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ON l-ADIC ITERATED INTEGRALS, II

FUNCTIONAL EQUATIONS AND l-ADIC POLYLOGARITHMS

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Abstract. We continue to study l-adic iterated integrals introduced in the first part. We shall show that the l-adic iterated integrals satisfy essentially the same functional equations as the classical complex iterated integrals. Next we are studying l-adic analogs of classical polylogarithms.

§9. Introduction to Part II

9.1. The classical complex iterated integrals satisfy functional equations (see [W1]). We shall show that l-adic iterated integrals satisfy the same functional equations as the classical complex iterated integrals.

First we introduce the following notation which we shall use in this paper. Let π (resp. L) be a group (resp. a Lie algebra). We denote by $\{\Gamma^k\pi\}_{k\geq 1}$ (resp. $\{\Gamma^k\mathrm{L}\}_{k\geq 1}$) the lower central series of the group π (resp. the Lie algebra L).

We set

$$gr^k_\Gamma\,\pi := \Gamma^k\pi/\Gamma^{k+1}\pi \quad (\text{resp. } \ gr^k_\Gamma\,\mathrm{L} := \Gamma^k\mathrm{L}/\Gamma^{k+1}\mathrm{L}).$$

Before we formulate our main result we make a following remark. Let $Y = \mathbf{P}^1_K \setminus \{b_1, \dots, b_{m+1}\}$. Then

$$\bigoplus_{k=1}^{\infty} gr^k_\Gamma\,\pi_1(Y(\mathbf{C}); x) \otimes \mathbf{Q}$$

is canonically isomorphic to a free Lie algebra over $\mathbf{Q}$ on m generators $Y_1, \dots, Y_m$, which we denote by $\mathrm{Lie}(Y_1, \dots, Y_m)$. Hence any linear form φ on $gr^k_\Gamma\,\pi_1(Y(\mathbf{C}); x) \otimes \mathbf{Q}$ corresponds to a linear form φ on $\mathrm{Lie}(Y_1, \dots, Y_m)$. Now we formulate our main result concerning functional equations.

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THEOREM D. *Let* $X = \mathbf{P}^1_K \setminus \{a_1, \dots, a_{n+1}\}$ *and let* $Y = \mathbf{P}^1_K \setminus \{b_1, \dots, b_{m+1}\}$. *Let* $z, v \in \hat{X}(K)$. *Let* $f_i : X \to Y$ *be a smooth morphism and let* $\varphi_i \in \mathrm{Lie}(Y_1, \dots, Y_m)^\diamond$ *be a linear form of degree* q *defined over* $\mathbf{Q}$ *for* $i = 1, \dots, N$. *Let* $n_1, \dots, n_N$ *be rational numbers. If*

$$\sum_{i=1}^{N} n_i \varphi_i \circ (f_i)_* = 0$$

in $\mathrm{Hom}\big(gr^q_\Gamma \pi_1(X(\mathbf{C}); v); \mathbf{Q}\big)$, *where*

$$(f_i)_* : gr^q_\Gamma \pi_1(X(\mathbf{C}); v) \longrightarrow gr^q_\Gamma \pi_1(Y(\mathbf{C}); f_i(v))$$

is the map induced by f_i *on fundamental groups for* $i = 1, \dots, N$, *then we have a functional equation*

$$\sum_{i=1}^{N} n_i \mathcal{L}^{\varphi_i}(f_i(z), f_i(v)) = 0.$$

Next we generalize well known formulas

$$\int_a^b \omega + \int_b^a \omega = 0 \quad \text{and} \quad \int_a^c \omega = \int_a^b \omega + \int_b^c \omega$$

from the elementary calculus (ω is a one-form). We show the following result.

THEOREM E. *Let* $z, y, v \in \hat{X}(K)$ *and let* $\varphi \in \mathrm{Lie}(X_1, \dots, X_n)^\diamond$. *Then we have*

$$\mathcal{L}^\varphi(z, v) + \mathcal{L}^\varphi(v, z) = 0$$

and

$$\mathcal{L}^\varphi(z, v) = \mathcal{L}^\varphi(z, y) + \mathcal{L}^\varphi(y, v).$$

Let ω_1, ω_2 be one-forms. The classical complex iterated integrals satisfy the following relations written here for two one-forms (see [Ch]).

i) $\int_\gamma \omega_1, \omega_2 + \int_\gamma \omega_2, \omega_1 = \int_\gamma \omega_1 \cdot \int_\gamma \omega_2$,

ii) $\int_{\alpha\beta} \omega_1, \omega_2 = \int_\alpha \omega_1, \omega_2 + \int_\alpha \omega_1 \cdot \int_\beta \omega_2 + \int_\beta \omega_1, \omega_2$,

iii) $\int_\gamma \omega_1, \omega_2 = (-1)^2 \int_{\gamma^{-1}} \omega_2, \omega_1$.

The analog of the formula i) is satisfied by "l-adic iterated integrals" (coefficients of the power series $\Lambda_p(\sigma)$) by the very definition because the image of the inclusion map of the fundamental group into the algebra of formal non-commutative power series is of the form $\exp L(\mathbf{X})$, where $L(\mathbf{X})$ is the set of Lie elements in the algebra of formal non-commutative power series.

The formula

$$\mathfrak{f}_{pq}(\sigma) = q^{-1} \cdot \mathfrak{f}_p(\sigma) \cdot q \cdot \mathfrak{f}_q(\sigma)$$

(see Part I Lemma 1.0.6), which after using suitable embeddings implies

$$\Lambda_{pq}(\sigma) = \Lambda_p(\sigma) \cdot \Lambda_q(\sigma)$$

is the analog of the formula ii).

We do not know how to show an analog of the formula iii) for "l-adic iterated integrals" (coefficients of the power series $\Lambda_p(\sigma)$). To complete the picture we are still missing several l-adic analogs in the following table.

classical iterated integrals	**l-adic iterated integrals**
values of Riemann zeta function at positive integers	Soulé classes for $\mathbf{Q}$
multivalue zeta numbers	values of l-adic iterated integrals at 1 and at roots of 1
multivalue zeta functions	?
shuffle relations for multivalue zeta numbers and multivalue zeta functions	?

The classical polylogarithms are the most important examples of iterated integrals. In Section 11 we introduce l-adic polylogarithms and we study their properties. We prove a theorem saying when a linear combination of l-adic polylogarithms is a cocycle. The reader can compare our result with Proposition in Section 4.6 of [BD]. In Section 11 we study functional equations of l-adic polylogarithms. We show that the l-adic dilogarithm satisfies the distribution relation

$$m\left(\sum_{i=0}^{m-1} l_2(\xi_m^i z)\right) = l_2(z^m)$$

on the Galois group $G_{\mathbf{Q}(\mu_m)}$ and the Abel five term functional equation on $G_{\mathbf{Q}(\mu_{l^\infty})}$.

These results are stronger than those in Theorem D in the sense that we get functional equations on the Galois groups $G_{\mathbf{Q}(\mu_n)}$ and $G_{\mathbf{Q}(\mu_{l^\infty})}$, while the functional equations from Theorem D hold on the subgroup $\bigcap_{i=1}^N H_q(Y; f_i(z), f_i(v))$ of G_K.

