that $\mu(x,\delta) \approx \tau(x_0,\delta) \lesssim \varepsilon^{1/m+1}\varphi(x_0)^{2m/m+1} \ll \varphi(x_0)$, and hence it follows that

$$\lim_{\sigma\to 0} |\delta^{-1}\mu(x,\delta)D|_{k,W_b(x_0)} = 0.$$

Combining all these facts, we conclude that if $b \leq \sqrt{\sigma}$,

$$\lim_{\sigma\to 0}\left| L_1 - \left(\frac{\partial}{\partial y_1} - i\frac{\partial}{\partial y_2} + T_m(y)\frac{\partial}{\partial y_3}\right)\right|_{k,W_b(x_0)} = 0, \tag{3.27}$$

where $T_m(y) = \widetilde{T}_m(y_1, y_2, 0, 0)$, and that

$$\lim_{\sigma\to 0}\left| L_2 - \left(-i\frac{\partial}{\partial y_3} + \frac{\partial}{\partial y_4}\right)\right|_{k,W_b(x_0)} = 0.$$

Setting $W(x_0) = W_\sigma(x_0)$, for sufficiently small σ, we obtain (i) and (iii). By Proposition 3.2, it follows that $\tau(x_0,\delta) \approx \mu(x,\delta)$ for $x \in W(x_0)$. Since L_1, L_2 is orthonormal with respect to $\langle\ ,\ \rangle$, we conclude that if σ is small, then (3.19) and (3.20) hold.

To prove (3.18), we note that $\tau(x_0,\delta) \approx \tau(x,\delta)$ if $x \in W(x_0)$ and that $\tau(x_0,\delta)$ is defined independent (up to a universal constant) with respect to coordinate functions (Remark 3.4). These two facts give us (3.18). □

We need the following proposition to prove the subelliptic estimates for $\overline{\partial}$ equation in dilated coordinates y. We take the orthonormal frame $\{L_1, L_2\}$ and its dual frame $\{\omega^1, \omega^2\}$.

Proposition 3.6. *There exist a constant $c_0 > 0$, independent of x_0, and a list of vector fields $\{L^s, L^{s-1}, \ldots, L^1\}$, where $L^j = L_1$ or $\overline{L}_1$, $1 \leq j \leq s$, $s \leq m$, such that*

$$|\omega^2([L^s,[L^{s-1},\ldots,[L^2,L^1]\ldots])(x)| \geq 2c_0, \tag{3.28}$$

for all $x \in W(x_0)$.

Proof. Set $L^0 = L_1$ and $L^1 = \overline{L}_1$. For $(i_1 \cdots i_s)$ of an s-tuple of 0's and 1's, we define inductively by $L^{(i_1 \cdots i_s)} = [L^{i_s}, L^{(i_1 \cdots i_{s-1})}]$ and set

$$\lambda^{i_1 \cdots i_s}(y) = \omega^2(L^{(i_1 \cdots i_s)}), \text{ and} \qquad (3.29)$$
$$\mathcal{L}_{j,k}\omega^2(y) = L_1^{j-1}\overline{L}_1^k \omega^2([L_1, \overline{L}_1])(y).$$

Let I_2 be the ideal generated by $\lambda^{10} = \omega^2([L_1, \overline{L}_1])$, and I_s be the ideal generated by I_{s-1} and both $\lambda^{10\cdots i_s}$. By induction, it is not hard to show (see [8, 10]) that

$$\lambda^{10\cdots i_s}(y) = \mathcal{L}_{j,k}\omega^2(y), \text{ mod } I_{s-1}, \qquad (3.30)$$

where j is the number of 0's in $(10 \cdots i_s)$.

Set $\eta_\delta = \delta^{-1}\eta$ and set $w_1 = 1/2(y_1 - iy_2)$, $w_2 = 1/2(y_4 - iy_3)$, $D_k = \partial/\partial w_k$, $k = 1, 2$. Then it follows that $\eta_\delta(\partial/\partial y_3) \equiv 1$ along $M \cap V_\nu$, and $\mathcal{L}_{j,k}\omega^2(y) = i/2\mathcal{L}_{j,k}\eta_\delta(y)$. From the estimates in (3.8), we have:

$$|D_1^j \overline{D}_1^k \eta_\delta(\partial/\partial y_i)(x_0)| \leq C_{j,k}, \; i = 1, 2, \qquad (3.31)$$

for some constants $C_{j,k}$, independent of x_0. Note that L_1 has the representation as in (3.25). Therefore, as in the proof of Proposition 3.2, it follows that

$$\mathcal{L}_{j,k}\omega^2(y) = -\delta^{-1}\mu(x_0, \delta)\left[D_1^{j-1}\overline{D}_1^k(\operatorname{Im} D_1 \overline{A}_3)\eta_\delta(\partial/\partial y_3)\right] + E_{j+k-1},$$

where E_{j+k-1} satisfy, by virtue of (3.26) and (3.31), that

$$|D_1^s \overline{D}_1^t E_{j+k-1}(x_0)| \lesssim \mu(x_0, \delta), \; j + k + s + t \leq m. \qquad (3.32)$$

Note that we may write $A_3(y) = E(y) + iA(y)$, where $E(y)$ satisfies the estimates as in (3.32). If we combine the definition of $\tau(x_0, \delta)$ and the fact that $\tau(x_0, \delta) \approx \mu(x_0, \delta)$, it follows that there exist a constant $c_1 > 0$ and integers j_1, k_1, $(j_1 - 1) + k_1 = T(x_0, \delta)$, such that

$$|\delta^{-1}\mu(x_0, \delta) D_1^{j_1-1}\overline{D}_1^{k_1}(\operatorname{Im} D_1 \overline{A}_3)(x_0)| \geq 3c_1. \qquad (3.33)$$

Here $T(x_0, \delta)$ is defined as in (3.15). Combining (3.32) and (3.33), we get:

$$|\mathcal{L}_{j_1,k_1}\omega^2(0)| \geq 2c_1, \qquad (3.34)$$

provided that δ is sufficiently small. Set $j_1 + k_1 = T_1$ and assume that $g_1 \in I_{T_1-1}$. Then by virtue of (3.29) and (3.30), we can write

$$g_1 = \sum_{p=1}^{T_1-1} \sum_{j+k=p} f^p_{j,k}\mathcal{L}_{j,k}\omega^2(y), \tag{3.35}$$

where $f^p_{j,k}$'s are bounded (by $M > 0$) independent of x_0. If for all $j+k < T_1$,

$$|\mathcal{L}_{j,k}\omega^2(0)| < \frac{c_1}{sup_{W(x_0)}|f^p_{j,k}| \cdot 2^{T_1}},$$

then by (3.30), it follows that

$$|\lambda^{10\cdots i_s}(0)| \geq c_1,$$

for some list $\{L^s, L^{s-1}, \ldots, L^1\}$ of L_1 or $\overline{L}_1$. If not, then there exist j_2, k_2 with $j_2 + k_2 = T_2 < T_1$ such that

$$|\mathcal{L}_{j_2,k_2}\omega^2(0)| \geq \frac{c_1}{\sup_{W(x_0)} |f^p_{j_2,k_2}| \cdot 2^{T_1}} = 3c_2.$$

