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A HYBRID MEAN VALUE OF THE INVERSION OF L-FUNCTIONS AND GENERAL QUADRATIC GAUSS SUMS*

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Abstract. The main purpose of this paper is, using the estimates for character sums and the analytic method, to study the $2k$-th power mean of the inversion of Dirichlet L-functions with the weight of general quadratic Gauss sums, and give two interesting asymptotic formulas.

§1. Introduction

Let $q \geq 2$ be an integer, χ denotes a Dirichlet character modulo q. For any integer n, we define the general quadratic Gauss sums $G(n, \chi; q)$ as follows:

$$G(n, \chi; q) = \sum_{a=1}^{q} \chi(a) e\left(\frac{na^2}{q}\right), \tag{1}$$

where $e(y) = e^{2\pi i y}$. This sum is important, because it is a generalization of the classical Gauss sums and quadratic Gauss sums. But about the properties of $G(n, \chi; q)$, we know very little at present, we do not know even how large $|G(n, \chi; q)|$ is. The main purpose of this paper is, using the estimates for character sums and the analytic method, to study the asymptotic distribution of the $2k$-th power mean

$$\sum_{p \leq Q} \frac{1}{p \cdot (p-1)^2} \sum_{\chi \bmod p} \frac{|G(n, \chi; p)|^4}{|L(1, \chi)|^{2k}}, \tag{2}$$

and give a sharp asymptotic formula, where Q is any positive number with $Q \geq 2$, p denotes an odd prime, $L(s, \chi)$ denotes the Dirichlet L-function corresponding to character $\chi \bmod p$. In fact, we shall prove the following:

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THEOREM 1. *Let p denote an odd prime with $p \leq Q$. Then for any fixed positive integer k and integer n with $|n| \leq Q$, we have the asymptotic formula*

$$\sum_{p\leq Q} \frac{1}{p(p-1)^2} \sum_{\chi \bmod p} \frac{|G(n,\chi;p)|^4}{|L(1,\chi)|^{2k}} = 3 \cdot C(k) \cdot \pi(Q) + O\left(Q^{\frac{1}{2}+\epsilon}\right),$$

where $C(k) = \prod_p \left[1 + \frac{(C_k^1)^2}{p^2} + \frac{(C_k^2)^2}{p^4} + \cdots + \frac{(C_k^k)^2}{p^{2k}}\right]$, $\sum_{\chi \bmod p}$ *denotes the summation over all characters modulo p,* $\prod_p$ *denotes the product over all primes, $\pi(Q)$ denotes the number of all primes p satisfying $2 \leq p \leq Q$, ϵ denotes any fixed positive number, and* $C_k^i = \frac{k!}{(k-i)!i!}$.

THEOREM 2. *Let $Q \geq 3$ be a real number. Then for any fixed positive integer k and any integer n with $|n| \leq Q$, we have the asymptotic formula*

$$\sideset{}{'}\sum_{p\leq Q} \frac{1}{p^2(p-1)^2} \sum_{\chi \bmod p} \frac{|G(n,\chi;p)|^6}{|L(1,\chi)|^{2k}} = 10 \cdot C(k) \cdot \pi_1(Q) + O\left(Q^{\frac{1}{2}+\epsilon}\right),$$

where $\sum'$ *denotes the summation over all primes p with $p \leq Q$ and $p \equiv 3 \bmod 4$, $\pi_1(Q)$ denotes the number of all primes p satisfying $2 \leq p \leq Q$ and $p \equiv 3 \bmod 4$.*

§2. Some lemmas

In order to complete the proof of the theorems, we need following several lemmas.

LEMMA 1. *Let $y \geq 2$ be a real number, k be any positive integer. Then for*

$$r(n) = \sum_{n_1 n_2 \cdots n_k = n} \mu(n_1) \cdots \mu(n_k),$$

we have an estimate

$$\sum_{q\leq Q} \sup_{\substack{a,x \\ x\leq y}} \left| \sum_{\substack{n\leq x \\ n\equiv a(q)}} r(n) \right| \ll_{A,k} y(\ln y)^{-A} + y^{1-\frac{1}{2^k}} Q(\ln(yQ))^4,$$

where $\mu(n)$ is the Mőbius function, A is any positive number, $\ll_{A,k}$ denote the constants implied by the symbols $\ll$ depend only on parameter A and k.

Proof. (See reference [5]).

LEMMA 2. *Let p denote an odd prime with $p \le Q$, χ denote a Dirichlet character modulo p. Then for any positive integer k, we have an estimate*

$$\sum_{p\le Q}\frac{1}{p^{\frac{3}{2}}}\sum_{a=1}^{p-1}\left|\sum_{\chi \bmod p}\frac{\chi(a)}{|L(1,\chi)|^{2k}}\right| = O\left(Q^{\frac{1}{2}+\epsilon}\right),$$

where ϵ denotes any fixed positive number.