§10. Functional equations

10.0. Let $X = \mathbf{P}^1_K \setminus \{a_1, \dots, a_{n+1}\}$ and let $Y = \mathbf{P}^1_K \setminus \{b_1, \dots, b_{m+1}\}$. Let $f : X \to Y$ be a smooth morphism. Let $z, v \in \hat{X}(K)$. The morphism f induces

$$f_* : \pi_1(X_{\bar{K}}; v) \longrightarrow \pi_1(Y_{\bar{K}}; f(v))$$

and

$$f_* : \pi(X_{\bar{K}}; z, v) \longrightarrow \pi(Y_{\bar{K}}; f(z), f(v)).$$

Let us fix a path p from v to z. We recall that for $\sigma \in G_K$ we have defined

$$\mathfrak{f}_p(\sigma) := p^{-1} \cdot \sigma(p).$$

Then we have

$$f_*(\mathfrak{f}_p(\sigma)) = \mathfrak{f}_{f(p)}(\sigma). \tag{10.0.1}$$

Let $x = (x_1, \dots, x_{n+1})$ (resp. $y = (y_1, \dots, y_{m+1})$) be a sequence of geometric generators of $\pi_1(X(\mathbf{C}); v)$ (resp. $\pi_1(Y(\mathbf{C}); f(v))$. We set $\mathbf{X} := \{X_1, \dots, X_n\}$ and $\mathbf{Y} := \{Y_1, \dots, Y_n\}$. We recall that we have embeddings $k_x : \pi_1(X(\mathbf{C}); v) \to \mathbf{Q}_l\{\{\mathbf{X}\}\}$ and $k_y : \pi_1(Y(\mathbf{C}); f(v)) \to \mathbf{Q}_l\{\{\mathbf{Y}\}\}$ associated with a choice of sequences of geometric generators x of $\pi_1(X(\mathbf{C}); v)$ and y of $\pi_1(Y(\mathbf{C}); f(v))$. There is a homomorphism of $\mathbf{Q}_l$-algebras

$$f_\diamond : \mathbf{Q}_l\{\{\mathbf{X}\}\} \longrightarrow \mathbf{Q}_l\{\{\mathbf{Y}\}\}$$

such that

$$f_\diamond \circ k_x = k_y \circ f_* \quad \text{and} \quad f_\diamond \circ k_{x,p} = k_{y,f(p)} \circ f_*. \tag{10.0.2}$$

Let $\sigma \in G_{K(\mu_{l^\infty})}$. The equations (10.0.1) and (10.0.2) imply that

$$f_\diamond \circ \sigma_{x,p} = \sigma_{y,f(p)} \circ f_\diamond.$$

Hence we have

$$f_\diamond \circ \log \sigma_{x,p} = \log \sigma_{y,f(p)} \circ f_\diamond$$

and

$$(10.0.3) \qquad f_\diamond((\log \sigma_{x,p})(1)) = (\log \sigma_{y,f(p)})(1).$$

The map $f_\diamond$ induces a homomorphism of Lie algebras

$$f_\diamond : L(\mathbf{X}) \longrightarrow L(\mathbf{Y}).$$

Let

$$f_\bullet : \bigoplus_{i=1}^{\infty} gr^i_\Gamma\, L(\mathbf{X}) \longrightarrow \bigoplus_{i=1}^{\infty} gr^i_\Gamma\, L(\mathbf{Y})$$

be the map induced by $f_\diamond$ on associated graded Lie algebras. The associated graded Lie algebras are canonically isomorphic to $\mathrm{Lie}(\mathbf{X})$ and $\mathrm{Lie}(\mathbf{Y})$. Hence the map $f_\diamond$ induces

$$f_\bullet : \mathrm{Lie}(\mathbf{X}) \longrightarrow \mathrm{Lie}(\mathbf{Y}).$$

Let $\varphi \in \mathrm{Lie}(\mathbf{Y})^\diamond$ be a linear form of degree q. Let us set

$$a^{\varphi\circ f_\diamond}_{x,p} := \varphi(f_\diamond((\log \sigma_{x,p})(1))).$$

(In Part I we defined coefficients $a^{\varphi}_{x,p}$ only for homogenous forms, hence we introduce this new definition.) It follows from (10.0.3) that

$$(10.0.4) \qquad a^{\varphi\circ f_\diamond}_{x,p} = a^{\varphi}_{y,f(p)}.$$

The map $f_\diamond$ is not homogenous. Therefore we have

$$(10.0.5) \qquad a^{\varphi\circ f_\diamond}_{x,p} = a^{\varphi\circ f_\bullet}_{x,p} + \sum_{\deg \chi < q} a^{\chi}_{x,p}.$$

It follows from (10.0.4) and (10.0.5) that

$$(10.0.6) \qquad \mathcal{L}^{\varphi\circ f_\bullet}(z,v) = \mathcal{L}^{\varphi}(f(z), f(v))$$

on the subgroup $H_q(X; z, v)$ of G_K.

Below we shall use fundamental groups of X or Y with various base points. Sequences of geometric generators and embeddings into algebras of non-commutative formal power series will be chosen as above.

Let v and v' belong to $\hat{X}(K)$. If $x = (x_1, \ldots, x_{n+1})$ is a sequence of geometric generators of $\pi_1(X(\mathbf{C}); v)$ and q is a path from v' to v then

$q^{-1}\cdot x\cdot q := (q^{-1}\cdot x_1\cdot q, \dots, q^{-1}\cdot x_{n+1}\cdot q)$ is a sequence of geometric generators of $\pi_1(X(\mathbf{C}); v')$. Then we have embeddings

$$k_x : \pi_1(X(\mathbf{C}); v) \longrightarrow \mathbf{Q}_l\{\{\mathbf{X}\}\}$$

given by $k_x(x_i) = e^{X_i}$ for $i = 1, \dots, n$ and

$$k_{q^{-1}\cdot x\cdot q} : \pi_1(X(\mathbf{C}); v') \longrightarrow \mathbf{Q}_l\{\{\mathbf{X}\}\}$$

given by $k_{q^{-1}\cdot x\cdot q}(q^{-1}\cdot x_i\cdot q) = e^{X_i}$ for $i = 1, \dots, n$.

