

FAITHFUL REPRESENTATIONS OF LIE GROUPS II

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The present paper is a continuation of Part I.¹⁾ In the introduction of Part I we have explained our main problems and sketched their results. In the present part we shall proceed to give complete proofs of them.

Notations and definitions in Part I shall be retained.

Semi-simple f.r. Lie groups

5. Let Σ be a field and let a and b be matrices of degree n and m respectively with coefficients of Σ . We denote by $a \times b$ the Kronecker product of a and b , and by $a^{[k]}$ the k -times Kronecker product of a :

$$a^{[0]} = 1_1 \text{ (the unit matrix of degree 1),}$$

$$a^{[k]} = \overbrace{a \times a \times \dots \times a}^k \text{ for } k > 0.$$

Let now $\epsilon_1, \epsilon_2, \dots, \epsilon_n$ be the sequence of all eigen-values of a , repeated as many times as their multiplicities. Then $\epsilon_1^{i(1)} \epsilon_2^{i(2)} \dots \epsilon_n^{i(n)}$ is an eigen-value of $a^{[k]}$ if $i(1), i(2), \dots$ are non-negative integers such that $\sum_{j=1}^n i(j) = k$. Suppose now that $a^{[k]}$ is the unit matrix (of degree n). Since every eigen-value of $a^{[k]}$ is 1, we have $\epsilon_1 = \epsilon_2 = \dots = \epsilon_n = \epsilon$, where ϵ is a k -th root of unity. Let e_1 be an eigen-vector of a : $ae_1 = \epsilon e_1$, and let e_2 be a vector, linearly independent of e_1 . We operate $a^{[k]}$ to $e_2 \times e_1 \times \dots \times e_1$ and get $ae_2 = \epsilon e_2$ easily. Hence if $a^{[k]}$ is the unit matrix, then $a = \epsilon 1_n$ where $\epsilon^k = 1$. Using this fact we shall prove the following algebraic

LEMMA 8. *Let Σ be an algebraically closed field, and let \mathcal{G} be a group composed of non-singular matrices with coefficients of Σ . Let now \mathcal{D} be a finite central invariant subgroup of \mathcal{G} . Then we can construct a faithful representation of \mathcal{G}/\mathcal{D} , which is an induced representation of \mathcal{G} on a certain tensor space, where we assume that the order of \mathcal{D} is indivisible by the characteristic of Σ .*

Proof. It is clearly sufficient to prove the lemma in case when the order q of \mathcal{D} is a prime number. Let $d (\neq e)$ be an element of \mathcal{D} . Since \mathcal{D} is comp-

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¹⁾ Gotô [22].

letely reducible because q does not divide the characteristic of Σ , we may assume that d is of the form

$$d = \begin{pmatrix} \varepsilon^{i(1)} 1_{j(1)} & & & 0 \\ & \varepsilon^{i(2)} 1_{j(2)} & & \\ & & \ddots & \\ 0 & & & \varepsilon^{i(k)} 1_{j(k)} \end{pmatrix},$$

where ε is a q -th root of unity and $i(s) \not\equiv i(t) \pmod{q}$ if $s \neq t$. Since $gd = dg$ for any element g of \mathfrak{G} , every eigen-space of d is allowable by \mathfrak{G} :

$$\mathfrak{G} \ni g = \begin{pmatrix} g_1 & & & 0 \\ & g_2 & & \\ & & \ddots & \\ 0 & & & g_h \end{pmatrix}.$$

Clearly, we can determine non-negative integers a_{ls} ($l = 1, 2, \dots, h-1$; $s = 1, 2, \dots, h$) such that a solution (x_1, \dots, x_h) of the congruence equations

$$a_{l1}x_1 + a_{l2}x_2 + \dots + a_{lh}x_h \equiv 0 \pmod{q} \quad l = 1, 2, \dots, h-1$$

satisfies the relations

$$x_s \equiv mi(s) \pmod{q} \quad s = 1, 2, \dots, h$$

for a suitably chosen integer m .

Now an induced representation of \mathfrak{G} is given by

$$g \rightarrow \begin{pmatrix} g^* & 0 \\ 0 & g^{**} \end{pmatrix},$$

where

$$g^* = \begin{pmatrix} g_1^{[q]} & & & 0 \\ & g_2^{[q]} & & \\ & & \ddots & \\ 0 & & & g_h^{[q]} \end{pmatrix}.$$

and

$$g^{**} = \begin{pmatrix} g_1^{[a_{11}]} \times \dots \times g_h^{[a_{1h}]} & & & \\ & g_1^{[a_{21}]} \times \dots \times g_h^{[a_{2h}]} & & 0 \\ & & \ddots & \\ 0 & & & g_1^{[a_{h-11}]} \times \dots \times g_h^{[a_{h-1h}]} \end{pmatrix}$$

Let \mathfrak{B} be the kernel of the representation. Since $z_s^{[q]}$ is the unit matrix for $z \in \mathfrak{B}$, we have

$$z = \begin{pmatrix} e^{f(1)}1_{j(1)} & & & 0 \\ & e^{f(2)}1_{j(2)} & & \\ & & \ddots & \\ 0 & & & e^{f(h)}1_{j(h)} \end{pmatrix}.$$

Now since z^{**} is also the unit matrix we get that

$$a_{1l}f(1) + a_{2l}f(2) + \dots + a_{lh}f(h) \equiv 0 \pmod{q}, \quad l = 1, 2, \dots, h - 1$$

so that from the definition of a_{ls} there exists an integer m such that

$$f(s) \equiv mi(s) \pmod{q} \quad s = 1, 2, \dots, h.$$

Hence we have $z \in \mathfrak{D}$, i.e. $\mathfrak{Z} \subseteq \mathfrak{D}$. That $\mathfrak{D} \subseteq \mathfrak{Z}$ is obvious, q.e.d.

