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# ON CLASSIFICATION OF $\mathbb{Q}$-FANO 3-FOLDS OF GORENSTEIN INDEX 2. I

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**Abstract.** We formulate a generalization of K. Takeuchi's method to classify smooth Fano 3-folds and use it to give a list of numerical possibilities of $\mathbb{Q}$-Fano 3-folds $X$ with $\operatorname{Pic} X = \mathbb{Z}(-2K_X)$ and $h^0(-K_X) \geq 4$ containing index 2 points $P$ such that $(X, P) \simeq (\{xy + z^2 + u^a = 0\}/\mathbb{Z}_2(1,1,1,0), o)$ for some $a \in \mathbb{N}$. In particular we prove that then $(-K_X)^3 \leq 15$ and $h^0(-K_X) \leq 10$. Moreover we show that such an $X$ is birational to a simpler Mori fiber space.

## Notation and Conventions

$\mathbb{N}$: The set of positive integers.
$\sim$: Linear equivalence.
$\equiv$: Numerical equivalence.
$\mathbb{F}_n$: Segre-del Pezzo scroll of degree $n$.
$\mathbb{F}_{n,0}$: Surface obtained by contracting the negative section of $\mathbb{F}_n$.
$Q_3$: Smooth quadric 3-fold.
ODP: Ordinary double point, i.e., singularity analytically isomorphic to $\{xy + z^2 + u^2 = 0 \subset \mathbb{C}^4\}$.
QODP: Singularity analytically isomorphic to $\{xy + z^2 + u^2 = 0 \subset \mathbb{C}^4/\mathbb{Z}_2(1,1,1,0)\}$.
$B_i$ $(1 \leq i \leq 5)$: Factorial Gorenstein terminal Fano 3-fold of Fano index 2, and with Picard number 1 and $(-K)^3 = 8i$, where $K$ is the canonical divisor.
$A_{2g-2}$ $(1 \leq g \leq 12$ and $g \neq 11)$: Factorial Gorenstein terminal Fano 3-fold of Fano index 1, and with Picard number 1 and genus $g$.

## §0. Introduction

In this paper we work over $\mathbb{C}$, the complex number field.

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Definition 0.0. ($\mathbb{Q}$-Fano variety) Let $X$ be a normal projective variety. $X$ is said to be a *terminal* (resp. *canonical*, *klt*, etc.) *$\mathbb{Q}$-Fano variety* if $X$ has only terminal (resp. canonical, Kawamata log terminal, etc.) singularities and $-K_X$ is ample. By replacing 'ample' with 'nef and big', *terminal* (resp. *canonical*, *klt*, etc.) *weak $\mathbb{Q}$-Fano varieties* are similarly defined. If $X$ has only terminal singularities, then we say that $X$ is a *$\mathbb{Q}$-Fano variety* for short and if $X$ has only Gorenstein terminal (resp. canonical, klt, etc.) singularities, we say that $X$ is a *Gorenstein terminal* (resp. *canonical*, *klt*, etc.) *Fano variety*.

Let $I(X) := \min\{I \mid IK_X \text{ is a Cartier divisor}\}$ and we call $I(X)$ the *Gorenstein index* of $X$.

Write $I(X)(-K_X) \equiv r(X)H(X)$, where $H(X)$ is a primitive Cartier divisor and $r(X) \in \mathbb{N}$. (Note that $H(X)$ is unique since $\operatorname{Pic} X$ is torsion free.) Then we call $r(X)/I(X)$ the *Fano index* of $X$ and denote it by $F(X)$.

G. Fano started the study of Fano 3-folds to prove the irrationality of smooth cubic 3-folds. Since then many people studied smooth Fano 3-folds. The minimal model program asserts that every projective variety is birationally equivalent to a minimal variety or a variety having a $\mathbb{Q}$-Fano fiber space structure. So it is important to study $\mathbb{Q}$-Fano varieties, which is a generalization of Fano varieties.

Here we mention the known results about the classification of $\mathbb{Q}$-Fano 3-folds:

(1) G. Fano started the classification of smooth Fano 3-folds and it was completed by V. A. Iskovskih [Isk77], [Isk78], [Isk79] and [Isk90], V. V. Shokurov [Sho79b], [Sho79a], T. Fujita [Fuj80], [Fuj81] and [Fuj84], S. Mori and S. Mukai [MM81], [MM83] and [MM85].
(2) S. Mukai [Muk95] classified indecomposable Gorenstein canonical Fano 3-folds by using vector bundles.
(3) T. Sano [San95] and independently F. Campana and H. Flenner [CF93] classified non-Gorenstein Fano 3-folds of Fano index $> 1$.
(4) T. Sano [San96] classified non-Gorenstein Fano 3-folds of Fano index 1 and with only cyclic quotient terminal singularities. Recently T. Minagawa [Min99] proved that non-Gorenstein $\mathbb{Q}$-Fano 3-folds with Fano index 1 can be deformed to one with only cyclic quotient terminal singularities.
(5) A. R. Fletcher [Fle00] gave the classification of $\mathbb{Q}$-Fano 3-folds which are weighted complete intersections of codimension 1 or 2. Recently

S. Altınok [Alt] (see also [Reid96]) obtained a list of $\mathbb{Q}$-Fano 3-folds which are subvarieties in a weighted projective space of codimension 3 or 4.

On the other hand, K. Takeuchi [Take89] simplified and amplified V. A. Iskovskih's method of classification by simple numerical calculations based on the theory of the extremal rays. In particular he reproved Shokurov's theorem [Sho79a], the existence of lines on a smooth Fano 3-fold of Fano index 1 and with Picard number 1.

In this paper, we formulate a generalization of Takeuchi's construction for a $\mathbb{Q}$-factorial $\mathbb{Q}$-Fano 3-fold $X$ with $\rho(X) = 1$, and use it to classify $\mathbb{Q}$-factorial $\mathbb{Q}$-Fano 3-folds $X$ with the following properties.

Main Assumption 0.1. (1) $\rho(X) = 1$,
(2) $I(X) = 2$,
(3) $F(X) = 1/2$,
(4) $h^0(-K_X) \geq 4$, and
(5) there exists an index 2 point $P$ such that

$$(X, P) \simeq (\{xy + z^2 + u^a = 0\}/\mathbb{Z}_2(1,1,1,0), o)$$

for some $a \in \mathbb{N}$.

The intent of the assumption (5) is explained in the end of 0.2.

A generalized Takeuchi's construction 0.2. Here we explain a generalization of Takeuchi's construction. Let $X$ be a $\mathbb{Q}$-factorial $\mathbb{Q}$-Fano 3-fold with $\rho(X) = 1$. Suppose that we are given a birational morphism $f\colon Y \to X$ with the following properties.

(1) $Y$ is a weak $\mathbb{Q}$-Fano 3-fold, and
(2) $f$ is an extremal contraction such that $E := \operatorname{exc} f$ is a prime $\mathbb{Q}$-Cartier divisor.

Then we obtain the following diagram (see §3).

$$\begin{array}{ccccccc} & Y_0 := Y & \overset{g_0}{\dashrightarrow} & Y_1 \overset{g_1}{\dashrightarrow} \cdots & \overset{g_{k-1}}{\dashrightarrow} & Y_k & \\ {}^{f}\swarrow & & & & & & \searrow^{f'} \\ X & & & & & & X', \end{array}$$

where

(1) $Y_0 \dashrightarrow Y_1$ is a flop or a flip and $Y_i \dashrightarrow Y_{i+1}$ is a flip for $i \geq 1$, and
(2) $f'$ is a crepant divisorial contraction (in this case, $k = 0$) or an extremal contraction which is not isomorphic in codimension 1.

We use the following notation.

- $Y' := Y_k$,
- $E_i :=$ the strict transform of $E$ on $Y_i$,
- $\widetilde{E} :=$ the strict transform of $E$ on $Y'$,
- $e := E^3 - {E_1}^3$ if $Y_0 \dashrightarrow Y_1$ is a flop or $e := 0$ otherwise, and
- $d_i := (-K_{Y_i})^3 - (-K_{Y_{i+1}})^3$ (resp. $a_i := \frac{E_i \cdot l_i}{(-K_{Y_i}) \cdot l_i}$) if $Y_i \dashrightarrow Y_{i+1}$ is a flip with a flipping curve $l_i$, or $d_i := 0$ (resp. $a_i := 0$) otherwise.

We define rational numbers $z$ and $u$ as follows. In case $f'$ is birational, the $f'$-exceptional divisor $E'$ satisfies $E' \equiv z(-K_{Y'}) - u\widetilde{E}$. Otherwise the pull-back $L$ of the ample generator of $\operatorname{Pic} X' \simeq \mathbb{Z}$ satisfies $L \equiv z(-K_{Y'}) - u\widetilde{E}$.

We note the following.

(1)

$$(-K_{Y'})^2\widetilde{E} = (-K_Y)^2 E - \sum a_i d_i,$$
$$(-K_{Y'})\widetilde{E}^2 = (-K_Y)E^2 - \sum {a_i}^2 d_i,$$
$$\widetilde{E}^3 = E^3 - e - \sum {a_i}^3 d_i$$

(See Lemma 3.1 for details).

(2) On the other hand, the properties of $f'$ in various cases restrict the relation of $(-K_{Y'})^3$, $(-K_{Y'})^2\widetilde{E}$, $(-K_{Y'})\widetilde{E}^2$ and $\widetilde{E}^3$. For example, assume that $\dim X' = 1$. Then we have

$$(-K_{Y'})L^2 = z^2(-K_{Y'})^3 - 2zu(-K_{Y'})^2\widetilde{E} + u^2(-K_{Y'})\widetilde{E}^2 = 0,$$
$$\widetilde{E}L^2 = z^2(-K_{Y'})^2\widetilde{E} - 2zu(-K_{Y'})\widetilde{E}^2 + u^2\widetilde{E}^3 = 0,$$
$$(-K_{Y'})^2L = z(-K_{Y'})^3 - u(-K_{Y'})^2\widetilde{E} = \deg F,$$

where $F$ is a general fiber of $f'$ and $\deg F := (-K_F)^2$.

(1) and (2) give equations of Diophantine type. In this paper, we show that under Assumption 0.1, Construction 0.2 works for $X$ and the weighted blow-up $f$ with weights $\frac{1}{2}(1,1,1,2)$ at an index 2 point satisfying Assumption 0.1 (5). By the description of the weighted blow-up $f$ and

the flips $Y_i \dashrightarrow Y_{i+1}$, the index of $Y_i$'s are not greater that 2. Hence the equations of Diophantine type can be solved and we obtain the following possibilities of $X$.

Main Theorem 0.3. (see Theorem 5.0) *Let $X$ be as in Main Assumption 0.1, and $f\colon Y \to X$ the weighted blow-up at $P$ with weights $\frac{1}{2}(1,1,1,2)$. Then $Y$ is a weak $\mathbb{Q}$-Fano 3-fold with $I(Y)=2$.*

*Consider the diagram as in 0.2. Then the possibilities of $X$ are classified as in Tables 1–5 and Tables 1′–5′ with the notation of 0.2 and the following additional notation (the possibilities in Tables 1′–5′ are excluded in the forthcoming paper [Taka02]). In particular we have $(-K_X)^3 \leq 15$ and $h^0(-K_X) \leq 10$.*

*$h := h^0(-K_X)$,*
*$N$ is the number of $\frac{1}{2}(1,1,1)$-singularities obtained by deforming non-Gorenstein points of $X$ locally, and*
*$n$ is the sum of the number of $\frac{1}{2}(1,1,1)$-singularities obtained by deforming non-Gorenstein points on flipping curves of $Y_i$ locally, where the sum is taken over all $i$ such that $Y_i \dashrightarrow Y_{i+1}$ is a flip.*

*Table 1. $f'$ is of $(2,1)$-type. I*

| No. | $h$ | $(-K_X)^3$ | $N$ | $e$ | $n$ | $z$ | $\deg C$ | $g(C)$ | $X'$ |
|---|---|---|---|---|---|---|---|---|---|
| 1.1 | 6 | 7 | 2 | 7 | 0 | 4 | 7 | 8 | [5] |
| 1.2 | 6 | 15/2 | 3 | 7 | 0 | 2 | 3 | 0 | $[2], I(X')=2$ |
| 1.3 | 6 | 15/2 | 3 | 6 | 1 | 4 | 6 | 3 | [5] |
| 1.4 | 7 | 17/2 | 1 | 6 | 0 | 3 | 9 | 9 | $\mathbb{P}^3$ |
| 1.5 | 7 | 9 | 2 | 6 | 0 | 2 | 6 | 3 | [3] |
| 1.6 | 7 | 9 | 2 | 5 | 1 | 3 | 8 | 5 | $\mathbb{P}^3$ |
| 1.7 | 7 | 19/2 | 3 | 5 | 1 | 2 | 5 | 0 | [3] |
| 1.8 | 7 | 19/2 | 3 | 4 | 2 | 3 | 7 | 1 | $\mathbb{P}^3$ |
| 1.9 | 8 | 21/2 | 1 | 6 | 0 | 1 | 3 | 0 | $B_3$ |
| 1.10 | 8 | 21/2 | 1 | 5 | 0 | 2 | 9 | 6 | $Q_3$ |
| 1.11 | 8 | 11 | 2 | 4 | 1 | 2 | 8 | 3 | $Q_3$ |
| 1.12 | 9 | 25/2 | 1 | 5 | 0 | 1 | 5 | 1 | $B_4$ |
| 1.13 | 10 | 29/2 | 1 | 4 | 0 | 1 | 7 | 2 | $B_5$ |
| 1.14 | 10 | 15 | 2 | 3 | 1 | 1 | 6 | 0 | $B_5$ |

*Table* $1'$. $f'$ *is of* $(2,1)$*-type. I*

| $h$ | $(-K_X)^3$ | $N$ | $e$ | $n$ | $z$ | $\deg C$ | $g(C)$ | $X'$ |
|---|---|---|---|---|---|---|---|---|
| 8 | 23/2 | 3 | 3 | 2 | 2 | 7 | 0 | $Q_3$ |

*Notation and Remarks for Table* 1 *and Table* $1'$.

$C := f'(E')$,

$\deg C := (H(X') \cdot C)$ (*see Definition* 0.0 *for the definition of* $H(X')$),

$g(C) :=$ *the genus of* $C$ *in case* $X$ *has only* $\frac{1}{2}(1,1,1)$*-singularities,*

*see Theorem* 1.6 *for the definition of* $[i]$,

$u = z + 1$.

*Table* 2. $f'$ *is of* $(2,1)$*-type. II*

| *No.* | $(-K_X)^3$ | $N$ | $e$ | $\deg C$ | $X'$ |
|---|---|---|---|---|---|
| 2.1 | 7/2 | 3 | 10 | 1 | $A_6$ |
| 2.2 | 4 | 4 | 8 | 2 | $A_8$ |
| 2.3 | 9/2 | 5 | 6 | 3 | $A_{10}$ |
| 2.4 | 5 | 6 | 4 | 4 | $A_{12}$ |

*Table* $2'$. $f'$ *is of* $(2,1)$*-type. II*

| $(-K_X)^3$ | $N$ | $e$ | $\deg C$ | $X'$ |
|---|---|---|---|---|
| 11/2 | 7 | 2 | 5 | $A_{14}$ |

*Notation and Remarks for Table* 2 *and Table* $2'$.