THEOREM 10.0.7. *Let $f_i : X \to Y$ be a smooth morphism and let $\varphi_i \in L(Y_1, \dots, Y_m)^\diamond$ be a linear form of degree q defined over $\mathbf{Q}$ for $i = 1, \dots, N$. Let $z, v \in \hat{X}(K)$. Let $n_1, \dots, n_N$ be rational numbers. If*

$$\sum_{i=1}^{N} n_i\varphi_i \circ (f_i)_* = 0$$

in $\mathrm{Hom}\big(gr^q_\Gamma\,\pi_1(X(\mathbf{C}); v); \mathbf{Q}\big)$, *where*

$$(f_i)_* : gr^q_\Gamma\,\pi_1(X(\mathbf{C}); v) \longrightarrow gr^q_\Gamma\,\pi_1(Y(\mathbf{C}); f_i(v))$$

is the map induced by f_i for $i = 1, \dots, N$, then we have functional equations

$$\sum_{i=1}^{N} n_i\mathcal{L}^{\varphi_i}(f_i(z); f_i(v)) = 0$$

on the subgroup $H_q(X; z, v)$ of G_K and

$$\sum_{i=1}^{N} n_i a^{\varphi_i}_{y_i, f_i(p)} = \text{lower degree terms}$$

on G_K, where "lower degree terms" means a linear combination of $a^\chi_{x,p}$ with degree of χ strictly smaller than q and y_i is a sequence of geometric generators of $\pi_1(Y(\mathbf{C}); f_i(v))$ for $i = 1, \dots, N$.

Proof. It follows from (10.0.6) that

$$\sum_{i=1}^{N} n_i\mathcal{L}^{\varphi_i}(f_i(z); f_i(v)) = \sum_{i=1}^{N} n_i\mathcal{L}^{\varphi_i\circ(f_i)_\bullet}(z, v) = \mathcal{L}^{\sum_{i=1}^{N} n_i\varphi_i\circ(f_i)_\bullet}(z, v) = 0.$$

It follows from (10.0.4) and (10.0.5) that

$$\sum_{i=1}^{N} n_i a^{\varphi_i}_{y_i,f_i(p)} = \sum_{i=1}^{N} n_i a^{\varphi_i\circ(f_i)_\diamond}_{x,p} = \sum_{i=1}^{N} n_i a^{\varphi_i\circ(f_i)_\bullet}_{x,p} + \text{lower degree terms}$$
$$= a^{\sum_{i=1}^{N} n_i\varphi_i\circ(f_i)_\bullet}_{x,p} + \text{lower degree terms} = \text{lower degree terms}.$$

10.1. Let p be a path from v to z. Let $x = (x_1, \dots, x_{n+1})$ be a sequence of geometric generators of $\pi_1(X(\mathbf{C}); v)$. Then $x' := (p\cdot x_1 \cdot p^{-1}, \dots, p\cdot x_{n+1}\cdot p^{-1})$ is a sequence of geometric generators of $\pi_1(X(\mathbf{C}); z)$. The action of $\sigma_{p^{-1}}$ on $\pi_1(X_{\bar K}; z)$ can be expressed in the following way by the action of σ_p on $\pi_1(X_{\bar K}; v)$. Let $\omega \in \pi_1(X_{\bar K}; z)$. Then $\sigma_{p^{-1}}(\omega) = p\cdot\sigma(p^{-1}\cdot\omega\cdot p)\cdot \mathfrak{f}_p(\sigma)^{-1}\cdot p^{-1}$. This implies that on $\mathbf{Q}_l\{\{\mathbf{X}\}\}$ we have

(10.1.0) $$\sigma_{x,p} = L_{\Lambda_p(\sigma)}\circ\sigma_x \quad\text{and}\quad \sigma_{x',p^{-1}} = R_{\Lambda_p(\sigma)^{-1}}\circ\sigma_x.$$

LEMMA 10.1.1. *Let D be a derivation of the algebra $\mathbf{Q}_l\{\{\mathbf{X}\}\}$ and let $\omega\in L(\mathbf{X})$. Then*

$$L_\omega \bigcirc D = L_\zeta + D \quad\textit{and}\quad R_{-\omega}\bigcirc D = R_{-\zeta} + D$$

for some $\zeta \in L(\mathbf{X})$.

Proof. The lemma follows from the identities

$$[L_\omega, D] = L_{-D(\omega)},\quad [R_{-\omega}, D] = R_{D(\omega)}$$

and

$$[L_\omega, L_{-D(\omega)}] = L_{-[\omega,D(\omega)]},\quad [R_{-\omega}, R_{D(\omega)}] = R_{[\omega,D(\omega)]}.$$

THEOREM 10.1.2. *Let $z, v\in \hat X(K)$ and let p be a path from v to z. Then we have*

$$i)\ \ \mathcal{L}^e(z,v) + \mathcal{L}^e(v,z) = 0,$$
$$ii)\ \ a^e_{x,p} + a^e_{pxp^{-1},p^{-1}} = 0.$$

Proof. It follows from (10.1.0) that

$$(\log\sigma_{x',p^{-1}})(1) = (R_{-\log\Lambda_p(\sigma)}\bigcirc\log\sigma_x)(1).$$

It follows from Lemma 10.1.1 that

$$(R_{-\log\Lambda_p(\sigma)}\bigcirc\log\sigma_x)(1) = -(L_{\log\Lambda_p(\sigma)}\bigcirc\log\sigma_x)(1).$$

Hence we get that

$$(\log \sigma_{x',p^{-1}})(1) = -(\log \sigma_{x,p})(1).$$

Evaluating a linear form on both sides of the equation we get the theorem.

THEOREM 10.1.3. *Let $z, y, v \in \hat{X}(K)$. Then we have*

$$\mathcal{L}^e(z,v) = \mathcal{L}^e(z,y) + \mathcal{L}^e(y,v).$$

Proof. Let p be a path from v to y, let r be a path from y to z and let $q = r \cdot p$. We have $\sigma_p = L_{\mathfrak{f}_p(\sigma)} \circ \sigma$ and $\sigma_q = L_{\mathfrak{f}_q(\sigma)} \circ \sigma$ on $\pi_1(X_{\bar{K}}; v)$ and $\sigma_r = L_{\mathfrak{f}_r(\sigma)} \circ \sigma$ on $\pi_1(X_{\bar{K}}; y)$. It follows from Lemma 1.0.6 that $\sigma_q = L_{p^{-1}\mathfrak{f}_r(\sigma)p} \circ \sigma_p$. Let us choose a sequence x of geometric generators of $\pi_1(X_{\bar{K}}; y)$. Then $x' = p^{-1} \cdot x \cdot p$ is a sequence of geometric generators of $\pi_1(X_{\bar{K}}; v)$. Observe that

$$\sigma_{x',q} = \sigma_{x,r} \circ \sigma_x^{-1} \circ \sigma_{x',p}.$$

Hence we get

$$\log \sigma_{x',q} = \log \sigma_{x,r} \bigcirc \log \sigma_x^{-1} \bigcirc \log \sigma_{x',p}.$$

Let σ belongs to the degree m step of the filtration defined in Section 3, i.e., $\sigma \in \mathcal{K}_m^T(X)$ for some finite subset $T \subset \hat{X}(K)^2$. Then

$$(\log \sigma_{x',q})(1) \equiv (\log \sigma_{x,r})(1) + (\log \sigma_{x',p})(1) \mod \Gamma^{m+1}L(\mathbf{X}).$$

Evaluating a linear form of degree m on both sides of the congruence we get the theorem.

10.2. It follows from Proposition 7.1.10 that relations between functions $\mathcal{L}^e(z,v)$ imply relations between symbols $\{z,v\}_e$. Hence we get the following result.