For $g_2 \in I_{T_2}$, we represent g_2 as in (3.35) and proceed as above with c_1, T_1 replaced by c_2 and T_2 respectively. Note that if we iterate down to 1, then the required inequality vacuously holds. Therefore there exist a constant $c_0 > 0$, independent of x_0, and a list $\{L^s, \ldots, L^1\}$ of L_1 and $\overline{L}_1$ such that

$$|\omega^2([L^s, [L^{s-1}, \ldots, [L^2, L^1] \ldots])(x_0)| \geq 3c_0.$$

Now, by a simple Taylor's theorem argument, it follows that (3.28) holds for all $x \in W_\sigma(x_0)$ provided that σ is sufficiently small. □

Using the special frames constructed above, we now want to define L^2-operators with mixed boundary conditions. We first define nearby almost complex structures in terms of these special frames. We define a norm $|A|_{k,W(x_0)}$ for the restriction of A to $W(x_0)$ by writing $A = \sum_{j,l=1}^2 A_{jl}\overline{\omega}^l \otimes L_j$ and then by defining

$$|A(y)|_k = \sum_{|\alpha|\leq k} \sum_{j,l=1}^{2} |D_y^\alpha A_{jl}(y)|,$$

and $|A|_{k,W(x_0)} = \sup\{|A(y)|_k;\ y \in W(x_0)\}$. It is obvious that there exists $\varepsilon_0 > 0$ so that if $|A|_{0,W(x_0)} < \varepsilon_0$, then we can define an almost-complex structure in $W(x_0)$ by

$$\overline{\mathcal{L}}^A = \{\overline{L} + A(\overline{L});\ \overline{L} \in \mathcal{L}_z^0,\ z \in S_{\varepsilon,\sigma}\}.$$

In terms of the frame L_1, L_2, ω^1, ω^2 in $W(x_0)$, we define

$$X_j^A = L_j + \overline{A}(L_j),\ j = 1, 2,$$

and let η_A^l be the dual frame. Set

$$\begin{aligned} L_1^A &= X_1^A - (X_1^A r)(X_2^A r)^{-1} X_2^A,\ L_2^A = X_2^A, \text{ and} \\ \omega_A^1 &= \eta_A^1,\ \omega_A^2 = \left(\eta_A^2 + (X_1^A r)(X_2^A r)^{-1}\eta_A^1\right). \end{aligned} \tag{3.36}$$

Obviously, the frame ω_A^l, $l = 1, 2$, is dual to L_j^A, $j = 1, 2$, and $L_1^A r \equiv 0$. If we set

$$h^A = (X_1^A r)(X_2^A r)^{-1} = (X_1^A y_4)(X_2^A y_4)^{-1} = \frac{\overline{A}_{21}(\overline{L}_2 y_4)}{L_2 y_4 + \overline{A}_{22}(\overline{L}_2 y_4)},$$

then it follows that

$$\begin{aligned} L_1^A &= L_1 - h^A L_2 + \sum_{j=1}^{2} (\overline{A}_{j1} - h^A \overline{A}_{j2}) \overline{L}_j, \text{ and,} \\ L_2^A &= L_2 + \sum_{j=1}^{2} \overline{A}_{j2} \overline{L}_j. \end{aligned} \tag{3.37}$$

In order to measure how L_j^A, $j = 1, 2$ depend on A, we define

$$P_k(y; A) = \sum_{\substack{k_1,\ldots,k_N \\ |k_1|+\cdots+|k_N| \le k}} \prod_{\nu=1}^{N} |A(y)|_{k_\nu}. \tag{3.38}$$

LEMMA 3.7. *If A satisfies $|A|_{0,W(x_0)} < \varepsilon_0$ for sufficiently small ε_0, then the following pointwise estimates hold for $y \in W(x_0)$:*

$$|L_j^A - L_j|_k \le C_k P_k(A; y), \text{ and} \tag{3.39}$$

$$|\omega_A^l - \omega^l|_k \le C_k P_k(A; y),\ j, l = 1, 2. \tag{3.40}$$

Proof. If we differentiate the expressions in (3.37), then we obtain sums of terms, each of which contains a finite product of derivatives of A, as in (3.38). Hence we get (3.39). Similarly, we can get (3.40). □

Suppose that A satisfies

$$|A|_{m+5,W(x_0)} \leq \varepsilon_0. \tag{3.41}$$

Then it is clear that there is an independent constant $C > 0$ such that

$$|L_j^A|_{m+5,W(x_0)} \leq C, \ |\omega_A^l|_{m+5,W(x_0)} \leq C, \ j, l = 1, 2.$$

In terms of L_1^A, L_2^A, and ω^1, ω^2 frame, we define inductively by

$$L_A^{(i_1 \cdots i_s)} = [L_A^{i_s}, L_A^{(i_1 \cdots i_{s-1})}], \text{ and, } \lambda_A^{i_1 \cdots i_s}(y) = \omega_A^2(L_A^{(i_1 \cdots i_s)}),$$

where $L_A^0 = L_A$, $L_A^1 = \overline{L}_A$. Using Proposition 3.6 and Lemma 3.7, we prove the following proposition which is crucial in proving subelliptic estimates in Section 4.

PROPOSITION 3.8. *Assume that* (3.41) *holds for sufficiently small* $\varepsilon_0 > 0$. *Then there exist a constant* $c_0 > 0$, *independent of* x_0, *and* $T = T(x_0)$, $2 \leq T \leq m$, *such that for some* $j + k = T$ *we have:*

$$|\lambda_A^{10 \cdots i_T}(y)| \geq c_0, \ y \in W(x_0). \tag{3.42}$$

Proof. By Lemma 3.7, it follows that we can write, for each $s \geq 1$, as:

$$|\lambda_A^{i_1 \cdots i_s}(y) - \lambda^{i_1 \cdots i_s}(y)| \leq C_s P_s(y; A),$$

where $\lambda^{i_1 \cdots i_s}(y)$ is defined as in (3.29). From Proposition 3.6, there is $T = T(x_0)$, $2 \leq T \leq m$, such that $|\lambda^{10 \cdots i_T}(y)| \geq 2c_0$ for all $y \in W(x_0)$. Hence (3.42) follows provided that $\varepsilon_0 > 0$ is sufficiently small. □

Next, we show that there exists a smooth Hermitian metric on $S_{\varepsilon,\sigma}$ such that for all $x_0 \in M$ the frame L_1^A, L_2^A given by (3.24) is orthonormal. For $L \in \mathcal{L}^0$ and $A \in \Gamma^{0,1}(S_{\varepsilon,\sigma})$ satisfying (3.31), define a bundle isomorphism $P_A: \mathcal{L}^0 \to \mathcal{L}^A$ by $P_A(L) = L + A(L)$. Define a homomorphism $H_A: \mathcal{L}^A \to \mathcal{R}^A$, where $\mathcal{R}^A = \{L \in \mathcal{L}^A;\ Lr = 0\}$, by

$$H_A(L) = L - \frac{Lr}{X_2^A r} X_2^A = L - \frac{Ly_4}{L_2^A y_4} L_2^A.$$

Then $H_A \circ P_A$ is an isomorphism of $\mathcal{R}$ onto $\mathcal{R}^A$. We define a metric $\langle\ ,\ \rangle_A$ on $\mathcal{L}^A$ by

$$\begin{aligned}&\langle (H_A \circ P_A)\overline{L}_1, (H_A \circ P_A)\overline{L}_1\rangle_A = \langle \overline{L}_1, \overline{L}_1\rangle,\ \overline{L}_1, \in \mathcal{R},\\ &\langle L_2^A, L_2^A\rangle_A = 1, \text{ and}\\ &\langle (H_A \circ P_A)\overline{L}_1, L_2^A\rangle_A = 0,\ \overline{L}_1 \in \mathcal{R}.\end{aligned}$$

Note that L_2^A is actually globally defined, so that the above conditions determine a metric on $\mathcal{L}^A$. Since L_j, $j = 1, 2$, defined in (3.24) are an orthonormal basis of $\mathcal{L}$, it follows that L_j^A, $j = 1, 2$ are an orthonormal basis of $\mathcal{L}^A$ with respect to $\langle\ ,\ \rangle_A$.