Proof. For convenience, firstly we write

$$A(\chi,y)=\sum_{\frac{p}{a}<n\le y}\chi(n)r(n), \qquad B(\chi,y)=\sum_{p<n\le y}\chi(n)r(n),$$

where a is any positive integer with $1 \le a < p$. If $\mathrm{Re}(s) > 1$ and $\chi \ne \chi_0$ (principal character mod p), then we have

$$\frac{1}{L^k(s,\chi)}=\sum_{n=1}^{\infty}\frac{\chi(n)r(n)}{n^s}. \tag{3}$$

From (3) and Abel's identity we have

$$\frac{1}{L^k(s,\chi)}=\sum_{n\le N}\frac{\chi(n)r(n)}{n^s}+s\int_N^{+\infty}\frac{A(\chi,y)}{y^{s+1}}dy. \tag{4}$$

By the property of Dirichlet L-function and $L(1,\chi)\ne 0$ we know that (4) is also correct if $s = 1$. So we have

$$\begin{aligned}\frac{1}{L^k(1,\chi)} &= \sum_{1\le n<\frac{p}{a}}\frac{\chi(n)r(n)}{n}+\int_{\frac{p}{a}}^{+\infty}\frac{A(\chi,y)}{y^2}dy \\ &= \sum_{1\le n<p}\frac{\chi(n)r(n)}{n}+\int_{p}^{+\infty}\frac{B(\chi,y)}{y^2}dy.\end{aligned} \tag{5}$$

Hence from (3), (5) and note that $\frac{1}{L(1,\chi_0)}=0$ we get

$$\sum_{\chi \bmod p}\frac{\chi(a)}{|L(1,\chi)|^{2k}}=\sideset{}{'}\sum_{\chi \bmod p}\chi(a)\left|\sum_{n=1}^{\infty}\frac{\chi(n)r(n)}{n}\right|^2$$

$$
\begin{aligned}
&= \sideset{}{'}\sum_{\chi \bmod p} \chi(a) \left(\sum_{1\le n<\frac{p}{a}} \frac{\chi(n)r(n)}{n} + \int_{\frac{p}{a}}^{+\infty} \frac{A(\chi,y)}{y^2}dy \right) \\
&\qquad \times \left(\sum_{1\le m<p} \frac{\overline{\chi}(m)r(m)}{m} + \int_{p}^{+\infty} \frac{B(\overline{\chi},y)}{y^2}dy \right) \\
&= \sideset{}{'}\sum_{\chi \bmod p} \chi(a) \left(\sum_{1\le n<\frac{p}{a}} \frac{\chi(n)r(n)}{n} \right) \left(\sum_{1\le m<p} \frac{\overline{\chi}(m)r(m)}{m} \right) \\
&\qquad + \sideset{}{'}\sum_{\chi \bmod p} \chi(a) \left(\sum_{1\le n<\frac{p}{a}} \frac{\chi(n)r(n)}{n} \right) \left(\int_{p}^{+\infty} \frac{B(\overline{\chi},y)}{y^2}dy \right) \\
&\qquad + \sideset{}{'}\sum_{\chi \bmod p} \chi(a) \left(\sum_{1\le m<p} \frac{\overline{\chi}(m)r(m)}{m} \right) \left(\int_{\frac{p}{a}}^{+\infty} \frac{A(\chi,y)}{y^2}dy \right) \\
&\qquad + \sideset{}{'}\sum_{\chi \bmod p} \chi(a) \left(\int_{\frac{p}{a}}^{+\infty} \frac{A(\chi,y)}{y^2}dy \right) \left(\int_{p}^{+\infty} \frac{B(\overline{\chi},y)}{y^2}dy \right) \\
&\equiv M_1 + M_2 + M_3 + M_4,
\end{aligned}
$$

where $\sideset{}{'}\sum_{\chi \bmod p}$ denotes the summation over all nonprincipal characters mod p. So from the above we have

$$
\begin{aligned}
(6)\qquad &\sum_{p\le Q} \frac{1}{p^{\frac{3}{2}}} \sum_{a=1}^{p-1} \left| \sum_{\chi \bmod p} \frac{\chi(a)}{|L(1,\chi)|^{2k}} \right| \\
&\qquad\qquad \le \sum_{p\le Q} \frac{1}{p^{\frac{3}{2}}} \sum_{a=1}^{p-1} \left(|M_1| + |M_2| + |M_3| + |M_4| \right).
\end{aligned}
$$

Now we shall estimate each term in expression (6). From the orthogonality relation for characters modulo p we know that for $(p, mn) = 1$,

$$
\sum_{\chi \bmod p} \chi(n)\overline{\chi}(m) = \begin{cases} p-1, & \text{if } n \equiv m \bmod p; \\ 0, & \text{otherwise.} \end{cases}
$$