COROLLARY 10.2.1. *Assume that Conjectures D_n are true for all n. Assume that for all n the maps realization* : $\mathrm{Ext}^1_{\mathcal{MM}_K}(\mathbf{Q}(0), \mathbf{Q}(n)) \otimes \mathbf{Q} \to H^1(G_K, \mathbf{Q}_l(n))$ *are injective. Then we have*

$$\{z,v\}_e + \{v,z\}_e = 0$$

and

$$\{z,v\}_e = \{z,y\}_e + \{y,v\}_e$$

in $\mathcal{L}^K(X)$.

Proof. The corollary follows from Theorems 10.1.2 and 10.1.3 and Proposition 7.1.10.

10.3. Let π be a group. If π is nilpotent then we denote by $\pi \otimes \mathbf{Q}$ its rationalization. For an arbitrary group π, $\pi \otimes \mathbf{Q} := \varprojlim_n ((\pi/\Gamma^n\pi) \otimes \mathbf{Q})$ is a rational completion of π. The group $\pi_1(X_{\bar{K}}; v)$ is equipped with a pro-finite topology. Hence every quotient $\pi_1(X_{\bar{K}}; v)/\Gamma^n\pi_1(X_{\bar{K}}; v)$ is equipped with a pro-finite topology. Therefore rationalization $(\pi_1(X_{\bar{K}}; v)/\Gamma^n\pi_1(X_{\bar{K}}; v)) \otimes \mathbf{Q}$ is a $\mathbf{Q}_l$-Lie group. Hence $\pi_1(X_{\bar{K}}; v) \otimes \mathbf{Q} = \varprojlim_n ((\pi_1(X_{\bar{K}}; v)/\Gamma^n\pi_1(X_{\bar{K}}; v)) \otimes \mathbf{Q})$ is equipped with a topology of the inverse limit of $\mathbf{Q}_l$-Lie groups. The action of G_K on $\pi_1(X_{\bar{K}}; v)$ extends uniquely to a continous action of G_K on $\pi_1(X_{\bar{K}}; v) \otimes \mathbf{Q}$.

Now we shall define a rational completion of $\pi_1(X_{\bar{K}}; v)$-torsor $\pi(X_{\bar{K}}; z, v)$. We introduce an equivalence relation on the product $\pi(X_{\bar{K}}; z, v) \times \pi_1(X_{\bar{K}}; v) \otimes \mathbf{Q}$. We say that a pair (p, S) is equivalent to a pair (q, T) and we write $(p, S) \sim (q, T)$ if $S = (p^{-1} \cdot q) \cdot T$ in $\pi_1(X_{\bar{K}}; v) \otimes \mathbf{Q}$.

We set

$$\pi(X_{\bar{K}}; z, v) \otimes \mathbf{Q} := \big(\pi(X_{\bar{K}}; z, v) \times \pi_1(X_{\bar{K}}; v) \otimes \mathbf{Q}\big)/\sim.$$

The Galois group G_K acts on the product $\pi(X_{\bar{K}}; z, v) \times \pi_1(X_{\bar{K}}; v) \otimes \mathbf{Q}$ component wise. The group $\pi_1(X_{\bar{K}}; v) \otimes \mathbf{Q}$ acts on the product $\pi(X_{\bar{K}}; z, v) \times \pi_1(X_{\bar{K}}; v) \otimes \mathbf{Q}$ by the right multiplication on the second factor. Both actions are compatible with the equivalence relation $\sim$ and continous. Hence G_K acts on the set of equivalence classes $\pi(X_{\bar{K}}; z, v) \otimes \mathbf{Q}$. The action of $\pi_1(X_{\bar{K}}; v) \otimes \mathbf{Q}$ on the product $\pi(X_{\bar{K}}; z, v) \times \pi_1(X_{\bar{K}}; v) \otimes \mathbf{Q}$ induces a structure of $\pi_1(X_{\bar{K}}; v) \otimes \mathbf{Q}$-torsor on the set of equivalence classes $\pi(X_{\bar{K}}; z, v) \otimes \mathbf{Q}$. Elements of $\pi(X_{\bar{K}}; z, v) \otimes \mathbf{Q}$ have the form $p \cdot S$, where p is in $\pi(X_{\bar{K}}; z, v)$ and $S \in \pi_1(X_{\bar{K}}; v) \otimes \mathbf{Q}$. We shall call them $\mathbf{Q}_l$-paths.

LEMMA 10.3.1. *The embedding* $k_x : \pi_1(X_{\bar{K}}; v) \to \mathbf{Q}_l\{\{\mathbf{X}\}\}$ *extends uniquely to a continous multiplicative embedding* $\bar{k}_x : \pi_1(X_{\bar{K}}; v) \otimes \mathbf{Q} \to \mathbf{Q}_l\{\{\mathbf{X}\}\}$.

Proof. The image of k_x is contained in $\mathbf{Q}_l\{\{\mathbf{X}\}\}^*$. The group $\mathbf{Q}_l\{\{\mathbf{X}\}\}^*$ is a pro-unipotent group with exponents in $\mathbf{Q}_l$. Hence k_x extends to $\bar{k}_x : \pi_1(X_{\bar{K}}; v) \otimes \mathbf{Q} \to \mathbf{Q}_l\{\{\mathbf{X}\}\}$.

Further we shall denote the embedding $\bar{k}_x$ by k_x. One shows that the formulas

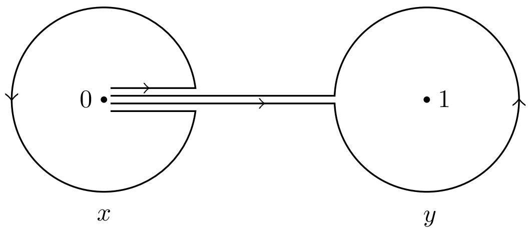
$$\begin{aligned}
&\mathfrak{f}_{p\cdot q}(\sigma) = q^{-1} \cdot \mathfrak{f}_p(\sigma) \cdot q \cdot \mathfrak{f}_q(\sigma),\\
&\Lambda_{p\cdot q}(\sigma) = \Lambda_p(\sigma) \cdot \Lambda_q(\sigma),\\
&\Lambda_p(\tau \cdot \sigma) = \Lambda_p(\tau) \cdot \tau(\Lambda_p(\sigma))
\end{aligned}$$

and $g_*(\mathfrak{f}_p) = \mathfrak{f}_{g(p)}$, where $g : X_K \to X_K$ is a regular map, are valid also for $\mathbf{Q}_l$-paths p and q.

§11. l-adic polylogarithms

11.0. In this subsection we introduce l-adic polylogarithms. We give sufficient conditions when a linear combination of l-adic polylogarithms is a cocycle. Next we are studying a relative version of l-adic polylogarithms. We also show that l-adic polylogarithms are special case of l-adic iterated integrals introduced in Section 5.

Let K be a number field. Let $V = \mathbf{P}^1_K \setminus \{0, 1, \infty\}$. Let x and y be standard generators of $\pi_1(V_{\bar{K}}; \overrightarrow{01})$ – loops around 0 and 1 respectively (see the Picture 1).