$C := f'(E')$,

$\deg C := (-K_{X'} \cdot C)$,

$z = u = 1$,

$h = 4$ *and* $n = 0$.

*Table* 3. $f'$ *is* $(2,0)$*-type or crepant divisorial.*

| *No.* | $h$ | $(-K_X)^3$ | $N$ | $e$ | $n$ | *type of* $f'$ |
|---|---|---|---|---|---|---|
| 3.1 | 4 | 5/2 | 1 | 15 | 0 | $(2,0)_4$ |
| $3.1'$ | 4 | 5/2 | 1 | / | / | *crep. div.* |
| 3.2 | 4 | 3 | 2 | 12 | 0 | $(2,0)_8$ |
| 3.3 | 4 | 4 | 4 | 9 | 3 | $(2,0)_1$ |
| 3.4 | 4 | 9/2 | 5 | 8 | 3 | $(2,0)_5$ |

<u>*Remarks for Table* 3.</u>

$z = u = 1$,

(No. 3.1) $X'$ *also belongs to this class,*

(No. 3.1$'$) $X'$ *is a Fano* 3*-fold of* $(-K_{X'})^3 = 2$ *and with a canonical singularity along the image of* $f'$*-exceptional divisor,*

(No. 3.2) $X' \simeq A_4$ *with one Gorenstein terminal singularity,*

(No. 3.3) $X'$ *is smooth, isomorphic to* $A_{10}$,

(No. 3.4) $X'$ *is smooth, isomorphic to* $A_{16}$.

*Table* 4. $f'$ *is of* $(3,2)$*-type.*

| *No.* | $h$ | $(-K_X)^3$ | $N$ | $e$ | $n$ | $\deg \Delta$ |
|---|---|---|---|---|---|---|
| 4.1 | 5 | 11/2 | 3 | 8 | 0 | 8 |
| 4.2 | 5 | 6 | 4 | 7 | 1 | 6 |
| 4.3 | 6 | 13/2 | 1 | 7 | 0 | 7 |
| 4.4 | 6 | 7 | 2 | 6 | 1 | 6 |
| 4.5 | 6 | 15/2 | 3 | 5 | 2 | 5 |
| 4.6 | 6 | 8 | 4 | 4 | 3 | 4 |
| 4.7 | 6 | 17/2 | 5 | 3 | 4 | 3 |
| 4.8 | 10 | 29/2 | 1 | 6 | 0 | 0 |

*Table* 4$'$. $f'$ *is of* $(3,2)$*-type.*

| $h$ | $(-K_X)^3$ | $N$ | $e$ | $n$ | $\deg \Delta$ |
|---|---|---|---|---|---|
| 5 | 13/2 | 5 | 6 | 2 | 4 |
| 5 | 7 | 6 | 5 | 3 | 2 |
| 5 | 15/2 | 7 | 4 | 4 | 0 |
| 6 | 9 | 6 | 2 | 5 | 2 |
| 6 | 19/2 | 7 | 1 | 6 | 1 |

<u>*Notation and Remarks for Table* 4 *and Table* 4$'$.</u>

$\Delta :=$ *the discriminant divisor of* $f'$,

$\deg \Delta$ *is measured by the ample generator of* $\operatorname{Pic} X'$,

*in case* $h = 5$, $z = u = 2$ *and* $X' \simeq \mathbb{F}_{2,0}$,

*in case* $h = 6$, $z = u = 1$ *and* $X' \simeq \mathbb{P}^2$,

*in case* $h = 10$, $z = 1$, $u = 2$ *and* $X' \simeq \mathbb{P}^2$.

*Table* 5. *$f'$ is of* $(3,1)$*-type.*

| *No.* | $h$ | $(-K_X)^3$ | $N$ | $e$ | $n$ | $\deg F$ |
|---|---|---|---|---|---|---|
| 5.1 | 4 | 9/2 | 5 | 9 | 0 | 6 |
| 5.2 | 5 | 9/2 | 1 | 9 | 0 | 3 |
| 5.3 | 5 | 5 | 2 | 8 | 1 | 4 |
| 5.4 | 5 | 11/2 | 3 | 7 | 2 | 5 |
| 5.5 | 5 | 6 | 4 | 6 | 3 | 6 |

*Table* 5′. *$f'$ is of* $(3,1)$*-type.*

| $h$ | $(-K_X)^3$ | $N$ | $e$ | $n$ | $\deg F$ |
|---|---|---|---|---|---|
| 4 | 5 | 6 | 8 | 1 | 8 |

*Notation and Remarks for Table* 5 *and Table* 5′*.*
$F :=$ *a general fiber of* $f'$,
*in case* $h = 4$, $z = u = 2$,
*in case* $h = 5$, $z = u = 1$.

In the forthcoming paper [Taka02], we will study the geometric realization of the diagram in Construction 0.2.

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## §1. Preliminaries

Theorem 1.0. (Vanishing theorem) *Let $f\colon X \to Y$ be a projective morphism from a normal variety $X$ with only Kawamata log terminal singularities. Let $D$ be a $\mathbb{Q}$-Cartier integral Weil divisor such that $D - K_X$ is $f$-nef and $f$-big. Then $R^i f_* \mathcal{O}_X(D) = 0$ for all $i > 0$.*

*We quote this theorem as KKV vanishing theorem.*

*Proof.* See [Kod53], [Kaw82] and [Vie82]. ∎

Definition 1.1. (Axial weight) Let $(X, P)$ be a germ of 3-dimensional terminal singularity of index $> 1$. By the classification of such singularities [Mor85], we can easily see that a general deformation of $(X, P)$ has only cyclic quotient singularities. The number of these cyclic quotient singularities is said to be the *axial weight* of $(X, P)$ and denote it by $\mathrm{aw}(X, P)$. Let $X$ be a 3-fold with only terminal singularities. We define $\mathrm{aw}(X) := \sum \mathrm{aw}(X, P)$, where the summation takes place over points of index $> 1$.

Theorem 1.2. (Special case of the singular Riemann-Roch Theorem) *Let $X$ be a 3-fold with at worst index 2 terminal singularities and $D$ an integral Weil divisor on $X$. Then the following formula holds.*

$$\chi(\mathcal{O}_X(D)) = \chi(\mathcal{O}_X) + \frac{1}{12} D(D-K_X)(2D-K_X) + \frac{1}{12} D \cdot c_2(X) + \sum c_Q(D),$$

*where the summation takes place over index 2 points where $D$ is not Cartier and $\sum c_Q(D) = -n/8$ for some non-negative integer $n$. (See* [Reid87, Theorem 10.2] *for the definition of $c_Q(D)$.)*

*Proof.* See [Reid87, Theorem 10.2]. ∎

Theorem 1.3. *Let $X$ be a projective 3-fold with at worst index 2 terminal singularities. Then $-K_X \cdot c_2(X) = 24 - 3N/2$, where $N := \mathrm{aw}(X)$. Moreover assume that $X$ is a $\mathbb{Q}$-factorial $\mathbb{Q}$-Fano 3-fold with $\rho(X) = 1$. Then $-K_X \cdot c_2(X) > 0$. In particular $N \leq 15$.*

*Proof.* See [Kaw86, Lemma 2.2 and Lemma 2.3] and [Kaw92, Proposition 1]. ∎

Corollary 1.4. *Let $X$ be a weak $\mathbb{Q}$-Fano 3-fold with $I(X) = 2$. Then $h^0(-K_X) = 3 + \frac{1}{2}(-K_X)^3 - \frac{N}{4}$, where $N := \mathrm{aw}(X)$.*

*Proof.* This follows directly from Theorem 1.0, Theorem 1.2 and Theorem 1.3. ∎

Proposition 1.5. *Let $X$ be a 3-fold with only terminal singularities and $D$ a Cartier divisor on $X$. Let $f\colon X \to (Y, Q)$ be a $D$-flopping contraction (i.e., a flopping contraction such that $-D$ is $f$-ample) to a germ $(Y, Q)$ and $f^+\colon X^+ \to Y$ the $D$-flop constructed as in* [Kol89, Theorem 2.4]. *Then if $(Y, Q)$ is not exceptional type of index 4 (the type (2) of* [Reid87, Theorem (6.1)]*), the strict transform $D^+$ of $D$ on $X^+$ is a Cartier divisor.*

*Proof.* By passing to the analytic category and taking algebraization [Art69, Theorem 3.8], we may assume that $C := \operatorname{exc} f$ is irreducible. Moreover since we can deform $X$ to a 3-fold with only cyclic quotient terminal singularities [Mor88, (1b.8.2) Corollary] and such a deformation lifts to that of $f\colon X \to Y$ [KM92, (11.4) Proposition], we may assume that $X$ has only cyclic quotient terminal singularities. Let $H'$ be a general hyperplane section through $Q$ and $H := f^*H'$. Then it is well known that

(1.5.1) $H'$ and $H$ have only canonical singularities and $H$ is dominated by the minimal resolution of $H'$.

We show that $X$ has at most 2 singularities on $C$. Assume the contrary. Then $X$ has 3 singularities on $C$, and they coincide with the singularities of $H$ on $C$ by (1.5.1). Let $p\colon \widetilde{Y} \to Y$ be the index 1 cover, $\widetilde{X} := X \times_Y \widetilde{Y}$, $\widetilde{C}$ (resp. $\widetilde{H}'$, $\widetilde{H}$) the pull-back of $C$ (resp. $H'$, $H$) on $\widetilde{X}$ and $\tilde{f}\colon \widetilde{X} \to \widetilde{Y}$ the induced morphism. Then $\widetilde{X}$ is smooth and $\tilde{f}$ is also a flopping contraction. We prove that $\widetilde{C}$ is irreducible. If $\widetilde{C}$ is reducible, then there are components which intersect at 3 points since $(Y, Q)$ is not of exceptional type, a contradiction to $R^1\tilde{f}_*\mathcal{O}_{\widetilde{X}} = 0$. Hence $\widetilde{C}$ is irreducible. By [Reid87, (4.10)], $\widetilde{H}$ must be smooth. Hence $\widetilde{H}'$ has only ODP whence $H'$ has a canonical singularity of type A. But then $H$ has at most 2 singularities, a contradiction. So we have the assertion.

Now also $H$ has exactly two singularities. For otherwise $\operatorname{aw}(Y, Q) = 1$ since $\operatorname{aw}(Y, Q) = \operatorname{aw}(X)$. Hence $Q$ is a cyclic quotient singularity but then there is no flopping contraction to $Q$, a contradiction. We can prove as above that $\widetilde{C}$ is irreducible. Let $r$ be the index of $Q$. Let $P$ be a non-Gorenstein point on $C$ and $\widetilde{P}$ the inverse image on $\widetilde{X}$. Then $P$ is also of index $r$ and by [Reid87, (4.10)], we have locally analytically

$$(\widetilde{P} \in \widetilde{C} \subset \widetilde{X}) \simeq (o \in \{x = y = 0\} \subset \mathbb{C}^3),$$

where $x$, $y$, $z$ are coordinates of $\mathbb{C}^3$ which are semi-invariants of the $\mathbb{Z}_r$-action. Let $\widetilde{E}$ be a Cartier divisor which is localized to $z = 0$ and $E$ the image of $\widetilde{E}$ on $X$. Then we have $E \cdot C = 1/r$. Since $rE$ is a Cartier divisor and $\operatorname{Pic} X \simeq \operatorname{Pic} C$, we have $D \sim r(D \cdot C)E$. Then $D^+ \sim r(D \cdot C)E^+$, where $E^+$ is the strict transform of $E$ on $X^+$ because linear equivalence is preserved by a flop. Since the analytic types of $X$ and $X^+$ are the same by [Kol89, Theorem 2.4], $r(D \cdot C)E^+$ is Cartier and so is $D^+$. □

THEOREM 1.6. *Let $X$ be a $\mathbb{Q}$-Fano $d$-fold of $F(X) > d-2$, $I := I(X)$ and $H := H(X)$. Then $(X, H)$ be one of the following.*

[1] $((6) \subset \mathbb{P}(1^2, 2, 3, I^{d-2}), \mathcal{O}(I))$ *with* $I = 2, 3, 4, 5, 6$ *and* $d \geq 3$.
[2] $((4) \subset \mathbb{P}(1^3, 2, I^{d-2}), \mathcal{O}(I))$ *with* $I = 2, 3$ *and* $d \geq 3$.
[3] $((3) \subset \mathbb{P}(1^4, 2^{d-2}), \mathcal{O}(2))$ *with* $I = 2$ *and* $d \geq 3$.
[4] $((2) \subset \mathbb{P}(1^5, 2^{d-3}), \mathcal{O}(2))$ *with* $I = 2$ *and* $d \geq 4$, *and the defining equation does not contain the coordinate of weight* 2.
[5] $(\mathbb{P}(1^3, 2^{d-2}), \mathcal{O}(2))$ *with* $I = 2$ *and* $d \geq 3$.
[6] $(\mathbb{P}(1^3, 2, 4^{d-3}), \mathcal{O}(4))$ *with* $I = 4$ *and* $d \geq 4$.
[7] $(\mathbb{P}(1^4, 3^{d-3}), \mathcal{O}(3))$ *with* $I = 3$ *and* $d \geq 4$.
[8] $(\mathbb{P}(1^5, 2^{d-4}), \mathcal{O}(2))$ *with* $I = 2$ *and* $d \geq 5$.

*Proof.* See [San96]. □

## §2. Extremal contractions from 3-folds with only terminal singularities of index 2

The results in this section are well known to the experts, except Proposition 2.3.

DEFINITION 2.0. (Extremal contraction) Let $X$ be an analytic 3-fold with only terminal singularities and $f \colon X \to (Y, Q)$ a projective morphism onto a germ of a normal variety with only connected fibers. Let $\operatorname{exc} f$ be the locus where $f$ is not isomorphic. Assume that $-K_X$ is $f$-ample.

(1) If $\dim Y = 3$ and $\dim \operatorname{exc} f = 1$, then we say that $f$ is *a flipping contraction.*
(2) Only in this case, we assume that $-K_X$ is $f$-numerically trivial instead that $-K_X$ is $f$-ample. If $\dim Y = 3$ and $\dim \operatorname{exc} f = 1$, then we say that $f$ is a *flopping contraction.*
(3) Assume that $\dim Y = 3$, $\operatorname{exc} f$ is purely 2-dimensional and every component of the exceptional divisor $E$ is contracted to a curve. Let $C := f(E)$. Assume moreover that over a general point of every component of $C$, $f$ coincides with the blow-up along $C$ and $-E$ is $f$-ample. Then we say that $f$ is an *extremal contraction of* $(2,1)$*-type.* We say

$f$ is an *extremal divisorial contraction* if $f$ is an extremal contraction of $(2,1)$-type or $(2,0)$-type.