LEMMA 9.²⁾ *Let \mathfrak{G} be a semi-simple f.r.³⁾ Lie group and let \mathfrak{G}^* be a Lie group homomorphic with \mathfrak{G} . Then \mathfrak{G}^* is also f.r.*

Proof. We may suppose that $\mathfrak{G}/\mathfrak{N} \cong \mathfrak{G}^*$, where \mathfrak{N} is a closed invariant subgroup of \mathfrak{G} . Let G be the Lie algebra of \mathfrak{G} , and let \mathfrak{N}_0 be the connected component of \mathfrak{N} containing e . As the Lie algebra N of \mathfrak{N} is an ideal of semi-simple G , there exists an ideal M of G such that

$$G = M + N, \quad M \cap N = 0.$$

We denote by \mathfrak{M} the subgroup of \mathfrak{G} generated by M . Then since M is semi-simple as an ideal of a semi-simple Lie algebra and \mathfrak{G} is f.r., Corollary to Theorem 2 in Part I implies that \mathfrak{M} is closed. Now from

$$\mathfrak{M}/\mathfrak{M} \cap \mathfrak{N}_0 \cong \mathfrak{G}/\mathfrak{N}_0 \quad \text{and} \quad \mathfrak{G}/\mathfrak{N}_0/\mathfrak{N}/\mathfrak{N}_0 \cong \mathfrak{G}/\mathfrak{N},$$

\mathfrak{G}^* is homomorphic with \mathfrak{M} : $\mathfrak{M} \sim \mathfrak{G}^*$. On the other hand since the Lie algebra of \mathfrak{M} is isomorphic to that of \mathfrak{G}^* , \mathfrak{M} and \mathfrak{G}^* are locally isomorphic, so that the kernel \mathfrak{D} of the homomorphism $\mathfrak{M} \sim \mathfrak{G}^*$ is discrete. Then in virtue of Lemma 5 in Part I our assertion is a direct consequence of Lemma 8, q.e.d.

6. A Lie algebra G is called *complex* if G is isomorphic with a Lie algebra over the field C of complex numbers, and a Lie group is called *complex* when its Lie algebra is complex. When the structure of a Lie algebra G is given by

$$G = Px_1 + Px_2 + \dots + Px_r, \quad [x_i, x_j] = \sum_s \gamma_{ijs} x_s, \quad \gamma_{ijs} \in P,$$

where x_1, \dots, x_r constitute a basis of G , the complex Lie algebra G_c defined by

$$G_c = Cx_1 + Cx_2 + \dots + Cx_r, \quad [x_i, x_j] = \sum_s \gamma_{ijs} x_s$$

²⁾ Cf. Malcev [13].

³⁾ A connected Lie group is called faithfully representable (f.r.) if it admits a continuous faithful linear representation. See Part I.

is called the *complex form* of G , and conversely G is called a *real form* of G_c . A complex Lie algebra may have no real form at all, or have several real forms. Note that a Lie algebra is semi-simple if and only if its complex form is semi-simple.

Let G^* be a complex semi-simple Lie algebra. H. Weyl has, in his fundamental papers on semi-simple Lie groups,⁴⁾ proved the following important results: We can select a suitable basis x_1, x_2, \dots, x_r with respect to C so that

$$G^* = Cx_1 + Cx_2 + \dots + Cx_r, \quad [x_i, x_j] = \sum_s r_{ijs} x_s, \quad r_{ijs} \in P$$

and moreover any Lie group which is generated by the real form

$$K = Px_1 + Px_2 + \dots + Px_r$$

is always compact. Such a real form, called a *compact form* of G^* , is unique up to inner automorphisms.

Now consider a general real form of G^* .⁵⁾ Take an involutive automorphism τ , $\tau^2 = 1_r$, of a compact form K , and decompose K into eigen-spaces of τ :

$$K = K_1 + K_{-1},$$

where K_1, K_{-1} are eigen-spaces of τ belonging to eigen-values 1, -1 respectively. Now the subalgebra of G^* given by

$$G = K_1 + \sqrt{-1} K_{-1}$$

is a real form of G^* , and conversely all real forms of G^* can be obtained in such a way. Now isomorphic real forms of G^* , considered as subalgebras of G^* , are conjugated with respect to automorphisms. Therefore since any local automorphism of a simply connected Lie group can be extended to an automorphism in the large, *in the simply connected complex semi-simple Lie group generated by G^* , isomorphic real forms of G^* generate isomorphic subgroups.*

Next, we shall call a complex semi-simple Lie algebra *complex simple* if it has no proper complex ideal distinct from 0. It is easy to see that a complex simple Lie algebra, as well as a real form of a complex simple Lie algebra, is simple. Let G be a simple Lie algebra and G_c the complex form of G . Then only the following two cases are possible.⁶⁾

Case 1. G_c is simple. Then G is a real form of G_c , which is complex simple.

Case 2. G_c is not simple. In this case G itself is complex simple.

From now on we shall call a real form of a complex simple Lie algebra *real*

⁴⁾ Weyl [23].

⁵⁾ Cartan [20]. See also Gantmacher [21].

⁶⁾ E. g. Gantmacher [21].

simple. A *real*, or *complex simple* Lie group is defined correspondingly.

Let now \mathfrak{G}^* be a complex Lie group and G^* the Lie algebra of \mathfrak{G}^* . We shall call a subgroup \mathfrak{G} of \mathfrak{G}^* generated by a real form G of G^* a *real form* of \mathfrak{G}^* , and call \mathfrak{G}^* a *complex form* of \mathfrak{G} . While a Lie algebra always has one and only one complex form, a Lie group may have no complex form at all, or have several complex forms.

LEMMA 10. *Let \mathfrak{G} be a linear Lie group. If \mathfrak{G} is compact or real simple, then there exists a complex-linear Lie group \mathfrak{G}_c which contains \mathfrak{G} as a real form.* Here we shall call a linear Lie group *complex-linear* if its linear Lie algebra contains any complex multiple of its matrix.

Proof. Let $G = Px_1 + \dots + Px_r$ be the linear Lie algebra of \mathfrak{G} . It is clearly sufficient to show that x_i 's are linearly independent with respect to C .

First suppose that \mathfrak{G} is compact. Then since any compact linear group is equivalent to a subgroup of the unitary group, there exists a matrix a such that $a^{-1}Ga$ is composed of skew hermitian matrices. Suppose now that $\alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_r x_r = 0$ $\alpha_i \in C$. Then we have $\alpha_1 a^{-1}x_1 a + \alpha_2 a^{-1}x_2 a + \dots + \alpha_r a^{-1}x_r a = 0$. Now since every $a^{-1}x_i a$ is skew hermitian, we have $\bar{\alpha}_1 a^{-1}x_1 a + \bar{\alpha}_2 a^{-1}x_2 a + \dots + \bar{\alpha}_r a^{-1}x_r a = 0$, so that $\bar{\alpha}_1 x_1 + \bar{\alpha}_2 x_2 + \dots + \bar{\alpha}_r x_r = 0$, where $\bar{\alpha}$ denotes the conjugate complex number of α . Then the linear independence of x_i 's with respect to P implies that $\alpha_1 = \alpha_2 = \dots = \alpha_r = 0$.

Next let G be real simple. We know that its complex form G_c is simple. Hence the complex-linear Lie algebra spanned by $x_i, Cx_1 + Cx_2 + \dots + Cx_r$, forms a faithful representation of G_c , q.e.d.

7. Let G be a Lie algebra and let $\hat{\mathfrak{G}}$ be a corresponding connected Lie group. If $\hat{\mathfrak{G}}$ is f.r. and any f.r. Lie group \mathfrak{G} , generated by G , is homomorphic with $\hat{\mathfrak{G}}$, we call $\hat{\mathfrak{G}}$ a *linear covering group* of G , or of an f.r. Lie group generated by G , or merely with respect to its structure.