Picture 1

Let $k : \pi_1(V_{\bar{K}}; \overrightarrow{01}) \otimes \mathbf{Q} \to \mathbf{Q}_l\{\{X, Y\}\}$ be a multiplicative continous embedding given by $k(x) = e^X$ and $k(y) = e^Y$. We denote by $\mathrm{Lie}(X, Y)$ a free Lie algebra over $\mathbf{Q}_l$ on X and Y and by $L(X, Y)$ a completion of $\mathrm{Lie}(X, Y)$ with respect to the lower central series. We identify $L(X, Y)$ with the Lie algebra of Lie elements in $\mathbf{Q}_l\{\{X, Y\}\}$.

Let us set $E_1 := Y$, $E_{k+1} := [E_k, X]$. Let $\mathcal{B}$ be a base of $\mathrm{Lie}(X, Y)$ given by basic Lie elements. We assume that $E_k \in \mathcal{B}$ for $k = 1, 2, \dots$.

Let $z \in \hat{V}(K)$ and let p be a $\mathbf{Q}_l$-path from $\overrightarrow{01}$ to z. We recall that $\mathfrak{f}_p(\sigma) = p^{-1} \cdot \sigma(p) \in \pi_1(V_{\bar{K}}; \overrightarrow{01}) \otimes \mathbf{Q}$ and $\Lambda_p(\sigma) := k(\mathfrak{f}_p(\sigma)) \in \mathbf{Q}_l\{\{X, Y\}\}$ for any $\sigma \in G_K$.

If $e \in \mathcal{B}$ we denote by e^* the dual linear form to e with respect to $\mathcal{B}$.

DEFINITION 11.0.1. Let $\sigma \in G_K$. We set

$$l_n(z)(\sigma) := E_n^*(\log \Lambda_p(\sigma)) \quad \text{and} \quad l(z)(\sigma) := X^*(\log \Lambda_p(\sigma)).$$

The coefficient $l_n(z)$ is an l-adic polylogarithm (n-th order l-adic polylogarithm) evaluated at z. It is a function from G_K to $\mathbf{Q}_l(n)$. It depends on a choice of p in $\pi(V_{\bar{K}}; z, \overrightarrow{01}) \otimes \mathbf{Q}$. The coefficient $l(z)$ is an l-adic logarithm evaluated at z. If we are using various paths and it is important to indicate the dependence of $l_n(z)$ (resp. $l(z)$) on a path p we shall write $l_n(z)_p$ (resp. $l(z)_p$).

DEFINITION 11.0.2. We set

$$\mathcal{L}_n(z) := l_n(z)_{|H_n(V;z,\overrightarrow{01})}.$$

Observe that $\mathcal{L}_n(z)$ depends only on z.

Let us set $e_1 := y$ and $e_{k+1} := (e_k, x)$. Observe that any element $g \in \pi_1(V_{\bar{K}}; \overrightarrow{01}) \otimes \mathbf{Q}$ can be written in the following form

$$g = x^{\alpha^0(g)} \cdot y^{\alpha^1(g)} \cdot e_2^{\alpha^2(g)} \cdot e_3^{\alpha^3(g)} \cdot ((y,x)y)^{\beta(g)} \cdot e_4^{\alpha^4(g)} \cdot f_4 \cdot \dots \cdot e_n^{\alpha^n(g)} \cdot f_n \cdot \dots ,$$

where the exponents are in $\mathbf{Q}_l$ and each f_n is a product of powers of commutators of length n, which contain y at least twice.

DEFINITION 11.0.3. Let $\sigma \in G_K$. We define functions $\kappa_z^n : G_K \to \mathbf{Q}_l$ by the identity

$$\mathfrak{f}_p(\sigma) = x^{\kappa_z^0(\sigma)} \cdot y^{\kappa_z^1(\sigma)} \cdot e_2^{\kappa_z^2(\sigma)} \cdot e_3^{\kappa_z^3(\sigma)} \cdot f_3 \cdot e_4^{\kappa_z^4(\sigma)} \cdot f_4 \cdot \dots \cdot e_n^{\kappa_z^n(\sigma)} \cdot f_n \cdot \dots .$$

Let $n \geq 1$. Then κ_z^n we view as a function from G_K to $\mathbf{Q}_l(n)$. κ_z^0 we view as a function from G_K to $\mathbf{Q}_l(1)$. We shall also use the notation $\kappa_0(z) := \kappa_z^0$ and $\kappa_1(z) := \kappa_z^1$. If we are using various paths and it is important to indicate the dependence of $\kappa_z^n(\sigma)$ on a path p we shall write $\kappa_z^n(\sigma)_p$.

We shall express l-adic polylogarithms in terms of functions κ_z^n.

Let $f \in L(X,Y)$. We define a derivation $ad\, f$ of $L(X,Y)$ setting $(ad\, f)(g) = [f,g]$ for any $g \in L(X,Y)$.

Let I_k be a Lie ideal of $L(X,Y)$ generated topologically by Lie brackets which contain Y at least k-times.

LEMMA 11.0.4. *We have*

$$\log(k(e_{n+1})) = (-1)^n \sum_{k_1,\dots,k_n=1}^{\infty} \frac{1}{k_1! \cdots k_n!} (ad\, X)^{k_1+\dots+k_n}(Y) \mod I_2.$$

Lemma 11.0.5. (see [B] chapitre II) *We have*

$$\log(e^X \cdot e^Y) = X + Y + \frac{1}{2}[X,Y] + \sum_{n=1}^{\infty} \frac{1}{(2n)!} B_{2n} (ad\, X)^{2n}(Y) \mod I_2.$$

Proposition 11.0.6. *Let $\sigma \in G_K$. We have*

$$\begin{aligned}
\log \Lambda_p(\sigma) &= \kappa_z^0(\sigma) X \\
&+ \sum_{i=1}^{\infty} (-1)^{i-1} \kappa_z^i(\sigma) \Bigg(\sum_{k_1,\dots,k_{i-1}=1}^{\infty} \frac{1}{k_1! \cdots k_{i-1}!} (ad\, X)^{k_1+\dots+k_{i-1}}(Y) \Bigg) \\
&+ \frac{1}{2}\Bigg[\kappa_z^0(\sigma) X,\ \sum_{i=1}^{\infty} (-1)^{i-1} \kappa_z^i(\sigma) \\
&\qquad \times \Bigg(\sum_{k_1,\dots,k_{i-1}=1}^{\infty} \frac{1}{k_1! \cdots k_{i-1}!} (ad\, X)^{k_1+\dots+k_{i-1}}(Y) \Bigg) \Bigg] \\
&+ \sum_{n=1}^{\infty} \frac{(\kappa_z^0(\sigma))^{2n}}{(2n)!} B_{2n} (ad\, X)^{2n} \Bigg(\sum_{i=1}^{\infty} (-1)^{i-1} \kappa_z^i(\sigma) \\
&\qquad \times \Bigg(\sum_{k_1,\dots,k_{i-1}=1}^{\infty} \frac{1}{k_1! \cdots k_{i-1}!} (ad\, X)^{k_1+\dots+k_{i-1}}(Y) \Bigg) \Bigg) \mod I_2.
\end{aligned}$$

Proof. The proposition follows from Lemmas 11.0.4 and 11.0.5.