Let dV denote the volume form associated with the Riemannian metric $\langle\ ,\ \rangle$. In the coordinates $(y_1, \ldots, y_4)$ in $W(x_0)$, we can write $dV = V(y)dy$, where $dy = dy_1 \cdots dy_4$, and where V satisfies

$$|V|_{k,W(x_0)} \le C_k, \text{ and } \inf_{y \in W(x_0)} V(y) > c > 0,$$

where c is independent of σ, ε, and x_0. We will define the inner product for two functions g, $h \in C^\infty(S_{\varepsilon,\sigma})$ by

$$(g, h) = \int g\overline{h}\, dV.$$

Then the following lemma follows from the Divergence Theorem.

LEMMA 3.9. *Let L_1^A, L_2^A be the frame constructed in $W(x_0)$. Then there exist functions $e_j \in C^\infty(W(x_0))$, $j = 1, 2$, and a function $P = \langle L_2^A, \nu\rangle \in C^\infty(W(x_0))$, ν a unit normal vector, such that for all g, $h \in C^\infty(W(x_0))$,*

$$(L_1^A g, h) = -(g, \overline{L}_1^A h) - (e_1 g, h), \text{ and} \tag{3.43}$$

$$(L_2^A g, h) = -(g, \overline{L}_2^A h) - (e_2 g, h) - \int_{M_0} Pg\overline{h}\, dS + \int_{M_\sigma} Pg\overline{h}\, dS, \tag{3.44}$$

where $dS = V\, ds$, $M_0 = \{z;\ r(z) = 0\}$ and $M_\sigma = \{z;\ r(z) = \varepsilon\sigma^{3 \cdot 2^{m-1}}\}$. The function P satisfies $c < P(y) < C$, $y \in W(x_0)$, where c and C are independent of ε, σ, and x_0.

Let $\Lambda^{0,q}(S_{\varepsilon,\sigma}; A)$ denote the space of $(0,q)$-forms with respect to $\mathcal{L}^A$ on $S_{\varepsilon,\sigma}$, and set

$$\Gamma^{0,q}(S_{\varepsilon,\sigma}; A) = \Lambda^{0,q}(S_{\varepsilon,\sigma}; A) \otimes \mathcal{L}^A.$$

Now let us define, for a given structure $\mathcal{L}^A$ satisfying (3.41) for small ε_0, the L^2-operators corresponding to D_2 and its adjoint. We define $\mathcal{E}_c^{0,q}(S_{\varepsilon,\sigma}; A)$ to be the set of smooth sections U of $\Gamma^{0,q}(S_{\varepsilon,\sigma}; A)$ such that support of U is a compact subset of $S_{\varepsilon,\sigma}$. Let $\mathcal{E}_0^{0,q}(S_{\varepsilon,\sigma}; A)$ denote the set of sections of $\mathcal{E}_c^{0,q}(S_{\varepsilon,\sigma}; A)$ with compact support in the interior of $S_{\varepsilon,\sigma}$. Suppose that $U = \sum_{l=1}^{2} \sum_{|J|=q} U_l^J \overline{\omega}_A^J \cdot L_l^A$ is an element of $\Gamma^{0,q}(S_{\varepsilon,\sigma}; A)$ with compact support in $W(x_0)$. We define

$$\|U\|^2 = \int_{S_{\varepsilon,\sigma}} \sum_{l=1}^{2} \sum_{|J|=q} |U_l^J|^2 \, dV, \tag{3.45}$$

where dV is the volume form given by the metric of $\mathcal{L}^0$. Since L_1^A, L_2^A is an orthonormal frame, the quantity in (3.45) is independent of the frame neighborhood $W(x_0)$. Thus, by using a partition of unity, it follows that the norm in (3.45) extends to all of $\Gamma^{0,q}(S_{\varepsilon,\sigma}; A)$. Let $L_q^2(S_{\varepsilon,\sigma}, T_A^{1,0})$ denote the set of sections of $\Gamma^{0,q}(S_{\varepsilon,\sigma}; A)$ such that (3.45) is finite.

Define $\mathcal{B}_-^q(S_{\varepsilon,\sigma}; A)$ to be the set of forms in $\mathcal{E}_c^{0,q}(S_{\varepsilon,\sigma}; A)$ such that U_l^J vanishes on M_0 whenever $2 \notin J$. (This is also independent of the frame neighborhood $W(x_0)$.) Similarly, define $\mathcal{B}_+^q(S_{\varepsilon,\sigma}; A)$ to be the set of forms in $\mathcal{E}_c^{0,q}(S_{\varepsilon,\sigma}; A)$ such that U_l^J vanishes on M_σ whenever $2 \in J$. We now define the formal adjoint D_q' of D_q on $\mathcal{E}_c^{0,q}(S_{\varepsilon,\sigma}; A)$ by $D_q' U = G \in \mathcal{E}_c^{0,q-1}(S_{\varepsilon,\sigma}; A)$ if for all $V \in \mathcal{E}_0^{0,q-1}(S_{\varepsilon,\sigma}; A)$,

$$(U, D_q V) = (G, V),$$

where (,) corresponds to the norm in (3.45). Also, by D_2 we obviously mean the operator defined in (2.3) for the structure $\mathcal{L}^A$. By combining (2.3) and (3.43)–(3.44), it follows that if $U = \sum_\nu U_\nu \overline{\omega}_{12} \cdot L_\nu^A \in \Gamma^{0,2}(S_\varepsilon; A)$ is supported in $W(x_0)$, then

$$D_2' U = \sum_{\nu=1}^{2} \left(\overline{\partial}^* U_\nu - \sum_{\mu=1}^{2} \left[\partial \overline{\omega}_A^\mu (L_1^A, \overline{L}_\nu^A) \overline{\omega}_A^2 + \partial \overline{\omega}_A^\mu (L_2^A, \overline{L}_\nu^A) \overline{\omega}_A^1 \right] \right) L_\nu^A, \tag{3.46}$$

where

$$\overline{\partial}^* U_\nu = -(L_1^A U_\nu + e_1 U_\nu)\overline{\omega}_A^2 - (L_2^A U_\nu + e_2 U_\nu)\overline{\omega}_A^1 - \sum_{l=1}^{2} \omega_A^l([L_1^A, L_2^A])U_\nu \overline{\omega}_A^l. \tag{3.47}$$

