Y. Ding and S. Lu
Nagoya Math. J.
Vol. 167 (2002), 17–33

BOUNDEDNESS OF HOMOGENEOUS FRACTIONAL INTEGRALS ON L^p FOR $N/\alpha \leq P \leq \infty$

YONG DING AND SHANZHEN LU

Abstract. In this paper we study the map properties of the homogeneous fractional integral operator $T_{\Omega,\alpha}$ on $L^p(\mathbb{R}^n)$ for $n/\alpha \leq p \leq \infty$.

We prove that if Ω satisfies some smoothness conditions on S^{n-1}, then $T_{\Omega,\alpha}$ is bounded from $L^{n/\alpha}(\mathbb{R}^n)$ to $BMO(\mathbb{R}^n)$, and from $L^p(\mathbb{R}^n)\,(n/\alpha < p \leq \infty)$ to a class of the Campanato spaces $\mathcal{L}_{l,\lambda}(\mathbb{R}^n)$, respectively. As the corollary of the results above, we show that when Ω satisfies some smoothness conditions on S^{n-1}, the homogeneous fractional integral operator $T_{\Omega,\alpha}$ is also bounded from $H^p(\mathbb{R}^n)\,(n/(n+\alpha) \leq p \leq 1)$ to $L^q(\mathbb{R}^n)$ for $1/q = 1/p - \alpha/n$. The results are the extensions of Stein-Weiss (for $p = 1$) and Taibleson-Weiss's (for $n/(n+\alpha) \leq p < 1$) results on the boundedness of the Riesz potential operator I_α on the Hardy spaces $H^p(\mathbb{R}^n)$.

§1. Introduction and results

It is well-known that the Hardy-Littlewood-Sobolev theorem showed that the Riesz potential operator I_α is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for $0 < \alpha < n$, $1 < p < n/\alpha$ and $1/q = 1/p - \alpha/n$. Here

$$I_\alpha f(x) = \frac{1}{\gamma(\alpha)} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\,dy, \quad \text{and} \quad \gamma(\alpha) = \frac{\pi^{n/2} 2^\alpha \Gamma(\frac{\alpha}{2})}{\Gamma(\frac{n-\alpha}{2})}.$$

In 1960, Stein and Weiss [11] used the theory of the harmonic functions of several variables to prove that I_α is bounded from $H^1(\mathbb{R}^n)$ to $L^{n/(n-\alpha)}(\mathbb{R}^n)$. In 1980, using the molecular characterization of the real Hardy spaces, Taibleson and Weiss [12] proved that I_α is also bounded from $H^p(\mathbb{R}^n)$ to $H^q(\mathbb{R}^n)$, where $0 < p < 1$ and $1/q = 1/p - \alpha/n$.

Moreover, for the extreme case $p = n/\alpha$, it is easy to verify that I_α is not bounded from $L^{n/\alpha}(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$. However, as its substitute, we know

Received September 19, 1998.
2000 Mathematics Subject Classification: 42B20, 47G10.
The research was supported by National Natural Science Foundation (Grant: 19971010), Doctoral Programme Foundation of Institution of Higher Education and National 973 Project of China (G 19990751).

that I_α is bounded from $L^{n/\alpha}(\mathbb{R}^n)$ to $\mathrm{BMO}(\mathbb{R}^n)$. In 1974, Muckenhoupt and Wheeden [8] gave the weighted boundedness of I_α from $L^{n/\alpha}(\mathbb{R}^n)$ to $\mathrm{BMO}(\mathbb{R}^n)$.

On the other hand, it has appeared about the investigations of the various map properties of the homogeneous fractional integral operators $T_{\Omega,\alpha}$, which is defined by

$$T_{\Omega,\alpha}f(x)=\int_{\mathbb{R}^n}\frac{\Omega(x-y)}{|x-y|^{n-\alpha}}f(y)\,dy,$$

where $0<\alpha<n$, Ω is homogeneous of degree zero on $\mathbb{R}^n$ with $\Omega\in L^s(S^{n-1})\,(s\geq 1)$ and S^{n-1} denotes the unit sphere of $\mathbb{R}^n$. For instance, the weighted (L^p,L^q)-boundedness of $T_{\Omega,\alpha}$ for $1<p<n/\alpha$ had been studied in [7] (for power weights) and in [2] (for $A(p,q)$ weights). The weak boundedness of $T_{\Omega,\alpha}$ when $p=1$ can be found in [1] (unweighted) and in [4] (with power weights). Moreover, for $p=n/\alpha$, an exponential integral inequality of $T_{\Omega,\alpha}$ had been given in [3].

In comparison with the map properties of the Riesz potential operator I_α, it is natural to ask under what conditions, the homogeneous fractional integral operator $T_{\Omega,\alpha}$ has the same map properties on $H^p(\mathbb{R}^n)$ as the Riesz potential operator I_α.

The aim of this paper is to answer the question above. First we shall prove that if Ω satisfies some smoothness conditions on S^{n-1}, then $T_{\Omega,\alpha}$ is bounded from $L^{n/\alpha}(\mathbb{R}^n)$ to $BMO(\mathbb{R}^n)$ and from $L^p(\mathbb{R}^n)\,(n/\alpha<p\leq\infty)$ to a class of the Campanato spaces $\mathcal{L}_{l,\lambda}(\mathbb{R}^n)$, respectively. As its corollary, then we verify that Stein-Weiss's conclusion (for $p=1$) and Taibleson-Weiss's conclusion (for $n/(n+\alpha)\leq p<1$) hold still for $T_{\Omega,\alpha}$ instead of I_α.

It is worth pointing out that in the proof of our results, we use only the dual theory on the real Hardy spaces, while the atomic-molecular decomposition of $H^p(\mathbb{R}^n)$ is not used. Therefore, our method gives indeed another way proving Stein-Weiss and Taibleson-Weiss's results on I_α.

Before stating our results, let us give some definitions.

Suppose that $Q=Q(x_0,d)$ is a cube with its sides parallel to the coordinate axes and center at x_0, diameter $d>0$. For $1\leq l\leq\infty$, $-n/l\leq\lambda\leq 1$, we denote

$$\|f\|_{\mathcal{L}_{l,\lambda}}=\sup_Q\frac{1}{|Q|^{\lambda/n}}\left(\frac{1}{|Q|}\int_Q|f(x)-f_Q|^l\,dx\right)^{1/l},$$

where $f_Q = \frac{1}{|Q|}\int_Q f(y)dy$. Then the Campanato spaces $\mathcal{L}_{l,\lambda}(\mathbb{R}^n)$ is defined by

$$\mathcal{L}_{l,\lambda}(\mathbb{R}^n) = \{f \in L^l_{loc}(\mathbb{R}^n) : \|f\|_{\mathcal{L}_{l,\lambda}} < \infty\}.$$

If we identify functions that differ by a constant, then $\mathcal{L}_{l,\lambda}$ becomes a Banach space with the norm $\|\cdot\|_{\mathcal{L}_{l,\lambda}}$. It is well-known that

$$\mathcal{L}_{l,\lambda}(\mathbb{R}^n) \sim \begin{cases} \mathrm{Lip}_\lambda(\mathbb{R}^n), & \text{for } 0 < \lambda < 1, \\ \mathrm{BMO}(\mathbb{R}^n), & \text{for } \lambda = 0, \\ \text{Morrey space } L^{p,n+l\lambda}(\mathbb{R}^n), & \text{for } -n/l \le \lambda < 0. \end{cases}$$

On the other properties of the spaces $\mathcal{L}_{l,\lambda}(\mathbb{R}^n)$, we refer the reader to [9].