THEOREM 3. *For any simple Lie algebra G there exists one and only one linear covering group $\hat{\mathfrak{G}}$. Furthermore any local isomorphism of $\hat{\mathfrak{G}}$ to any f.r. Lie group can be extended to a homomorphism in the large, in particular a local automorphism of $\hat{\mathfrak{G}}$ can be extended to an automorphism.*

The linear covering group of a complex simple Lie algebra is simply connected, and that of a real simple Lie algebra can be obtained as a real form of the simply connected group generated by its complex form.

Proof. A compact Lie group is f.r.⁷⁾ in virtue of the theory of almost pe-

⁷⁾ E. g. Chevalley [8], Chap. VI.

riodic functions. Let G^* be a complex (semi-) simple Lie algebra and K its compact form. Since K always generate a compact group by the above mentioned theorem of Weyl, there exists a simply connected linear Lie group \mathfrak{R} generated by K . Now by Lemma 10 there exists a complex-linear Lie group \mathfrak{R}_c which contains \mathfrak{R} as a real form, whence the Lie algebra of \mathfrak{R}_c is isomorphic with G^* . The space of \mathfrak{R}_c , the direct product of that of \mathfrak{R} and the Euclidean space of a certain dimension, is also simply connected. Therefore a simply connected complex (semi-) simple Lie group is f.r.

Next let G be a real simple Lie algebra, and let $\tilde{\mathfrak{G}}$ be the simply connected Lie group corresponding to G and \mathfrak{Z} the center of $\tilde{\mathfrak{G}}$. Denote by $\tilde{\mathfrak{G}}_c$ the simply connected Lie group corresponding to the complex form G_c of G , and by $\hat{\mathfrak{G}}$ a real form of $\tilde{\mathfrak{G}}_c$ whose Lie algebra is G . The structure of $\hat{\mathfrak{G}}$ is then determined uniquely. (See §6.) Let now $\hat{\mathfrak{Z}}$ be a discrete invariant subgroup of $\tilde{\mathfrak{G}}$ so that $\tilde{\mathfrak{G}}/\hat{\mathfrak{Z}} \cong \hat{\mathfrak{G}}$. Since $\hat{\mathfrak{G}}$ is f.r. and the center of $\hat{\mathfrak{G}}$ is isomorphic to $\mathfrak{Z}/\hat{\mathfrak{Z}}$, the index $[\mathfrak{Z} : \hat{\mathfrak{Z}}]$ must be finite by Lemma 5 in Part I.

Now let \mathfrak{G} be an f.r. Lie group corresponding to G . The existence of a complex form \mathfrak{G}_c of \mathfrak{G} can be assured by Lemma 10. Since in the homomorphism $\tilde{\mathfrak{G}}_c \sim \mathfrak{G}_c$ there corresponds \mathfrak{G} to $\hat{\mathfrak{G}}$, we have $\hat{\mathfrak{G}} \sim \mathfrak{G}$. Hence $\hat{\mathfrak{G}}$ is a linear covering group for G . We note also the fact that the order of the center of $\hat{\mathfrak{G}}$ is not less than that of \mathfrak{G} .

Suppose now that a linear representation $f, \tilde{\mathfrak{G}} \ni \tilde{g} \rightarrow f(\tilde{g})$, gives a faithful representation of $\hat{\mathfrak{G}}$, and let σ be an automorphism of $\tilde{\mathfrak{G}}$. The kernel of the representation of $\tilde{\mathfrak{G}}$ given by

$$\tilde{g} \rightarrow \begin{pmatrix} f(\tilde{g}) & 0 \\ 0 & f(\tilde{g}^\sigma) \end{pmatrix}$$

is clearly $\hat{\mathfrak{Z}} \cap \hat{\mathfrak{Z}}^\sigma = \mathfrak{D}$. Since $\tilde{\mathfrak{G}}/\mathfrak{D}$ is f.r., we have that $[\mathfrak{Z} : \mathfrak{D}] \leq [\mathfrak{Z} : \hat{\mathfrak{Z}}] < \infty$. Since \mathfrak{D} is contained in \mathfrak{Z} we get $\hat{\mathfrak{Z}} = \hat{\mathfrak{Z}}^\sigma = \mathfrak{D}$, namely $\hat{\mathfrak{Z}}$ is characteristic. The uniqueness of the linear covering group of G may be proved by a similar argument.

Next let \mathfrak{G} be an f.r. Lie group generated by G , and let φ_0 be a local isomorphism of $\hat{\mathfrak{G}}$ to \mathfrak{G} . Since $\hat{\mathfrak{Z}}$ is a characteristic subgroup of $\tilde{\mathfrak{G}}$, there exists a homomorphism φ so that $\varphi = \varphi_0$ locally, q.e.d.

In virtue of Lemma 9 and Theorem 3 we can count up all simple f.r. Lie groups, since all simply connected complex simple Lie groups are known.

For example in the four grand classes A_n, B_n, C_n ($n = 1, 2, \dots$) and D_n ($n = 3, 4, \dots$) of complex simple Lie algebras given by Lie-Killing-Cartan,³⁾

³⁾ E. g. Cartan [3].

we are familiar with the simply connected groups for the classes A_n and C_n i.e. the unimodular groups and the symplectic groups,⁹⁾ and for B_n and D_n the classes of the proper orthogonal groups, the so-called spin representations¹⁰⁾ give us simply connected groups.

The necessity of the second example of non-f.r. Lie groups given by Cartan,¹¹⁾ i.e. an arbitrary proper covering group of $\mathfrak{S}(2, P)$, is now clarified, because $\mathfrak{S}(2, P)$ is, as a real form of the complex unimodular group, which is simply connected, a linear covering group of itself.

THEOREM 4. *For any semi-simple Lie algebra G there exists one and only one linear covering group $\hat{\mathfrak{G}}$, and $\hat{\mathfrak{G}}$ has the property stated in the first half of Theorem 3 for a simple Lie algebra.*

Moreover $\hat{\mathfrak{G}}$ is decomposed into a direct product of simple Lie groups. If G is complex, $\hat{\mathfrak{G}}$ is simply conneted.

Proof. Decompose G into simple ideals: $G = G_1 + \dots + G_s$, and let $\hat{\mathfrak{G}}_i$ be the linear covering group of G_i . Then the direct product

$$\hat{\mathfrak{G}} = \hat{\mathfrak{G}}_1 \times \dots \times \hat{\mathfrak{G}}_s$$

is clearly the uniquely determined linear covering group of G , and other assertions are easy to prove by an analogous argument as in Theorem 3, q.e.d.