We now extend the definition of the operator D_q and D'_q to the L^2-spaces. We define an operator

$$T: L^2_{q-1}(S_{\varepsilon,\sigma}; T_A^{1,0}) \to L^2_q(S_{\varepsilon,\sigma}; T_A^{1,0})$$

by the condition that $U \in \mathrm{Dom}(T)$ and $TU = F \in L^2_q(S_{\varepsilon,\sigma}, T_A^{1,0})$ if for all $V \in \mathcal{B}^q_-(S_{\varepsilon,\sigma}; A)$, we have

$$(U, D'_q V) = (F, V).$$

Similarly, if $U \in L^2_q(S_{\varepsilon,\sigma}; T_A^{1,0})$, then $U \in \mathrm{Dom}(S)$ and $SU = G \in L^2_{q+1}(S_{\varepsilon,\sigma}; T_A^{1,0})$ if for all $V \in \mathcal{B}^{q+1}_-(S_{\varepsilon,\sigma}; A)$,

$$(U, D'_{q+1} V) = (G, V).$$

Note that these definitions imply that if $U \in \mathrm{Dom}(T)$ (or $\mathrm{Dom}(S)$), then $TU = D_q U$ (or $SU = D_{q+1} U$) as in the sense of distribution theory. Let $T^*: L^2_q(S_{\varepsilon,\sigma}; T_A^{1,0}) \longrightarrow L^2_{q-1}(S_{\varepsilon,\sigma}; T_A^{1,0})$ and $S^*: L^2_{q+1}(S_{\varepsilon,\sigma}; T_A^{1,0}) \longrightarrow L^2_q(S_{\varepsilon,\sigma}; T_A^{1,0})$ be the Hilbert space adjoints of T and S respectively. It follows that if $U \in \mathrm{Dom}(T^*)$, then $T^*U = D'_q U$ and that if $U \in \mathrm{Dom}(S^*)$, then $S^*U = D'_{q+1} U$, as in the sense of distributions. Therefore it follows that

$$\mathcal{E}_c^{0,q-1}(S_{\varepsilon,\sigma}; A) \cap \mathrm{Dom}(T) = \mathcal{B}^{q-1}_+(S_{\varepsilon,\sigma}; A), \text{ and,}$$
$$\mathcal{E}_c^{0,q}(S_{\varepsilon,\sigma}; A) \cap \mathrm{Dom}(T^*) = \mathcal{B}^q_-(S_{\varepsilon,\sigma}; A).$$

Similar relations hold for S. Set

$$\mathcal{B}^q(S_{\varepsilon,\sigma}; A) = \mathcal{B}^q_+(S_{\varepsilon,\sigma}; A) \cap \mathcal{B}^q_-(S_{\varepsilon,\sigma}; A).$$

Then we can approximate $U \in \mathrm{Dom}(S) \cap \mathrm{Dom}(T^*)$ by $U_\mu \in \mathcal{B}^q(S_{\varepsilon,\sigma}; A)$ in the graph norm of S and T^* [4, Lemma 6.4]:

LEMMA 3.10. *Let* $U \in \mathrm{Dom}(S) \cap \mathrm{Dom}(T^*)$. *Then there exists* $U_\mu \in \mathcal{B}^q(S_{\varepsilon,\sigma}; A)$ *such that*

$$\lim_{\mu\to\infty} (\|U_\mu - U\| + \|SU_\mu - SU\| + \|T^*U_\mu - T^*U\|) = 0.$$

Finally suppose that we have proved the estimate

$$\|U\|^2 \le C(\|T^*U\|^2 + \|SU\|^2) \tag{3.48}$$

for all $U \in \mathcal{B}^q(S_{\varepsilon,\sigma}; A)$. Then Lemma 3.10 shows that (3.48) holds for all $U \in \mathrm{Dom}\, T^* \cap \mathrm{Dom}\, S$. Then from the usual $\bar\partial$-Neumann theory it follows that for all $G \in L^2_q(S_{\varepsilon,\sigma}; T^{1,0}_A)$, there exists an element $NG \in \mathrm{Dom}(T^*) \cap \mathrm{Dom}(S)$ such that

$$\|NG\| \le C^2\|G\|,$$

and

$$(G, V) = (T^*(NG), T^*V) + (SNG, SV), \quad V \in \mathrm{Dom}(T^*) \cap \mathrm{Dom}(S).$$

We will call N the Neumann operator associated with D_q.

§4. The Subelliptic Estimate for D_2

In this section we prove a subelliptic estimate for the D_2-Neumann problem with almost-complex structure $\mathcal{L}^A$.

We first define tangential norms that will be used in the estimates. For any $s \in \mathbb{R}$, set

$$|||f|||_s^2 = \int_0^{\sigma^{3\cdot 2^{m-1}}} \int_{\mathbb{R}^3} |\hat f(\xi, y_4)|^2 (1 + |\xi|^2)^s \, d\xi \, dy_4,$$

where $\hat f(\xi, y_4) = \int_{\mathbb{R}^3} e^{-iy'\cdot\xi} f(y', y_4)\, dy'$. For any integer $k \ge 0$ and any $s \in \mathbb{R}$, set

$$\|f\|_{s,k}^2 = \sum_{j=0}^{k} \left|\left|\left| \frac{\partial^j f}{\partial y_4^j} \right|\right|\right|_{s-j}^2 .$$

Finally for any integer $m \ge 0$ and $f \in C^\infty(W(x'))$, set

$$\|f\|_m^2 = \sum_{|\alpha|\le m} \|D_y^\alpha f\|^2.$$

By using the coefficients of U, we can easily define all of the above norms for any section U of $\Gamma^{0,q}$. We define $\mathcal{A}(S_{\varepsilon,\sigma})$ to be the space of sections $A \in \Gamma^{0,1}(S_{\varepsilon,\sigma}; 0)$ such that along M_0, $A(\overline{L}) = 0$ whenever $\overline{L} \in T^{0,1} \cap \mathbb{C}TM_0$. Then the goal of this section is to prove the following subelliptic estimate:

THEOREM 4.1. *Suppose $T(\overline{M}) = m < \infty$ and that A is a section of $\mathcal{A}(S_{\varepsilon,\sigma})$ that satisfies (3.41) for some small $\varepsilon_0 > 0$. Then there exist small positive constants σ_1 and ε_1 so that if $\varepsilon < \varepsilon_1$, if $\sigma < \sigma_1$, and if $|A|_{m+5,W(x_0)} \leq \varepsilon$, then the D_2-Neumann problem on $S_{\varepsilon,\sigma}$ for the almost-complex structure $\mathcal{L}^A$ satisfies the following estimate for all forms $U \in \mathcal{B}^2(S_{\varepsilon,\sigma}; A)$ that are compactly supported in $W(x_0)$:*

$$\sigma^{-3}\|U\|^2 + L^A(U) + \|U\|^2_{2^{-m},1} \leq C(\|SU\|^2 + \|T^*U\|^2), \tag{4.1}$$

where $L^A(U)$ is defined by

$$L^A(U) = \|L_1^A U\|^2 + \|\overline{L}_1^A U\|^2 + \|L_2^A U\|. \tag{4.2}$$

Now set $X_1 = \operatorname{Re} L_1^A = \sum_{k=1}^3 a_{1k}\frac{\partial}{\partial y_k}$, $X_2 = \operatorname{Im} L_1^A = \sum_{k=1}^3 a_{2k}\frac{\partial}{\partial y_k}$, and $\|a^i\|_r = \sum_{k=1}^3 \|a_{ik}\|_r$, $i = 1, 2$. Assume that A satisfies (3.41). Then the restriction of L_1^A to the level set $y_4 = \lambda$ is a C^{m+5}-vector field uniformly in λ.