From this identity we can easily get

$$
(7)\quad M_1 = \sum_{\chi \bmod p} \chi(a) \left(\sum_{1\le n<\frac{p}{a}} \frac{\chi(n)r(n)}{n} \right) \left(\sum_{1\le m<p} \frac{\overline{\chi}(m)r(m)}{m} \right) + O\left(p^{\epsilon}\right)
$$

$$= (p-1)\sum_{1\le n<\frac{p}{a}}\ \sum_{\substack{1\le m<p\\ an\equiv m(p)}}\frac{r(n)r(m)}{nm}+O\left(p^{\epsilon}\right)$$
$$= (p-1)\sum_{1\le n<\frac{p}{a}}\frac{r(n)r(an)}{an^2}+O\left(p^{\epsilon}\right)$$
$$\le \frac{pr(a)}{a}\sum_{1\le n<\frac{p}{a}}\frac{|r(n)|^2}{n^2}+O\left(p^{\epsilon}\right)$$
$$\le \frac{pr(a)}{a}\sum_{\substack{n=1\\(n,p)=1}}^{\infty}\frac{|r(n)|^2}{n^2}+O(p^{\epsilon}).$$

Note that

$$\sum_{\substack{n=1\\(n,p)=1}}^{\infty}\frac{r^2(n)}{n^2}=\prod_{\substack{p_1\\p_1\ne p}}\left[1+\frac{(C_k^1)^2}{p_1^2}+\frac{(C_k^2)^2}{p_1^4}\cdots+\frac{(C_k^k)^2}{p_1^{2k}}\right]\ll C,$$

an absolute constant, and the estimate $r(n)\le d_k(n)\ll p^{\epsilon}$, from (7) we immediately get

$$\sum_{p\le Q}\frac{1}{p^{\frac{3}{2}}}\sum_{a=1}^{p-1}|M_1|\ll\sum_{p\le Q}\frac{1}{p^{\frac{1}{2}}}\sum_{a=1}^{p-1}\frac{r(a)}{a}+\sum_{p\le Q}\frac{1}{p^{\frac{3}{2}}}\sum_{a=1}^{p-1}p^{\epsilon}\ll Q^{\frac{1}{2}+\epsilon}. \tag{8}$$

$$\sum_{p\le Q}\frac{1}{p^{\frac{3}{2}}}\sum_{a=1}^{p-1}|M_2| \tag{9}$$
$$=\sum_{p\le Q}\frac{1}{p^{\frac{3}{2}}}\sum_{a=1}^{p-1}\left|\sideset{}{'}\sum_{\chi \bmod p}\chi(a)\left(\sum_{n<\frac{p}{a}}\frac{\chi(n)r(n)}{n}\right)\left(\int_p^{+\infty}\frac{B(\overline{\chi},y)}{y^2}dy\right)\right|$$
$$\le\sum_{p\le Q}\frac{1}{p^{\frac{3}{2}}}\sum_{a=1}^{p-1}\left|\sideset{}{'}\sum_{\chi \bmod p}\chi(a)\left(\sum_{n<\frac{p}{a}}\frac{\chi(n)r(n)}{n}\right)\left(\int_p^{Q^{2^k}}\frac{B(\overline{\chi},y)}{y^2}dy\right)\right|$$
$$+\sum_{p\le Q}\frac{1}{p^{\frac{3}{2}}}\sum_{a=1}^{p-1}\left|\sideset{}{'}\sum_{\chi \bmod p}\chi(a)\left(\sum_{n<\frac{p}{a}}\frac{\chi(n)r(n)}{n}\right)\left(\int_{Q^{2^k}}^{+\infty}\frac{B(\overline{\chi},y)}{y^2}dy\right)\right|$$
$$\le\sum_{p\le Q}\frac{1}{p^{\frac{3}{2}}}\sum_{a=1}^{p-1}\left|\sum_{\chi \bmod p}\chi(a)\left(\sum_{n<\frac{p}{a}}\frac{\chi(n)r(n)}{n}\right)\left(\int_p^{Q^{2^k}}\frac{\sum_{p<m\le y}\overline{\chi}(m)r(m)}{y^2}dy\right)\right|$$

$$+\sum_{p\le Q}\frac{1}{p^{\frac12}}\sum_{a=1}^{p-1}\sum_{n<\frac{p}{a}}\frac{|r(n)|}{n}\left|\int_{Q^{2^k}}^{+\infty}\frac{1}{y^2}\left(\sum_{\substack{p<m\le y\\ m\equiv na(p)}}r(m)\right)dy\right|+O\left(Q^{\frac12+\epsilon}\right)$$

$$\le\sum_{p\le Q}\frac{1}{p^{\frac32}}\sum_{a=1}^{p-1}\left|\sum_{\chi \bmod p}\chi(a)\left(\sum_{n<\frac{p}{a}}\frac{\chi(n)r(n)}{n}\right)\left(\int_p^{Q^{2^k}}\frac{\sum_{p<m\le y}\overline{\chi}(m)r(m)}{y^2}dy\right)\right|$$

$$+O\left(Q^{\frac12+\epsilon}\int_{Q^{2^k}}^{+\infty}\frac{1}{y^2}\sum_{p\le Q}\sup_{\substack{a,z\\ z\le y}}\left|\sum_{\substack{m\le z\\ m\equiv a(p)}}r(m)\right|dy\right)+O\left(Q^{\frac12+\epsilon}\right).$$