Now we can easily prove the following

COROLLARY 1. *A connected semi-simple Lie group is f.r. if and only if every (closed) simple invariant Lie sudgroup is f.r.*

Next let \mathfrak{G} be a topological group. In the introduction of Part I we defined a notion of an (*l*)-group. We repeat the definition: If there exists a set of closed invariant subgroups $\{\mathfrak{N}_\alpha\}$ of \mathfrak{G} such that

$$1) \mathfrak{G}/\mathfrak{N}_\alpha \text{ is f.r.} \quad \text{and} \quad 2) \bigcap \mathfrak{N}_\alpha = e,$$

then \mathfrak{G} is called an (*l*)-group.

COROLLARY 2. *A connected semi-simple Lie group is f.r. if it is an (*l*)-group.*

Proof. Let \mathfrak{G} be a connected semi-simple (*l*)-group and $\tilde{\mathfrak{G}}$ the universal covering group of \mathfrak{G} , and let $\hat{\mathfrak{G}}$ be the linear covering group of the Lie algebra of \mathfrak{G} . We may suppose that $\mathfrak{G} = \tilde{\mathfrak{G}}/\mathfrak{D}$ and $\hat{\mathfrak{G}} = \tilde{\mathfrak{G}}/\mathfrak{Z}$. Let z be an element of \mathfrak{Z} . Since $\mathfrak{G}/\mathfrak{N}_\alpha$ is f.r. there corresponds to z the unit element in the homomorphism $\tilde{\mathfrak{G}}(\sim \mathfrak{G}) \sim \mathfrak{G}/\mathfrak{N}_\alpha$, namely $z\mathfrak{D} \subseteq \mathfrak{N}_\alpha$, whence $z\mathfrak{D} \subseteq \mathfrak{D}$ because $\bigcap \mathfrak{N}_\alpha$ is the unit

⁹⁾ E. g. Weyl [24].

¹⁰⁾ Cartan [6].

¹¹⁾ Cartan [6]. Cf. §4 in Part I.

element. Therefore we have $\mathfrak{B} \subseteq \mathfrak{D}$, namely \mathfrak{G} is f.r., q.e.d

Solvable f.r. Lie groups

8. We first prove the following

LEMMA 11.¹²⁾ *A simply connected solvable Lie group is f.r.*

Proof. Let G be a solvable Lie algebra. Then there exists a linear Lie algebra G_1 isomorphic with G . Next since $G/D(G)$ is commutative, we may easily prove that there exists a linear Lie algebra G_2 , composed of nilpotent matrices, which is isomorphic with $G/D(G)$. Then the representation of G given by

$$\begin{pmatrix} G_1 & 0 \\ 0 & G_2 \end{pmatrix}$$

is obviously faithful. Furthermore in this representation there corresponds a nilpotent matrix to any element of $D(G)$ by a theorem of Lie, and to any element which does not belong to $D(G)$ there corresponds a non-zero nilpotent matrix in the representation G_2 . Hence the linear Lie algebra generates a simply connected linear Lie group by Lemma 6 in Part I, q.e.d.

LEMMA 12. *Let \mathfrak{G} be a linear Lie group and let \mathfrak{N} be a closed invariant subgroup, \mathfrak{H} a closed vector subgroup of \mathfrak{G} . Suppose that $\mathfrak{G} = \mathfrak{H}\mathfrak{N}$, $\mathfrak{H} \cap \mathfrak{N} = e$. If a discrete subgroup \mathfrak{D} of \mathfrak{H} is contained in the center of \mathfrak{G} , then $\mathfrak{G}/\mathfrak{D}$ is f.r.*

Proof. Let m be the dimension of \mathfrak{H} . It is clearly sufficient to prove the lemma when $m = 1$. We may suppose that \mathfrak{H} is a one-parameter group $\exp \lambda x$, where x is a matrix and λ a real parameter, furthermore that \mathfrak{D} is generated by $\exp x = d$. We decompose the vector space M on which \mathfrak{G} operates into eigen-spaces with respect to x :

$$M = M_\alpha + M_\beta + \dots,$$

where α, β, \dots denote eigen-values of x . Next we sum up eigen-spaces whose eigen-values are the same mod. $2\pi\sqrt{-1}$:

$$\begin{aligned} M &= M_1 + M_2 + \dots \\ \left\{ \begin{array}{l} M_1 = M_{\alpha(1)} + M_{\beta(1)} + \dots \quad \alpha(1) \equiv \beta(1) \equiv \dots \pmod{2\pi\sqrt{-1}}, \\ M_2 = M_{\alpha(2)} + M_{\beta(2)} + \dots \quad \alpha(2) \equiv \beta(2) \equiv \dots \pmod{2\pi\sqrt{-1}}, \\ \dots \end{array} \right. \end{aligned}$$

$M = M_1 + M_2 + \dots$ is then clearly the eigen-space decomposition with respect to $d = \exp x$, whose eigen-values are $\exp \alpha(1) = \exp \beta(1) = \dots, \exp \alpha(2), \dots$

¹²⁾ Cartan [5].

Since d is contained in the center, the eigen-spaces $M_l (l = 1, 2, \dots)$ are allowable by \mathfrak{G} , and hence \mathfrak{G} is decomposed into the representations induced on M_l .

Now define matrices x_1 and x_3 by the equations

$$\begin{aligned} x_1 e_l &= \alpha(l) e_l \quad \text{for } e_l \in M_l \quad l = 1, 2, \dots, \\ x_3 e_\rho &= \rho e_\rho \quad \text{for } e_\rho \in M_\rho \quad \rho = \alpha, \beta, \dots, \end{aligned}$$

and put

$$x_0 = x - x_3, \quad x_2 = x_3 - x_1,$$

then we have

$$x = x_0 + x_1 + x_2,$$

where x 's are mutually commutative, x_0 is nilpotent, and x_1 and x_2 are semi-simple.¹³⁾ Since the eigen-values of x_2 are all integral multiple of $2\pi\sqrt{-1}$, we have $\exp x_2 = 1_n$ where n denotes the dimension of M , and hence

$$d = \exp x_0 \exp x_1.$$

Then since d is contained in the center, we can easily prove that the matrix x_0 is commutative with any matrix of \mathfrak{G} . Hence we have

$$[\exp \lambda(x_0 + x_1), \mathfrak{G}] = 0,$$

for any real number λ . Therefore the correspondence defined by

$$\mathfrak{G} \ni g = \exp \lambda x \cdot h \rightarrow \exp \lambda x_2 \cdot h \quad h \in \mathfrak{N}$$

gives a representation \mathfrak{G}^* of \mathfrak{G} . Let \mathfrak{Z} be the kernel of the homomorphism. The relations $\mathfrak{N} \cap \mathfrak{Z} = e$ and $\mathfrak{D} \subseteq \mathfrak{Z}$ are now obvious.