Using Proposition 11.0.6 we can easily calculate l-adic polylogarithms in terms of functions κ_z^n. For example in small degrees we get the following result.

Corollary 11.0.7. *We have*

$$l(z) = \kappa_z^0, \quad l_1(z) = \kappa_z^1, \quad l_2(z) = \kappa_z^2 - \frac{1}{2}\kappa_z^0 \cdot \kappa_z^1$$

and

$$l_3(z) = \kappa_z^3 - \frac{1}{2}\kappa_z^0 \cdot \kappa_z^2 + \frac{1}{12}(\kappa_z^0)^2 \cdot \kappa_z^1 - \frac{1}{2}\kappa_z^2.$$

Proposition 11.0.8. *Let $\zeta \in \hat{V}(K)$ and let p be a $\mathbf{Q}_l$-path from $\overrightarrow{01}$ to ζ. Let q be the standard path from $\overrightarrow{01}$ to $\overrightarrow{10}$ (an interval $[0,1]$). Let $g: V_K \to V_K$ be given by $g(z) = 1 - z$. Then we have*

$$l_1(\zeta)_p = l(1-\zeta)_{g(p)\cdot q}.$$

Proof. It follows from Corollary 11.0.7 that

$$\mathfrak{f}_p \equiv x^{l(\zeta)_p} \cdot y^{l_1(\zeta)_p} \mod \Gamma^2(\pi_1(V_{\bar{K}}; \overrightarrow{01}) \otimes \mathbf{Q}).$$

Observe that $g(p) \cdot q$ is a $\mathbf{Q}_l$-path from $\overrightarrow{01}$ to $1 - \zeta$. Hence we have

$$\mathfrak{f}_{g(p)\cdot q} \equiv x^{l(1-\zeta)_{g(p)\cdot q}} \cdot y^{l_1(1-\zeta)_{g(p)\cdot q}} \mod \Gamma^2(\pi_1(V_{\bar{K}}; \overrightarrow{01}) \otimes \mathbf{Q}).$$

On the other side

$$\begin{aligned}\mathfrak{f}_{g(p)\cdot q} &= q^{-1} \cdot \mathfrak{f}_{g(p)} \cdot q \cdot \mathfrak{f}_q = q^{-1} \cdot g_*(\mathfrak{f}_p) \cdot q \cdot \mathfrak{f}_q \\ &\equiv x^{l_1(\zeta)_p} \cdot y^{l(\zeta)_p} \mod \Gamma^2(\pi_1(V_{\bar{K}}; \overrightarrow{01}) \otimes \mathbf{Q})\end{aligned}$$

because $q^{-1} \cdot g_*(x) \cdot q = y$, $q^{-1} \cdot g_*(y) \cdot q = x$ and $\mathfrak{f}_q \equiv 1 \mod \Gamma^2(\pi_1(V_{\bar{K}}; \overrightarrow{01}) \otimes \mathbf{Q})$. The proposition follows from the last two congruences.

THEOREM 11.0.9. *Let $z_i \in \hat{V}(K)$, let $p_i \in \pi(V_{\bar{\mathbf{Q}}}; z_i, \overrightarrow{01}) \otimes \mathbf{Q}$ and let $n_i \in \mathbf{Q}_l$ for $i = 1, \dots, N$. Let us assume that l-adic polylogarithms $l_k(z_i)$ calculated along the $\mathbf{Q}_l$-paths p_i for $i = 1, \dots, N$ satisfy the following conditions*

i) $\sum_{i=1}^N n_i (l(z_i)(\tau))^\alpha \cdot (l(z_i)(\sigma))^\beta \cdot \big(l(z_i)(\tau) \cdot l_1(z_i)(\sigma) - l(z_i)(\sigma) \cdot l_1(z_i)(\tau)\big) = 0$ *for any $\tau, \sigma \in G_K$ and for any α and β such that $\alpha + \beta = n - 2$,*

ii) $\sum_{i=1}^N n_i (l(z_i)(\tau))^\alpha \cdot (l(z_i)(\sigma))^\beta \cdot l_k(z_i)(\sigma) = 0$ *for any $\tau, \sigma \in G_K$, for $k = 2, \dots, n-1$ and for any α and β such that $\alpha + \beta = n - k$.*

Then $\sum_{i=1}^N n_i l_n(z_i)$ is a cocycle on G_K with values in $\mathbf{Q}_l(n)$.

Proof. The equality $\Lambda_p(\tau\sigma) = \Lambda_p(\tau) \cdot \tau(\Lambda_p(\sigma))$ implies

$$\begin{aligned}\log \Lambda_p(\tau\sigma) &= \log \Lambda_p(\tau) + \log \tau(\Lambda_p(\sigma)) + \frac{1}{2}[\log \Lambda_p(\tau), \log(\tau(\Lambda_p(\sigma)))] \\ &- \frac{1}{12}[[\log \Lambda_p(\tau), \log(\tau(\Lambda_p(\sigma)))], \log \Lambda_p(\tau)] \\ &+ \frac{1}{12}[[\log \Lambda_p(\tau), \log(\tau(\Lambda_p(\sigma)))], \log(\tau(\Lambda_p(\sigma)))] \\ &- \frac{1}{24}[[[\log \Lambda_p(\tau), \log(\tau(\Lambda_p(\sigma)))], \log(\tau(\Lambda_p(\sigma)))], \log \Lambda_p(\tau)] + \cdots .\end{aligned}$$

Comparing coefficients at E_n we get

$$\begin{aligned}
l_n(z)(\tau\sigma) &= l_n(z)(\tau) + \chi(\tau)^n l_n(z)(\sigma) \\
&+ \frac{1}{2}\big(l_{n-1}(z)(\tau)\chi(\tau)l(z)(\sigma) - \chi(\tau)^{n-1}l_{n-1}(z)(\sigma)l(z)(\tau)\big) \\
&- \frac{1}{12}\big(l_{n-2}(z)(\tau)\chi(\tau)l(z)(\tau)l(z)(\sigma) - \chi(\tau)^{n-2}l_{n-2}(z)(\sigma)(l(z)(\tau))^2\big) \\
&+ \frac{1}{12}\big(l_{n-2}(z)(\tau)\chi(\tau)^2(l(z)(\sigma))^2 - \chi(\tau)^{n-1}l_{n-2}(z)(\sigma)l(z)(\tau)l(z)(\sigma)\big) \\
&- \frac{1}{24}\big(l_{n-3}(z)(\tau)\chi(\tau)^2 l(z)(\tau)(l(z)(\sigma))^2 \\
&\qquad - \chi(\tau)^{n-2}l_{n-3}(z)(\sigma)(l(z)(\tau))^2 l(z)(\sigma)\big) + \cdots .
\end{aligned}$$

The assumptions of the theorem imply that

$$\sum_{i=1}^N n_i l_n(z_i)(\tau\sigma) = \sum_{i=1}^N n_i l_n(z_i)(\tau) + \chi(\tau)^n \sum_{i=1}^N n_i l_n(z_i)(\sigma).$$

The l-adic polylogarithm $l_n(z)_p$ depends on a choice of a $\mathbf{Q}_l$-path from $\overrightarrow{01}$ to z. We have the following elementary result.