So from Lemma 1 (taking $A=2$) and (9) we have

$$\sum_{p\le Q}\frac{1}{p^{\frac32}}\sum_{a=1}^{p-1}|M_2| \tag{10}$$

$$\le\sum_{p\le Q}\frac{1}{p^{\frac12}}\int_p^{Q^{2^k}}\frac{1}{y^2}\left[\sum_{a=1}^{p-1}\sum_{n<\frac{p}{a}}\sum_{\substack{p<m\le y\\ an\equiv m(p)}}\frac{r(n)r(m)}{n}\right]dy+O\left(Q^{\frac12+\epsilon}\right)$$

$$+O\left(Q^{\frac12+\epsilon}\int_{Q^{2^k}}^{+\infty}\frac{1}{y^2}\left[y(\ln y)^{-A}+y^{1-\frac{1}{2^k}}Q(\ln(yQ))^4\right]dy\right)$$

$$=O\left(\sum_{p\le Q}\frac{1}{p^{\frac12}}\sum_{a=1}^{p-1}\sum_{n<\frac{p}{a}}\frac1n\int_p^{Q^{2^k}}\frac{y\cdot\frac1p\cdot p^{\epsilon}}{y^2}dy\right)$$

$$+O\left(Q^{\frac12+\epsilon}\int_{Q^{2^k}}^{+\infty}\left[y^{-1}(\ln y)^{-A}+y^{-1-\frac{1}{2^k}}Q(\ln(yQ))^4\right]dy\right)$$

$$=O\left(\sum_{p\le Q}p^{-\frac12+\epsilon}\right)+O\left(Q^{\frac12+\epsilon}\right)$$

$$=O\left(Q^{\frac12+\epsilon}\right).$$

Applying the same method of proving (10), we also have the estimate

$$\sum_{p\le Q}\frac{1}{p^{\frac32}}\sum_{a=1}^{p-1}|M_3| \tag{11}$$

$$=\sum_{p\le Q}\frac{1}{p^{\frac32}}\sum_{a=1}^{p-1}\left|{\sum_{\chi \bmod p}}'\chi(a)\left(\sum_{m\le p}\frac{\overline{\chi}(m)r(m)}{m}\right)\left(\int_{\frac{p}{a}}^{+\infty}\frac{A(\chi,y)}{y^2}dy\right)\right|$$

$$\leq \sum_{p\leq Q}\frac{1}{p^{\frac{3}{2}}}\sum_{a=1}^{p-1}\left|\sum_{\chi \bmod p}{}' \chi(a)\left(\sum_{m\leq p}\frac{\overline{\chi}(m)r(m)}{m}\right)\left(\int_{\frac{p}{a}}^{Q^{2^k}}\frac{A(\chi,y)}{y^2}dy\right)\right|$$

$$+\sum_{p\leq Q}\frac{1}{p^{\frac{3}{2}}}\sum_{a=1}^{p-1}\left|\sum_{\chi \bmod p}{}' \chi(a)\left(\sum_{m\leq p}\frac{\overline{\chi}(m)r(m)}{m}\right)\left(\int_{Q^{2^k}}^{+\infty}\frac{A(\chi,y)}{y^2}dy\right)\right|$$

$$\leq \sum_{p\leq Q}\frac{1}{p^{\frac{3}{2}}}\sum_{a=1}^{p-1}\left|\sum_{\chi \bmod p}\chi(a)\left(\sum_{m<p}\frac{\overline{\chi}(m)r(m)}{m}\right)\left(\int_{\frac{p}{a}}^{Q^{2^k}}\frac{\sum\limits_{\frac{p}{a}<n\leq y}\chi(n)r(n)}{y^2}dy\right)\right|$$

$$+\sum_{p\leq Q}\frac{1}{p^{\frac{1}{2}}}\sum_{a=1}^{p-1}\int_{Q^{2^k}}^{+\infty}\frac{1}{y^2}\left|\sum_{m<p}\frac{r(m)}{m}\sum_{\substack{\frac{p}{a}<n\leq y\\ an\equiv m(p)}}r(n)\right|dy+O\left(Q^{\frac{1}{2}+\epsilon}\right)$$

$$\leq \sum_{p\leq Q}\frac{1}{p^{\frac{1}{2}}}\sum_{a=1}^{p-1}\int_{\frac{p}{a}}^{Q^{2^k}}\frac{1}{y^2}\left|\sum_{m<p}\sum_{\substack{\frac{p}{a}<n\leq y\\ an\equiv m(p)}}\frac{r(m)r(n)}{m}\right|dy+O\left(Q^{\frac{1}{2}+\epsilon}\right)$$

$$+O\left(Q^{\frac{1}{2}+\epsilon}\int_{Q^{2^k}}^{+\infty}\frac{1}{y^2}\sum_{p\leq Q}\sup_{\substack{a,z\\ z\leq y}}\left|\sum_{\substack{n\leq z\\ n\equiv a(p)}}r(n)\right|dy\right)$$

$$=O\left(\sum_{p\leq Q}\frac{p}{p^{\frac{1}{2}}}\sum_{m<p}\frac{1}{m}\int_{\frac{p}{a}}^{Q^{2^k}}\frac{y\cdot\frac{1}{p}\cdot p^{\epsilon}}{y^2}dy\right)+O\left(Q^{\frac{1}{2}+\epsilon}\right)$$