We say that Ω satisfies the L^s−Dini condition if Ω is homogeneous of degree zero on $\mathbb{R}^n$ with $\Omega \in L^s(S^{n-1})$ $(s \ge 1)$, and

$$\int_0^1 \omega_s(\delta)\frac{d\delta}{\delta} < \infty,$$

where $\omega_s(\delta)$ denotes the integral modulus of continuity of order s of Ω defined by

$$\omega_s(\delta) = \sup_{|\rho|<\delta}\left(\int_{S^{n-1}} |\Omega(\rho x') - \Omega(x')|^s\, dx'\right)^{1/s}$$

and ρ is a rotation in $\mathbb{R}^n$ and $|\rho| = \|\rho - I\|$.

A nonnegative locally integrable function pair (u, ν) on $\mathbb{R}^n$ is said to belong to $A(p,\infty)$ $(1 < p < \infty)$, if there is a constant $C > 0$ such that for any cube Q in $\mathbb{R}^n$

$$\left(\operatorname*{ess\,sup}_{x\in Q} \nu(x)\right)\left(\frac{1}{|Q|}\int_Q u(x)^{-p'}dx\right)^{1/p'} \le C < \infty,$$

where $p' = p/(p-1)$.

For a nonnegative locally integrable function $w(x)$ on $\mathbb{R}^n$, let us consider the function class $BMO_w(\mathbb{R}^n)$, the weighted version of $BMO(\mathbb{R}^n)$. We say a function $g \in BMO_w(\mathbb{R}^n)$, if there is a constant $C > 0$ such that for any cube $Q \in \mathbb{R}^n$,

$$\|g\|_{BMO_w} := \left(\operatorname*{ess\,sup}_{x\in Q} w(x)\right)\left(\frac{1}{|Q|}\int_Q |g(x) - g_Q|dx\right) \le C < \infty,$$

where $g_Q = \frac{1}{|Q|}\int_Q g(y)dy$.

Now, let us formulate our results as follows. The first conclusion is about the weighted boundedness of $T_{\Omega,\alpha}$ from $L^{n/\alpha}(u^{n/\alpha},\mathbb{R}^n)$ to $BMO_\nu(\mathbb{R}^n)$.

THEOREM 1. *Let $0<\alpha<n$, $s>n/(n-\alpha)$. If Ω satisfies the L^s-Dini condition and $(u^{s'},\nu^{s'})\in A(n/\alpha s',\infty)$, then there is a $C>0$ such that for any cube $Q\in\mathbb{R}^n$,*

$$\|T_{\Omega,\alpha}f\|_{BMO_\nu}\le C\|f\|_{L^{n/\alpha}(u^{n/\alpha})}. \tag{1.1}$$

Remark 1. Obviously, Theorem 7 in [8] is the especial example of Theorem 1 when $\Omega\equiv 1$, $s=\infty$ and $u(x)=\nu(x)$.

The following two theorems show that $T_{\Omega,\alpha}$ is bounded map from $L^p(\mathbb{R}^n)\,(n/\alpha<p\le\infty)$ to the Campanato spaces $\mathcal{L}_{l,\lambda}(\mathbb{R}^n)$ for appropriate indices $\lambda>0$ and $l\ge 1$.

THEOREM 2. *Let $0<\alpha<1$, $n/\alpha<p<\infty$ and $s>n/(n-\alpha)$. If for some $\beta>\alpha-n/p$, the integral modulus of continuity $\omega_s(\delta)$ of order s of Ω satisfies*

$$\int_0^1\omega_s(\delta)\frac{d\delta}{\delta^{1+\beta}}<\infty, \tag{1.2}$$

then there is a $C>0$ such that for $1\le l\le n/(n-\alpha)$, $\|T_{\Omega,\alpha}f\|_{\mathcal{L}_{l,n(\frac{\alpha}{n}-\frac{1}{p})}}\le C\|f\|_{L^p}$.

THEOREM 3. *Let $0<\alpha<1$ and $s>n/(n-\alpha)$. If the integral modulus of continuity $\omega_s(\delta)$ of order s of Ω satisfies*

$$\int_0^1\omega_s(\delta)\frac{d\delta}{\delta^{1+\alpha}}<\infty, \tag{1.3}$$

then there is a $C>0$ such that for $1\le l\le n/(n-\alpha)$, $\|T_{\Omega,\alpha}f\|_{\mathcal{L}_{l,\alpha}}\le C\|f\|_{L^\infty}$.

Having the conclusions above, by the dual theory on real Hardy spaces, we can obtain the boundedness of the operator $T_{\Omega,\alpha}$ acting on some real Hardy spaces.

THEOREM 4. *Let $0<\alpha<n$, $s>n/(n-\alpha)$. If Ω satisfies the L^s-Dini condition, then there is a $C>0$ such that*

$$\|T_{\Omega,\alpha}f\|_{L^{n/(n-\alpha)}}\le C\|f\|_{H^1}.$$

THEOREM 5. *Let $0<\alpha<1$, $n/(n+\alpha)<p<1$, $1/q=1/p-\alpha/n$ and $s>n/(n-\alpha)$. If for $\beta>n(1/p-1)$, the integral modulus of continuity $\omega_s(\delta)$ of order s of Ω satisfies* (1.2), *then there is a $C>0$ such that*

$$\|T_{\Omega,\alpha}f\|_{L^q}\le C\|f\|_{H^p}.$$

THEOREM 6. *Let $0<\alpha<1$, $p=n/(n+\alpha)$ and $s>n/(n-\alpha)$. If the integral modulus of continuity $\omega_s(\delta)$ of order s of Ω satisfies* (1.3), *then there is a $C>0$ such that*

$$\|T_{\Omega,\alpha}f\|_{L^1}\le C\|f\|_{H^{n/(n+\alpha)}}.$$

Below the letter C will denote a constant not necessarily the same at each occurrence.

§2. Boundedness of $T_{\Omega,\alpha}$ acting on $L^p(\mathbb{R}^n)$ for $n/\alpha\le p\le\infty$

In this section we shall give the proofs of Theorems 1 through 3. Let us begin with giving a lemma.