(4) Assume that $\dim Y = 3$, $\operatorname{exc} f$ is an irreducible divisor $E$ and $f(E)$ is a point. Then we say that $f$ is an *extremal contraction of* $(2,0)$*-type.*

(5) If $\dim Y = 2$ and every fiber is 1-dimensional, then we say that $f$ is an *extremal contraction of* $(3,2)$*-type.*

(6) If $\dim Y = 1$ and $f^{-1}(Q)_{\mathrm{red}}$ is irreducible, then we say that $f$ is an *extremal contraction of* $(3,1)$*-type.*

Proposition 2.1. (Flipping contraction) *Let $X$ be an analytic 3-fold with only index 2 terminal singularities and $f\colon X \to (Y,Q)$ a flipping contraction to a germ $(Y,Q)$. Let $C$ be its exceptional curve. (Since $(Y,Q)$ is a germ, $C$ is connected.) Then we have the following.*

(1) *$C \simeq \mathbb{P}^1$ and there is only one index 2 singularity on $C$ and $-K_X \cdot C = 1/2$.*

(2) *Let $P$ be the unique index 2 singularity on $C$. Then locally analytically $(P \in C \subset X) \simeq (o \in \{x_2 = x_3 = x_4 = 0\} \subset \{x_1x_2 + p({x_3}^2, x_4) = 0\}/\mathbb{Z}_2(1,1,1,0))$.*

(3) *Let $p(0,x_4) = a{x_4}^k$, where $a$ is a unit in $\mathbb{C}\{x_1,x_2,x_3,x_4\}$ and $k \in \mathbb{N}$ (note that $k = \mathrm{aw}(X,P)$). Then there is a deformation $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{Y}$ of $f$ over a 1-dimensional disc $(\Delta,0)$ such that for $t \neq 0$, $\mathfrak{X}_t$ has only $k\frac{1}{2}(1,1,1)$-singularities and $\mathfrak{f}_t\colon \mathfrak{X}_t \to \mathfrak{Y}_t$ is a bimeromorphic morphism which is localized to $k$ flipping contractions.*

(4) *Assume that $P$ is a $\frac{1}{2}(1,1,1)$-singularity. Then we can construct the flip of $f$ as follows. Let $g\colon X_1 \to X$ be the blow-up of $P$ and $E_1$ the exceptional divisor. Let $h\colon X_2 \to X_1$ be the blow-up along the strict transform $C_1$ of $C$ on $X_1$ and $E_2$ the exceptional divisor. Then $E_2 \simeq \mathbb{P}^1 \times \mathbb{P}^1$ and we can blow it down to another direction. Let $i\colon X_2 \to {X_1}^+$ be the blow-down and ${E_1}^+$ the strict transform of $E_1$ on ${X_1}^+$. Then ${E_1}^+ \simeq \mathbb{F}_1$ and we can blow it down to the ruling direction. Let $j\colon {X_1}^+ \to X^+$ be the blow-down. Then $X \dashrightarrow X^+$ is the flip.*

(5) *If $X$ is projective and $f$ is an algebraic flipping contraction, then $(-K_{X^+})^3 = (-K_X)^3 - \frac{n}{2}$, where $n = \sum \mathrm{aw}(X,P)$ and the summation is taken over the index 2 points on flipping curves.*

*Proof.* As for (1), (2) and (4), see [KM92, (4.2) and (4.4.5)]. We prove (3). Construct $Y'$ as in [ibid. (4.3)]. Then $Y' = \{y_1y_3 + y_2p({y_2}^2, y_4) = 0\}$ as in [ibid. (4.4.2)]. Then $f$ is obtained by blow-up of $Y'$ along $\{y_2 = y_3 = 0\}$

and dividing by the $\mathbb{Z}_2$ action. Let $\mathfrak{Y}' = \{y_1y_3+y_2(p({y_2}^2, y_4)+ty_4) = 0\}$ be a deformation of $Y'$ over a 1-dimensional disc $(\Delta, 0)$. Then by blow-up of $\mathfrak{Y}'$ along $\{y_2 = y_3 = 0\}$ and dividing by the induced $\mathbb{Z}_2$ action, we obtain the desired $\mathfrak{f}$. Next we prove (5). If we compactify $\mathfrak{X}$ in (3), then (5) holds by (4) and the invariance of $(-K)^3$ in a flat family. Since $(-K_X)^3-(-K_{X^+})^3$ can be expressed by an intersection number of the pull-back of $(-K_X)$ with exceptional divisors on a simultaneous resolution of $X^+$ and $X$ (and hence it is determined locally around flipping curves), the general case follows. □

Proposition 2.2. (Contraction of $(2,1)$-type) *Let $X$ be an analytic 3-fold with only index 2 terminal singularities and $f\colon X \to (Y, Q)$ an extremal contraction of $(2,1)$-type to a germ $(Y, Q)$. Let $E$ be the exceptional divisor and $C := f(E)$. Let $l$ be the fiber over $Q$.*

(1) *Assume that $l$ contains no index 2 point. Then $Q$ is a smooth point and $f$ is the blow-up along $C$.*
(2) *Assume that $l$ contains an index 2 point. Then $l$ contains only one index 2 point (we denote it by $P$) and every component $l'$ of $l$ passes through $P$ and satisfies $-K_X \cdot l' = 1/2$. Moreover $Y$ is Gorenstein.*
(3) *Assume that $X$ is projective. Then the following formula holds.*

$$(-K_E)^2 = 8(1-g(\overline{C})) - 2m,$$

*where $\overline{C}$ is the normalization of $C$ and $m$ is a non-negative integer.*
(4) *Assume that $X$ has only $\frac{1}{2}(1,1,1)$-singularities. Then*
   (4a) *$C$ is a smooth curve.*
   (4b) *$(Q \in Y) \simeq (o \in (\{xy+zw = 0\} \subset \mathbb{C}^4))$ or $(o \in (\{xy+z^2+w^3 = 0\} \subset \mathbb{C}^4))$.*
   (4c) *$f$ is constructed as follows. Let $g\colon Z \to Y$ be the blow-up of $Y$ at $Q$ and $F$ the exceptional divisor. Let $h\colon W \to Z$ the blow-up of $Z$ along the transform $C'$ of $C$ and $G$ the exceptional divisor. Since $C$ is smooth, $C \cap F$ is a smooth point of $F$. So if $Y \simeq (\{xy + zw = 0\} \subset \mathbb{C}^4)$, then the transforms $l_1$ and $l_2$ of two rulings of $F \simeq \mathbb{P}^1 \times \mathbb{P}^1$ through $C \cap F$ are the flopping curves (resp. if $Y \simeq (\{xy + z^2 + w^3 = 0\} \subset \mathbb{C}^4)$, then the transform $l$ of a ruling $F \simeq \mathbb{F}_{2,0}$ through $F \cap C$ is the flopping curve). Let $W \dashrightarrow W^+$ be the flop and $F'$ the strict transform of $F$ on $W^+$. Then $(F, -F|_F) \simeq (\mathbb{P}^2, \mathcal{O}(2))$. Hence we can contract it. Let $h'\colon W^+ \to X$ be the contraction and $f\colon X \to Y$ the natural morphism.*

(4d) *In the former case of* (b), $\operatorname{Sing} E \cap l = \{P\}$, *$P$ is an ordinary double point of $E$ and $l$ is a reducible conic. In the latter case of* (b), $\operatorname{Sing} E \cap l = \{P, P'\}$, *$P$, $P'$ are ordinary double points of $E$ and $l$ is a double line.*

(4e) *If $X$ is projective, then $m$ is the number of $\frac{1}{2}(1,1,1)$-singularities contained in $E$.*

*Proof.* See [Mor82, Theorem 3.3] for (1) and [KM92, (4.6), (4.7)] for (2).

Assume that $X$ is projective. Let $\mu\colon \overline{E} \to E$ be the normalization and define a $\mathbb{Q}$-divisor $Z$ by $K_{\overline{E}} = \mu^* K_E - Z$. Then $Z$ is effective and its support is contained in fibers. Hence $Z.(-K_{\overline{E}}) \geq 0$ and $(-K_E)^2 \leq (-K_{\overline{E}})^2 \leq 8(1 - g(\overline{C}))$. Since $-K_X - E \sim f^*(-K_Y) - 2E$, $(-K_E)^2 = (-K_X - E)^2 E = 2(2E^3 - 2f^*(-K_Y)E^2) \in 2\mathbb{Z}$. Hence we have the formula as in (3).

Assume that $X$ has only $\frac{1}{2}(1,1,1)$-singularities.

(4a) Let $\widetilde{X} := \boldsymbol{Spec}(\mathcal{O}_X \oplus \mathcal{O}_X(K_X))$, where we define a ring structure of $\mathcal{O}_X \oplus \mathcal{O}_X(K_X)$ by a smooth general element $G$ of $|-2K_X|$. Let $\widetilde{E}$ be the pull-back of $E$. Note that $\widetilde{X}$ is smooth. Then there is a natural crepant contraction of $\widetilde{E}$ from $\widetilde{X}$ which contracts $\widetilde{E}$ to a curve $\widetilde{C} \simeq C$. Note that $\widetilde{E}$ is negative for exceptional curves of the crepant contraction and the contraction coincides with the blow-up of $\widetilde{C}$ at a general point of $\widetilde{C}$. By these and the proof of [Wil93, Theorem 2.2] and [Wil97, Proposition 3.1], we know that $\widetilde{C}$ (and hence $C$) is smooth.

(4b) By (4a), we know that Case 1 in [KM92, (4.8.3)] does not occur by [KM92, Proposition 4.10.1], and (4b) follows from [KM92, (4.8.4) and (4.8.5)].

(4c) It is clear that $f$ constructed as in the statement satisfies the assumption of Proposition 2.2. By the uniqueness of such a contraction, (c) follows.

(4d), (4e) This easily follows from (4c).

□

Proposition 2.3. (Contraction of $(2,0)$-type) *Let $X$ be a 3-fold with only index 2 terminal singularities and $f\colon X \to (Y, Q)$ an extremal contraction of $(2,0)$-type to a germ $(Y, Q)$ which contracts a divisor $E$ to $Q$.*

(1) *Assume that $E$ contains no index 2 point. Then one of the following holds.*

$(2,0)_1$ : $(E, -E|_E) \simeq (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$ *and* $Q$ *is a smooth point.*

$(2,0)_2$ : $(E, -E|_E) \simeq (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^3}(1)|_{\mathbb{P}^1 \times \mathbb{P}^1})$ *and* $(Y, Q) \simeq ((\{xy + zw = 0\} \subset \mathbb{C}^4), o)$.

$(2,0)_3$ : $(E, -E|_E) \simeq (\mathbb{F}_{2,0}, \mathcal{O}_{\mathbb{P}^3}(1)|_{\mathbb{F}_{2,0}})$ *and* $(Y, Q) \simeq ((\{xy + z^2 + u^a = 0\} \subset \mathbb{C}^4), o)$ *with* $a \geq 3$.

$(2,0)_4$ : $(E, -E|_E) \simeq (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ *and* $Q$ *is a* $\frac{1}{2}(1,1,1)$*-singularity.*

*Moreover in any case,* $f$ *is the blow-up of* $Q$.

(2) *Assume that* $E$ *contains an index* 2 *point. Then one of the following holds*:

$(2,0)_5$ : $(E, -E|_E) \simeq (\mathbb{F}_{2,0}, l)$, *where* $l$ *is a ruling of* $\mathbb{F}_{2,0}$. $Q$ *is a smooth point and* $f$ *is a weighted blow-up with weight* $(2,1,1)$. *In particular we have* $K_X = f^*K_Y + 3E$.

$(2,0)_6$ : $K_X = f^*K_Y + E$ *and* $Q$ *is a Gorenstein singular point.* $E^3 = 1/2$.

$(2,0)_7$ : $K_X = f^*K_Y + E$ *and* $Q$ *is a Gorenstein singular point.* $E^3 = 1$.

$(2,0)_8$ : $K_X = f^*K_Y + E$ *and* $Q$ *is a Gorenstein singular point.* $E^3 = 3/2$.

$(2,0)_9$ : $K_X = f^*K_Y + E$ *and* $Q$ *is a Gorenstein singular point.* $E^3 = 2$.

$(2,0)_{10}$ : $(E, -E|_E) \simeq ((\{xy + z^2 = 0\} \subset \mathbb{P}(1,1,1,2)), \mathcal{O}(2))$, *and* $(Y, Q) \simeq ((\{xy + z^2 + u^a = 0\} \subset \mathbb{C}^4/\mathbb{Z}_2(1,1,1,0)), o)$, $a \geq 2$. $f$ *is the weighted blow-up with weights* $\frac{1}{2}(1,1,1,2)$. *In particular we have* $K_X = f^*K_Y + \frac{1}{2}E$.

$(2,0)_{11}$ : $(E, -E|_E) \simeq (\mathbb{F}_{2,0}, 3l)$. $Q$ *is a* $\frac{1}{3}(2,1,1)$*-singularity and* $f$ *is a weighted blow-up with a weight* $\frac{1}{3}(2,1,1)$. *In particular we have* $K_X = f^*K_Y + \frac{1}{3}E$.

*Proof.* (1) is proved in [Mor82, Theorem 3.4 and Corollary 3.5] and [Cut88] and in case $Q$ is a non-Gorenstein point, (2) is proved in [Luo98, Corollary 2.5 and Theorem 2.6]. We prove here that if $E$ contains an index 2 point and $Q$ is a Gorenstein point, $f$ is of $(2,0)_5$–$(2,0)_9$-type. Let $a$ be the discrepancy for $E$. Since $Q$ is assumed to be Gorenstein, $a$ is a positive integer.