$$=O\left(Q^{\frac{1}{2}+\epsilon}\right).$$

where we have used the estimate $r(n)\ll n^{\epsilon}$ and for any fixed positive integers h and m, the number of the solutions of equation $an=hp+m$ (for a and n) is $\ll d(hp+m)\ll p^{\epsilon}$. Similarly, we can also get the estimate

$$\sum_{p\leq Q}\frac{1}{p^{\frac{3}{2}}}\sum_{a=1}^{p-1}\left|\sum_{\chi \bmod p}{}' \chi(a)\int_{\frac{p}{a}}^{+\infty}\frac{A(\chi,y)}{y^2}dy\int_{p}^{+\infty}\frac{B(\overline{\chi},z)}{z^2}dz\right|=O\left(Q^{\frac{1}{2}+\epsilon}\right). \tag{12}$$

Combining (6), (8), (10), (11) and (12) we obtain the estimate

$$\sum_{p\leq Q}\frac{1}{p^{\frac{3}{2}}}\sum_{a=1}^{p-1}\left|\sum_{\chi \bmod p}\frac{\chi(a)}{|L(1,\chi)|^{2k}}\right|=O\left(Q^{\frac{1}{2}+\epsilon}\right).$$

This proves Lemma 2.

LEMMA 3. *Let $Q \geq 2$, then for any fixed integer k, we have the asymptotic formula*

$$\sum_{p \leq Q} \frac{1}{\phi(p)} \sum_{\substack{\chi \neq \chi_0 \\ \chi(-1)=1}} \frac{1}{|L(1,\chi)|^{2k}} = \frac{1}{2} \cdot C(k) \cdot \pi(Q) + O\left(Q^{\frac{1}{2}+\epsilon}\right).$$

Proof. For prime power p^α, note that $\mu(1) = 1$, $\mu(p) = -1$ and $\mu(p^i) = 0$ if $i > 1$. So that we have the identity

$$r(p^\alpha) = \sum_{n_1 n_2 \cdots n_k = p^\alpha} \mu(n_1)\mu(n_2)\cdots\mu(n_k) = \begin{cases} (-1)^\alpha C_k^\alpha, & \text{if } \alpha \leq k; \\ 0, & \text{otherwise.} \end{cases}$$

and the asymptotic formula

$$\begin{aligned} \sum_{\substack{n=1 \\ (n,p)=1}}^{\infty} \frac{r^2(n)}{n^2} &= \prod_{\substack{p_1 \\ p_1 \neq p}} \left[1 + \frac{(C_k^1)^2}{p_1^2} + \frac{(C_k^2)^2}{p_1^4} + \cdots + \frac{(C_k^k)^2}{p_1^{2k}}\right] \\ &= C(k) + O\left(\frac{1}{p^2}\right), \end{aligned}$$

from the orthogonality relation for characters and the method of proving Lemma 2 we can easily get

$$\begin{aligned} &\sum_{\substack{\chi \neq \chi_0 \\ \chi(-1)=1}} \left| \sum_{n \leq Q^{2k}} \frac{r(n)\chi(n)}{n} \right|^2 \\ &= \frac{\phi(p)}{2} \sum_{\substack{n \leq Q^{2k} \\ (n,p)=1}} \frac{r^2(n)}{n^2} + \frac{\phi(p)}{2} \sum_{n \leq Q^{2k}} \sum_{\substack{m \leq Q^{2k} \\ m \equiv \pm n(p) \\ m \neq n}} \frac{r(n)r(m)}{nm} + O\left(Q^\epsilon\right) \\ &= \frac{\phi(p)}{2} \cdot C(k) + O\left(Q^\epsilon\right), \end{aligned}$$

$$\sum_{p \leq Q} \frac{1}{\phi(p)} \sum_{\substack{\chi \neq \chi_0 \\ \chi(-1)=1}} \left(\sum_{n \leq Q^{2k}} \frac{r(n)\overline{\chi}(n)}{n} \right) \int_{Q^{2k}}^{\infty} \frac{A(y,\chi)}{y^2} dy \ll Q^{\frac{1}{2}+\epsilon}$$

and

$$\sum_{p \leq Q} \frac{1}{\phi(p)} \sum_{\substack{\chi \neq \chi_0 \\ \chi(-1)=1}} \left| \int_{Q^{2k}}^{\infty} \frac{A(y,\chi)}{y^2} dy \right|^2 \ll Q^{\frac{1}{2}+\epsilon},$$

where $A(y,\chi)=\sum\limits_{Q^{2k}<n\le y} r(n)\chi(n)$. Note that the identity

$$\frac{1}{L^k(1,\chi)}=\sum_{n=1}^{+\infty}\frac{r(n)\chi(n)}{n}=\sum_{n\le Q^{2k}}\frac{r(n)\chi(n)}{n}+\int_{Q^{2k}}^{\infty}\frac{A(y,\chi)}{y^2}dy.$$