LEMMA 1. *Suppose that $0<\alpha<n$, $s>1$, Ω satisfies the L^s–Dini condition. There is a constant $0<a_0<1/2$ such that if $|x|<a_0R$, then*

$$\left(\int_{R<|y|<2R}\left|\frac{\Omega(y-x)}{|y-x|^{n-\alpha}}-\frac{\Omega(y)}{|y|^{n-\alpha}}\right|^s dy\right)^{1/s} \le CR^{n/s-(n-\alpha)}\left\{\frac{|x|}{R}+\int_{|x|/2R<\delta<|x|/R}\omega_s(\delta)\frac{d\delta}{\delta}\right\}.$$

Using the similar method as proving Lemma 5 in [5], we can prove Lemma 1. We omit the detail here.

Proof of Theorem 1. Fix a cube $Q\subset\mathbb{R}^n$, we denote the center and the diameter of Q by x_0 and d, respectively. Writing

$$\begin{aligned} T_{\Omega,\alpha}f(x) &= \int_B \frac{\Omega(x-y)}{|x-y|^{n-\alpha}}f(y)dy+\int_{\mathbb{R}^n\setminus B}\frac{\Omega(x-y)}{|x-y|^{n-\alpha}}f(y)dy \\ &:= T_1f(x)+T_2f(x), \end{aligned}$$

where $B=\{y\in\mathbb{R}^n;|y-x_0|<d\}$. It is sufficient to prove (1.1) for $T_1f(x)$ and $T_2f(x)$, respectively. Below we denote briefly $\operatorname{ess\,sup}_{x\in Q}\nu(x)$ by E.

First let us consider $T_1 f(x)$. We have

$$\begin{aligned}
&\frac{E}{|Q|}\int_Q |T_1 f(x) - (T_1 f)_Q| dx \\
&\le \frac{E}{|Q|}\int_Q \int_B \frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}}|f(y)|dydx \\
&\qquad + \frac{E}{|Q|}\int_Q \bigg(\frac{1}{|Q|}\int_Q\int_B \frac{|\Omega(z-y)|}{|z-y|^{n-\alpha}}|f(y)|dydz\bigg)dx \\
&\le \frac{2E}{|Q|}\int_B |f(y)| \int_Q \frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}}dxdy \\
&\le \frac{2E}{|Q|}\int_B |f(y)| \int_{|x-y|<2d} \frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}}dxdy.
\end{aligned}$$

Note that $\Omega(x') \in L^s(S^{n-1})$, we get

$$\int_{|x-y|<2d} \frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}}dx \le Cd^\alpha \|\Omega\|_{L^s(S^{n-1})} \le C|Q|^{\alpha/n}\|\Omega\|_{L^s(S^{n-1})}.$$

On the other hand, by Hölder's inequality,

$$\int_B |f(y)|dy \le \bigg(\int_B |f(y)u(y)|^p dy\bigg)^{1/p}\bigg(\int_B u(y)^{-p'}dy\bigg)^{1/p'}.$$

Here and below we denote $p = n/\alpha$ in the proof of Theorem 1. Since $p' < s'(p/s')'$, using Hölder's inequality again, we have

$$\begin{aligned}
(2.1)\qquad &\frac{E}{|Q|}\int_Q |T_1 f(x) - (T_1 f)_Q| dx \\
&\le CE|Q|^{-1+\alpha/n}\bigg(\int_B |f(y)u(y)|^p dy\bigg)^{1/p}\bigg(\int_B u(y)^{-p'}dy\bigg)^{1/p'} \\
&\le CE\bigg(\int_B |f(y)u(y)|^p dy\bigg)^{1/p}\bigg(\frac{1}{|2\sqrt{n}Q|}\int_{2\sqrt{n}Q} u(y)^{-p'}dy\bigg)^{1/p'} \\
&\le CE\bigg(\int_B |f(y)u(y)|^p dy\bigg)^{1/p}\bigg(\frac{1}{|2\sqrt{n}Q|}\int_{2\sqrt{n}Q} u(y)^{-s'(p/s')'}dy\bigg)^{1/[s'(p/s')']},
\end{aligned}$$

where $2\sqrt{n}Q$ denotes the cube with the center at x_0 and the diameter $2\sqrt{n}d$. By the condition $(u(x)^{s'}, \nu(x)^{s'}) \in A(p/s', \infty)$, we get

$$(2.2)\quad E\bigg(\frac{1}{|2\sqrt{n}Q|}\int_{2\sqrt{n}Q} u(x)^{-s'(p/s')'}dx\bigg)^{1/[s'(p/s')']}$$

$$\le \left\{ \left(\operatorname{ess}\sup_{x\in 2\sqrt{n}Q} \nu(x)^{s'} \right) \left(\frac{1}{|2\sqrt{n}Q|} \int_{2\sqrt{n}Q} (u(x)^{s'})^{-(p/s')'} dx \right)^{1/(p/s')'} \right\}^{1/s'}$$
$$\le C < \infty.$$

Therefore, by (2.1) and (2.2) we obtain

$$\frac{E}{|Q|} \int_Q |T_1 f(x) - (T_1 f)_Q| dx \le C \left(\int_{\mathbb{R}^n} |f(x)u(x)|^p dy \right)^{1/p}. \tag{2.3}$$

Now, let us turn to the estimation for $T_2 f(x)$. In this case we have

$$\begin{aligned} &\frac{E}{|Q|} \int_Q |T_2 f(x) - (T_2 f)_Q| dx \\ &= \frac{E}{|Q|} \int_Q \left| \frac{1}{|Q|} \int_Q \left\{ \int_{|y-x_0|\ge d} f(y) \left[\frac{\Omega(x-y)}{|x-y|^{n-\alpha}} - \frac{\Omega(z-y)}{|z-y|^{n-\alpha}} \right] dy \right\} dz \right| dx \\ &\le \frac{E}{|Q|} \int_Q \frac{1}{|Q|} \int_Q \left\{ \sum_{j=0}^{\infty} \int_{2^j d\le |y-x_0|<2^{j+1}d} |f(y)| \left| \frac{\Omega(x-y)}{|x-y|^{n-\alpha}} - \frac{\Omega(z-y)}{|z-y|^{n-\alpha}} \right| dy \right\} dz dx. \end{aligned} \tag{2.4}$$

By Hölder's inequality, we get

$$\begin{aligned} &\int_{2^j d\le |y-x_0|<2^{j+1}d} |f(y)| \left| \frac{\Omega(x-y)}{|x-y|^{n-\alpha}} - \frac{\Omega(z-y)}{|z-y|^{n-\alpha}} \right| dy \\ &\le \left(\int_{2^j d\le |y-x_0|<2^{j+1}d} |f(y)|^{s'} dy \right)^{1/s'} \\ &\quad \times \left(\int_{2^j d\le |y-x_0|<2^{j+1}d} \left| \frac{\Omega(x-y)}{|x-y|^{n-\alpha}} - \frac{\Omega(z-y)}{|z-y|^{n-\alpha}} \right|^s dy \right)^{1/s}. \end{aligned} \tag{2.5}$$