First assume that $a \geq 2$. Let $L := -2E$. Then $L$ is free by [AW93] since $K_X + \frac{a}{2}L \equiv 0$ and $a/2 \geq 1$. Let $D$ be a general member of $|L|$ and

$C := E|_D$. Since $-K_D \equiv -(a-2)E|_D$ is nef and big, $C$ is a tree of $\mathbb{P}^1$ by KKV vanishing theorem. Let $\mu\colon \widetilde{E} \to E$ be the normalization of $E$. If $C$ is reducible, then $\mu^*C$ is not connected, a contradiction to the ampleness of $\mu^*C$. Hence $C \simeq \mathbb{P}^1$. By this we know that $E$ is normal since $E$ satisfies $S_2$ condition. Since $C$ is ample and isomorphic to $\mathbb{P}^1$, $E \simeq \mathbb{P}^2$, $\mathbb{F}_n$ $(n \geq 1)$ or $\mathbb{F}_{n,0}$ $(n \geq 2)$ by a classical result (see for example [Băd84]). But if former 2 cases occur, $X$ is smooth, a contradiction to the assumption of (2). Hence $E \simeq \mathbb{F}_{n,0}$ $(n \geq 2)$. We prove that $n = 2$. Let $v$ be the vertex of $E$. Then $v$ is the unique singularity on $E$ and hence it is of index 2. If $E$ is Cartier at $v$, then for a exceptional divisor $F$ over $v$ with discrepancy $1/2$ (such an $F$ exists by [Kaw93]), the discrepancy of $F$ for $K_Y$ is not an integer, a contradiction. Hence $K_X + E$ is a Cartier divisor and hence $K_E$ is Cartier at $v$. So $n$ must be 2. Moreover by $K_E = (a+1)E|_E$, $a = 3$ since $a \geq 2$ and $E \simeq \mathbb{F}_{2,0}$. By taking the canonical cover near $v$ of $X$, we know that $v$ is a $\frac{1}{2}(1,1,1)$-singularity. We prove that $Q$ is smooth and $f$ is a weighted blow-up with a weight $(2,1,1)$. Let $\overline{X} \to X$ be the blow-up at $v$. We see that the strict transform $\overline{E}$ of $E$ on $\overline{X}$ is contracted to a curve and let $\overline{X} \to \overline{X'}$ the contraction. Then next we can contract the strict transform of $F$ to a smooth point, which is no other than $Q$. We can easily show that a weighted blow-up with a weight $(2,1,1)$ is decomposed into contractions as above. So we are done.

Next we assume that $a = 1$. Let $P$ be an index 2 point on $X$. If $E$ is Cartier at $P$, then for a exceptional divisor $F$ over $P$ with discrepancy $1/2$, the discrepancy of $F$ for $K_Y$ is not an integer, a contradiction. Hence $E$ is not Cartier at $P$ whence $M := -K_X - E$ is an ample Cartier divisor. So $E$ is a Gorenstein (possibly non normal) del Pezzo surface since $-K_E = M|_E$. Since $\chi(\mathcal{O}_E) = 1$ by [Sak84, Theorem (5.1)] and [Reid94, Corollary 4.10], $\operatorname{Pic} E$ is torsion free. So $-K_X + E|_E \sim 0$ and hence $-K_X + E \sim 0$ by $\operatorname{Pic} X \simeq \operatorname{Pic} E$. So we have $M \sim -2K_X$. Since $(-K_E)^2 = 4E^3 \geq 2$, $|-K_E|$ is free by [Reid94, Corollary 4.10] and [Fuj90, Corollary 1.5]. By the exact sequence

$$0 \longrightarrow \mathcal{O}_X(-2E - K_X) \longrightarrow \mathcal{O}_X(-E - K_X) \longrightarrow \mathcal{O}_E(-K_E) \longrightarrow 0$$

and the KKV vanishing theorem, $|M|$ is also free. Let $G$ be a general member of $|M|$, $l := E|_G$ and $G' := f(G)$. Then $Q$ is a minimally elliptic singularity of $G'$ by the formula $K_G = f|_G{}^* K_{G'} - l$ and [Lau77, Theorem 3.4]. On the other hand, the embedded dimension of $G'$ at $Q$ is at most 4 since $Q$ is a cDV singularity on $Y$. Hence we have $-(l^2)_G \leq 4$ by

[Lau77, Theorem 3.13] whence $(-K_E)^2 = -2(l^2)_G = 2, 4, 6, 8$. These correspond to type $E_6 \sim E_9$ respectively. □

PROPOSITION 2.4. (Contraction of $(3,2)$-type) *Let $X$ be an analytic 3-fold with only index 2 terminal singularities and $f\colon X \to (Y, Q)$ an extremal contraction of $(3,2)$-type to a germ of surface. Let $l$ be the fiber over $Q$. Then $Q$ is a smooth point or an ordinary double point. Moreover the following description holds.*

(1) *If $l$ contains no index 2 point, $Q$ is a smooth point and $f$ is a usual conic bundle.*
(2) *If $l$ contains an index 2 point and $Q$ is a smooth point, $l$ contains only one index 2 point and every component $l'$ of $l$ passes through it. Moreover $-K_X \cdot l' = 1/2$.*
(3) *If $l$ contains an index 2 point and $Q$ is an ordinary double point, $f$ is analytically isomorphic to one of the following.*
   (3-1) *Let $\mathbb{P}^1 \times (\mathbb{C}^2, o) \to (\mathbb{C}^2, o)$ be the natural projection. Define the action of the group $\mathbb{Z}_2$ on ${\mathbb{P}^1}_{x_0,x_1} \times {\mathbb{C}^2}_{u,v}$:*

$$(x_0, x_1; u, v) \longmapsto (x_0, -x_1; -u, -v).$$

   *Set $X = \mathbb{P}^1 \times \mathbb{C}^2/\mathbb{Z}_2$ and $(Y, Q) = (\mathbb{C}^2/\mathbb{Z}_2, o)$.*
   *In particular $X$ has two $\frac{1}{2}(1,1,1)$-singularities on $l$ and $l_{\mathrm{red}} \simeq \mathbb{P}^1$ and $-K_X \cdot l_{\mathrm{red}} = 1$.*
   (3-2) *Let $X'$ be a hypersurface in ${\mathbb{P}^2}_{x_0,x_1,x_2} \times {\mathbb{C}^2}_{u,v}$ defined by the equation ${x_0}^2 + {x_1}^2 + {x_2}^2\phi(u,v) = 0$, where $\phi(u,v)$ has no multiple factors and contains only monomials of even degree. Let $f'\colon X' \to \mathbb{C}^2$ be the natural projection. Define the action of the group $\mathbb{Z}_2$ on $X'$ as follows.*

$$(x_0, x_1, x_2; u, v) \longmapsto (-x_0, x_1, x_2; -u, -v).$$

   *Set $X := X'/\mathbb{Z}_2$ and $(Y, Q) = (\mathbb{C}^2/\mathbb{Z}_2, o)$.*
   *In particular $P$ is the unique index 2 point and $\mathrm{aw}(X, P) = 2$. If $\mathrm{mult}_{(0,0)}(\phi) = 2$, then $(X, P)$ is a $cA/2$ point or if $\mathrm{mult}_{(0,0)}(\phi) \geq 4$, then $(X, P)$ is a $cAx/2$ point.*

*Proof.* See [Mor82, Theorem 3.5] for (1) and [Pro97, Theorems 3.1, 3.15 and Examples 2.1 and 2.3] for (2) and (3). □

Proposition 2.5. (Contraction of $(3,1)$-type) *Let $X$ be an analytic 3-fold with only index 2 terminal singularities and $f\colon X \to (C, Q)$ be an extremal contraction of $(3,1)$-type to a germ of a curve. Let $F$ be the fiber over $Q$. Then $Q$ is a smooth point and the following description holds.*

(1) *If $F$ contains no index 2 point, then all fibers are irreducible and reduced and (possibly non-normal) Gorenstein del Pezzo surfaces. Moreover if $(-K_F)^2 = 9$, we can write $-K_X \sim 3A$ for some relatively ample divisor $A$ and $X = \mathbb{P}(f_*\mathcal{O}_X(A))$ which is a $\mathbb{P}^2$-bundle. If $(-K_F)^2 = 8$, we can write $-K_X \sim 2A$ for some relatively ample divisor $A$ and $X$ is embedded in $\mathbb{P}^3$-bundle $\mathbb{P}(f_*\mathcal{O}_X(A))$ as a quadric bundle (the last means all fibers are quadrics in $\mathbb{P}^3$). The case $(-K_F)^2 = 7$ does not occur.*

(2) *If $F$ contains an index 2 point, then $F$ is irreducible and reduced, or $F = 2F_{\mathrm{red}}$ and $F_{\mathrm{red}}$ is irreducible. $F_{\mathrm{red}}$ is a del Pezzo surface of Gorenstein index $\leq 2$.*

*Proof.* See [Mor82, Theorem 3.5] for (1). (2) follows from the existence of a section [CT86]. □

## §3. A generalization of Takeuchi's construction

In this section, we explain the construction as in 0.2 in a more general setting. The situation of Set up 3.3 is closer to that of 0.2. We use slight different notation to 0.2 for unified treatment of several cases. The differences between the notation of this section and that of 0.2 are as follows. $D$, $D_i$ and $\widetilde{D}$ of this section correspond to $E$, $E_i$ and $\widetilde{E}$ in 0.2 respectively. $D'$ of this section corresponds to $E'$ in 0.2 in case $f'$ is birational, or $L$ in 0.2 in case $f'$ is not birational.

Set up 3.0. Let $Y$ be a $\mathbb{Q}$-factorial terminal with $\rho(Y) = 2$. Assume that there exists a diagram as follows.

$$\begin{array}{ccccccccc} Y_0 := Y & \overset{g_0}{\dashrightarrow} & Y_1 & \overset{g_1}{\dashrightarrow} & \cdots & \overset{g_{k-1}}{\dashrightarrow} & Y_k & & \\ & & & & & & & \searrow^{f'} & \\ & & & & & & & & X', \end{array}$$

where

(1) $Y_i \dashrightarrow Y_{i+1}$ is a flop or a flip.

(2) $f'$ is an extremal contraction which is not isomorphic in codimension 1, or a crepant divisorial contraction.

We define $Y' := Y_k$.

We do calculations which are similar to ones Kiyohiko Takeuchi did in [Take89]. The following lemma is basic for our computations.

LEMMA 3.1. *We use the notation of Set up* 3.0. *Let $D$ be a divisor on $Y$. Let $\gamma_i$ be an irreducible component of the flipping* (*or flopping*) *curve for $g_i$ and $D_i$ the strict transform of $D$ on $Y_i$* (*we set $D_0 = D$*). *Then*

(1) *If $Y_i \overset{g_0}{\dashrightarrow} Y_{i+1}$ is a flop, then $(-K_{Y_{i+1}})^3 = (-K_{Y_i})^3$, $(-K_{Y_{i+1}})^2 D_{i+1} = (-K_{Y_i})^2 D_i$, $(-K_{Y_{i+1}}) {D_{i+1}}^2 = (-K_{Y_i}) {D_i}^2$ and $e_i := {D_i}^3 - {D_{i+1}}^3 \in \mathbb{Z}/s^3$, where $s$ is a positive rational number such that $sD_i$ is numerically equivalent to a Cartier divisor relatively with respect to the flopping contraction. The sign of $e_i$ is the same as one of $(D_i \cdot l_i)$.*

(2) *If $Y_i \overset{g_i}{\dashrightarrow} Y_{i+1}$ is a flip, let $d_i := (-K_{Y_i})^3 - (-K_{Y_{i+1}})^3$. Then $d_i > 0$ and $(-K_{Y_{i+1}})^2 D_{i+1} = (-K_{Y_i})^2 D_i - a_i d_i$, $(-K_{Y_{i+1}}) {D_{i+1}}^2 = (-K_{Y_i}) {D_i}^2 - {a_i}^2 d_i$ and ${D_{i+1}}^3 = {D_i}^3 - {a_i}^3 d_i$, where $a_i := \frac{D_i \cdot \gamma_i}{(-K_{Y_i}) \cdot \gamma_i}$* (*note that this number $a_i$ is well defined since flipping curves are numerically proportional*).

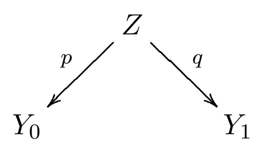
(3) *We define $e_i$* (*resp. $a_i$ and $n_i$*) *to be* 0 *if $Y_i \dashrightarrow Y_{i+1}$ is not a flop* (*resp. a flip*). *Then we have*

$$(-K_{Y'})^2 \widetilde{D} = (-K_Y)^2 D - \sum a_i d_i,$$
$$(-K_{Y'}) \widetilde{D}^2 = (-K_Y) D^2 - \sum {a_i}^2 d_i,$$
$$\widetilde{D}^3 = D^3 - \sum e_i - \sum {a_i}^3 d_i.$$

(4) *If $D$ is a non-zero effective divisor and $D \cdot \gamma_0 > 0$, then $D_i \cdot \gamma_i > 0$ for any $i$.*

*Proof.*

(1) Let



be the common resolution of $Y_i$ and $Y_{i+1}$. Then by the negativity lemma ([K$^+$92, Lemma 2.19]), we can easily see that $p^*K_{Y_i} = q^*K_{Y_{i+1}}$ (for example, see [Kol89, Proof of Lemma 4.3] or below argument). Thus, the former 3 equalities follow. Since $sD_{i+1}$ is numerically equivalent to a Cartier divisor relatively by Proposition 1.5, we have $e_i \in \mathbb{Z}/s^3$. Let

$$p^{-1}D_i = p^*D_i - R = q^*D_{i+1} - R',$$

where $R$ and $R'$ are effective divisors which are exceptional for $p$ and $q$. Rewrite this as

$$-p^*D_i = -q^*D_{i+1} + R' - R.$$

We only treat the case that $D_i \cdot l_i > 0$. Then $-q^*D_{i+1}$ is $p$-nef. Hence we see that $R' - R > 0$ and $p_*(R' - R) \neq 0$ by the negativity lemma. So we can write $p^*D_i = q^*D_{i+1} - F$, where $F := R' - R$ is an effective divisor. Consider the identity $(p^*D_i)(q^*D_{i+1})^2 = (q^*D_{i+1} - F)(q^*D_{i+1})^2$. Its right side is equal to ${D_{i+1}}^3$. Its left side is equal to $(p^*D_i)(p^*D_i + F)^2 = D^3 + D.p_*(F^2)$. By $p_*F \neq 0$, we know that $-p_*(F^2)$ is a non-zero effective 1-cycle. Hence $D_i.p_*(F^2) < 0$ and we are done.

(2) The proof is very similar to one of (1). Let

$$\begin{array}{ccc} & Z & \\ {}^{p}\swarrow & & \searrow^{q} \\ Y_i & & Y_{i+1} \end{array}$$

be the common resolution of $Y_i$ and $Y_{i+1}$. By the definition of $a_i$,

(a) $$H_i := a_i(-K_{Y_i}) - D_i$$

is numerically trivial for the flipping curves. Let ${H_i}^+$ be the strict transform of $H_i$. By the negativity lemma, we can easily see that $p^*H_i = q^*{H_i}^+$ and $p^*(-K_{Y_i}) = q^*(-K_{Y_{i+1}}) - G$, where $G$ is an effective divisor which is exceptional for $p$ and $q$. $d_i > 0$ can be proved similarly to the proof of positivity of $e$. Consider the following identities.

(b) $$\begin{aligned}(-K_{Y_i})^2H_i = (p^*(-K_{Y_i}))^2p^*H_i &= (q^*(-K_{Y_{i+1}}) - G)^2q^*{H_i}^+ \\ &= (-K_{Y_{i+1}})^2{H_i}^+\end{aligned}$$

and similarly

(c) $$(-K_{Y_i})H_i{}^2 = (-K_{Y_{i+1}})H_i{}^{+2}$$

and

(d) $$H_i{}^3 = H_i^{+3}.$$

By (a)–(d) and the definition of $d_i$, we obtain the assertion.