Combining the above asymptotic formula and the estimates we have

$$\begin{aligned}&\sum_{p\le Q}\frac{1}{\phi(p)}\sum_{\substack{\chi\ne\chi_0\\ \chi(-1)=1}}\frac{1}{|L(1,\chi)|^{2k}}\\ &=\sum_{p\le Q}\frac{1}{\phi(p)}\sum_{\substack{\chi\ne\chi_0\\ \chi(-1)=1}}\left|\sum_{n\le Q^{2k}}\frac{r(n)\chi(n)}{n}\right|^2\\ &\quad+\sum_{p\le Q}\frac{1}{\phi(p)}\sum_{\substack{\chi\ne\chi_0\\ \chi(-1)=1}}\left(\sum_{n\le Q^{2k}}\frac{r(n)\chi(n)}{n}\right)\int_{Q^{2k}}^{\infty}\frac{A(y,\overline{\chi})}{y^2}dy\\ &\quad+\sum_{p\le Q}\frac{1}{\phi(p)}\sum_{\substack{\chi\ne\chi_0\\ \chi(-1)=1}}\left(\sum_{n\le Q^{2k}}\frac{r(n)\overline{\chi}(n)}{n}\right)\int_{Q^{2k}}^{\infty}\frac{A(y,\chi)}{y^2}dy\\ &\quad+\sum_{p\le Q}\frac{1}{\phi(p)}\sum_{\substack{\chi\ne\chi_0\\ \chi(-1)=1}}\left|\int_{Q^{2k}}^{\infty}\frac{A(y,\chi)}{y^2}dy\right|^2\\ &=\frac{1}{2}\cdot C(k)\cdot\pi(Q)+O\left(Q^{\frac{1}{2}+\epsilon}\right).\end{aligned}$$

This proves Lemma 3.

LEMMA 4. *For any integer $q\ge 1$, we have the calculation formula*

$$G(1;q)=\frac{1}{2}\sqrt{q}(1+i)\left(1+e^{\frac{-\pi iq}{2}}\right)=\begin{cases}\sqrt{q} & \textit{if}\quad q\equiv 1 \pmod 4;\\ 0 & \textit{if}\quad q\equiv 2 \pmod 4;\\ i\sqrt{q} & \textit{if}\quad q\equiv 3 \pmod 4;\\ (1+i)\sqrt{q} & \textit{if}\quad q\equiv 0 \pmod 4,\end{cases}$$

where $G(1;q)=\sum\limits_{a=1}^{q} e\left(\dfrac{a^2}{q}\right)$.

Proof. This is a remarkable formula of Gauss. See the formula (30) of [1].

LEMMA 5. *Let p be an odd prime, χ be any nonprincipal even character (i.e. $\chi(-1)=1$ and $\chi\neq\chi_0$) mod p. Then for any integer n with $(n,p)=1$, we have the identity*

$$|G(n,\chi;p)|^2 = 2p + \left(\frac{n}{p}\right) G(1;p) \sum_{a=1}^{p-1} \chi(a) \left(\frac{a^2-1}{p}\right),$$

where $\left(\frac{n}{p}\right)$ is the Legendre symbol.

Proof. First we let $G(n;p) = \sum_{a=1}^{p} e\left(\frac{na^2}{p}\right)$. Then for $p \nmid n$, from the formula (29) of [1] we know that

$$G(n;p) = \left(\frac{n}{p}\right) G(1;p). \tag{13}$$

Applying (13) we can get that if χ is a nonprincipal even character mod p, then

$$\begin{aligned}
|G(n,\chi;p)|^2 &= \sum_{a=1}^{p-1}\sum_{b=1}^{p-1} \chi(a)\overline{\chi}(b) e\left(\frac{na^2-nb^2}{p}\right) \\
&= \sum_{a=1}^{p-1}\sum_{b=1}^{p-1} \chi(a\overline{b}) e\left(\frac{n(a^2-b^2)}{p}\right) \\
&= \sum_{a=1}^{p-1}\sum_{b=1}^{p-1} \chi(a) e\left(\frac{nb^2(a^2-1)}{p}\right) \\
&= \sum_{a=1}^{p-1} \chi(a) \left[\sum_{b=1}^{p} e\left(\frac{nb^2(a^2-1)}{p}\right) - 1\right] \\
&= \sum_{a=1}^{p-1} \chi(a) G(n(a^2-1);p) - \sum_{a=1}^{p-1}\chi(a) \\
&= 2G(0;p) + \sum_{a=2}^{p-2} \chi(a) G(n(a^2-1);p) \\
&= 2p + \left(\frac{n}{p}\right) G(1;p) \sum_{a=1}^{p-1} \chi(a) \left(\frac{a^2-1}{p}\right).
\end{aligned}$$

This proves Lemma 5.