Since

$$\begin{aligned} &\left| \frac{\Omega(x-y)}{|x-y|^{n-\alpha}} - \frac{\Omega(z-y)}{|z-y|^{n-\alpha}} \right| \\ &\le \left| \frac{\Omega(x-y)}{|x-y|^{n-\alpha}} - \frac{\Omega(y-x_0)}{|y-x_0|^{n-\alpha}} \right| + \left| \frac{\Omega(z-y)}{|z-y|^{n-\alpha}} - \frac{\Omega(y-x_0)}{|y-x_0|^{n-\alpha}} \right|, \end{aligned}$$

we have

$$\begin{aligned}&\left(\int_{2^jd\le|y-x_0|<2^{j+1}d}\left|\frac{\Omega(x-y)}{|x-y|^{n-\alpha}}-\frac{\Omega(z-y)}{|z-y|^{n-\alpha}}\right|^s dy\right)^{1/s}\\ &\le\left(\int_{2^jd\le|y-x_0|<2^{j+1}d}\left|\frac{\Omega(x-y)}{|x-y|^{n-\alpha}}-\frac{\Omega(y-x_0)}{|y-x_0|^{n-\alpha}}\right|^s dy\right)^{1/s}\\ &\qquad+\left(\int_{2^jd\le|y-x_0|<2^{j+1}d}\left|\frac{\Omega(z-y)}{|z-y|^{n-\alpha}}-\frac{\Omega(y-x_0)}{|y-x_0|^{n-\alpha}}\right|^s dy\right)^{1/s}\\ &:=J_1+J_2.\end{aligned}$$

Let us give the estimations of J_1 and J_2, respectively. Writing J_1 as

$$\left(\int_{2^jd\le|y|<2^{j+1}d}\left|\frac{\Omega((x-x_0)-y)}{|(x-x_0)-y|^{n-\alpha}}-\frac{\Omega(y)}{|y|^{n-\alpha}}\right|^s dy\right)^{1/s}.$$

Note that $x\in Q$, if taking $R=2^jd$, then $|x-x_0|<\frac{1}{2^{j+1}}R$. Applying Lemma 1 to J_1, we get

$$\begin{aligned}J_1&\le C(2^jd)^{n/s-(n-\alpha)}\left\{\frac{|x-x_0|}{2^jd}+\int_{|x-x_0|/2^{j+1}d<\delta<|x-x_0|/2^jd}\omega_s(\delta)\frac{d\delta}{\delta}\right\}\\ &\le C(2^jd)^{n/s-(n-\alpha)}\left\{\frac{1}{2^{j+1}}+\int_{|x-x_0|/2^{j+1}d}^{|x-x_0|/2^jd}\omega_s(\delta)\frac{d\delta}{\delta}\right\}.\end{aligned}$$

By $z\in Q$ and using similar method, we have

$$J_2\le C(2^jd)^{n/s-(n-\alpha)}\left\{\frac{1}{2^{j+1}}+\int_{|z-x_0|/2^{j+1}d}^{|z-x_0|/2^jd}\omega_s(\delta)\frac{d\delta}{\delta}\right\}.$$

Since $p=n/\alpha$ and $n/s-(n-\alpha)=-n/[s'(p/s')']$, we get

$$(2^jd)^{n/s-(n-\alpha)}\le C|2^{j+1}\sqrt{n}Q|^{-1/[s'(p/s')']}.$$

Thus, with the estimations for J_1 and J_2, we have

$$\begin{aligned}&\left(\int_{2^jd\le|y-x_0|<2^{j+1}d}\left|\frac{\Omega(x-y)}{|x-y|^{n-\alpha}}-\frac{\Omega(z-y)}{|z-y|^{n-\alpha}}\right|^s dy\right)^{1/s}\\ &\le C|2^{j+1}\sqrt{n}Q|^{-1/[s'(p/s')']}\left\{\frac{1}{2^j}+\int_{|x-x_0|/2^{j+1}d}^{|x-x_0|/2^jd}\omega_s(\delta)\frac{d\delta}{\delta}\right.\\ &\qquad\left.+\int_{|z-x_0|/2^{j+1}d}^{|z-x_0|/2^jd}\omega_s(\delta)\frac{d\delta}{\delta}\right\}.\end{aligned}\tag{2.6}$$

On the other hand, using Hölder's inequality again we have

$$\begin{aligned}
(2.7)\quad &\left(\int_{2^j d\le|y-x_0|<2^{j+1}d}|f(y)|^{s'}dy\right)^{1/s'}\\
&\le\left(\int_{2^j d\le|y-x_0|<2^{j+1}d}|f(y)u(y)|^{p}dy\right)^{1/p}\\
&\qquad\times\left(\int_{2^j d\le|y-x_0|<2^{j+1}d}u(y)^{-s'(p/s')'}dy\right)^{1/[s'(p/s')']}\\
&\le\left(\int_{\mathbb{R}^n}|f(y)u(y)|^{p}dy\right)^{1/p}\left(\int_{2^{j+1}\sqrt{n}Q}u(y)^{-s'(p/s')'}dy\right)^{1/[s'(p/s')']}.
\end{aligned}$$

Since $(u(x)^{s'},\nu(x)^{s'})\in A(p/s',\infty)$, it is easy to see that there is a $C>0$ such that for any $j\ge 0$,

$$\begin{aligned}
(2.8)\quad &E\left(\frac{1}{|2^{j+1}\sqrt{n}Q|}\int_{2^{j+1}\sqrt{n}Q}u(x)^{-s'(p/s')'}dx\right)^{1/[s'(p/s')']}\\
&\le\left\{\left(\operatorname*{ess\ sup}_{x\in 2^{j+1}\sqrt{n}Q}\nu(x)^{s'}\right)\right.\\
&\qquad\left.\times\left(\frac{1}{|2^{j+1}\sqrt{n}Q|}\int_{2^{j+1}\sqrt{n}Q}u(x)^{-s'(p/s')'}dx\right)^{1/(p/s')'}\right\}^{1/s'}\\
&\le C<\infty.
\end{aligned}$$

From (2.5),(2.6),(2.7) and (2.8), we obtain

$$\begin{aligned}
&\sum_{j=0}^{\infty}E\int_{2^j d\le|y-x_0|<2^{j+1}d}|f(y)|\left|\frac{\Omega(x-y)}{|x-y|^{n-\alpha}}-\frac{\Omega(z-y)}{|z-y|^{n-\alpha}}\right|dy\\
&\le C\sum_{j=0}^{\infty}\left(\int_{\mathbb{R}^n}|f(y)u(y)|^{p}dy\right)^{1/p}E\left(\int_{2^{j+1}\sqrt{n}Q}u(y)^{-s'(p/s')'}dy\right)^{1/[s'(p/s')']}\\
&\quad\times|2^{j+1}\sqrt{n}Q|^{-1/[s'(p/s')']}\\
&\quad\times\left\{\frac{1}{2^j}+\int_{|x-x_0|/2^{j+1}d}^{|x-x_0|/2^j d}\omega_s(\delta)\frac{d\delta}{\delta}+\int_{|z-x_0|/2^{j+1}d}^{|z-x_0|/2^j d}\omega_s(\delta)\frac{d\delta}{\delta}\right\}\\
&\le C\left(\int_{\mathbb{R}^n}|f(y)u(y)|^{p}dy\right)^{1/p}\\
&\quad\times\sum_{j=0}^{\infty}\left\{\frac{1}{2^j}+\int_{|x-x_0|/2^{j+1}d}^{|x-x_0|/2^j d}\omega_s(\delta)\frac{d\delta}{\delta}+\int_{|z-x_0|/2^{j+1}d}^{|z-x_0|/2^j d}\omega_s(\delta)\frac{d\delta}{\delta}\right\}
\end{aligned}$$

$$\leq C\Big(\int_{\mathbb{R}^n}|f(y)u(y)|^p dy\Big)^{1/p}\Big\{2+2\int_0^1 \omega_s(\delta)\frac{d\delta}{\delta}\Big\}$$
$$\leq C\Big(\int_{\mathbb{R}^n}|f(y)u(y)|^p dy\Big)^{1/p}.$$