PROPOSITION 4.2. *Let X_1, X_2 be smooth compactly supported vector fields in $\mathbb{R}^4$ and suppose that there exists a set $K \Subset \mathbb{R}^4$ and a constant $c > 0$ and vector fields $X^1, \ldots, X^m$, $X^i = X_1$ or X_2, $i = 1, 2, \ldots, m$, so that for all $x \in K$,*

$$\inf\left\{\sum_{j=1}^{2} |\eta(X_j)| + |\eta([X^m, X^{m-1}, \ldots, [X^2, X^1]\ldots])|; \quad \eta \in T_x^*,\ \eta\left(\frac{\partial}{\partial y_4}\right) = 0,\ |\eta| = 1\right\} > c. \tag{4.3}$$

Then there exists a constant C independent of X_1, X_2 so that for all $U \in C_0^\infty(\mathbb{R}^4)$ with $\operatorname{supp} U \subset K$,

$$\|U\|^2_{2^{-m}} \leq C\left(1 + \sum_{j=1}^{2} \|a^j\|^2_{m+5}\right)^{2m} \left(\|X_1 U\|^2 + \|X_2 U\|^2 + \|U\|^2\right). \tag{4.4}$$

Proof. The proof is similar to that of [7]. We just observe carefully how the coefficient functions depend. Then we can show, by induction, that the coefficient functions a^j of X_1, X_2 appear as in the right hand side of (4.4).

If we combine Proposition 3.8 and Proposition 4.2, we have the following corollary.

COROLLARY 4.3. *Assume that* $T(\overline{M}) \leq m$ *and that* (3.41) *holds for a sufficiently small* $\varepsilon_0 > 0$. *Then for all* $f \in C_0^\infty(W(x'))$,

$$|||f|||_{2^{-m}}^2 \leq C(\|L_1^A f\|^2 + \|\overline{L}_1^A f\|^2) + C\|f\|^2, \tag{4.5}$$

where C *is independent of* x' *and* ε_0.

Proof. Since we are assuming (3.41), the coefficients a_{ik} of X_i, $i = 1, 2$ satisfy $\|a_{ik}\|_{m+5} \leq C'$. Therefore by virtue of the estimates in (3.42), the corollary follows from Proposition 4.2.

For convenience, in all that follows in this section, we omit the notation A from the frames L_1^A, L_2^A, and ω_A^1, ω_A^2. Note that in $W(x_0)$, we have technically chosen so that $y_4 = 0$ and $y_4 = \sigma^{3\cdot 2^{m-1}}$ coincide with $r = 0$ and $r = \varepsilon\sigma^{3\cdot 2^{m-1}}$, respectively, the boundaries of $S_{\varepsilon,\sigma}$. Then the following lemma can be proved by modifying the proof of Lemma 7.7 in [4].

LEMMA 4.4. *Suppose that* $f \in C_0^\infty(W(x_0))$ *and that* f *vanishes on* M_0 *or on* M_σ. *If* σ *is sufficiently small, say* $\sigma < \sigma_1$, *then there exists a constant* C *independent of* ε, σ, *and* x_0 *so that for all* $f \in C_0^\infty(W(x_0))$,

$$\sigma^{-3}\|f\|^2 \leq C(\|\overline{L}_2 f\|^2 + \|L_1 f\|^2 + \|\overline{L}_1 f\|^2), \text{ and} \tag{4.7}$$

$$\sigma^{-3}\|f\|^2 \leq C(\|L_2 f\|^2 + \|L_1 f\|^2 + \|\overline{L}_1 f\|^2). \tag{4.8}$$

We now return to the proof of Theorem 4.1. If $U \in \mathcal{B}^2(S_\varepsilon, A)$, then U can be written as $U = \sum_{l=1}^2 U_l \overline{\omega}^1 \wedge \overline{\omega}^2 \cdot L_l$, where $U_l = 0$ on M_σ, $l = 1, 2$. This fact makes us easy to handle the boundary terms occuring when we integrate by parts. Assume that supp$U \Subset W(x_0)$. Then it is obvious that $SU = 0$, and it follows from (3.46) and (3.47) that

$$T^*U = D_2' U = BU + \mathcal{O}(|U|),$$

where

$$BU = -\sum_{l=1}^2 (L_1 U_l \overline{\omega}^2 + L_2 U_l \overline{\omega}^1) \cdot L_l. \tag{4.9}$$

Hence it follows that

$$\|BU\|^2 \leq 2\|T^*U\|^2 + C\|U\|^2, \tag{4.10}$$

and we conclude from (4.9) that

$$\|BU\|^2 = \sum_{l=1}^{2}\sum_{j=1}^{2} \|L_j U_l\|^2.$$

If we use Lemma 3.9 and the boundary conditions, we get, for $U = U_l$, that

$$\begin{aligned}\|L_1U\|^2 &= (L_1U, L_1U) = -(\overline{L}_1L_1U, U) - (L_1U, e_1U)\\ &= -(L_1\overline{L}_1U, U) + ([L_1, \overline{L}_1]U, U) - (L_1U, e_1U)\\ &= (\overline{L}_1U, \overline{L}_1U) + (\overline{L}_1U, \overline{e}_1U) - (L_1U, e_1U) + ([L_1, \overline{L}_1]U, U).\end{aligned}$$

Note that we can write

$$[L_1, \overline{L}_1] = \sum_{i=1}^{2} \omega^i([L_1, \overline{L}_1])L_i + \sum_{i=1}^{2} \overline{\omega}^i([L_1, \overline{L}_1])\overline{L}_i.$$

Set $c_{11}^i = \omega^i([L_1, \overline{L}_1])$, and $d_{11}^i = \overline{\omega}^i([L_1, \overline{L}_1])$. Then

$$([L_1, \overline{L}_1]U, U) = \sum_{i=1}^{2}(c_{11}^i L_iU, U) + \sum_{i=1}^{2}(d_{11}^i \overline{L}_iU, U),$$

and hence

$$\|L_1U\|^2 = \|\overline{L}_1U\|^2 + (c_{11}^2 L_2U, U) + (d_{11}^2 \overline{L}_2U, U) + \mathcal{O}((\|L_1U\| + \|\overline{L}_1U\|)\|U\|).$$