(3) This follows from (1) and (2).

(4) We prove this by induction on $i$. Assume that $D_i \cdot \gamma_i > 0$ is proved. Then $D_{i+1}.\gamma_i{}^+ < 0$, where $\gamma_i{}^+$ is the flopped or flipped curve corresponding to $\gamma_i$. If $D_{i+1} \cdot \gamma_{i+1} \leq 0$, then $D_{i+1}$ is non-positive for two extremal rays of $Y_{i+1}$ and hence $D_{i+1}$ is non positive for all effective curves on $Y_{i+1}$ since $\rho(Y_{i+1}) = 2$. But this contradicts the effectivity of $D_{i+1}$ and $D_{i+1} \neq 0$.

□

From now on, we divide $f'$ into several cases.

Case 1. $f'$ is an extremal contraction of $(2,1)$-type.
Case 2. $f'$ is an extremal contraction of $(2,0)$-type.
Case 3. $f'$ is an extremal contraction of $(3,2)$-type.
Case 4. $f'$ is an extremal contraction of $(3,1)$-type.
Case 5. $f'$ is a crepant divisorial contraction.

Assume that $D$ and $-K_Y$ are numerically independent. Let $\widetilde{D}$ be the strict transform of $D$ on $Y'$. In case $f'$ is birational (resp. $f'$ is not birational), let $D'$ be the exceptional divisor of $f'$ (resp. the pull-back of the ample generator of $\operatorname{Pic} X'$). By $\rho(Y') = 2$, we can write

(3.1) $$D' \equiv z(-K_{Y'}) - u\widetilde{D}.$$

In case $f'$ is birational and $-K_{Y'}$ is $f'$-ample, let $d'/r'$ be the discrepancy of $D'$ for $K_{X'}$, where $r'$ is 1 in Case 1, or the index of $f'(D')$ in Case 2. Note that $d' = 1$ in Case 1.

Claim 3.2. *$zd' + r' = uk$ for some $k \in \mathbb{Z}$.*

*Proof.* By (3.1) and $-K_{Y'} = f'^*(-K_{X'}) - \frac{d'}{r'}D'$, we have $(zd'+r')D' \equiv r'zf'^*(-K_{X'}) - ur'\widetilde{D}$. Since $r'f'(\widetilde{D})$ is Cartier divisor along $f'(D')$ outside a finite set of points in Case 1 (resp. at $f'(D')$ in Case 2), $\frac{zd'+r'}{u}$ is an integer. □

**Case 1.** Let $C := f'(D')$. We have the following.

$$\text{(3-1-1)}\quad (-K_{Y'}+D')^2(-K_{Y'}) = u^2\bigl\{k^2(-K_{Y'})^3 - 2k(-K_{Y'})^2\widetilde{D} + (-K_{Y'})\widetilde{D}^2\bigr\} = (-K_{X'})^3.$$

$$\text{(3-1-2)}\quad (-K_{Y'}+D')^2\widetilde{D} = u^2\bigl\{k^2(-K_{Y'})^2\widetilde{D} - 2k(-K_{Y'})\widetilde{D}^2 + \widetilde{D}^3\bigr\} = \frac{z}{u}(-K_{X'})^3.$$

$$\text{(3-1-3)}\quad z(-K_{Y'}+D')^2(-K_{Y'}) - (z+1)(-K_{Y'}+D')D'(-K_{Y'}) = u^2\bigl\{k(-K_{Y'})^2\widetilde{D} - (-K_{Y'})\widetilde{D}^2\bigr\} = z(-K_{X'})^3 - (z+1)(-K_{X'}\cdot C).$$

$$\text{(3-1-4)}\quad z^2(-K_{Y'}+D')^2(-K_{Y'}) - (z+1)^2(-K_{Y'})D'^2 = 2zu(z+1)(-K_{Y'})^2\widetilde{D} - (2z+1)u^2(-K_{Y'})\widetilde{D}^2 = z^2(-K_{X'})^3 - (z+1)^2\Bigl\{2(g(\overline{C})-1) + \frac{m}{2}\Bigr\},$$

where $\overline{C}$ is the normalization of $C$ and $m \in \mathbb{N}$. (The last equality of (3-1-4) can be proved similarly to that of Proposition 2.2 (3).)

We rewrite these by using Lemma 3.1 as follows.

$$\text{(3-1-1}'\text{)}\quad \Bigl\{k^2(-K_Y)^3 - 2k(-K_Y)^2D + (-K_Y)D^2 - \sum d_i(a_i-k)^2\Bigr\}u^2 = (-K_{X'})^3.$$

$$\text{(3-1-2}'\text{)}\quad e + \sum d_i a_i(a_i-k)^2 = k^2(-K_Y)^2D - 2k(-K_Y)D^2 + D^3 - \frac{z}{u^3}(-K_{X'})^3.$$

$$\text{(3-1-3}'\text{)}\quad u^2\Bigl\{k(-K_Y)^2D - (-K_Y)D^2 + \sum d_i a_i(a_i-k)\Bigr\} = z(-K_{X'})^3 - (z+1)(-K_{X'}\cdot C).$$

$$\text{(3-1-4}'\text{)}\quad 2zu(z+1)(-K_Y)^2D - (2z+1)u^2(-K_Y)D^2 + \sum d_i a_i u^2\{a_i(2z+1) - 2zk\} = z^2(-K_{X'})^3 - (z+1)^2\Bigl\{2(g(\overline{C})-1) + \frac{m}{2}\Bigr\}.$$

**Case 2.** We have the following.

(3-2-1) $$z^3(-K_{Y'})^3 - 3z^2u(-K_{Y'})^2\widetilde{D} + 3zu^2(-K_{Y'})\widetilde{D}^2 - u^3\widetilde{D}^3 = {D'}^3.$$

(3-2-2) $$\widetilde{D}{D'}^2 = z^2(-K_{Y'})^2\widetilde{D} - 2zu(-K_{Y'})\widetilde{D}^2 + u^2\widetilde{D}^3 = -\frac{k}{r'}{D'}^3.$$

(3-2-3) $$(-K_{Y'})D'\widetilde{D} = z(-K_{Y'})^2\widetilde{D} - u(-K_{Y'})\widetilde{D}^2 = \frac{d'k}{{r'}^2}{D'}^3.$$

We rewrite these by using Lemma 3.1 as follows.

(3-2-1′) $$z^3(-K_Y)^3 - 3z^2u(-K_Y)^2D + 3zu^2(-K_Y)D^2 - u^3D^3 + \sum d_i(ua_i - z)^3 + u^3e = {D'}^3.$$

(3-2-2′) $$\sum d_ia_i(a_iu - z)^2 + u^2e = z^2(-K_Y)^2D - 2zu(-K_Y)D^2 + u^2D^3 + \frac{k}{r'}{D'}^3.$$

(3-2-3′) $$z(-K_Y)^2D - u(-K_Y)D^2 + \sum d_ia_i(a_iu - z) = \frac{d'k}{{r'}^2}{D'}^3.$$

**Case 3.** We have the following.

(3-3-1) $${D'}^3 = z^3(-K_{Y'})^3 - 3z^2u(-K_{Y'})^2\widetilde{D} + 3zu^2(-K_{Y'})\widetilde{D}^2 - u^3\widetilde{D}^3 = 0.$$

(3-3-2) $$\widetilde{D}{D'}^2 = z^2(-K_{Y'})^2\widetilde{D} - 2zu(-K_{Y'})\widetilde{D}^2 + u^2\widetilde{D}^3 = \frac{2z}{u}l^2.$$

(3-3-3) $$z(-K_{Y'})^3 - u(-K_{Y'})^2\widetilde{D} = (-K_{Y'})^2D'.$$

We set $u = mz$ and $l = f'_*D'$. We rewrite these by using Lemma 3.1 as follows.

(3-3-1′) $$(-K_Y)^3 - 3m(-K_Y)^2D + 3m^2(-K_Y)D^2 - m^3D^3 + \sum d_i(ma_i - 1)^3 + m^3e = 0.$$

(3-3-2′) $$z^2\Big\{\sum d_ia_i(ma_i - 1)^2 + m^2e\Big\} = z^2\{(-K_Y)^2D - 2m(-K_Y)D^2 + m^2D^3\} - \frac{2}{m}l^2.$$

(3-3-3′) $$z\Big\{(-K_Y)^3 - m(-K_Y)^2D + \sum d_i(ma_i - 1)\Big\} = (-K_{Y'})^2D'.$$

If $l$ is free, then $(-K_{Y'})^2D' = 8(1-g(l)) - \Delta\cdot l + 4l^2$.

**Case 4.** We calculate the following.

(3-4-1) $$(-K_{Y'})D'^2 = z^2(-K_{Y'})^3 - 2zu(-K_{Y'})^2\widetilde{D} + u^2(-K_{Y'})\widetilde{D}^2 = 0.$$

(3-4-2) $$\widetilde{D}D'^2 = z^2(-K_{Y'})^2\widetilde{D} - 2zu(-K_{Y'})\widetilde{D}^2 + u^2\widetilde{D}^3 = 0.$$

(3-4-3) $$(-K_{Y'})^2D' = z(-K_{Y'})^3 - u(-K_{Y'})^2\widetilde{D} = \deg F,$$

where $F$ is a general fiber of $f'$ and $\deg F := (-K_F)^2$.

We set $u = mz$. We rewrite these by using Lemma 3.1 as follows.

(3-4-1′) $$2m(-K_Y)^2D - m^2(-K_Y)D^2 + \sum d_i(ma_i - 1)^2 = (-K_Y)^3.$$

(3-4-2′) $$\sum d_ia_i(ma_i - 1)^2 + m^2e = (-K_Y)^2D - 2m(-K_Y)D^2 + m^2D^3.$$

(3-4-3′) $$z\Big\{(-K_Y)^3 - m(-K_Y)^2D + \sum d_i(ma_i - 1)\Big\} = \deg F.$$

**Case 5.** Since $-K_{Y'}\cdot l = 0$ and $D'\cdot l = -2$ for a general fiber $l$ of $D'$, we have $u(D\cdot l) = 2$. By $(-K_Y)^2D' = 0$, we have

(3-5-1′) $$\sum d_i(z - ua_i) = z(-K_Y)^3 - u(-K_Y)^2D$$

Set up 3.3. From now on we moreover assume that $Y$ is a weak $\mathbb{Q}$-Fano 3-fold and there exists an extremal contraction $f\colon Y \to X$ which is not isomorphic in codimension 1. In case $\operatorname{excep} f$ is a divisor, let $D$ be the exceptional divisor, or the pull-back of the ample generator of $\operatorname{Pic} X$ otherwise. Let $R$ be the extremal ray other than one associated to $f$. If $R$ is a ray associated to a contraction which is not isomorphic in codimension 1, denote the contraction by $f'\colon Y := Y_0 \to X'$. If $R$ is a flopping ray, then after the flop $Y_0 \dashrightarrow Y_1$, another extremal ray of $Y_1$ is $K_{Y_1}$-negative because $K_{Y_1}$ is not nef and $\rho(Y_1) = 2$. By this consideration, we see that we can run the minimal model program from $Y_0$ or $Y_1$ and we obtain the diagram as in Set up 3.0. Note that if $Y_i \dashrightarrow Y_{i+1}$ is a flop, then $i = 0$, and if $f'$ is a crepant contraction, then $Y = Y'$ and $\dim X' = 3$. We denote $e_0$ by $e$ for simplicity.

Claim 3.4. (1) *In case $f'$ is of $(3,2)$-type, then $X'$ is a log del Pezzo surface with $\rho(X') = 1$.*

(2) *In case $f'$ is of $(3,1)$-type, then $X' \simeq \mathbb{P}^1$ and hence $D' = f'^*\mathcal{O}_{\mathbb{P}^1}(1)$.*

*Proof.*

(1) By [Pro97, Lemma 1.10], $X'$ has only cyclic quotient singularities. By the general theory of the conic bundle, $-4K_{X'} \equiv f'_*(-K_{Y'}{}^2) + \Delta$, where $\Delta$ is the discriminant divisor of $f'$. Hence $-K_{X'}.A > 0$ for any ample divisor $A$ on $X'$ since $-K_{Y'}$ is big. Hence $X'$ is a log del Pezzo surface with $\rho(X') = 1$.

(2) By the edge sequence of the Leray spectral sequence

$$0 \longrightarrow H^1(X', \mathcal{O}_{X'}) \longrightarrow H^1(Y', \mathcal{O}_{Y'}) \qquad \text{(exact)}$$

and $H^1(Y', \mathcal{O}_{Y'}) = 0$, we have $H^1(X', \mathcal{O}_{X'}) = 0$, i.e., $X' \simeq \mathbb{P}^1$ and hence $D' = f'^*\mathcal{O}_{\mathbb{P}^1}(1)$.

∎

CLAIM 3.5. *$D$ and $-K_Y$ are numerically independent.*

*Proof.* $D$ and $-K_Y$ are non-zero and $\mathbb{Q}$-effective. Hence they are positive for general curves. On the other hand $D$ is $f$-semi-negative and $-K_Y$ is $f$-ample. Hence we have the assertion. ∎

CLAIM 3.6. (1) *Assume that $f$ is birational. Write $-K_X \equiv qS$, where $S$ is the positive generator of $Z^1(X)/\equiv$ and $q$ is a positive integer. Let $d/r$ be the discrepancy of $D$ for $K_X$, where $r$ is $1$ in case $f(D)$ is a curve, or the index of $f(D)$ in case $f(D)$ is a point. Note that $d = 1$ in the former case. Then $z \in \mathbb{N}/q$, $u > 0$ and $dz + ru \in \mathbb{N}$.*

(2) *Assume that $f$ is of $(3,2)$-type. Then $z \in \mathbb{N}/2$ and $u > 0$. Assume moreover that there exists a degenerate fiber contained in* $\operatorname{Reg} Y$. *Then $z \in \mathbb{N}$.*

(3) *Assume that $f$ is of $(3,1)$-type. Then $u > 0$. Let $F$ be a general fiber. Then*

(1-1) *in case $F \not\simeq \mathbb{P}^1 \times \mathbb{P}^1, \mathbb{P}^2$, we have $z \in \mathbb{N}$.*

(1-2) *In case $F \simeq \mathbb{P}^1 \times \mathbb{P}^1$, we have $z \in \mathbb{N}/2$.*

(1-3) *In case $F \simeq \mathbb{P}^2$, we have $z \in \mathbb{N}/3$.*

*Proof.* Let $\overline{D'}$ be the strict transform of $D'$ on $Y$. Assume that $f$ is birational. Then on $X$, $f(\overline{D'}) \equiv zqS$. So $z \in \mathbb{N}/q$. By (3.1), we have $\overline{D'} \equiv zf^*(-K_X) - \left(\frac{dz}{r} + u\right)E$. Hence $\frac{dz}{r} + u \in \mathbb{N}/r$.