Combining this with (2.4), we have

$$\frac{E}{|Q|}\int_Q |T_2f(x)-(T_2f)_Q|dx \leq C\Big(\int_{\mathbb{R}^n}|f(x)u(x)|^p dy\Big)^{1/p}. \tag{2.9}$$

By (2.3) and (2.9), we complete the proof of Theorem 1.

Proof of Theorem 2. As the proof of Theorem 1, We need only to prove (1.3) for T_1 and T_2, respectively. First let us consider $T_1f(x)$. We have

$$\frac{1}{|Q|^{\alpha/n-1/p}}\Big(\frac{1}{|Q|}\int_Q |T_1f(x)-(T_1f)_Q|^l dx\Big)^{1/l}$$
$$\leq \frac{2}{|Q|^{\alpha/n-1/p}}\Big(\frac{1}{|Q|}\int_Q |T_1f(x)|^l dx\Big)^{1/l}$$
$$= \frac{2}{|Q|^{\alpha/n-1/p}}\Big(\frac{1}{|Q|}\int_Q \Big|\int_B \frac{\Omega(x-y)}{|x-y|^{n-\alpha}}f(y)\,dy\Big|^l dx\Big)^{1/l}$$
$$\leq \frac{2}{|Q|^{\alpha/n-1/p}}\frac{1}{|Q|^{1/l}}\int_B |f(y)|\Big(\int_{|y-x|<2d}\Big(\frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}}\Big)^l dx\Big)^{1/l}dy.$$

Note that $\Omega(x')\in L^s(S^{n-1})$ and $s>n/(n-\alpha)\geq l$, hence

$$\begin{aligned}\Big(\int_{|x-y|<2d}\Big(\frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}}\Big)^l dx\Big)^{1/l} &\leq Cd^{n/l-(n-\alpha)}\|\Omega\|_{L^s(S^{n-1})}\\ &\leq C|Q|^{1/l-(1-\alpha/n)}\|\Omega\|_{L^s(S^{n-1})}.\end{aligned} \tag{2.10}$$

On the other hand, by Hölder's inequality,

$$\int_B |f(y)|dy \leq C|Q|^{1/p'}\Big(\int_B |f(y)|^p dy\Big)^{1/p} \leq C|Q|^{1/p'}\|f\|_p.$$

Thus,

$$\begin{aligned}&\frac{1}{|Q|^{\alpha/n-1/p}}\Big(\frac{1}{|Q|}\int_Q |T_1f(x)-(T_1f)_Q|^l\,dx\Big)^{1/l}\\ &\quad\leq C|Q|^{1/p-\alpha/n-1/l+1/p'+1/l-(1-\alpha/n)}\|\Omega\|_{L^s(S^{n-1})}\|f\|_p \leq C\|f\|_p.\end{aligned} \tag{2.11}$$

Now, let us turn to the estimation for $T_2 f(x)$. In this case we have

$$\begin{aligned}(2.12)\qquad &\frac{1}{|Q|^{\alpha/n-1/p}}\left(\frac{1}{|Q|}\int_Q |T_2 f(x)-(T_2 f)_Q|^l\,dx\right)^{1/l}\\ &= \frac{1}{|Q|^{\alpha/n-1/p}}\left(\frac{1}{|Q|}\int_Q\left|\frac{1}{|Q|}\int_Q\left\{\sum_{j=0}^{\infty}\int_{2^j d\le|y-x_0|<2^{j+1}d} f(y)\right.\right.\right.\\ &\qquad\left.\left.\left.\times\left[\frac{\Omega(x-y)}{|x-y|^{n-\alpha}}-\frac{\Omega(z-y)}{|z-y|^{n-\alpha}}\right]dy\right\}dz\right|^l dx\right)^{1/l}.\end{aligned}$$

By (2.5) and $s' < n/\alpha < p$,

$$\begin{aligned}(2.13)\qquad &\int_{2^j d\le|y-x_0|<2^{j+1}d}|f(y)|\left|\frac{\Omega(x-y)}{|x-y|^{n-\alpha}}-\frac{\Omega(z-y)}{|z-y|^{n-\alpha}}\right|dy\\ &\le\left(\int_{2^j d\le|y-x_0|<2^{j+1}d}|f(y)|^{s'}dy\right)^{1/s'}(J_1+J_2)\\ &\le C\|f\|_p(2^j d)^{n/[s'(p/s')']}(J_1+J_2).\end{aligned}$$

Since the integral modulus of continuity $\omega_s(\delta)$ of order s of Ω satisfies (1.2) and

$$\int_0^1\omega_s(\delta)\frac{d\delta}{\delta}<\int_0^1\omega_s(\delta)\frac{d\delta}{\delta^{1+\beta}}<\infty,$$

we know that Ω satisfies also the L^s-Dini condition. From Lemma 1 and the proof of Theorem 1,

$$\begin{aligned}(2.14)\quad J_1+J_2 &\le C(2^j d)^{n/s-(n-\alpha)}\\ &\times\left\{\frac{1}{2^j}+\int_{|x-x_0|/2^{j+1}d}^{|x-x_0|/2^j d}\omega_s(\delta)\frac{d\delta}{\delta}+\int_{|z-x_0|/2^{j+1}d}^{|z-x_0|/2^j d}\omega_s(\delta)\frac{d\delta}{\delta}\right\}.\end{aligned}$$

Note that

$$(2^j d)^{n/[s'(p/s')']+n/s-(n-\alpha)}=(2^j d)^{n(\alpha/n-1/p)}\le C|Q|^{\alpha/n-1/p}2^{jn(\alpha/n-1/p)}.$$

Moreover,

$$\begin{aligned}(2.15)\qquad &2^{jn(\alpha/n-1/p)}\int_{|x-x_0|/2^{j+1}d}^{|x-x_0|/2^j d}\omega_s(\delta)\frac{d\delta}{\delta}\\ &\le 2^{jn(\alpha/n-1/p)}(|x-x_0|/2^j d)^{\beta}\int_{|x-x_0|/2^{j+1}d}^{|x-x_0|/2^j d}\omega_s(\delta)\frac{d\delta}{\delta^{1+\beta}}\\ &\le C2^{j[n(\alpha/n-1/p)-\beta]}\int_0^1\omega_s(\delta)\frac{d\delta}{\delta^{1+\beta}}.\end{aligned}$$