Next since $\mathfrak{G}/\mathfrak{D}\mathfrak{N}$ is a one-dimensional toroidal group, there exists a linear representation \mathfrak{G}^{**} of \mathfrak{G} such that the kernel coincides with $\mathfrak{D}\mathfrak{N}$. Hence the representation of \mathfrak{G} given by

$$\begin{pmatrix} \mathfrak{G}^* & 0 \\ 0 & \mathfrak{G}^{**} \end{pmatrix}$$

is clearly a faithful representation of $\mathfrak{G}/\mathfrak{D}$, q.e.d.

LEMMA 13. *Let \mathfrak{G} be a connected Lie group and \mathfrak{N} a nilpotent Lie invariant subgroup. Then any compact subgroup of \mathfrak{N} is central in \mathfrak{G} .*

Proof. Let G be the Lie algebra of \mathfrak{G} , and N the subalgebra of G corresponding to \mathfrak{N} . Then an inner derivation of G induced by an element of N is always nilpotent. Hence the linear group \mathfrak{N}^* , corresponding to \mathfrak{N} in the adjoint representation of \mathfrak{G} , is simply connected, by Lemma 6 in Part I, and contains no compact subgroup except e . Hence a compact subgroup \mathfrak{R} is contained in the kernel of the adjoint representation, i.e. the center, q.e.d.

¹³⁾ A matrix is called semi-simple if it has simple elementary divisors.

THEOREM 5.¹⁴⁾ *Let \mathfrak{G} be a connected solvable Lie group. The following conditions are all necessary and sufficient for \mathfrak{G} to be f.r.: 1) $\mathfrak{G} = \mathfrak{U}\mathfrak{N}$ $\mathfrak{U} \cap \mathfrak{N} = e$, where \mathfrak{U} is a (maximal) compact subgroup and \mathfrak{N} is a closed simply connected invariant subgroup. 2) $C(\mathfrak{G})(= \overline{D(\mathfrak{G})})$ is simply connected. 3) The center of $C(\mathfrak{G})$ is connected and simply connected. 4) $\mathfrak{U} \cap D(\mathfrak{G}) = e$, where \mathfrak{U} is a maximal compact group of \mathfrak{G} . We note here the fact that a compact connected solvable Lie group is commutative, and hence a toroidal group.*

Proof. Let G be the Lie algebra of \mathfrak{G} and A a subalgebra of G corresponding to a maximal compact subgroup \mathfrak{U} . First we shall proceed to prove the following implications: (\mathfrak{G} is f.r.) \rightarrow 4) \rightarrow 1) \rightarrow (\mathfrak{G} is f.r.)

Let \mathfrak{G} be f.r. Since there corresponds a semi-simple matrix for an element of A , and a nilpotent matrix for an element of $D(G)$ in any representation of \mathfrak{G} , we have $A \cap D(G) = 0$. On the other hand the commutator group $D(\mathfrak{G})$ is closed and simply connected by Lemma 7 in Part I. Hence we get $\mathfrak{U} \cap D(\mathfrak{G}) = e$, i.e. 4).

Next from 4) we have $A \cap D(G) = 0$, and hence there exists a subspace N of G such that $N \cong D(G)$, $G = A + N$ $A \cap N = 0$. N is then obviously an ideal. Let $\tilde{\mathfrak{G}}$ be the universal covering group of \mathfrak{G} and let $\tilde{\mathfrak{U}}, \tilde{\mathfrak{N}}$ be (closed) subgroups of $\tilde{\mathfrak{G}}$ generated by A, N respectively. Then the simple connectedness of the factor group $\tilde{\mathfrak{G}}/\tilde{\mathfrak{N}}$ by Lemma 3 in Part I implies that $\tilde{\mathfrak{U}} \cap \tilde{\mathfrak{N}} = e$. Let \mathfrak{D} be the kernel of the homomorphism $\tilde{\mathfrak{G}} \sim \mathfrak{G}$. Then \mathfrak{D} is obviously contained in $\tilde{\mathfrak{U}}$ because \mathfrak{U} is a maximal compact subgroup of \mathfrak{G} . Hence N generates a closed simply connected group \mathfrak{N} in \mathfrak{G} . That $\mathfrak{U} \cap \mathfrak{N} = e$ follows immediately. Thus we get 1).

Now suppose that the condition 1) is satisfied. Then by Lemma 11 the universal covering group of \mathfrak{G} is f.r., and using Lemma 12 we get a faithful representation of \mathfrak{G} . Thus we have the equivalence of the faithful representability, 1) and 4).

Now from the simple connectedness of the adjoint group of a connected nilpotent Lie group and by Lemma 3 in Part I we can easily prove that 2) is equivalent to 3). Next from 1) we get 2) because $C(\mathfrak{G})$ is contained in \mathfrak{N} . Suppose now 2) is satisfied for \mathfrak{G} . Then $D(\mathfrak{G})$ is, as a Lie subgroup of the simply connected $C(\mathfrak{G})$, closed, and hence coincides with $C(\mathfrak{G})$, q.e.d.

From Theorem 5 and Lemma 13 we get the following

COROLLARY 1.¹⁵⁾ *A connected nilpotent Lie group is f.r. if and only if it is*

¹⁴⁾ Cf. Malcev [11], [13].

¹⁵⁾ Cf. Birkhoff [19].

a direct product of a compact group and a simply connected Lie group.

COROLLARY 2. A connected solvable Lie (*l*)-group is f.r.

Proof. Let \mathfrak{G} be a connected solvable Lie group. Suppose that \mathfrak{G} is not f.r. Then the center of $C(\mathfrak{G})$ contains a compact group $\mathfrak{R}(\neq e)$ by Theorem 5. Since \mathfrak{R} is mapped into the unit matrix in any linear representation of \mathfrak{G} , \mathfrak{G} is not an (*l*)-group. q.e.d.

Remark. A connected non-f.r. solvable Lie group has no locally isomorphic (univalent) representation.

F.r. Lie groups

9. Let G be a Lie algebra and R the radical of G , and let R_1 be the intersection of R and $D(G)$. Then the factor algebra G/R_1 is decomposed into a direct sum of a semi-simple ideal S and the center Z :

$$G/R_1 = S + Z \quad S \cap Z = 0 \quad [S, Z] = 0$$

Let $\hat{\mathfrak{S}}$ be the linear covering group of S , and Z the m -dimensional vector group, where m denotes the dimension of Z . The direct product $\hat{\mathfrak{S}} \times \mathfrak{Z}$ is then clearly f.r., and furthermore there exists a faithful representation of $\hat{\mathfrak{S}} \times \mathfrak{Z}$ such that the center of the corresponding linear Lie algebra consists of nilpotent matrices. Let G_1 be the linear Lie algebra of such a representation. Let now G_2 be an arbitrary faithful representation of G . Then the representation of G defined by

$$G^* = \begin{pmatrix} G_1 & 0 \\ 0 & G_2 \end{pmatrix}$$

is of course faithful.