Assume that $f$ is not birational. Let $l$ be a curve in a fiber of $f$. Then $0 \le \overline{D'} \cdot l = z(-K_Y) \cdot l$. Hence $z \ge 0$. If $z = 0$, then $\overline{D'} \equiv -uD$ and $u < 0$ whence $\overline{D'}$ is positive for the extremal ray different from one associated to $f$. Hence by Lemma 3.1 (4), $D'$ is $f'$-ample, a contradiction. Thus we have $z > 0$.

In case $f$ is $(3,2)$-type, let $l$ be a general fiber. Then $\overline{D'} \cdot l = 2z \in \mathbb{N}$. Assume that there exists a degenerate fiber contained in $\operatorname{Reg} Y$ and let $l'$ be a component of it. Then $\overline{D'} \cdot l' = z \in \mathbb{N}$.

In case $f$ is of $(3,1)$-type, let $F$ be a general fiber and $l \subset F$ a $(-1)$-curve in case $F \not\simeq \mathbb{P}^1 \times \mathbb{P}^1, \mathbb{P}^2$ or a ruling $\subset F$ if $F \simeq \mathbb{P}^1 \times \mathbb{P}^1$ or a line $\subset F$ if $F \simeq \mathbb{P}^2$. Then we obtain the similar assertion.

If $u \le 0$, then $D'$ is big, a contradiction. □

Claim 3.7. *(1) If $Y_i \overset{g_i}{\dashrightarrow} Y_{i+1}$ and $Y_{i+1} \overset{g_{i+1}}{\dashrightarrow} Y_{i+2}$ are flips, then $a_{i+1} < a_i$.*

*(2) If $Y_i \dashrightarrow Y_{i+1}$ is a flip, then $k < d'a_i$ in case $f'$ is birational (resp. $ma_i > 1$ in case $f'$ is not birational).*

*Proof.* We use the notation of Lemma 3.1 and let ${\gamma_i}^+$ be a flipped curve on $Y_{i+1}$.

(1) By $(a_i(-K_{Y_{i+1}}) - D_{i+1}).{\gamma_i}^+ = 0$ and $(a_i(-K_{Y_{i+1}}) - D_{i+1}) \cdot m > 0$ for a general curve $m$ on $Y_{i+1}$, we have $(a_i(-K_{Y_{i+1}}) - D_{i+1}) \cdot \gamma_{i+1} > 0$. On the other hand we have $(a_{i+1}(-K_{Y_{i+1}}) - D_{i+1}) \cdot \gamma_{i+1} = 0$. Hence we are done.

(2) If $Y_i \dashrightarrow Y_{i+1}$ is a flip and $k \ge d'a_i$ (resp. $ma_i \le 1$) for some $i$, then $(k(-K_{Y_i}) - d'D_i) \cdot \gamma_i \ge 0$ and hence $(k(-K_{Y_{i+1}}) - d'D_{i+1}).{\gamma_i}^+ \le 0$ (resp. $(-K_{Y_i} - mD_i) \cdot \gamma_i \ge 0$ and hence $(-K_{Y_{i+1}} - mD_{i+1}).{\gamma_i}^+ \le 0$). Note that $f'^*(-K_{X'}) \equiv \frac{u}{r'}\{k(-K_{Y'}) - d'\widetilde{D}\}$ in case $f$ is birational (resp. $D' \equiv z(-K_{Y'} - m\widetilde{D})$ in case $f$ is not birational). Hence $k(-K_{Y_i}) - d'D_i$ (resp. $-K_{Y'} - m\widetilde{D}$) is a non-zero $\mathbb{Q}$-effective divisor for any $i$. Thus by $\rho(Y_{i+1}) = 2$, $k(-K_{Y_{i+1}}) - d'D_{i+1}$ (resp. $-K_{Y_i} - mD_i$) is positive for another extremal ray of $Y_{i+1}$. So $k(-K_{Y'}) - d'\widetilde{D}$ (resp. $-K_{Y_i} - mD_i$) is positive for a fiber of $f'$. But this is absurd.

□

By an additional assumption that $|-K_Y - D| \ne \phi$, the relation of $u$ and $z$ is restricted as follows.

Claim 3.8. *If* $|-K_Y - D| \neq \phi$, *then* $z \leq u$. *Moreover in Case* 3, $m = 1$ *or* 2, *or in Case* 4, $m = 1$ *or* $m = 2$ *and* $F \simeq \mathbb{P}^1 \times \mathbb{P}^1$ *or* $m = 3/2$ *or* 3 *and* $F \simeq \mathbb{P}^2$.

*Proof.* By (3.1), we have

$$D' \equiv (z-u)(-K_{Y'}) + u(-K_{Y'} - \widetilde{D}). \tag{3.2}$$

By the assumption, $|-K_{Y'} - \widetilde{D}| \neq \phi$. Hence if $z > u$, then $D'$ is big by (3.2), a contradiction. So $z \leq u$.

In Case 3, for a general fiber $l$, we have $\widetilde{D} \cdot l = \frac{2z}{u} \in \mathbb{N}$. So $\frac{2z}{u} = 1$ or 2 since $z \leq u$. In Case 4, let $l$ be a $(-1)$-curve in $F$ if $F \not\simeq \mathbb{P}^1 \times \mathbb{P}^1, \mathbb{P}^2$ or a ruling if $F \simeq \mathbb{P}^1 \times \mathbb{P}^1$ or a line if $F \simeq \mathbb{P}^2$. By calculating $\widetilde{D} \cdot l$, we obtain the assertion similarly to Case 3. □

## §4. Existence of a weak ℚ-Fano blow-up for a ℚ-Fano 3-fold with $I(X) = 2$

Definition 4.0. Let $X$ be a ℚ-Fano variety. We say that a birational morphism $f \colon Y \to X$ is a *weak* ℚ*-Fano blow-up* if the following hold.

(1) $Y$ is a weak ℚ-Fano variety.
(2) $f$ is an extremal contraction whose exceptional locus is a prime ℚ-Cartier divisor.

Theorem 4.1. *Let $X$ be a klt weak ℚ-Fano 3-fold. Assume the following.*

(1) $I(X) \leq 2$,
(2) *there are only a finite number of non-Gorenstein points on $X$, and*
(3) $(-K_X)^3 \geq 1$ *and* $h^0(-K_X) \geq 1$.

*Then* $|-2K_X|$ *is free.*

*Proof.* By replacing $X$ by its anti-canonical model, we can assume that $X$ is a klt ℚ-Fano 3-fold. Let $S$ be a general member of $|-2K_X|$. By [Amb99, Theorem 1.2], $S$ has only log terminal singularities. By the exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(-2K_X) \longrightarrow \mathcal{O}_S(-2K_X|_S) \longrightarrow 0$$

and $h^1(\mathcal{O}_X) = 0$, we have $|-2K_X|_S| = |-2K_X||_S$ and $\mathrm{Bs}|-2K_X| = \mathrm{Bs}|-2K_X|_S|$. Note that $-K_X|_S = K_S$. Hence it suffices to prove that

$|K_S + K_S|$ is free. Assume that $|2K_S|$ is not free. Let $y$ be a base point of $|K_S + K_S|$. Assume that $y$ is worse than canonical. By [Kawa00, Theorem 9], $y$ is a cyclic quotient singularity of index 2. So Kawachi's invariant $\delta'$ defined in [Kawa00] is $1/2$ at $y$. On the other hand, by the assumption that $(-K_X)^3 \geq 1$, ${K_S}^2 \geq 2$ holds. So ${K_S}^2 > \delta_y$ holds ($\delta_y$ is defined in [KaM98]). But by (1), we have $K_S \cdot C = -K_X \cdot C \geq 1/2$ for any curve $C$ whence by [ibid.], $y$ cannot be a base point of $|2K_S|$, a contradiction. So we may assume that $S$ does not contain a non-Gorenstein point of $X$ by (2) and has only canonical singularities. Let $\mu\colon \widetilde{S} \to S$ be the minimal resolution. Since $h^0(K_{\widetilde{S}}) = h^0(K_S) = h^0(-K_X) \geq 1$, $|2K_{\widetilde{S}}|$ is free by [Fra91] and hence $|2K_S|$ is free, a contradiction again.

Hence $|K_S + K_S|$ is free and also $|-2K_X|$ is free. □

Proposition 4.2. *Let $X$ be a weak $\mathbb{Q}$-Fano 3-fold with $I(X) = 2$ such that $|-2K_X|$ is free. Let $P$ be an index 2 point such that there is no curve $l$ through $P$ such that $-K_X \cdot l = 0$. Let $f\colon Y \to X$ an extremal contraction of $(2,0)$-type from a 3-fold $Y$ with only terminal singularities such that*

1. *$f$-exceptional divisor is a prime $\mathbb{Q}$-Cartier divisor. We call it $E$,*
2. *$P := f(E)$ and $-K_Y = f^*(-K_X) - \frac{1}{2}E$, and*
3. *$(-K_Y)^3 > 0$.*

*Then $Y$ is a weak $\mathbb{Q}$-Fano 3-fold.*

*Proof.* By the assumption that there is no curve $l$ through $P$ such that $-K_X \cdot l = 0$, $\mathrm{Bs}|-2K_X - P|$ is a finite set of points near $P$. So by $H^0(-2K_Y) \simeq H^0(\mathcal{O}(-2K_X) \otimes m_P)$, we know $-K_Y$ is nef. So by (3), it is also big and we are done. □

We need the following technical lemma.

Lemma 4.3. *Let $X$ be a $\mathbb{Q}$-factorial $\mathbb{Q}$-Fano 3-fold with $\rho(X) = 1$, $I(X) = 2$ and $F(X) = 1/2$. Let $f\colon Y \to X$ be a weak $\mathbb{Q}$-Fano blow-up with $I(Y) = 2$ and $E$ the $f$-exceptional divisor. Assume that*

1. *$f(E)$ is a point,*
2. *$|-2K_Y|$ is free,*
3. *$h^0(-K_Y - E) > 0$, and*
4. *there is no divisor contracted to a point by a multi-anti-canonical morphism.*

*Then* $H^0(\mathcal{O}_Y(-2K_Y)) \to H^0(\mathcal{O}_E(-2K_Y|_E))$ *is surjective.*

*Proof.* We are inspired by [Reid80, p.29, Step 4]. It suffices to prove that $h^1(\mathcal{O}_Y(-2K_Y - E)) = 0$. Take a general member $T \in |-2K_Y|$. Then by the exact sequence

$$0 \longrightarrow \mathcal{O}_Y(-E) \longrightarrow \mathcal{O}_Y(-2K_Y - E) \longrightarrow \mathcal{O}_T(-2K_Y - E|_T) \longrightarrow 0$$

and $h^i(\mathcal{O}_Y(-E)) = 0$ for $i = 1, 2$ (these vanishing easily follows from

$$0 \longrightarrow \mathcal{O}_Y(-E) \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{O}_E \longrightarrow 0$$

since by Proposition 2.3, $h^1(\mathcal{O}_E) = 0$), we obtain $h^1(\mathcal{O}_Y(-2K_Y - E)) = h^1(\mathcal{O}_T(-2K_Y - E|_T))$. By Serre duality, we have $h^1(\mathcal{O}_T(-2K_Y - E|_T)) = h^1(\mathcal{O}_T(2K_T - E|_T)) = h^1(\mathcal{O}_T(K_Y + E|_T))$. We prove that $h^1(\mathcal{O}_T(K_Y + E|_T)) = 0$ below. Take a member $F \in |-K_Y - E| \neq \phi$. Then since $\rho(X) = 1$ and $-K_X$ is a positive generator of $Z^1(X)/\equiv$, we can write $F = F' + rE$, where $F'$ is a prime divisor and $r$ is a non-negative integer. Since $|-2K_Y|$ is free and $T$ is general, we may assume that $F'|_T$ and $E|_T$ is irreducible by (4). Note that $(F' + rE)|_T \cdot E|_T = (-K_Y - E)E(-2K_Y) > 0$ and $(E|_T)^2 < 0$. Hence if $r > 0$, for every integer $b \in [1, r]$, we have $(F'|_T + (r-b)E|_T)(bE|_T) > 0$, which means $F|_T$ is numerically 1-connected. So by [Ram72, Lemma 3], we have $H^0(\mathcal{O}_{F|_T}) \simeq \mathbb{C}$. Hence by the exact sequence

$$0 \longrightarrow \mathcal{O}_T(-F|_T) \longrightarrow \mathcal{O}_T \longrightarrow \mathcal{O}_{F|_T} \longrightarrow 0,$$

we have $h^1(\mathcal{O}_T(-F|_T)) = 0$ which is exactly what we want. □

## §5. Solution of the equations of Diophantine type for a ℚ-Fano 3-fold with $I(X) = 2$

We will prove the following theorem in this section, which is a slight generalization of the main theorem (compare the assumption (3)).

THEOREM 5.0. *Let $X$ be a ℚ-factorial ℚ-Fano 3-fold with the following properties.*

(1) $\rho(X) = 1$,
(2) $I(X) = 2$,
(3) $-K_X$ *is the positive generator of* $Z^1(X)/\equiv$,
(4) $h^0(-K_X) \geq 4$, *and*

(5) *there exists an index* 2 *point* $P$ *such that*

$$(X,P) \simeq (\{xy+z^2+u^a=0\}/\mathbb{Z}_2(1,1,1,0), o)$$

*for some* $a \in \mathbb{N}$.

*Let* $f\colon Y \to X$ *be the weighted blow-up at* $P$ *with weight* $\frac{1}{2}(1,1,1,2)$ *and* $E$ *the exceptional divisor. Then* $Y$ *is a weak* $\mathbb{Q}$*-Fano* 3*-fold with* $I(Y)=2$. *Run the program as in Set up* 3.3. *Then* $z \le u$ *and* $Y_i \dashrightarrow Y_{i+1}$ *is a flip for at most one* $i$ *and* $a_i = 2$ *for such* $i$ *(we use the notation as in Set up* 3.3*). Moreover we figure out the solutions of equations in Section* 3 *as in Tables* 1–5 *and Tables* 1′–5′ *of the main theorem with the following additional possibilities for the case that* $F(X)=1$.

$f'$ *is of* $(2,1)$*-type.*

| $(-K_X)^3$ | $N$ | $e$ | $\deg C$ | $X'$ |
|---|---|---|---|---|
| 6 | 8 | 0 | 6 | $B_2$ |

$$z=u=1.$$

$f'$ *is of* $(3,2)$*-type.*

| $h$ | $(-K_X)^3$ | $N$ | $e$ | $n$ | $\deg\Delta$ |
|---|---|---|---|---|---|
| 6 | 10 | 8 | 0 | 7 | 0 |

*Proof.* By (4) and Corollary 1.4, we have $(-K_X)^3 > 2$. Moreover $(-K_Y)^3 = (-K_X)^3 - \frac{1}{2} > 0$. Hence by Proposition 4.2, $Y$ is a weak $\mathbb{Q}$-Fano 3-fold. We can easily check that $I(Y)=2$ by calculating the weighted blow-up (here we need the assumption (5)).