By $0 < \alpha < 1$ and $\beta > \alpha - n/p$, we have $n(\alpha/n - 1/p) - 1 < 0$ and $n(\alpha/n - 1/p) - \beta < 0$, respectively. Thus, by (2.13)–(2.15) and (1.2),

$$\begin{aligned}&\sum_{j=0}^{\infty}\int_{2^j d\le|y-x_0|<2^{j+1}d} f(y)\left[\frac{\Omega(x-y)}{|x-y|^{n-\alpha}}-\frac{\Omega(z-y)}{|z-y|^{n-\alpha}}\right]dy\\ &\le C\|f\|_p|Q|^{\alpha/n-1/p}\sum_{j=0}^{\infty}\left\{2^{j[n(\alpha/n-1/p)-1]}+C2^{j[n(\alpha/n-1/p)-\beta]}\int_0^1\omega_s(\delta)\frac{d\delta}{\delta^{1+\beta}}\right\}\\ &\le C\|f\|_p|Q|^{\alpha/n-1/p}.\end{aligned}$$

Combining this with (2.12), we have

$$\frac{1}{|Q|^{\alpha/n-1/p}}\left(\frac{1}{|Q|}\int_Q|T_2f(x)-(T_2f)_Q|^l\,dx\right)^{1/l}\le C\|f\|_p. \tag{2.16}$$

By (2.11) and (2.16), we complete the proof of Theorem 2.

Proof of Theorem 3. For $T_1f(x)$, by $f\in L^\infty$ and (2.10) we get

$$\begin{aligned}&\frac{1}{|Q|^{\alpha/n}}\left(\frac{1}{|Q|}\int_Q|T_1f(x)-(T_1f)_Q|^l dx\right)^{1/l}\\ &\le\frac{2}{|Q|^{\alpha/n}}\frac{1}{|Q|^{1/l}}\int_B|f(y)|\left(\int_{|y-x|<2d}\left(\frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}}\right)^l dx\right)^{1/l}dy\\ &\le C|Q|^{-\alpha/n-1/l+1+1/l-(1-\alpha/n)}\|\Omega\|_{L^s(S^{n-1})}\|f\|_\infty\le C\|f\|_\infty.\end{aligned}\tag{2.17}$$

On the other hand, by $f\in L^\infty$ and (2.13) and (2.14),

$$\begin{aligned}&\int_{2^j d\le|y-x_0|<2^{j+1}d}|f(y)|\left|\frac{\Omega(x-y)}{|x-y|^{n-\alpha}}-\frac{\Omega(z-y)}{|z-y|^{n-\alpha}}\right|dy\\ &\le\left(\int_{2^j d\le|y-x_0|<2^{j+1}d}|f(y)|^{s'}dy\right)^{1/s'}(J_1+J_2)\\ &\le C\|f\|_\infty(2^jd)^{n/s'}(2^jd)^{n/s-(n-\alpha)}\\ &\qquad\times\left\{\frac{1}{2^j}+\int_{|x-x_0|/2^{j+1}d}^{|x-x_0|/2^jd}\omega_s(\delta)\frac{d\delta}{\delta}+\int_{|z-x_0|/2^{j+1}d}^{|z-x_0|/2^jd}\omega_s(\delta)\frac{d\delta}{\delta}\right\}.\end{aligned}\tag{2.18}$$

Note that $(2^jd)^{n/s'+n/s-(n-\alpha)}\le C|Q|^{\alpha/n}2^{j\alpha}$, by (2.18) and (1.3),

$$\sum_{j=0}^{\infty}\int_{2^j d\le|y-x_0|<2^{j+1}d} f(y)\left[\frac{\Omega(x-y)}{|x-y|^{n-\alpha}}-\frac{\Omega(z-y)}{|z-y|^{n-\alpha}}\right]dy \tag{2.19}$$

$$\leq C\|f\|_\infty |Q|^{\alpha/n}$$
$$\times \sum_{j=0}^{\infty} \left\{ 2^{j(\alpha-1)} + \left(\int_{|x-x_0|/2^{j+1}d}^{|x-x_0|/2^j d} \omega_s(\delta) \frac{d\delta}{\delta^{1+\alpha}} + \int_{|z-x_0|/2^{j+1}d}^{|z-x_0|/2^j d} \omega_s(\delta) \frac{d\delta}{\delta^{1+\alpha}} \right) \right\}$$
$$\leq C\|f\|_\infty |Q|^{\alpha/n}.$$

Now,we may give the estimate of $T_2 f(x)$. By (2.12) (taking $p = \infty$) and (2.19), we have

$$\frac{1}{|Q|^{\alpha/n}} \left(\frac{1}{|Q|} \int_Q |T_2 f(x) - (T_2 f)_Q|^l \, dx \right)^{1/l} \leq C\|f\|_\infty. \tag{2.20}$$

Thus, Theorem 3 follows from (2.17) and (2.20).

§3. Boundedness of $T_{\Omega,\alpha}$ acting on $H^p(\mathbb{R}^n)$ for $n/(n+\alpha) \leq p \leq 1$

Before giving the proofs of Theorems 4 through 6, let us recall some definitions. Assume that $0 < p \leq 1 \leq q \leq \infty$, $p \neq q$, and s be a nonnegative integer with $s \geq [n(1/p - 1)]$. Then a function $a(x) \in L^q(\mathbb{R}^n)$ is called a (p, q, s) atom, if there is a cube $Q \subset \mathbb{R}^n$ such that $a(x)$ satisfies the following conditions: (i) supp$a \subset Q$; (ii) $\|a\|_{L^q} \leq |Q|^{\frac{1}{q}-\frac{1}{p}}$; and (iii) $\int a(x) x^\gamma dx = 0$ for all multi-indices γ of order $|\gamma| \leq s$. The atom Hardy spaces $H_a^{p,q,s}(\mathbb{R}^n)$ is defined by

$$H_a^{p,q,s}(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n) : f(x)$$
$$= \sum_k \lambda_k a_k(x), \text{ each } a_k \text{ is a } (p,q,s) \text{ atom and } \sum_k |\lambda_k|^p < \infty\},$$

where $\mathcal{S}'(\mathbb{R}^n)$ denotes the tempered distribution class, and the equality in the definition above is in the sense of distribution. Setting $H_a^{p,q,s}(\mathbb{R}^n)$ norm of f by

$$\|f\|_{H_a^{p,q,s}} = \inf(\sum_k |\lambda_k|^p)^{1/p},$$

where the infimum is taken over all decompositions of $f(x) = \sum_k \lambda_k a_k(x)$. Then by the theory of atomic decomposition on real Hardy spaces $H^p(\mathbb{R}^n)$ (see [6] or [10], for example), we know that

$$H_a^{p,q,s}(\mathbb{R}^n) = H^p(\mathbb{R}^n), \quad \text{in the sense } \|f\|_{H_a^{p,q,s}} \sim \|f\|_{H^p}. \tag{3.1}$$