§3. Proof of the theorems

In this section, we shall complete the proof of the theorems. First note that if χ is an odd character modulo p (or $\chi \neq \chi_0$ and $p|n$), then

$$G(n,\chi;p) = \sum_{a=1}^{p} \chi(a) e\left(\frac{a^2 n}{p}\right) = 0.$$

So for any integer n, from Lemma 5 we have

$$\begin{aligned}
(14)\quad & \sum_{p\le Q} \frac{1}{p\phi^2(p)} \sum_{\chi \bmod p} \frac{|G(n,\chi;p)|^4}{|L(1,\chi)|^{2k}} \\
&= \sum_{\substack{p\le Q\\(p,n)=1}} \frac{1}{p\phi^2(p)} \sum_{\substack{\chi\neq\chi_0\\ \chi(-1)=1}} \frac{|G(n,\chi;p)|^4}{|L(1,\chi)|^{2k}} \\
&= \sum_{\substack{p\le Q\\(p,n)=1}} \frac{1}{p\phi^2(p)} \sum_{\substack{\chi\neq\chi_0\\ \chi(-1)=1}} \left[2p + \left(\frac{n}{p}\right) G(1;p) \sum_{a=1}^{p-1} \chi(a) \left(\frac{a^2-1}{p}\right)\right]^2 \\
&\qquad \cdot \frac{1}{|L(1,\chi)|^{2k}} \\
&= \sum_{\substack{p\le Q\\(p,n)=1}} \frac{1}{p\phi^2(p)} \sum_{\substack{\chi\neq\chi_0\\ \chi(-1)=1}} \left[4p^2 + 4p \left(\frac{n}{p}\right) G(1;p) \sum_{a=1}^{p-1} \chi(a) \left(\frac{a^2-1}{p}\right)\right] \\
&\qquad \cdot \frac{1}{|L(1,\chi)|^{2k}} \\
&\quad + \sum_{\substack{p\le Q\\(p,n)=1}} \frac{G^2(1;p)}{p\phi^2(p)} \sum_{\substack{\chi\neq\chi_0\\ \chi(-1)=1}} \sum_{a=1}^{p-1}\sum_{b=1}^{p-1} \chi(ab) \left(\frac{a^2-1}{p}\right)\left(\frac{b^2-1}{p}\right) \\
&\qquad \cdot \frac{1}{|L(1,\chi)|^{2k}}.
\end{aligned}$$

On the other hand, we have the identities

$$\begin{aligned}
(15)\quad & \sum_{a=1}^{p-1}\sum_{b=1}^{p-1} \chi(ab) \left(\frac{a^2-1}{p}\right)\left(\frac{b^2-1}{p}\right) \\
&= \sum_{a=1}^{p-1}\sum_{b=1}^{p-1} \chi(a) \left(\frac{a^2\overline{b}^2-1}{p}\right)\left(\frac{b^2-1}{p}\right)
\end{aligned}$$

$$= \sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi(a)\left(\frac{a^2-b^2}{p}\right)\left(\frac{b^2-1}{p}\right)$$
$$= 2\left(\frac{-1}{p}\right)(p-3) + \sum_{a=2}^{p-2}\sum_{b=1}^{p-1}\chi(a)\left(\frac{a^2-b^2}{p}\right)\left(\frac{b^2-1}{p}\right),$$

$$\sum_{a=2}^{p-2}\left(\frac{a^2-b^2}{p}\right)\sum_{\chi(-1)=-1}\frac{\chi(a)}{|L(1,\chi)|^{2k}}$$
$$= \sum_{a=1}^{p-1}\left(\frac{a^2-1}{p}\right)\sum_{\chi(-1)=-1}\frac{\chi(a)}{|L(1,\chi)|^{2k}} = 0.$$

From the Lemma 1 of [8] we can also deduce the estimate

$$\sum_{b=1}^{p-1}\left(\frac{b^2-a^2}{p}\right)\left(\frac{b^2-1}{p}\right) \ll \sqrt{p}, \quad a^2 \not\equiv 1 \bmod p. \tag{16}$$

Now combining (14), (15), (16), Lemma 2, Lemma 3 and Lemma 4 we have

$$\sum_{p\le Q}\frac{1}{p\phi^2(p)}\sum_{\chi \bmod p}\frac{|G(n,\chi;p)|^4}{|L(1,\chi)|^{2k}}$$
$$= \sum_{\substack{p\le Q\\(p,n)=1}}\frac{1}{p\phi^2(p)}\left(4p^2 + G^2(1;p)\cdot 2\left(\frac{-1}{p}\right)(p-3)\right)\sum_{\substack{\chi\ne\chi_0\\ \chi(-1)=1}}\frac{1}{|L(1,\chi)|^{2k}}$$
$$+ \sum_{\substack{p\le Q\\(p,n)=1}}\frac{1}{p\phi^2(p)}4p\left(\frac{n}{p}\right)G(1;p)\sum_{a=1}^{p-1}\left(\frac{a^2-1}{p}\right)\sum_{\substack{\chi\ne\chi_0\\ \chi(-1)=1}}\frac{\chi(a)}{|L(1,\chi)|^{2k}}$$
$$+ \sum_{\substack{p\le Q\\(p,n)=1}}\frac{G^2(1;p)}{p\phi^2(p)}\sum_{a=2}^{p-2}\sum_{b=1}^{p-1}\left(\frac{a^2-b^2}{p}\right)\left(\frac{b^2-1}{p}\right)\sum_{\substack{\chi\ne\chi_0\\ \chi(-1)=1}}\frac{\chi(a)}{|L(1,\chi)|^{2k}}$$
$$= \sum_{p\le Q}\frac{6p^2-6p}{p(p-1)^2}\sum_{\substack{\chi\ne\chi_0\\ \chi(-1)=1}}\frac{1}{|L(1,\chi)|^{2k}} + O\left(Q^{\epsilon}\right)$$
$$+O\left(\sum_{p\le Q}\frac{1}{p^{\frac{3}{2}}}\sum_{a=2}^{p-2}\left|\sum_{\chi\ne\chi_0}\frac{\chi(a)}{|L(1,\chi)|^{2k}}\right|\right)$$
$$= 3\cdot C(k)\cdot \pi(Q) + O(Q^{\frac{1}{2}+\epsilon}).$$