Let now $\hat{\mathfrak{G}}$ be the linear Lie group generated by G^* . Then we may easily prove that the radical $\hat{\mathfrak{R}}$ of $\hat{\mathfrak{G}}$ is simply connected in such a way as in the proof of Lemma 11. Let \mathfrak{S} be a maximal semi-simple Lie subgroup of $\hat{\mathfrak{G}}$. Since \mathfrak{S} is closed and the center of \mathfrak{S} is finite in virtue of § 4 in Part I, $\mathfrak{S} \cap \hat{\mathfrak{R}}$ is also a finite group, whence the simple connectedness of $\hat{\mathfrak{R}}$ implies that $\mathfrak{S} \cap \hat{\mathfrak{R}} = e$. On the other hand since $\hat{\mathfrak{G}}$ has a representation which is isomorphic to $\hat{\mathfrak{S}}$, we get that $\mathfrak{S} \cong \hat{\mathfrak{G}}/\hat{\mathfrak{R}} \sim \hat{\mathfrak{S}}$, whence we have $\mathfrak{S} \cong \hat{\mathfrak{S}}$ comparing the orders of the centers of \mathfrak{S} and $\hat{\mathfrak{S}}$.

Now we shall prove that our $\hat{\mathfrak{G}}$ is a linear covering group of G . First let $\tilde{\mathfrak{G}}$ be the universal covering group of $\hat{\mathfrak{G}}$ and let \mathfrak{G} be any f.r. Lie group corresponding to G . We may suppose that $\tilde{\mathfrak{G}}/\mathfrak{D} = \mathfrak{G}$ and $\tilde{\mathfrak{G}}/\mathfrak{B} = \hat{\mathfrak{G}}$. Now let $\tilde{\mathfrak{S}}$ be a maximal semi-simple Lie subgroup of $\tilde{\mathfrak{G}}$. Since the space of $\hat{\mathfrak{G}}$ is a direct product of the space of $\hat{\mathfrak{S}}$ and the Euclidean space, \mathfrak{B} is contained in $\tilde{\mathfrak{S}}$. Since

$\tilde{\mathfrak{D}}/\mathfrak{D}(\cong \tilde{\mathfrak{E}}/\tilde{\mathfrak{E}} \cap \mathfrak{D})$, a maximal semi-simple Lie subgroup of \mathfrak{G} , is f.r., we get that $\tilde{\mathfrak{E}} \cap \mathfrak{D} \cong \mathfrak{Z}$, whence \mathfrak{Z} is contained in \mathfrak{D} : $\mathfrak{Z} \subseteq \mathfrak{D}$. Hence \mathfrak{G} is homomorphic with $\hat{\mathfrak{G}}$, namely $\hat{\mathfrak{G}}$ is a linear covering group. Here we note the fact that the discrete invariant subgroup \mathfrak{Z} is characteristic, which will be seen easily. Now the uniqueness of the linear covering group for any Lie algebra may be proved as in the proof of Theorem 3.

Next let $\hat{\mathfrak{G}}$ be a linear covering group and \mathfrak{R} a Lie invariant subgroup of $\hat{\mathfrak{G}}$. An analogous argument as in Lemma 3 in Part I shows that \mathfrak{R} is closed and \mathfrak{R} and $\hat{\mathfrak{G}}/\mathfrak{R}$ are both linear covering groups.

From the above considerations we may easily prove the following

THEOREM 6. *For any Lie algebra G there exists one and only one linear covering group $\hat{\mathfrak{G}}$.*

The radical \mathfrak{R} of $\hat{\mathfrak{G}}$ is simply connected, and a maximal semi-simple Lie subgroup $\hat{\mathfrak{G}}$ is a linear covering group such that

$$\hat{\mathfrak{G}} = \hat{\mathfrak{E}}\mathfrak{R}, \quad \hat{\mathfrak{E}} \cap \mathfrak{R} = e.$$

In particular, a linear covering group of a complex Lie algebra is simply connected.¹⁶⁾

Any local homomorphism of a linear covering group onto any f.r. Lie group can be extended to a homomorphism in the large.

Remark. A connected Lie group is homomorphic with the linear covering group of its Lie algebra if its maximal semi-simple Lie subgroup is f.r.

THEOREM 7.¹⁷⁾ *Let \mathfrak{G} be a connected Lie group and \mathfrak{R} the radical of \mathfrak{G} . If a maximal semi-simple Lie subgroup \mathfrak{S} and \mathfrak{R} are f.r., then \mathfrak{G} is also f.r., and vice, versa.*

Proof. Let G be the Lie algebra of \mathfrak{G} and R the radical of G . Since the radical \mathfrak{R} is f.r. and solvable there exist closed subgroups \mathfrak{U} and \mathfrak{U}' of \mathfrak{R} such that

$$\mathfrak{R} = \mathfrak{U}\mathfrak{U}', \quad \mathfrak{U} \cap \mathfrak{U}' = e,$$

where \mathfrak{U} is a toroidal group and \mathfrak{U}' is simply connected and invariant in \mathfrak{R} . Let $h(\lambda)$ be an everywhere dense one-parameter subgroup of \mathfrak{U} : $\overline{h(\lambda)} = \mathfrak{U}$, and let x be an element of G which generates $h(\lambda)$. Now we decompose G and R into eigenspaces of the inner derivation δ_x induced by x . Since δ_x is semi-simple because of the compactness of \mathfrak{U} , we may put

¹⁶⁾ Matsushima [15].

¹⁷⁾ Malcev [13].

$$\begin{aligned} G &= G_0 + G_1, \quad \delta_x G_0 = 0, \quad \delta_x G_1 = G_1, \\ R &= R_0 + R_1, \quad \delta_x R_0 = 0, \quad \delta_x R_1 = R_1, \end{aligned}$$

where G_0 and R_0 are subalgebras. Since it is clear that $G_0 \cap R = R_0$ and $G_0 + R = G$, we have

$$G_0/R_0 \cong G/R \quad (\text{semi-simple}).$$

Hence R_0 is the radical of G_0 . Let S be a maximal semi-simple subalgebra of G . Then S is evidently maximal semi-simple in G . Let A be the Lie algebra of \mathfrak{U} . Since $\delta_x G_0 = 0$, we have $[A, G_0] = 0$, and in particular $[A, S] = 0$. Now let us consider R as an S -module. Then A and $D(R)$ are both S -modules such that $A \cap D(R) = 0$. Hence from the complete reducibility of R there exists a submodule N containing $D(R)$ such that

$$R = A + N, \quad A \cap N = 0$$

N is, as an ideal of R and an S -module, an ideal of G . Let now \mathfrak{N} be the subgroup of \mathfrak{G} generated by N . \mathfrak{N} is easily seen to be closed and simply connected. Thus we get the decomposition

$$\mathfrak{N} = \mathfrak{U}\mathfrak{N} \quad \mathfrak{U} \cap \mathfrak{N} = e,$$

where \mathfrak{U} is a toroidal group and \mathfrak{N} is a closed simply connected invariant subgroup of \mathfrak{G} .