Now let us give the definition of the dual spaces $(H_a^{p,q,s}(\mathbb{R}^n))^*$ of $H_a^{p,q,s}(\mathbb{R}^n)$ for $0 < p < 1$. Suppose that s is a nonnegative integer, $\mathcal{P}_s$ denotes the set of all polynomials with its degree $\le s$. Moreover, $\lambda \ge 0$, $1 \le l \le \infty$. Let

$$\|f\|_{\mathcal{L}_{l,\lambda,s}} = \sup_Q \frac{1}{|Q|^{\lambda/n}} \left(\frac{1}{|Q|} \int_Q |f(x) - (P_Q f)(x)|^l \, dx \right)^{1/l},$$

where $(P_Q f)(x)$ denotes the unique polynomial $P(x) \in \mathcal{P}_s$ satisfying

$$\int_Q [f(x) - P(x)] h(x) \, dx = 0, \quad \text{for any} \quad h(x) \in \mathcal{P}_s.$$

Then the Campanato space $\mathcal{L}_{l,\lambda,s}(\mathbb{R}^n)$ is defined by

$$\mathcal{L}_{l,\lambda,s}(\mathbb{R}^n) = \{ f \in L^l_{loc}(\mathbb{R}^n) : \|f\|_{\mathcal{L}_{l,\lambda,s}} < \infty \}.$$

The following conclusion shows that $\mathcal{L}_{l,\lambda,s}(\mathbb{R}^n)$ is the dual space of $H^p(\mathbb{R}^n)$.

Theorem A. ([6]) *Let $0 < p \le 1 \le q \le \infty$, $p \ne q$, $1/q + 1/q' = 1$ and s be a nonnegative integer with $s \ge [n(1/p - 1)]$. Then $(H_a^{p,q,s}(\mathbb{R}^n))^* = \mathcal{L}_{q',n(1/p-1),s}(\mathbb{R}^n)$.*

Thus, by Theorem A and (3.1) we get for $0 < p < 1$, $1 \le l \le \infty$ and $s \ge [n(1/p - 1)]$,

$$(H^p(\mathbb{R}^n))^* = \mathcal{L}_{l,n(1/p-1),s}(\mathbb{R}^n). \tag{3.2}$$

Below, let us consider another space $\mathcal{L}'_{l,\lambda,s}(\mathbb{R}^n)$, a version of $\mathcal{L}_{l,\lambda,s}(\mathbb{R}^n)$, which is defined by

$$\mathcal{L}'_{l,\lambda,s}(\mathbb{R}^n) = \{ f \in L^l_{loc}(\mathbb{R}^n) : \|f\|_{\mathcal{L}'_{l,\lambda,s}} < \infty \},$$

where s is a nonnegative integer, $\lambda \ge 0$, $1 \le l \le \infty$, and

$$\|f\|_{\mathcal{L}'_{l,\lambda,s}} = \sup_Q \frac{1}{|Q|^{\lambda/n}} \left(\inf_{P \in \mathcal{P}_s} \frac{1}{|Q|} \int_Q |f(x) - P(x)|^l \, dx \right)^{1/l}.$$

If we identify functions that differ by a polynomials with its degree $\le s$, then $\mathcal{L}'_{l,\lambda,s}(\mathbb{R}^n)$ becames a Banach space with the norm $\|\cdot\|_{\mathcal{L}'_{l,\lambda,s}}$.

In [12], it was proved that the space $\mathcal{L}'_{l,\lambda,s}(\mathbb{R}^n)$ is equal to the space $\mathcal{L}_{l,\lambda,s}(\mathbb{R}^n)$ in the sense

$$\|f\|_{\mathcal{L}'_{l,\lambda,s}} \sim \|f\|_{\mathcal{L}_{l,\lambda,s}}.$$

From this and (3.2), for $0 < p < 1$, $1 \le l \le \infty$ and $s \ge [n(1/p-1)]$, we have

$$(H^p(\mathbb{R}^n))^* = \mathcal{L}'_{l,n(1/p-1),s}(\mathbb{R}^n). \tag{3.3}$$

On the other hand, from the definitions of $\mathcal{L}_{l,\lambda}(\mathbb{R}^n)$ and $\mathcal{L}'_{l,\lambda,s}(\mathbb{R}^n)$, it is easy to verify that for any nonnegative integer s and $\lambda > 0$, $1 \le l \le \infty$

$$\mathcal{L}_{l,\lambda}(\mathbb{R}^n) \subset \mathcal{L}'_{l,\lambda,s}(\mathbb{R}^n), \quad \text{and} \quad \|f\|_{\mathcal{L}'_{l,\lambda,s}} \le \|f\|_{\mathcal{L}_{l,\lambda}} \quad \text{for} \quad f \in \mathcal{L}_{l,\lambda}(\mathbb{R}^n). \tag{3.4}$$

Therefore, by (3.3) and (3.4) we get for $0 < p < 1$ and $1 \le l \le \infty$

$$\mathcal{L}_{l,n(\frac{1}{p}-1)}(\mathbb{R}^n) \subset (H^p(\mathbb{R}^n))^*. \tag{3.5}$$

Now let us turn to the proofs of Theorem 4 through 6.

Proof of Theorem 4. Note that the dual relations $(L^{n/(n-\alpha)}(\mathbb{R}^n))^* = L^{n/\alpha}(\mathbb{R}^n)$, and $(H^1(\mathbb{R}^n))^* = BMO(\mathbb{R}^n)$, by (1.1) (taking $u(x) = \nu(x) \equiv 1$), for any $f \in H^1(\mathbb{R}^n)$ we have

$$\begin{aligned}\|T_{\Omega,\alpha}f\|_{L^{n/(n-\alpha)}} &= \sup_g \left| \int_{\mathbb{R}^n} T_{\Omega,\alpha}f(x)g(x)\,dx \right| \\ &= \sup_g \left| \int_{\mathbb{R}^n} f(x)(T_{\Omega,\alpha})^* g(x)\,dx \right|,\end{aligned}$$

where the supremum is taken over all $g \in L^{n/\alpha}(\mathbb{R}^n)$ with $\|g\|_{L^{n/\alpha}} \le 1$, and $(T_{\Omega,\alpha})^*$ denotes the adjoint operator of $T_{\Omega,\alpha}$. Obviously, we have $(T_{\Omega,\alpha})^* = T_{\widetilde{\Omega},\alpha}$, where $\widetilde{\Omega}(x) = \overline{\Omega(-x)}$. It is easy to see that $\overline{\Omega(-x)}$ satisfies the same conditions as $\Omega(x)$. Thus, we know that under the conditions of Theorem 4, the conclusion of Theorem 1 holds also for $\widetilde{\Omega}(x)$. Therefore,

$$\begin{aligned}\|T_{\Omega,\alpha}f\|_{L^{n/(n-\alpha)}} &= \sup_g \left| \int_{\mathbb{R}^n} f(x)(T_{\Omega,\alpha})^* g(x)\,dx \right| \\ &\le \sup_g \|f\|_{H^1} \|(T_{\Omega,\alpha})^* g\|_{BMO} \\ &\le C \sup_g \|f\|_{H^1} \|g\|_{L^{n/\alpha}} \le C\|f\|_{H^1}.\end{aligned}$$

This is (1.5).