Note that

$$(d_{11}^2 \overline{L}_2U, U) = -(U, L_2(\overline{d}_{11}^2U)) - (\bar{e}_2U, \overline{d}_{11}^2U) - \int_{M_0} d_{11}^2|U|^2\, dS,$$

because $U = 0$ on M_σ. Therefore it follows that

$$\frac{1}{2}\|L_1U\|^2 = \frac{1}{2}\|\overline{L}_1U\|^2 - \frac{1}{2}\int_{M_0} d_{11}^2|U|^2\, dS + \mathcal{O}(\sigma L^A(U)) + \mathcal{O}(\sigma^{-1}\|U\|^2),$$

and hence from (4.2) we have

$$\begin{aligned}\|BU\|^2 &= \frac{1}{2}\|L_1U\|^2 + \frac{1}{2}\|\overline{L}_1U\|^2 + \|L_2U\|^2\\ &\quad - \frac{1}{2}\int_{M_0} d_{11}^2|U|^2\, ds + \mathcal{O}(\sigma L^A(U)) + \mathcal{O}(\sigma^{-1}\|U\|^2)\\ &\geq \frac{1}{3}L^A(U) - \frac{1}{2}\int_{M_0} d_{11}^2|U|^2\, ds - C\sigma^{-1}\|U\|^2,\end{aligned}$$

provided that σ is sufficiently small. Note that $d_{11}^2 = -c_{11}^2 = -\omega^2([L_1, \overline{L}_1]) \leq 0$ on M_0 because M_0 is pseudoconvex. Therefore we get

$$\|BU\|^2 \geq \frac{1}{3}L^A(U) - C\sigma^{-1}\|U\|^2. \tag{4.11}$$

By combining (4.10) and (4.11) we get

$$\frac{1}{3}L^A(U) - C\sigma^{-1}\|U\|^2 \leq 2\|T^*U\|^2. \tag{4.12}$$

From (4.5) and Lemma 4.4, it follows that

$$|||U|||_{2-m}^2 + \sigma^{-3}\|U\|^2 \leq CL^A(U). \tag{4.13}$$

If we combine (4.10), (4.12) and (4.13) we obtain for sufficiently small σ that

$$\sigma^{-3}\|U\|^2 + L^A(U) + |||U|||_{2-m}^2 \leq C(\|T^*U\|^2 + \|SU\|^2). \tag{4.14}$$

For the estimates of the non-tangential derivatives of U, we note that $L_2^A = \frac{\partial}{\partial y_4} + X$, where $X = \sum_{j=1}^3 b_j(y)\frac{\partial}{\partial y_j}$. Therefore a standard argument yields the inequality

$$|||\frac{\partial f}{\partial y_4}|||_{-1+2-m}^2 \leq C(1 + \sum_{j=1}^{3} |b_j|_{\widetilde{W}(x_0),5}^2)(|||f|||_{2-m}^2 + \|\overline{L}_2 f\|^2 + \|f\|^2), \tag{4.15}$$

for all $f \in C_0^\infty(\widetilde{W}(x_0))$, where $\widetilde{W}(x_0)$ is a neighborhood containing $W(x_0)$. This inequality can be applied with $f = U_l$ and one obtains (4.1) combining (4.13)–(4.15). This completes the proof of Theorem 4.1.

We now define Sobolev spaces for sections of $\Gamma^{0,q}(S_{\varepsilon,\sigma}; A)$. Recall that the open sets $B_b(x_0)$ satisfy (3.19) and (3.20) for each $x_0 \in M$. Choose a set $T_\sigma = \{x_i^\sigma \in M,\ i \in I\}$ such that the sets $B_{c\sigma/2}(x_i^\sigma)$, $i \in I$, cover $S_{\varepsilon,\sigma}$, and such that no two points x_i^σ and x_j^σ satisfy $|x_i^\sigma - x_j^\sigma| \leq c\sigma/4$ where $|\ |$ is the distance function on $S_{\varepsilon,\sigma}$. It follows that the sets $W(x_i^\sigma)$, $i \in I$, cover $S_{\varepsilon,\sigma}$ and that there exists an integer $\widetilde{N}$ such that no point of $S_{\varepsilon,\sigma}$ lies in more than $\widetilde{N}$ of the open sets $W(x_i^\sigma)$. Furthermore, there exist functions ζ_i, ζ_i' (that are independent of y_{2n}) $\in C_0^\infty(W(x_i^\sigma))$ such that $\sum_{i\in I}\zeta_i^2 \equiv 1$, such that if $x \in \operatorname{supp}\zeta_i$, then

$$\zeta_i' \equiv 1 \text{ in } B_{c'\sigma}(x), \tag{4.16}$$

and such that both ζ_i and ζ_i' satisfy

(4.17) $$|\zeta_i|_{k,W(x_i^\sigma)} + |\zeta_i'|_{k,W(x_i^\sigma)} \leq C_k\sigma^{-k}.$$

Now let F be any section of $\Gamma^{0,q}(S_{\varepsilon,\sigma}; A)$. We define

$$\|F\|_{k,A}^2 = \sum_{i\in I} \|\zeta_i F\|_{k,A,W(x_i^\sigma)}^2,$$

where

$$\|\zeta_i F\|_{k,A,W(x_i^\sigma)}^2 = \sum_{j=1}^{2} \sum_{|J|=q} \|\zeta_i F_j^J\|_{k,W(x_i^\sigma)}^2,$$

and where $F = \sum_{j=1}^{2}\sum_{|J|=q} F_j^J \overline{\omega}_A^J \cdot L_j^A$ is the decomposition of F in terms of the L_1^A, L_2^A, ω_A^1, ω_A^2 frame of $W(x_i^\sigma)$. Moreover, the Sobolev norm $\|\ \|_{k,W(x_i^\sigma)}$ is taken with respect to the y-coordinates of $W(x_i^\sigma)$. We define $H_k^{0,q}(S_{\varepsilon,\sigma}; T_A^{1,0})$ to be the set of all sections F of $\Gamma^{0,q}(S_{\varepsilon,\sigma}; A)$ for which $\|F\|_{k,A} < \infty$. If we define $L_q^2(S_{\varepsilon,\sigma}; T_A^{1,0})$ to be the set of all $F \in \Gamma^{0,q}(S_{\varepsilon,\sigma}; A)$ such that $\|F\|^2 < \infty$, then it is obvious that the norms $\|\ \|$ and $\|\ \|_{0,A}$ are equivalent on $L_q^2(S_{\varepsilon,\sigma}; T_A^{1,0})$. We also define $\mathcal{A}(S_{\varepsilon,\sigma})$ to be the space of sections $A \in \Gamma^{0,1}(S_{\varepsilon,\sigma}; 0)$ such that along M_0, $A(\overline{L}) = 0$ whenever $\overline{L} \in T^{0,1} \cap \mathbb{C}TM_0$. Since $\mathcal{A}(S_{\varepsilon,\sigma}) \subset \Gamma^{0,1}(S_{\varepsilon,\sigma}; 0)$, we define $\|A\|_k = \|A\|_{k,0}$, and we define $H_k(S_{\varepsilon,\sigma}; \mathcal{A})$ to be the set of $A \in \mathcal{A}(S_{\varepsilon,\sigma})$ such that $\|A\|_k < \infty$.