We run the program as in Set up 3.3. We follow the notation in 0.2. The differences between the notation of 0.2 and Set up 3.3 are explained in the beginning of Section 3.

By the assumption (4) and the description of the extraction $f$ as in Proposition 2.3 $(2,0)_{10}$, the exact sequence

$$0 \longrightarrow \mathcal{O}_Y(-K_Y-E) \longrightarrow \mathcal{O}_Y(-K_Y) \longrightarrow \mathcal{O}_E(1) \longrightarrow 0,$$

yields $|-K_Y-E| \neq \phi$. Hence by Claim 3.8, we have $z \le u$.

Claim 5.1. *$E_i$ is a Cartier divisor for any $i$. In particular $a_i$ is an even integer.*

*Proof.* Assume that $g_0$ is a flop. By Proposition 1.5, $E_1$ is a Cartier divisor since $E$ is a Cartier divisor. If $g_i$ is a flip, there is no non-Gorenstein point on the flipped curves. Hence $E_i$ is Cartier by induction on $i$. The latter half follows from Proposition 2.1 (1). ∎

We follow the case division in Section 3. Note that by Claim 3.7 and Claim 5.1, once we prove that $a_i = 2$ if $a_i > 0$, we see that there is at most one flip.

**Case 1.** In this case we first show that $F(X') \geq 1$. In fact by (3.1), we have $-K_{X'} \equiv \frac{u}{z} f'(\widetilde{E})$. Since $f'(\widetilde{E})$ is Cartier by Proposition 2.2 (2) and Claim 5.1, and $u \geq z$, the assertion holds. Moreover by [Isk79] and [San96], $F(X') = 1$, $3/2$, $2$, $5/2$, $3$ or $4$.

We note that by Proposition 2.2, we have $(-K_{E'})^2 = 8(1 - g(\overline{C})) - 2m$ with some non-negative integer $m$. By $z + 1 = uk$ and $z \leq u$, we have $z + 1 = u$ or $z = u = 1$.

First assume that $z + 1 = u$. Define $a \in \mathbb{Z}$ by the formula $f'(\widetilde{E}) = aH$, where $H$ is a primitive Cartier divisor of $X'$. Then $F(X') = a\frac{z+1}{z}$. Hence $z = 1, 2, 3, 4$ and if $z = 1$, then $F(X') = 2$ or $4$, if $z = 2$, then $F(X') = 3/2$ or $3$, if $z = 3$, then $F(X') = 4$, or if $z = 4$, then $F(X') = 5/2$. But we prove that the case that $z = 1$ and $F(X') = 4$ does not occur. For otherwise, let $H'$ be the strict transform of $f'^*H$ on $Y$. Then we have $-K_Y \equiv 2H' + E$ and hence $-K_X \equiv 2f(H')$, a contradiction to the assumption (3).

Assume $a_i \geq 4$ for some $i$. Note that $a_i u > z$ by $u \geq z$. By (3-1-2$'$), $e \leq (k+2)^2 - 2(4-k)^2 < 0$, a contradiction.

Set $n := 2\sum d_i$. We obtain the following.

(5-1-1) $$(-K_X)^3 = \frac{9+n}{2} + \frac{1}{u^2}(-K_{X'})^3$$

obtained by (3-1-1$'$),

(5-1-2) $$e + n = 9 - \frac{u-1}{u^3}(-K_{X'})^3$$

obtained by (3-1-2$'$),

(5-1-3) $$(-K_{X'} \cdot C) = \frac{u-1}{u}(-K_{X'})^3 - (3+n)u$$

obtained by (3-1-3$'$),

(5-1-4) $$-6u + 6 - 2nu + \frac{(u-1)^2}{u^2}(-K_{X'})^3 = 2g(\overline{C}) + \frac{m}{2}$$

obtained by (3-1-4′). We use (5-1-4) for the bound of $n$.

By (3.1), we have $\widetilde{E} \cdot l = 1$ for a general fiber $l$ of $E'$. If $E'$ contains an index 2 point, then there is a component $l'$ of a fiber such that $-K_{Y'} \cdot l' = 1/2$ by Proposition 2.2. So we have $\widetilde{E} \cdot l' = 1/2$. But this contradicts the fact that $\widetilde{E}$ is a Cartier divisor. Hence $E'$ contains no index 2 point. This fact and information from $X'$ determine $N$. Hence we can easily figure out the solutions as in Tables 1 and 1′.

Next assume $z = u = 1$. By Claim 3.7 (2) and Claim 5.1, $a_i \geq 4$ if $a_i > 0$. Assume that $a_i \geq 6$ for some $i$. By (3-1-2′), $e \leq (k+2)^2 - 3(6-k)^2 < 0$, a contradiction. Hence we must have $a_i = 4$ for all $i$ such that $Y_i \dashrightarrow Y_{i+1}$ is a flip. By setting $n := 2\sum d_i$, we obtain the following.

$$(5\text{-}1\text{-}1') \qquad (-K_X)^3 = 6 - \frac{1}{4}e - \frac{3}{2}n$$

obtained by (3-1-1′) and (3-1-2′),

$$(5\text{-}1\text{-}2') \qquad e + 8n = 16 - (-K_{X'})^3$$

obtained by (3-1-2′),

$$(5\text{-}1\text{-}3') \qquad (-K_{X'} \cdot C) = 6 - \frac{1}{2}e - 6n$$

obtained by (3-1-2′) and (3-1-3′),

$$(5\text{-}1\text{-}4') \qquad (-K_{X'})^3 - 2 - 16n = 8g(\overline{C}) + 2m.$$

By (5-1-3′) and $(-K_{X'} \cdot C) > 0$, we must have $n = 0$, i.e., there is no flip while $Y \dashrightarrow Y'$.

By (5-1-1′) and (5-1-2′), we deduce that $(-K_{X'})^3 = 16 - e > 0$. By (5-1-2′) and (5-1-3′), we have $(-K_{X'} \cdot C) = \frac{1}{2}(-K_{X'})^3 - 2 > 0$. Therefore, $(-K_{X'})^3 = 6, 8, 10, 12, 14, 16$.

Claim 5.2. $h^0(-K_X) = 4$.

*Proof.* By $\widetilde{E} \equiv -K_{Y'} - E'$, we have $E \equiv -K_Y - \widetilde{E'}$, where $\widetilde{E'}$ is the strict transform of $E'$. Since $E - (-K_Y - \widetilde{E'})$ is a Cartier divisor, we must have $E \sim -K_Y - \widetilde{E'}$ since $\operatorname{Pic} Y$ is torsion free. Hence $h^0(-K_Y - E) = 1$. But by the exact sequence

$$0 \longrightarrow \mathcal{O}_Y(-K_Y - E) \longrightarrow \mathcal{O}_Y(-K_Y) \longrightarrow \mathcal{O}_E(-K_Y|_E) \longrightarrow 0,$$

we have

$$h^0(-K_X) = h^0(-K_Y) \le h^0(-K_Y - E) + h^0(-K_Y|_E) = 4.$$

So $h^0(-K_X) = 4$. ∎

Hence we have $N = \frac{16-e}{2}$.

We prove that $X'$ is Gorenstein. Assume that $X'$ is non-Gorenstein. If $F(X') = 1$, then by [San96], $N - 1 \ge 8$, a contradiction. Since $(-K_{X'})^3 = 16 - e$, $F(X') > 1$ does not hold by [San95]. Hence $X'$ is Gorenstein.

Next we prove that $F(X') = 1, 2$ and if $F(X') = 2$, then $F(X) = 1$ and $N = 8$. By $(-K_{X'})^3 = 16 - e$, we clearly have $F(X') = 1, 2$. Assume that $F(X') = 2$. Let $H$ be the ample generator of $\operatorname{Pic} X'$ and $H' := f'^* H$. This is a Cartier divisor on $Y'$ and so is the strict transform $H''$ on $Y$ since $n = 0$. Since $H'' \equiv \frac{1}{2}(-K_Y + \widetilde{E'}) \equiv (-K_Y) - \frac{1}{2}E$, we have $f^* f_* H'' = H'' + E$. By this, we know $f_* H''$ is a Cartier divisor on $X$ ([KMM87, Lemma 3-2-5 (2)]). On the other hand, $f_* H'' \equiv -K_X$ and so $F(X)$ must be an integer. Hence $F(X) = 1$ by (3) of Main Assumption 0.1 and moreover by [San96], $N = 8$.

So we obtain the solutions as in Tables 2 and 2′.

**Case 2.** By Proposition 2.3, we obtain the values of $E'^3$, $(-K_{Y'})E'^2$ and $(-K_{Y'})^2 E'$ as follows.

| | $(2,0)_1$ | $(2,0)_{2,3}$ | $(2,0)_{4,10}$ | $(2,0)_5$ | $(2,0)_6$ | $(2,0)_7$ | $(2,0)_8$ | $(2,0)_9$ | $(2,0)_{11}$ |
|---|---|---|---|---|---|---|---|---|---|
| $E'^3$ | 1 | 2 | 4 | 1/2 | 1/2 | 1 | 3/2 | 2 | 9/2 |
| $(-K_{Y'})E'^2$ | −2 | −2 | −2 | −3/2 | −1/2 | −1 | −3/2 | −2 | −3/2 |
| $(-K_{Y'})^2 E'$ | 4 | 2 | 1 | 9/2 | 1/2 | 1 | 3/2 | 2 | 1/2 |

Assume that $f'$ is of $(2,0)_1$-type. By (3-2-3′), we have $z + 2u \le 2k$. On the other hand, we have $1 + 2z = uk \ge zk$. Hence $z = u = 1$ and $k = 3$ or $z = 1$, $u = 3/2$ and $k = 2$. First we treat the former case. By (3-2-3′) again, $\sum d_i a_i (a_i - 1) = 3$. Since $a_i \ge 2$ if $a_i > 0$, we have $a_i = 2$ if $a_i > 0$. By setting $n := 2 \sum d_i$, we have $n = 3$. We can easily see that $e = 9$, $(-K_X)^3 = 4$ and $(-K_{X'})^3 = 10$. By the assumption (4), we have $N = 4$. This also proves that $X'$ is Gorenstein and hence $X'$ is $A_{10}$.

Next we deny the latter case. If this case occurs, then for a flopped curve $l$ on $Y'$, we have $E' \cdot l = -\frac{3}{2} \widetilde{E} \cdot l = 3/2$ since by Lemma 4.3, $g(E) \simeq E$, where $g$ is the flopping contraction from $Y$. But this contradicts the fact that $E'$ is a Cartier divisor.

We prove that $f'$ cannot be of $(2,0)_2$-type or $(2,0)_3$-type. Assume that $f'$ is of $(2,0)_2$-type or $(2,0)_3$-type. Similarly to the above case, we have

$k = 2$ and $\sum d_i a_i(a_i - 1) = 1$ using (3-2-3′). But by Claim 3.7 (2), we have $a_i \geq 4$ if $a_i > 0$, a contradiction.

If $f'$ is of $(2,0)_4$–$(2,0)_{11}$-type, then we can figure out the solution similarly.

Therefore we can obtain the solutions as in Tables 3 and 3′.

**Case 3.** By Proposition 2.4, $X'$ has at worst ordinary double points as singularities. Hence $X' \simeq \mathbb{P}^2$ and $L = f'^*\mathcal{O}_{\mathbb{P}^2}(1)$, or $X' \simeq \mathbb{F}_{2,0}$ and $L = f'^*(\mathcal{O}_{\mathbb{P}^3}(1)|_{\mathbb{F}_{2,0}})$.

Assume $a_i \geq 4$ for some $i$. Note that $a_i u > z$ by $u \geq z$. By (3-3-2′), $m^2 e < (2m+1)^2 - 2(4m-1)^2 < 0$, a contradiction. Hence $a_i = 2$ for all $i$ such that $Y_i \dashrightarrow Y_{i+1}$ is a flip.

By setting $n := 2\sum d_i$, we have the following.

$$(-K_X)^3 = \frac{1}{2} + m(4m^2 + 6m + 3) - \frac{n}{2}(2m-1)^3 - m^3 e. \tag{5-3-1}$$

$$mz^2\{(2m+1)^2 - n(2m-1)^2 - m^2 e\} = 2l^2 = \overset{\mathbb{P}^2}{2},\ \overset{\mathbb{F}_{2,0}}{4}. \tag{5-3-2}$$

$$\begin{aligned} &mz\{2(2m+1)(m+1) - 2n(2m-1)(m-1) - m^2 e\} \\ &\quad = \overset{\mathbb{P}^2}{12 - \Delta \cdot l},\ \overset{\mathbb{F}_{2,0}}{16 - \Delta \cdot l}. \end{aligned} \tag{5-3-3}$$

By Claim 3.8, we have $m = 1$ or $2$.

If $m = 2$, we can easily figure out the solution.

Assume that $m = 1$. Then we have 3 sequences of solutions as follows.

(1) $X' \simeq \mathbb{P}^2$, $z = 1$, $n + e = 7$, $\Delta \cdot l = e$ and $h^0(-K_X) = \frac{25+n-N}{4}$.
(2) $X' \simeq \mathbb{F}_{2,0}$, $z = 1$, $n + e = 5$, $\Delta \cdot l = 4 + e$ and $h^0(-K_X) = \frac{29+n-N}{4}$.
(3) $X' \simeq \mathbb{F}_{2,0}$, $z = 2$, $n + e = 8$, $\Delta \cdot l = 2e - 8$ and $h^0(-K_X) = \frac{23+n-N}{4}$.

If $X' \simeq \mathbb{P}^2$ and $Y'$ has an index 2 point (resp. If $X' \simeq \mathbb{F}_{2,0}$ and $\mathrm{aw}(Y') > 2$), then there is a fiber containing a component $l$ such that $-K_{Y'} \cdot l = 1/2$ by Proposition 2.4. But these cases do not occur. For otherwise we have $\widetilde{E} \cdot l = z/2u < 1$, a contradiction to the fact that $\widetilde{E}$ is a Cartier divisor. Hence for (1) and (2) (resp. (3)), we have $N - n = 1$ (resp. $N - n = 3$) since $\mathrm{aw}(Y') = \mathrm{aw}(Y) - n = N - n - 1$. But if (2) and $N - n = 1$ hold, $Y'$ must be Gorenstein, a contradiction to Proposition 2.4. Hence we figure out the solutions as in Tables 4 and 4′.

**Case 4.** Similarly to Case 3, we can prove that $a_i = 2$ for all $i$ such that $Y_i \dashrightarrow Y_{i+1}$ is a flip using (3-4-2′).