Next let \mathfrak{S} be the subgroup generated by S . Since the center of \mathfrak{S} is finite in virtue of Lemma 5 in Part I because \mathfrak{S} is f.r., \mathfrak{S} is closed by Theorem 2 in Part I. Then the compactness of \mathfrak{U} implies that $\mathfrak{S}\mathfrak{U}$ is closed. Now since the center of $\mathfrak{S}\mathfrak{U}$ is compact and \mathfrak{N} contains no compact subgroup except e , we have $\mathfrak{S}\mathfrak{U} \cap \mathfrak{N} = e$, whence \mathfrak{G} is topologically a direct product of $\mathfrak{S}\mathfrak{U}$ and \mathfrak{N} : $\mathfrak{G} = (\mathfrak{S}\mathfrak{U})\mathfrak{N}$.¹⁸⁾

Let now $\hat{\mathfrak{G}}$ be the linear covering group of G . Then by the remark to Theorem 6, $\hat{\mathfrak{G}}$ is a covering group of \mathfrak{G} . Let $\hat{\mathfrak{S}}$, $\hat{\mathfrak{U}}$ and $\hat{\mathfrak{N}}$ be subgroups of $\hat{\mathfrak{G}}$ generated by S , A and N respectively. Then $\hat{\mathfrak{G}} = \hat{\mathfrak{S}}\hat{\mathfrak{U}}\hat{\mathfrak{N}}$ and the space of $\hat{\mathfrak{G}}$ is the direct product of the spaces of $\hat{\mathfrak{S}}$, $\hat{\mathfrak{U}}$ and $\hat{\mathfrak{N}}$. Since \mathfrak{N} is simply connected, we have $\mathfrak{D} \equiv \hat{\mathfrak{S}}\hat{\mathfrak{U}}$, where \mathfrak{D} denotes the kernel of the homomorphism $\hat{\mathfrak{G}} \sim \mathfrak{G}$.

¹⁸⁾ Matsushima in [15] proved the following theorem: *Let \mathfrak{G} be a connected Lie group and \mathfrak{H} and \mathfrak{N} closed subgroups. Let H, N be their Lie algebras. Then \mathfrak{G} is f.r. if the following conditions are satisfied. 1) \mathfrak{N} is a simply connected solvable invariant subgroup and \mathfrak{H} is f.r. 2) $\mathfrak{G} = \mathfrak{H}\mathfrak{N}$ $\mathfrak{H} \cap \mathfrak{N} = e$. 3) N is completely reducible as an H -module.* Using this theorem we get another proof of Theorem 7, because $\mathfrak{S}\mathfrak{U}$, which is a direct product of f.r. \mathfrak{S} and a toroidal group \mathfrak{U} mod. a finite group, is f.r. by Lemma 8, and our N is clearly a completely reducible $(S + A)$ -module. We note also the fact that our Theorem 7 and our decomposition $\mathfrak{G} = (\mathfrak{S}\mathfrak{U})\mathfrak{N}$ show the validity of the converse of the above Theorem of Matsushima.

Now let \mathfrak{B} be a maximal compact subgroup of \mathfrak{S} , and B its Lie algebra. Then $\mathfrak{K} = \mathfrak{B}\mathfrak{A}$ is obviously a maximal compact subgroup of $\mathfrak{S}\mathfrak{A}$, and the Lie algebra K of \mathfrak{K} coincides with $A + B$. Let \mathfrak{H} be the subgroup of $\hat{\mathfrak{G}}$ generated by B . Since $\hat{\mathfrak{S}}$ is a covering group of \mathfrak{S} of finite order, \mathfrak{H} is compact. Now since \mathfrak{K} is a maximal compact subgroup of $\mathfrak{S}\mathfrak{A}$, we may easily conclude that $\hat{\mathfrak{K}} = \mathfrak{H}\hat{\mathfrak{A}} \cong \mathfrak{D}$. Let A be of m dimensions. Then \mathfrak{A} is the m -dimensional vector group. Hence \mathfrak{D} is, as a discrete invariant subgroup of a direct product of a compact group and an m -dimensional vector group, of rank at most m . On the other hand, $\mathfrak{D}_1 = \mathfrak{D} \cap \hat{\mathfrak{A}}$ is obviously a free commutative group of rank m . Hence the factor group $\mathfrak{D}/\mathfrak{D}_1$ is finite. Now since $\hat{\mathfrak{S}}\hat{\mathfrak{A}}$ is a closed invariant subgroup and $\hat{\mathfrak{A}}$ is a vector group, Lemma 12 shows that $\hat{\mathfrak{G}}/\mathfrak{D}_1$ is f.r. Then, $\mathfrak{D}/\mathfrak{D}_1$, being a finite group, the relation

$$\mathfrak{G} \cong \hat{\mathfrak{G}}/\mathfrak{D} \cong \hat{\mathfrak{G}}/\mathfrak{D}_1 / \mathfrak{D}/\mathfrak{D}_1,$$

implies the faithful representability of \mathfrak{G} by Lemma 8, q.e.d.

COROLLARY 1. *Let \mathfrak{G} be an f.r. Lie group, and \mathfrak{K} a maximal compact subgroup of \mathfrak{G} . Then there exists a closed simply connected solvable subgroup \mathfrak{H} of \mathfrak{G} such that*

$$\mathfrak{G} = \mathfrak{K}\mathfrak{H} = \mathfrak{H}\mathfrak{K}, \quad \mathfrak{H} \cap \mathfrak{K} = e.$$

Proof. The assertion was proved by K. Iwasawa¹⁹⁾ for adjoint groups of semi-simple Lie groups. By our Lemma 5 in Part I it is also valid for semi-simple f.r. Lie groups. Then the decomposition of an f.r. Lie group \mathfrak{G} :

$$\mathfrak{G} = (\mathfrak{S}\mathfrak{A})\mathfrak{K}, \quad \mathfrak{S}\mathfrak{A} \cap \mathfrak{K} = e,$$

given in the proof of Theorem 7, easily implies our corollary, q.e.d.

Next let us call an f.r. group \mathfrak{G} *completely reducible* if any linear representation of \mathfrak{G} is completely reducible. Semi-simple, or compact f.r. groups are completely reducible, but a vector group is not.