We want to get an estimate in global form. Define $Q(U,U) = \|T^*U\|^2 + \|SU\|^2$. By using the partition of unity as defined above satisfying (4.16), (4.17), and the estimates in Theorem 4.1, we obtain:

COROLLARY 4.5. *Suppose that A satisfies* (3.41) *for all $x_0 \in \overline{M}$. Then there exists a fixed small σ and a constant $\varepsilon_1 > 0$ such that for all $\varepsilon, 0 < \varepsilon < \varepsilon_1$, and all $U \in$* Dom$(T^*) \cap$ Dom(S),

(4.18) $$\|U\|^2 \leq CQ(U,U).$$

Now let us fix $\sigma > 0$, satisfying Corollary 4.5 and set $W(x_0) = W_\sigma(x_0)$. Using Theorem 4.1 and the standard "bootstrap" method, we can get regularity estimates for the linearized equation. The proof follows the method similar to the proof in Section 9 of [4].

THEOREM 4.6. *Suppose that* (3.41) *holds and that U is the solution of $\Box U = G$, where $G \in H_k^{0,2}(S_\varepsilon; T_A^{1,0})$ for all $k > 0$. Then for all integers*

$k \geq 1$ and each pair of functions ζ, ζ' in $C_0^\infty(W(x_0))$ as in (4.16) and (4.17), U, $D_2^* U$ satisfy

$$\|\zeta U\|_k \lesssim \|\zeta' G\|_{k-2\delta} + (1 + \|A\|_{k+1})(\|\zeta' G\|_5 + \|\zeta' U\|) \text{ and} \tag{4.19}$$
$$\|\zeta D_2^* U\|_k \lesssim \|\zeta' G\|_k + (1 + \|A\|_{k+2})(\|\zeta' G\|_5 + \|\zeta' U\|).$$

Note that $N(j) = \{i \in I;\ W(x_i^\sigma) \cap W(x_j^\sigma) \neq \emptyset\}$ is bounded by a fixed number $\widetilde{N} \geq 1$. Also it follows from (3.18) and Lemma 3.7 that the frames $L_k^{A,j}$ in $W(x_j^\sigma)$ and $L_k^{A,i}$ in $W(x_i^\sigma)$, $k = 1, 2$, are related by

$$L_k^{A,j} = \sum_{l=1}^{2} B_{kl}^{A,ji} L_l^{A,i},\ k = 1, 2,$$

where $B_{kl}^{A,ji}$ satisfies

$$|D_{y^i}^m B_{kl}^{A,ji}| \lesssim 1 + P_{m,x_i^\sigma}(A). \tag{4.20}$$

Similarly if $w_{A,j}^k$, $j = 1, 2$, is the dual frame of $L_k^{A,j}$, then there exists a matrix $b_k^{A,ji}$ such that $\overline{w}_{A,j}^k = \sum_{l=1}^2 b_{k,l}^{A,ji} \overline{w}_{A,i}^l$, $k = 1, 2$, where $b_{k,l}^{A,ji}$ satisfies

$$|D_{y^i}^m b_{k,l}^{A,ji}| \lesssim 1 + P_{m,x_i^\sigma}(A). \tag{4.21}$$

Therefore it follows from (4.20) and (4.21) that for a section V in $\Gamma_A^{0,q}(S_\epsilon; A)$, $q = 1, 2$, and for functions ζ_j, $\zeta_j' \in C_0^\infty(W_j^\sigma)$ satisfying (4.16), (4.17), we have:

$$\|\zeta_j' V\|_{k,W(x_j^\sigma)}^2 \lesssim \sum_{i \in N(j)} (\|\zeta_i V\|_{k,W(x_i^\sigma)}^2 + \|A\|_k^2 \|\zeta_i V\|_{3,W(x_i^\sigma)}^2). \tag{4.22}$$

We now state the estimate (4.19) in global form.

THEOREM 4.7. *Assume that $\Box U = G$, where $G \in H_k^{0,2}(S_\varepsilon; T_A^{1,0})$ for all k and that A satisfies* (3.41). *Then*

$$\|D_2^* U\|_k \lesssim \|G\|_k + (1 + \|A\|_{k+2}) \|G\|_5. \tag{4.23}$$

Proof. Set $\zeta = \zeta_j \in C_0^\infty(W(x_j^\sigma))$ in (4.19) and sum up over j and then apply (4.22). Then we get

$$\|D_2^* U\|_k \lesssim \|G\|_k + (1 + \|A\|_{k+2})(\|G\|_5 + \|U\|).$$

Since (4.18) holds, it follows that

$$(1 + \|A\|_{k+2}) \|U\| \lesssim (1 + \|A\|_{k+2}) \|G\|_4,$$

and this proves (4.23). ☐

§5. Extension of *CR* structures

In this section we will prove Theorem 1.1 and Theorem 1.2 using the estimates in Section 4. If $A \in \mathcal{A}(S_{\varepsilon,\sigma})$ is sufficiently small and if we set $P_A(\overline{L}) = \overline{L} + A(\overline{L})$, then $\overline{\mathcal{L}}_A = \{P_A(\overline{L}); \overline{L} \in \overline{\mathcal{L}}\}$. If we set $Q_A(\omega) = \omega - A^*\omega$, then $\Lambda_A^{1,0} = \{Q_A(\omega);\ \omega \in \Lambda^{1,0}(\mathcal{L})\}$. We define a nonlinear operator $\Phi: \mathcal{A}(S_{\varepsilon,\sigma}) \to \Gamma^{0,2}(S_{\varepsilon,\sigma})$ as follows:

$$\Phi(A)(\overline{L}', \overline{L}'', \omega) = Q_A(\omega)([P_A(\overline{L}'), P_A(\overline{L}'')]). \tag{5.1}$$

Obviously, if $\Phi(A) = 0$, then $\mathcal{L}_A$ is an integrable almost complex structure on $S_{\varepsilon,\sigma}$.

Note that there is a natural map $\mathcal{P}_A: \Gamma_A^{0,2} \to \Gamma^{0,2}$, defined as follows: if $B \in \Gamma_A^{0,2}$, we define $\mathcal{P}_A B$ by

$$(\mathcal{P}_A B)(\overline{L}_1, \overline{L}_2, \omega) = B(P_A(\overline{L}_1), P_A(\overline{L}_2), Q_A(\omega)).$$

Therefore it follows from the definition of F^A in (2.5) that $\Phi(A) = \mathcal{P}_A(F^A)$. We note also that if d and A are small sections of $\mathcal{A}$ on $S_{\varepsilon,\sigma}$, then there exist sections $\Delta_{A,d}^+$ and $\Delta_{A,d}^-$ of $\Lambda_A^{0,1} \otimes T_A^{1,0}$ and $\Lambda_A^{0,1} \otimes T_A^{0,1}$, respectively, so that

$$P_{A+d}(\overline{L}) = P_A(\overline{L}) + \Delta_{A,d}^+(P_A(\overline{L})) + \Delta_{A,d}^-(P_A(\overline{L})).$$

Similarly, there exist sections $\delta_{A,\delta}^+$ and $\delta_{A,\delta}^-$ of $\mathrm{Hom}(\Lambda_A^{1,0}, \Lambda_A^{1,0})$ and $\mathrm{Hom}(\Lambda_A^{1,0}, \Lambda_A^{0,1})$, respectively, so that

$$Q_{A+d}(\omega) = Q_A(\omega) - \delta_{A,d}^+(Q_A(\omega)) - \delta_{A,d}^-(Q_A(\omega)).$$