Lemma 11.0.10. *Let p be a $\mathbf{Q}_l$-path from $\overrightarrow{01}$ to z and let $S \in \pi_1(V_{\bar K}, \overrightarrow{01}) \otimes \mathbf{Q}$. If $S \equiv x^\alpha \cdot y^\beta \mod \Gamma^2(\pi_1(V_{\bar K}; \overrightarrow{01}) \otimes \mathbf{Q})$ then $l(z)_{pS} = l(z)_p + \alpha(\chi - 1)$ and $l_1(z)_{pS} = l_1(z)_p + \beta(\chi - 1)$.*

Proof. We have $\mathfrak{f}_{pS}(\sigma) = S^{-1} \cdot \mathfrak{f}_p(\sigma) \cdot \sigma(S)$. Hence $\Lambda_{pS}(\sigma) = k(S)^{-1} \cdot \Lambda_p(\sigma) \cdot k(\sigma(S))$. Let $S = x^\alpha \cdot y^\beta \cdot e_2^{\beta_2} \cdot e_3^{\beta_3} \cdot f_3 \cdot e_4^{\beta_4} \cdot \cdots$. Therefore $\log \Lambda_{pS}(\sigma) = \big(-\log(e^{\alpha X} \cdot e^{\beta Y} \cdot (e^X \cdot e^Y \cdot e^{-X} \cdot e^{-Y})^{\beta_2} \cdot \cdots)\big) \bigcirc \log \Lambda_p(\sigma) \bigcirc \log\big(e^{\alpha\chi(\sigma)X} \cdot e^{\beta\chi(\sigma)Y} \cdot (e^{\chi(\sigma)X} \cdot e^{\chi(\sigma)Y} \cdot e^{-\chi(\sigma)X} \cdot e^{-\chi(\sigma)Y})^{\beta_2} \cdot \cdots\big) \equiv -\alpha X - \beta Y + l(z)_p(\sigma)X + l_1(z)_p(\sigma)Y + \alpha\chi(\sigma)X + \beta\chi(\sigma)Y \mod \Gamma^2 L(X,Y)$. The lemma follows from the congruence.

Theorem 11.0.11. *Let $z_i \in V(K)$, let $p_i \in \pi(V_{\bar K}; z_i, \overrightarrow{01}) \otimes \mathbf{Q}$ and let $n_i \in \mathbf{Q}$ for $i = 1, \ldots, N$. Let $\mathcal{S}$ be a subgroup of $K^* \otimes \mathbf{Q}$ generated by z_i and $1 - z_i$ for $i = 1, \ldots, N$. Assume that*

i) *the map $\varphi : \mathcal{S} \to Z^1(G_K; \mathbf{Q}_l(1))$ given by $\varphi(z_i) = l(z_i)_{p_i}$ and $\varphi(1 - z_i) = l_1(z_i)_{p_i}$ is well defined and it is a homomorphism;*

ii) *$\sum_{i=1}^N n_i \nu_1(z_i) \cdots \nu_{n-2}(z_i)(z_i) \wedge (1 - z_i) = 0$ in $(\mathcal{S} \wedge \mathcal{S}) \otimes \mathbf{Q}_l$ for any homomorphisms $\nu_1, \ldots, \nu_{n-2}$ from $\mathcal{S}$ to $\mathbf{Q}_l$;*

iii) $\sum_{i=1}^N n_i \cdot \nu_1(z_i)^\alpha \cdot \nu_2(z_i)^\beta \cdot l_k(z_i)(\sigma) = 0$ *for any homomorphisms* ν_1 *and* ν_2 *from* $\mathcal{S}$ *to* $\mathbf{Q}_l$, *for any* $\sigma \in G_K$, *for* $k = 2, \dots, n-1$ *and for any* α *and* β *such that* $\alpha + \beta = n - k$.

Then $\sum_{i=1}^N n_i l_n(z_i)_{p_i}$ *is a cocycle on* G_K *with values in* $\mathbf{Q}_l(n)$.

Proof. Let us fix $\tau \in G_K$. The map $\mathcal{S} \to \mathbf{Q}_l(1)$ given by $s \to \varphi(s)(\tau)$ $(z_i \to l(z_i)(\tau))$ is a homomorphism. Let us fix $\tau, \sigma \in G_K$. The map $\mathcal{S} \otimes \mathcal{S} \to \mathbf{Q}_l(2)$, $x \otimes y \to \varphi(x)(\tau) \cdot \varphi(y)(\sigma) - \varphi(x)(\sigma) \cdot \varphi(y)(\tau)$ $(z_i \otimes (1 - z_i) \to l(z_i)(\tau) \cdot l_1(z_i)(\sigma) - l(z_i)(\sigma) \cdot l_1(z_i)(\tau))$ factors through $\mathcal{S} \wedge \mathcal{S}$. Hence the theorem follows from Theorem 11.0.9.

COROLLARY 11.0.12. *Let* ξ_m *be a* m*-th root of* 1 *different from* 1. *There is a* $\mathbf{Q}_l$*-path* p *from* $\overrightarrow{01}$ *to* ξ_m *such that* $l_n(\xi_m)_p$ *is a cocycle on* $G_{\mathbf{Q}(\mu_m)}$. *If* l *does not divide* m *then one can choose the path* p *in* $\pi(V_{\bar{\mathbf{Q}}}; \xi_m, \overrightarrow{01})$.

Proof. Let $m = l^{k_0} \cdot r$, where l does not divide r. Let q be a path from $\overrightarrow{01}$ to ξ_m. There are α, β and γ in $\mathbf{Z}_l$ such that $(\xi_{l^{k_0+n}}^\alpha \cdot \xi_r^{\beta/l^n} \cdot \xi_{l^n}^\gamma)_{n \in \mathbf{N}}$ is a compatible family of l^n-th roots of ξ_m determined by the path q. Hence $l(\xi_m)_q = (\frac{\alpha}{k_0} + \gamma)(\chi - 1)$. Lemma 11.0.10 implies that there is a $\mathbf{Q}_l$-path p from $\overrightarrow{01}$ to ξ_m such that $l(\xi_m)_p = 0$. Theorem 11.0.9 implies that $l_n(\xi_m)_p$ is a cocycle. Observe that if $k_0 = 0$ then one can choose p in $\pi(V_{\bar{\mathbf{Q}}}; \xi_m, \overrightarrow{01})$.

The classical polylogarithms are iterated integrals defined by $\int_0^z \frac{dz}{1-z}, \frac{dz}{z}, \dots, \frac{dz}{z}$. The iterated integral $\int_a^b \frac{dz}{1-z}, \frac{dz}{z}, \dots, \frac{dz}{z}$ can be express by classical polylogarithms. Now we shall define a normalized analog of the iterated integral $\int_a^b \frac{dz}{1-z}, \frac{dz}{z}, \dots, \frac{dz}{z}$.