Proof of Theorem 5. By $n/(n+\alpha) < p < 1$ and $1/q = 1/p - \alpha/n$, we get $1 < q < n/(n-\alpha)$ and $n/\alpha < q' < \infty$. Moreover, it is easy to verify that $\beta > n(1/p-1)$ is equivalent to $\beta > \alpha - n/q'$. Thus, by Theorem 2 for $1 \le l \le n/(n-\alpha)$ and the adjoint operator $(T_{\Omega,\alpha})^*$ of $T_{\Omega,\alpha}$, we have

$$\|(T_{\Omega,\alpha})^* g\|_{\mathcal{L}_{l,n(\frac{1}{p}-1)}} = \|(T_{\Omega,\alpha})^* g\|_{\mathcal{L}_{l,n(\frac{\alpha}{n}-\frac{1}{q'})}} \le C\|g\|_{L^{q'}}. \tag{3.6}$$

On the other hand, by (3.5) we know that for $0 < p < 1$ and $1 \le l \le \infty$, $\mathcal{L}_{l,n(\frac{1}{p}-1)}(\mathbb{R}^n) \subset (H^p(\mathbb{R}^n))^*$. Thus, for any $f \in H^p(\mathbb{R}^n)$ $(n/(n+\alpha) < p < 1)$, if taking $1 \le l \le n/(n-\alpha)$ and using the idea above proving Theorem 4, then by (3.6) and (3.4) we get

$$\begin{aligned}\|T_{\Omega,\alpha} f\|_{L^q} &= \sup_g \left| \int_{\mathbb{R}^n} T_{\Omega,\alpha} f(x) g(x)\, dx \right| = \sup_g \left| \int_{\mathbb{R}^n} f(x) (T_{\Omega,\alpha})^* g(x)\, dx \right| \\ &\le \sup_g \|f\|_{H^p} \|(T_{\Omega,\alpha})^* g\|_{\mathcal{L}_{l,n(\frac{1}{p}-1),s}} \le \sup_g \|f\|_{H^p} \|(T_{\Omega,\alpha})^* g\|_{\mathcal{L}_{l,n(\frac{1}{p}-1)}} \\ &\le C \sup_g \|f\|_{H^p} \|g\|_{L^{q'}} \le C\|f\|_{H^p},\end{aligned}$$

where the supremum is taken over all $g \in L^{q'}(\mathbb{R}^n)$ with $\|g\|_{L^{q'}} \le 1$. Thus, we finish the proof of Theorem 5.

Proof of Theorem 6. Finally, let us apply the idea above to give the proof of Theorem 6. By Theorem 3, for $1 \le l \le n/(n-\alpha)$ and the adjoint operator $(T_{\Omega,\alpha})^*$ of $T_{\Omega,\alpha}$, we get

$$\|(T_{\Omega,\alpha})^* g\|_{\mathcal{L}_{l,\alpha}} \le C\|g\|_{L^\infty}. \tag{3.7}$$

By (3.5) we know that $\mathcal{L}_{l,\alpha}(\mathbb{R}^n) \subset (H^p(\mathbb{R}^n))^*$ for $1 \le l \le \infty$ and $p = n/(n+\alpha)$. Thus, for any $f \in H^p(\mathbb{R}^n)$ $(p = n/(n+\alpha))$, if taking $1 \le l \le n/(n-\alpha)$, by (3.4) and (3.7), we get

$$\begin{aligned}\|T_{\Omega,\alpha} f\|_{L^1} &= \sup_g \left| \int_{\mathbb{R}^n} T_{\Omega,\alpha} f(x) g(x)\, dx \right| = \sup_g \left| \int_{\mathbb{R}^n} f(x) (T_{\Omega,\alpha})^* g(x)\, dx \right| \\ &\le \sup_g \|f\|_{H^p} \|(T_{\Omega,\alpha})^* g\|_{\mathcal{L}_{l,\alpha,s}} \le \sup_g \|f\|_{H^p} \|(T_{\Omega,\alpha})^* g\|_{\mathcal{L}_{l,\alpha}} \\ &\le C \sup_g \|f\|_{H^p} \|g\|_{L^\infty} \le C\|f\|_{H^p},\end{aligned}$$

where the supremum is taken over all $g \in L^\infty(\mathbb{R}^n)$ with $\|g\|_{L^\infty} \le 1$. This is the conclusion of Theorem 6.

Acknowledgements. The authors would like to express their gratitude to the referee for his very valuable comments.

References

[1] S. Chanillo, D. Watson and R. L. Wheeden, *Some integral and maximal operators related to star-like*, Studia Math., **107** (1993), 223–255.

[2] Y. Ding and S. Z. Lu, *Weighted norm inequalities for fractional integral operators with rough kernel*, Canad. J. Math., **50** (1998), 29–39.

[3] Y. Ding and S. Z. Lu, *The* $L^{p_1} \times L^{p_2} \times \cdots \times L^{p_k}$ *boundedness for some rough operators*, J. Math. Anal. Appl., **203** (1996), 166–186.

[4] Y. Ding, *Weak type bounds for a class of rough operators with power weights*, Proc. Amer. Math. Soc., **125** (1997), 2939–2942.

[5] D. Kurtz and R. L. Wheeden, *Results on weighted norm inequalities for multipliers*, Trans. Amer. Math. Soc., **255** (1979), 343–362.

[6] S. Z. Lu, *Four lectures on real* H^p *spaces*, World Scientific Publishing Co. Pte. Ltd., 1995.

[7] B. Muckenhoupt and R. L. Wheeden, *Weighted norm inequalities for singular and fractional integrals*, Trans. Amer. Math. Soc., **161** (1971), 249–258.

[8] ———, *Weighted norm inequalities for fractional integrals*, Trans. Amer. Math. Soc., **192** (1974), 261–274.

[9] J. Peeter, *On the theory of* $\mathcal{L}_{p,\lambda}$ *spaces*, J. Funct. Anal., **4** (1969), 71–87.

[10] E. M. Stein, Harmonic Analysis: Real-Variable Methods, Orthogonality, and Ocillatory Integrals, Princeton Univ. Press, Princeton, N.J., 1993.

[11] E. M. Stein and G. Weiss, *On the theory of harmonic functions of several variables I: The theory of* H^p *spaces*, Acta Math., **103** (1960), 25–62.

[12] M. H. Taibleson and G. Weiss, *The molecular characterization of certain Hardy spaces*, Astérisque, **77** (1980), 67–149.

Yong Ding
Department of Mathematics
Beijing Normal University
Beijing, 100875
P.R.of China
dingy@bnu.edu.cn

Shanzhen Lu
Department of Mathematics
Beijing Normal University
Beijing, 100875
P.R.of China.
lusz@bnu.edu.cn