Then it follows that $\Delta_A^\pm(d) = \Delta_{A,d}^\pm$ both depend linearly on d and that the coefficients depend smoothly on A, and that the mapping $d \longrightarrow \Delta_A(d) = \Delta_A^+(d) + \Delta_A^-(d)$ is invertible. Then $\Phi'(A)(d)$, as an element of $\Gamma^{0,2}$, satisfies

$$\Phi'(A)(d) = (\mathcal{P}_A \circ D_2^A \circ \Delta_A^+)(d) - \mathcal{P}_A(h_A(d)(F^A)), \tag{5.2}$$

where $h_A(d): T_A^{1,0} \longrightarrow T_A^{1,0}$ denotes the adjoint of $\delta_A^+(d): \Lambda_A^{1,0} \longrightarrow T_A^{1,0}$. Since $\Phi(A) = \mathcal{P}_A(F^A)$, we let U_A be the solution of $\Box U_A = -F^A$ and then set $V_A = (D_2^A)^* U_A$ and then set $d_A = \Delta_A^{-1}(V_A)$. Since $D_3 = 0$, it follows that $D_2^A V_A = -F^A$. Hence we have from (5.2) that

$$\begin{aligned}\Phi(A) + \Phi'(A) d_A &= \mathcal{P}_A(F^A + D_2^A V_A) - \mathcal{P}_A(h_A(d_A)(F^A)) \\ &= -\mathcal{P}_A(h_A(d_A)(F^A)).\end{aligned} \tag{5.3}$$

Using the representations in (5.2) and (5.3), we can now obtain that (as in Section 11 in [4]), $\Phi(A)+\Phi'(A)(d_A)$ vanishes in second order in $\Phi(A)$. This is a key property in the Nash-Moser approximation process.

We recall that F^A vanishes in infinite order along M_0 (in x-coordinates!) This can be stated in y-coordinates as follows. The proof is similar to that of Lemma 6.2 in [4].

LEMMA 5.1. *Suppose that there exists a section* $F \in \Gamma^{0,2}(\overline{\Omega}^+)$ *where* $\overline{\Omega}^+ = \{(x,t) \in \Omega;\ 0 \le t < 1\}$ *such that* F *and all its derivatives vanish to infinite order along* M. *Then for all* k, $N = 0, 1, 2, \ldots$, *and all* $x_0 \in M$,

$$|F^0|_{k,W(x_0)} \le C_{k,N}\varepsilon^N \varphi(x_0)^N, \tag{5.4}$$

where F^0 *means that* F *is written out in* $W(x_0)$ *according to the frame* L_1^0, L_2^0, ω_0^1, ω_0^2 *of* $\mathcal{L}^0$ ($\mathcal{L}^A$ *with* $A = 0$).

We can now prove the main theorems of this paper:

Proof of Theorem 1.1. We will show that $\|\Phi(0)\|_D < b$ for the small $b > 0$ and the integer D which are appeared in the variant of Nash-Moser theorem [4, Theorem 13.1]. As in Section 11 of [4], the rest of the properties for the $\Phi(A)$ in the hypothesis of Nash-Moser theorem can be proved using the relations in (5.2) and (5.3), and the estimates for $\Box$ operator in Section 4.

Note that (4.17) and (5.4) imply that for each $i \in I$,

$$\|\zeta_i F^0\|_{k,0}^2 \le C_{k,N}\varepsilon^N \varphi(x_i^\sigma)^N$$

so that after summing up over x_i^σ,

$$\|F^0\|_{k,0,\Phi}^2 \le C_{k,N} \sum_{i\in I} \varphi(x_i^\sigma)^N \varepsilon^N. \tag{5.5}$$

Since the choice of the points that was made before (4.17) shows that the balls $B_{\frac{c\sigma}{8}}(x_i^\sigma)$, $i \in I$, are all disjoint, we can obtain an upper bound on $N(l)$, which is defined to be the number of $i \in I$ such that $2^{-l-1} \le \varphi(x_i^\sigma) < 2^{-l}$. In fact, in terms of the $\langle\ ,\ \rangle_0$-metric introduced in Section 2, the volume of $B_{\frac{c\sigma}{8}}(x_i^\sigma)$ is roughly bounded below by $\varepsilon^3\sigma^{3(1+2^{m-1})}\varphi(x_i^\sigma)^{6m} \sim \varepsilon^3\sigma^{3(1+2^{m-1})}2^{-6lm}$, and the $\langle\ ,\ \rangle_0$-volume of the region in $S_{\varepsilon,\sigma}$ with $2^{-l-1} \le$

$\varphi(x) \leq 2^{-l}$ is roughly bounded above by $\varepsilon\sigma^{3\cdot 2^{m-1}} \cdot 2^{-2ml}$. Thus, we conclude that

$$N(l) \lesssim \varepsilon^{-2}\sigma^{-3}2^{4ml}. \tag{5.6}$$

Thus (5.5) and (5.6) imply that if $N = 4ml + 1$, then

$$\|\Phi(A)\|_k = \|F^0\|_{k,0} \lesssim C_k \cdot \varepsilon$$

for sufficiently small ε. In particular, if we set $k = D$, and choose ε to be sufficiently small, then it follow that $\|\Phi(A)\|_D < b$. ∎

Proof of Theorem 1.2. Since $\overline{M} \subset bD$ is a compact pseudoconvex *CR* manifold of finite type, we conclude from Theorem 1.1 that there exist a continuous nonnegative function g and an integrable almost complex structure $\mathcal{L}^+$ on

$$S_g^+ = \{(x,t) \in M \times \mathbb{R};\ 0 \leq t \leq g(x)\}.$$

Moreover, since $\mathcal{L}^+$ is a small perturbation of $\mathcal{L}^0$, which satisfies $dt(\mathcal{J}_{\mathcal{L}^0}(X_0)) < 0$, it follows that $dt(\mathcal{J}_{\mathcal{L}^+}(X_0)) < 0$.

Let $\mathcal{L}^-$ be the integrable almost complex structure on D. We can smoothly extend $\mathcal{L}^+$ and $\mathcal{L}^-$ to $S_g = S_g^+ \cup S_g^-$, where $S_g^- = \{(x,t) \in M \times \mathbb{R};\ -g(x) \leq t \leq 0\}$. It follows that $\mathcal{L}^+$ and $\mathcal{L}^-$ are integrable to infinite order along $M \in bD$. Hence, Theorem 2.2 implies that there is a diffeomorphism $G\colon S_g \to S_g$ so that $G_*(\mathcal{L}^+) = \mathcal{L}^-$ to infinite order along M. Since $\mathcal{L}^{\pm}$ both satisfy $dt(\mathcal{J}_{\mathcal{L}^{\pm}}(X_0)) < 0$, the proof of Theorem 4.2 in [4] shows that G maps S_g^+ to S_g^+. Thus, if we define $\mathcal{L}$ on S_g by $\mathcal{L}_z = (G_*\mathcal{L}^+)_z$ if $z \in S_g^+$ and $\mathcal{L}_z = \mathcal{L}_z^-$ if $z \in S_g^-$, then $\mathcal{L}$ is integrable on S_g. ∎

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