Let \mathfrak{G} be a completely reducible f.r. group and G its Lie algebra. By a theorem of Cartan, an irreducible linear Lie algebra is a direct sum of a semi-simple ideal and its center which is composed of scalar matrices. Using this we can readily prove that G is decomposed into a direct sum of its center Z and a semi-simple ideal S : $G = S + Z$, $[S, Z] = 0$. Now let \mathfrak{S} and \mathfrak{Z} be subgroups of \mathfrak{G} generated by S and Z respectively. Assume that \mathfrak{Z} is not compact. Then the finiteness of the center of \mathfrak{S} by Lemma 5 in Part I obviously implies that there exists a closed vector invariant subgroup of \mathfrak{G} which is a direct factor. Hence \mathfrak{G} is not completely reducible, contrary to the assumption.

¹⁹⁾ Iwasawa [10].

Thus \mathfrak{G} must be compact. Since the converse proposition is clear, we have the following

LEMMA 14. *An f.r. group \mathfrak{G} is completely reducible if and only if the radical of \mathfrak{G} is compact.*

Now the following corollary is an immediate consequence of Theorem 7. (Cf. also Note ¹³⁾ of p. 103).

COROLLARY 2. *Let \mathfrak{G} be a connected Lie group. If there exist closed subgroups \mathfrak{I} and \mathfrak{N} of \mathfrak{G} such that $\mathfrak{G} = \mathfrak{I}\mathfrak{N}$ $\mathfrak{I} \cap \mathfrak{N} = e$, where \mathfrak{I} is a completely reducible f.r. group and \mathfrak{N} is a simply connected solvable invariant subgroup, then \mathfrak{G} is f.r., and vice versa.*

10. In this § three theorems shall be proved concerning f.r. Lie groups.

THEOREM 8. *A connected Lie group is f.r. if it is an (l)-group.*

Proof. Let \mathfrak{G} be a connected Lie group and let \mathfrak{S} and \mathfrak{R} be a maximal semi-simple Lie subgroup and the radical of \mathfrak{G} respectively. Suppose that \mathfrak{G} is an (l)-group. Then \mathfrak{S} and \mathfrak{R} are also (l)-groups. Now Corollary 2 to Theorem 4 and Corollary 2 to Theorem 5 imply that \mathfrak{S} and \mathfrak{R} are both f.r., whence \mathfrak{G} is f.r. by Theorem 7, q.e.d.

THEOREM 9. *Any f.r. Lie group is isomorphic with a closed subgroup of the general linear group of a certain degree.* In other words, for any f.r. Lie group there exists a topologically isomorphic linear representation with respect to the induced topology.

Proof. Let \mathfrak{G} be an f.r. Lie group and \mathfrak{R} the radical of \mathfrak{G} . The commutator subgroup $D(\mathfrak{G})$ is closed and the radical \mathfrak{R}_1 of $D(\mathfrak{G})$ is simply connected by Lemma 7 in Part I. Denote by \mathfrak{U} a maximal compact subgroup of \mathfrak{R} . Then there exists a closed simply connected invariant subgroup \mathfrak{N} such that $\mathfrak{R} = \mathfrak{U}\mathfrak{N}$, $\mathfrak{U} \cap \mathfrak{N} = e$, and $\mathfrak{N} \cong \mathfrak{R}_1$. As \mathfrak{U} is compact, $\mathfrak{U}D(\mathfrak{G})$ is a closed invariant subgroup. Let

$$\mathfrak{G} = (\mathfrak{S}\mathfrak{U})\mathfrak{N}, \quad \mathfrak{S}\mathfrak{U} \cap \mathfrak{N} = e$$

be a decomposition as in the proof of Theorem 7. Since $\mathfrak{S}\mathfrak{U}\mathfrak{N}_1$ coincides with $\mathfrak{U}D(\mathfrak{G})$ locally and they are both connected, we have

$$\mathfrak{S}\mathfrak{U}\mathfrak{N}_1 = \mathfrak{U}D(\mathfrak{G}),$$

whence $\mathfrak{U}D(\mathfrak{G}) \cap \mathfrak{N} = \mathfrak{N}_1$. Now from

$$\mathfrak{G}/\mathfrak{U}D(\mathfrak{G}) = \mathfrak{U}D(\mathfrak{G})\mathfrak{N}/\mathfrak{U}D(\mathfrak{G}) \cong \mathfrak{N}/\mathfrak{N} \cap \mathfrak{U}D(\mathfrak{G}) = \mathfrak{N}/\mathfrak{N}_1,$$

we see that $\mathfrak{G}/\mathfrak{U}D(\mathfrak{G})$ is a vector group. Hence there exists surely a representation of \mathfrak{G} such that the kernel coincides with $\mathfrak{U}D(\mathfrak{G})$ and whose linear Lie

algebra consists of nilpotent matrices. Let \mathfrak{G}^* be such a representation.

Next let \mathfrak{G}^{**} be a faithful representation of \mathfrak{G} . The representation defined by

$$\begin{pmatrix} \mathfrak{G}^* & 0 \\ 0 & \mathfrak{G}^{**} \end{pmatrix}$$

is of course faithful. On the other hand in this representation any non-zero element of the Lie algebra of \mathfrak{N} is represented by a matrix with non-zero 0-eigenspace, whence \mathfrak{N} is closed in the general linear group by Lemma 6 in Part I. Since \mathfrak{N} is compact, the radical $\mathfrak{R} = \mathfrak{N}\mathfrak{N}$ is also represented by a closed group. Thus Lemma 5 and Theorem 2 in Part I imply our assertion, q.e.d.

Remark. The above two theorems give us apparently weaker but equivalent definitions of f.r. groups.

Examples of non-f.r. Lie groups have been given by Cartan and Birkhoff. (See § 4 in Part I and § 7). Here we shall prove a theorem which shows the necessity of the above examples, namely the following

THEOREM 10. *A connected Lie group \mathfrak{G} is f.r. if and only if the following two conditions are satisfied.*

- 1) *All simple Lie subgroups are f.r.*
- 2) *The radical of $C(\mathfrak{G})$ is simply connected.*

Proof. If \mathfrak{G} is f.r., the above two conditions are obvious. Conversely let us suppose that 1) and 2) are satisfied. Let \mathfrak{S} be a maximal semi-simple Lie subgroup of \mathfrak{G} and \mathfrak{R} the radical of \mathfrak{G} . Since any simple invariant Lie subgroup of \mathfrak{S} is f.r., \mathfrak{S} is also f.r. by Corollary 1 to Theorem 4. Now since $C(\mathfrak{R})$ is contained in the radical of $C(\mathfrak{G})$, it is simply connected, whence \mathfrak{R} is f.r. by Theorem 5. Therefore the theorem follows from Theorem 7, q.e.d.

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