By setting $n := 2\sum d_i$, we rewrite (3-4-1′)–(3-4-3′) as follows.

$$(5\text{-}4\text{-}1)\qquad (-K_X)^3 = \frac{1}{2} + 2m(m+1) + \frac{1}{2}n(2m-1)^2.$$

$$(5\text{-}4\text{-}2)\qquad (2m+1)^2 = n(2m-1)^2 + m^2 e.$$

$$(5\text{-}4\text{-}3)\qquad z\{m(2m+1) + nm(2m-1)\} = \deg F.$$

By Claim 3.8, we have $m = 1$, $3/2$, $2$ or $3$.

We can easily see that there is no solution for $m = 3/2$, $2$ or $3$.

If $m = 1$, then we have $n + e = 9$, $(-K_X)^3 = \frac{n+9}{2}$ and $z(3+n) = \deg F$. Since $h^0(-K_X) = 3 + \frac{9+n-N}{4} \geq 4$, we have $N - n = 1$ or $5$. If $N - n = 1$, then $Y'$ is Gorenstein. Hence by the primitivity of $L$, $z = 1$. If $N - n = 5$ and $u = z = 1$ or $3$, $L \not\sim z(-K_{Y'} - \widetilde{E})$ since the right side is not Cartier. By Riemann-Roch theorem, $\chi(\mathcal{O}(L)) - \chi(O(z(-K_{Y'} - \widetilde{E}))) = 1/2$, a contradiction. Hence if $N - n = 5$, then $z = 2$ and so $n = 0$ or $1$ by $z(3+n) = \deg F$.

We prove $n \leq 3$. If $n = 4$, then $\deg F = 7$, a contradiction to Proposition 2.5. If $n = 5$, then $Y' \to X'$ is a quadric bundle over a $\mathbb{P}^1$ by Proposition 2.5. But then $(-K_{Y'})^3$ must be a multiple of 8, a contradiction. If $n = 6$, then $Y' \to X'$ is a $\mathbb{P}^2$-bundle over a $\mathbb{P}^1$ by Proposition 2.5. But then $(-K_{Y'})^3$ must be 54, a contradiction.

Hence we obtain the solutions as in Table 5 and 5′.

**Case 5.** Since $u \in \mathbb{N}/2$ and $E \cdot l \in \mathbb{N}$, we have $u = 1/2$, $1$, $2$ by $u(E \cdot l) = 2$. Moreover since $z(-K_Y)^3 = u$, $(-K_Y)^3 > 3/2$ and $z \leq u$, we have $z = 1$, $u = 2$ and $(-K_Y)^3 = 2$. Hence we are done in this case.

Finally we prove that if $N = 8$, then $F(X) = 1$.

The case $f'$ is of $(2,1)$-type. Note that $Y = Y'$ holds since $e = 0$. By the proof of Case 1 above, we have only to prove that $F(X') = 2$. Assume that $F(X') = 1$. By $\rho(X') = 1$, $I(X') = 1$, $F(X') = 1$ and the $\mathbb{Q}$-factoriality of $X'$, there exists a line $l$ intersecting $C$. Let $l'$ be the strict transform of $l$ on $Y$. By $-K_Y \cdot l' = -K_{X'} \cdot l - E' \cdot l'$ and the fact that $-K_Y$ is nef, we have $-K_Y \cdot l' = 0$ and $E' \cdot l' = 1$ or $-K_Y \cdot l' = 1/2$ and $E' \cdot l' = 1/2$. But the former case does not occur since $e = 0$. In the latter case $E \cap l' = \phi$ by $E \cdot l' = 0$. Hence $-K_X \cdot f(l') = 1/2$, which in turn show that for a $\mathbb{Q}$-Fano blow-up whose center is an index 2 point on $f(l')$, the resulting weak $\mathbb{Q}$-Fano 3-fold is not a $\mathbb{Q}$-Fano 3-fold. But by Tables 1–5 and 1′–5′ in the main theorem and additional possibilities in Theorem 5.0, we again fall into this case for a $\mathbb{Q}$-Fano blow-up at another index 2 point, a contradiction (the new $e$ must be 0). Hence we are done.

The case $f'$ is of $(3,2)$-type. In this case, $f'$ is a $\mathbb{P}^1$-bundle associated to some vector bundle $\mathcal{E}$ of rank 2 on $\mathbb{P}^2$. Let $T$ be its tautological divisor. By the adjunction formula $-K_{Y'} \sim 2T - (c_1(\mathcal{E}) - 3)L$, we have $6 = (-K_{Y'})^3 = 8T^3 - 6c_1(\mathcal{E})^2 + 54$ and hence $c_1(\mathcal{E})$ is an even. Hence $H' := 3T - (\frac{3}{2}c_1(\mathcal{E}) - 4)L$ is an integral Cartier divisor. Note that $H' \equiv -K_{Y'} + \frac{1}{2}\widetilde{E}$. Hence for a flipped curve $l_i{}^+$ on some $Y_i$ and the strict transform $H_i$ of $H'$ on $Y_i$, we have $H_i \cdot l_i{}^+ = -2$. Hence the strict transform $H$ of $H'$ on $Y$ is a Cartier divisor numerically equivalent to $-K_Y + \frac{1}{2}E$. Note that $H$ is $f$-numerically trivial. So by [KMM87, Lemma 3-2-3 (2)], $f(H)$ is a Cartier divisor and clearly numerically equivalent to $-K_X$.

We postpone to [Taka02] the proof of the nonexistence of a $\mathbb{Q}$-Fano 3-fold in Tables 1′–5′. See §5 of [Taka02]. □

*Remark.* If $X$ is a $\mathbb{Q}$-Fano 3-fold of $I(X) = 2$ and $F(X) = 1$, we see the case $N = 8$ in Table 2′ or Table 4′ actually occurs by [San95].

## References


[Alt] S. Altınok, *Graded rings corresponding to polarized K3 surfaces and $\mathbb{Q}$-fano 3-folds*, Univ. of Warwick, ph.D. thesis.

[Amb99] F. Ambro, *Ladders on Fano varieties*, J. Math. Sci., **94** (1999), 1126–1135.

[Art69] M. Artin, *Algebraic approximation of structure over complete local rings*, Inst. Hautes Études Sci. Publ. Math., **36** (1969), 23–58.

[AW93] M. Andreatta and J. Wiśniewski, *A note on nonvanishing and applications*, Duke Math. J., **72** (1993), 739–755.

[Băd84] L. Bădescu, *Hyperplane sections and deformations*, Lecture Notes in Math., vol. 1056, Springer-Verlag, Berlin-New York (1984), pp. 1–33.

[CF93] F. Campana and H. Flenner, *Projective threefolds containing a smooth rational surface with ample normal bundle*, J. reine angew. Math., **440** (1993), 77–98.

[CT86] J.-L. Colliot-Thélène, *Arithmétique des variétés rationnelles et problèmes birationnels*, Proc. Int. Conf. Math. (1986), 641–653.

[Cut88] S. Cutkosky, *Elementary contractions of Gorenstein threefolds*, Math. Ann., **280** (1988), 521–525.

[Fle00] A. R. Fletcher, *Working with weighted complete intersections*, Explicit birational geometry of 3-folds (2000), pp. 101–174.

[Fra91] P. Francia, *On the base points of the bicanonical system*, Symposia Math., **32** (1991), 141–150.

[Fuj80] T. Fujita, *On the structure of polarized manifolds with total deficiency one, part I*, J. Math. Soc. of Japan, **32** (1980), 709–725.

[Fuj81] T. Fujita, *On the structure of polarized manifolds with total deficiency one, part II*, J. Math. Soc. of Japan, **33** (1981), 415–434.

[Fuj84] T. Fujita, *On the structure of polarized manifolds with total deficiency one, part III*, J. Math. Soc. of Japan, **36** (1984), 75–89.

[Fuj90] T. Fujita, *On singular Del Pezzo varieties*, Lecture Notes in Math. vol. 1417, Springer-Verlag, Berlin-New York (1990), pp. 117–128.

[Isk77] V. A. Iskovskih, *Fano 3-folds 1*, Izv. Akad. Nauk SSSR Ser. Mat, **41** (1977); English transl. in Math. USSR Izv. **11** (1977), 485–527.

[Isk78] V. A. Iskovskih, *Fano 3-folds 2*, Izv. Akad. Nauk SSSR Ser. Mat, **42** (1978), 506–549; English transl. in Math. USSR Izv. **12** (1978), 469–506.

[Isk79] V. A. Iskovskih, *Anticanonical models of three-dimensional algebraic varieties*, Itogi Nauki i Tekhniki, Sovremennye Problemy Matematiki, **12** (1979), 59–157; English transl. in J. Soviet. Math. **13** (1980), 745–814.

[Isk90] V. A. Iskovskih, *Double projection from a line on Fano threefolds of the first kind*; English transl. in Math. USSR Sbornik **66** (1990), 265–284.

[K$^+$92] J. Kollár et al., Flips and abundance for algebraic threefolds, vol. 211, Astérisque, 1992.

[Kaw82] Y. Kawamata, *A generalization of Kodaira-Ramanujam's vanishing theorem*, Math. Ann., **261** (1982), 43–46.

[Kaw86] Y. Kawamata, *On the plurigenera of minimal algebraic 3-folds with* $K \equiv 0$, Math. Ann., **275** (1986), 539–546.

[Kaw92] Y. Kawamata, *Boundedness of $\mathbb{Q}$-Fano threefolds*, Proc. Int. Conf. Algebra, Contemp. Math., vol. **131**, Amer. Math. Soc., Providence, RI, (1992), pp. 439–445.

[Kaw93] Y. Kawamata, *The minimal discrepancy of a 3-fold terminal singularity*, Russian Acad. Sci. Izv. Math., **40** (1993), 193–195, Appendix to '3-fold log flips'.

[KaM98] T. Kawachi and V. Maşek, *Reider-type theorems on normal surfaces*, J. Alg. Geom., **7** (1998), 239–249.

[Kawa00] T. Kawachi, *On the base point freeness of adjoint bundles on normal surfaces*, Manuscripta Math., **101** (2000), 23–38.

[KM92] J. Kollár and S. Mori, *Classification of three dimensional flips*, J. of Amer. Math. Soc., **5** (1992), 533–703.

[KMM87] Y. Kawamata, K. Matsuda, and K. Matsuki, *Introduction to the minimal model problem*, Adv. St. Pure Math., vol. **10** (1987), pp. 287–360.

[Kod53] K. Kodaira, *On a differential method in the theory of analytic stacks*, Proc. Nat. Acad. Sci. USA., **39** (1953), 1268–1273.

[Kol89] J. Kollár, *Flops*, Nagoya Math. J., **113** (1989), 15–36.

[Lau77] H. Laufer, *On minimally elliptic singularities*, Amer. Jour. Math., **99** (1977), 1257–1295.

[Luo98] T. Luo, *Divisorial extremal contractions of threefolds: divisor to point*, Amer. J. of Math., **120** (1998), 441–451.

[Min99] T. Minagawa, *Deformations of $\mathbb{Q}$-Calabi-Yau 3-folds and $\mathbb{Q}$-Fano 3-folds of Fano index* 1, J. Math. Sci. Univ. Tokyo, **6** (1999), no. 2, 397–414.

[MM81] S. Mori and S. Mukai, *Classification of Fano 3-folds with* $b_2 \geq 2$, Manuscripta Math., **36** (1981), 147–162.

[MM83] S. Mori and S. Mukai, *On Fano 3-folds with* $b_2 \geq 2$, Algebraic and Analytic Varieties, Adv. Stud. in Pure Math., vol. **1** (1983), pp. 101–129.

[MM85] S. Mori and S. Mukai, *Classification of Fano 3-folds with* $b_2 \geq 2$*, I*, Algebraic and Topological Theories, 1985, to the memory of Dr. Takehiko MIYATA, 496–545.

[Mor82] S. Mori, *Threefolds whose canonical bundles are not numerically effective*, Ann. of Math., **116** (1982), 133–176.

[Mor85] S. Mori, *On 3-dimensional terminal singularities*, Nagoya Math. J., **98** (1985), 43–66.

[Mor88] S. Mori, *Flip theorem and the existence of minimal models for 3-folds*, J. of Amer. Math. Soc., **1** (1988), 117–253.

[Muk95] S. Mukai, *New development of the theory of Fano threefolds: Vector bundle method and moduli problem*, in Japanese, Sugaku, **47** (1995), 125–144.

[Pro97] Y. Prokhorov, *On the existence of complements of the canonical divisor for Mori conic bundles*; English transl. in Sbornik: Mathematics, **188** (1997), no. 11, 1665–1685.

[Ram72] C. P. Ramanujam, *Remarks on the Kodaira vanishing theorem*, J. of the Indian Math. Soc., **36** (1972), 41–51.

[Reid80] M. Reid, *Lines on Fano 3-folds according to Shokurov*, Tech. report, Mittag-Leffler Institute **11** (1980).

[Reid87] M. Reid, *Young person's guide to canonical singularities*, Algebraic Geometry, Bowdoin, 1985, Proc. Symp. Pure Math., vol. 46 (1987), pp. 345–414.

[Reid94] M. Reid, *Nonnormal del Pezzo surface*, Publ. RIMS Kyoto Univ., **30** (1994), 695–728.

[Reid96] M. Reid, *Graded rings over K3s*, abstracts of Matsumura memorial conference (1996).

[Sak84] F. Sakai, *Weil divisors on normal surfaces*, Duke Math. J., **51** (1984), no. 4, 877–887.

[San95] T. Sano, *On classification of non-Gorenstein* $\mathbb{Q}$*-Fano 3-folds of Fano index 1*, J. Math. Soc. Japan, **47** (1995), no. 2, 369–380.

[San96] T. Sano, *Classification of non-Gorenstein* $\mathbb{Q}$*-Fano* $d$*-folds of Fano index greater than* $d-2$, Nagoya Math. J., **142** (1996), 133–143.

[Sho79a] V. V. Shokurov, *The existence of a straight line on Fano 3-folds*, Izv. Akad. Nauk SSSR Ser. Mat, **43** (1979), 921–963; English transl. in Math. USSR Izv. **15** (1980), 173–209.

[Sho79b] V. V. Shokurov, *Smoothness of the anticanonical divisor on a Fano 3-folds*, Math. USSR. Izvestija, **43** (1979), 430–441; English transl. in Math. USSR Izv. **14** (1980), 395–405.

[Taka02] H. Takagi, *On classification of* $\mathbb{Q}$*-Fano 3-folds of Gorenstein index 2. II*, Nagoya Math. Journal, **167** (2002), 157–216.

[Take89] K. Takeuchi, *Some birational maps of Fano 3-folds*, Compositio Math., **71** (1989), 265–283.

[Vie82] E. Viehweg, *Vanishing theorems*, Journ. reine angew. Math., **335** (1982), 1–8.

[Wil93] P. M. H. Wilson, *The Kähler cone on Calabi-Yau threefolds* (*and Erratum*), Invent. Math., **107, 114** (1992, 1993), 561–583, 231–233.

[Wil97] P. M. H. Wilson, *Symplectic deformations of Calabi-Yau threefolds*, J. Diff. Geom., **45** (1997), 611–637.

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