This proves Theorem 1.

Now we prove Theorem 2. From Lemma 5 we have

$$\sum_{p\le Q}{}' \frac{1}{p^2\phi^2(p)} \sum_{\chi \bmod p} \frac{|G(n,\chi;p)|^6}{|L(1,\chi)|^{2k}} \tag{17}$$

$$= \sum_{\substack{p\le Q\\(p,n)=1}}{}' \frac{1}{p^2\phi^2(p)} \sum_{\substack{\chi\neq\chi_0\\ \chi(-1)=1}} \frac{|G(n,\chi;p)|^6}{|L(1,\chi)|^{2k}}$$

$$= \sum_{\substack{p\le Q\\(p,n)=1}}{}' \sum_{\substack{\chi\neq\chi_0\\ \chi(-1)=1}} \left[2p + \left(\frac{n}{p}\right) G(1;p) \sum_{a=1}^{p-1} \chi(a) \left(\frac{a^2-1}{p}\right)\right]^3 \frac{1}{|L(1,\chi)|^{2k}}$$

For $(p,n)=1$ and $p \equiv 3 \bmod 4$,

$$\sum_{\chi\neq\chi_0} \frac{|G(n,\chi;p)|^6}{|L(1,\chi)|^{2k}} = \sum_{\chi\neq\chi_0} \frac{|\overline{G(n,\chi;p)}|^6}{|\overline{L(1,\chi)}|^{2k}} \tag{18}$$

$$= \sum_{\chi\neq\chi_0} \left|\sum_{a=1}^{p-1} \overline{\chi}(a) e\left(\frac{-na^2}{p}\right)\right|^6 \cdot \frac{1}{|L(1,\overline{\chi})|^{2k}}$$

$$= \sum_{\chi\neq\chi_0} \left|\sum_{a=1}^{p-1} \chi(a) e\left(\frac{-na^2}{p}\right)\right|^6 \cdot \frac{1}{|L(1,\chi)|^{2k}}$$

$$= \sum_{\substack{\chi\neq\chi_0\\ \chi(-1)=1}} \left[2p + \left(\frac{-n}{p}\right) G(1;p) \sum_{a=1}^{p-1} \chi(a) \left(\frac{a^2-1}{p}\right)\right]^3 \cdot \frac{1}{|L(1,\chi)|^{2k}}.$$

Note that $\left(\frac{-1}{p}\right) = -1$, from (15), (16), (17), (18), Lemma 2, Lemma 3 and Lemma 4 we obtain

$$\sum_{p\le Q}{}' \frac{1}{p^2\phi^2(p)} \sum_{\chi \bmod p} \frac{|G(n,\chi;p)|^6}{|L(1,\chi)|^{2k}}$$

$$= \frac{1}{2}\left[\sum_{p\le Q}{}' \frac{1}{p^2\phi^2(p)} \sum_{\chi\neq\chi_0} \frac{|G(n,\chi;p)|^6}{|L(1,\chi)|^{2k}} + \sum_{p\le Q}{}' \frac{1}{p^2\phi^2(p)} \sum_{\chi\neq\chi_0} \frac{|\overline{G(n,\chi;p)}|^6}{|\overline{L(1,\chi)}|^{2k}}\right]$$

$$= \sum_{\substack{p\le Q\\(p,n)=1}}{}' \frac{1}{p^2\phi^2(p)} \sum_{\substack{\chi\ne\chi_0\\ \chi(-1)=1}} \left[(2p)^3 + 6pG^2(1;p)\left(\sum_{a=1}^{p-1}\chi(a)\left(\frac{a^2-1}{p}\right)\right)^2\right] \cdot \frac{1}{|L(1,\chi)|^{2k}}$$

$$= 10\cdot C(k)\cdot \pi_1(Q) + O\left(Q^{\frac{1}{2}+\epsilon}\right).$$

This completes the proof of Theorem 2.

Note. For general integer $m \ge 3$, whether there exists an asymptotic formula for

$$\sum_{p\le Q}\frac{1}{p^{m-1}(p-1)^2}\sum_{\chi \bmod p}\frac{|G(n,\chi;p)|^{2m}}{|L(1,\chi)|^{2k}}$$

is an unsolved problem.

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