Let $z, v \in \hat{V}(K)$. Let q be a path from $\overrightarrow{01}$ to v and let p be a path from v to z. We shall define relative polylogarithms $l_n(z, v)$. Let us set $x_1 := q \cdot x \cdot q^{-1}$, $y_1 := q \cdot y \cdot q^{-1}$. Observe that x_1, y_1 are generators of $\pi_1(V_{\bar{K}}; v)$. Let $G_{n+1} \subset \pi_1(V_{\bar{K}}; \overrightarrow{01})$ (resp. $G'_{n+1} \subset \pi_1(V_{\bar{K}}; v)$) be a closed normal subgroup generated by $\Gamma^{n+1}\pi_1(V_{\bar{K}}; \overrightarrow{01})$ (resp. $\Gamma^{n+1}\pi_1(V_{\bar{K}}; v)$) and all commutators which contain y (resp. y_1) at least twice. Let $\pi := \pi_1(V_{\bar{K}}; \overrightarrow{01})/G_{n+1}$ and $\pi' := \pi_1(V_{\bar{K}}; v)/G'_{n+1}$.

It follows from Proposition 2.2.1 that the action of G_K on π' is given by

$$\sigma(x_1) = (q \cdot \mathfrak{f}_q(\sigma) \cdot q^{-1}) \cdot x_1^{\chi(\sigma)} \cdot (q \cdot (\mathfrak{f}_q(\sigma))^{-1} \cdot q^{-1}) \mod G'_{m+1}$$

and

$$\sigma(y_1) = (q \cdot \mathfrak{f}_q(\sigma) \cdot q^{-1}) \cdot y_1^{\chi(\sigma)} \cdot (q \cdot (\mathfrak{f}_q(\sigma))^{-1} \cdot q^{-1}) \mod G'_{m+1}.$$

LEMMA 11.0.13. *The action of G_K on $\pi_1(V_{\bar{K}}; v)$ induced from the action on the torsor $\pi(V_{\bar{Q}}; z, v)$ by the isomorphism t_p (see Part I Section 1) is given by*

$$\sigma_p(w) = (q \cdot \mathfrak{f}_{pq}(\sigma) \cdot q^{-1}) \cdot \bar{\sigma}(w) \cdot (q \cdot (\mathfrak{f}_q(\sigma))^{-1} \cdot q^{-1}) \mod G'_{n+1},$$

where $\bar{\sigma}(x_1) = x_1^{\chi(\sigma)}$, $\bar{\sigma}(y_1) = y_1^{\chi(\sigma)}$ and $\bar{\sigma}$ is continous and multiplicative.

Proof. The formula for $\sigma_p(w)$ follows from Lemma 1.0.2 and Lemma 1.0.6.

Let I be the augmentation ideal of $\mathbf{Q}_l\{\{X, Y\}\}$ and let J_{n+1} be a closed ideal of $\mathbf{Q}_l\{\{X, Y\}\}$ generated by I^{n+1} and all monomials which contain Y at least twice. We define two maps

$$k : \pi_1(V_{\bar{K}}; \overrightarrow{01}) \longrightarrow \mathbf{Q}_l\{\{X, Y\}\}/J_{n+1} \quad \text{and}$$
$$k' : \pi_1(V_{\bar{K}}; v) \longrightarrow \mathbf{Q}_l\{\{X, Y\}\}/J_{n+1}$$

by $k(x) = e^X$, $k(y) = e^Y$ and $k'(x_1) = e^X$, $k'(y_1) = e^Y$.

Let $(\ )_p : G_K \to GL(\mathbf{Q}_l\{\{X, Y\}\}/J_{n+1})$ be the action of G_K induced from the action of G_K on the torsor $\pi(V_{\bar{\mathbf{Q}}}; z, v)$ by the isomorphism t_p and the embedding k'.

Let us set

$$\psi_p(\sigma) := \sigma_p \circ \rho(\chi(\sigma)^{-1}).$$

We recall that $E_1 := Y$ and $E_{k+1} := [E_k, X]$ for $k = 1, \ldots, n-1$. Then any Lie element of $\mathbf{Q}_l\{\{X, Y\}\}/J_{n+1}$ is a linear combination with $\mathbf{Q}_l$ coefficients of $X, E_1, \ldots, E_n$. If $g \in \pi'$ then $\log k'(g)$ is a Lie element of $\mathbf{Q}_l\{\{X, Y\}\}/J_{n+1}$.

DEFINITION 11.0.14. Let $\sigma \in G_K$. We set

$$(\log \psi_p(\sigma))(1) = l(z, v)_p(\sigma) X + \sum_{k=1}^{n} l_k(z, v)_p(\sigma) E_k.$$

PROPOSITION 11.0.15. *We have*

$$l_n(z, v)_p = l_n(z)_{pq} - l_n(v)_q.$$

Proof. Observe that $k'(q \cdot \mathfrak{f}_{pq}(\sigma) \cdot q^{-1}) = k(\mathfrak{f}_{pq}(\sigma)) = \Lambda_{pq}(\sigma)$ and $k'(q \cdot (\mathfrak{f}_q(\sigma))^{-1} \cdot q^{-1}) = k((\mathfrak{f}_q(\sigma))^{-1}) = (\Lambda_q(\sigma))^{-1}$. Let $\sigma \in G_K$. It follows from Lemma 11.0.13 that

$$\psi_p(\sigma) = L_{\Lambda_{pq}(\sigma)} \circ R_{(\Lambda_q(\sigma))^{-1}}.$$

This implies that

$$\log \psi_p(\sigma) = L_{\log \Lambda_{pq}(\sigma)} \bigcirc R_{-\log \Lambda_q(\sigma)}.$$

The operators $L_{\log \Lambda_{pq}(\sigma)}$ and $R_{-\log \Lambda_q(\sigma)}$ commute. Hence $\log \psi_p(\sigma) = L_{\log \Lambda_{pq}(\sigma)} + R_{-\log \Lambda_q(\sigma)}$. This implies the proposition.

COROLLARY 11.0.16. *We have*

$$l_n(z, \overrightarrow{01})_p = l_n(z)_p.$$

Proof. It follows from Proposition 11.0.15 that $l_n(z, \overrightarrow{01})_p = l_n(z)_p - l_n(\overrightarrow{01})_c$, where c is a constant path. For such a path $l_n(\overrightarrow{01})_c = 0$.

Remark. The relative polylogarithm $l_n(z, v)$ is the function $a_p^{E_n}$ from Section 5. Hence the l-adic polylogarithm $l_n(z)_p$ is also a special case of l-adic iterated integrals defined in Section 5.

We finish this subsection with a result expressing coefficiets of $\mathfrak{f}_p$ in degree one for an arbitrary X by l-adic logarithms.

PROPOSITION 11.0.17. *Let* $X = \mathbf{P}^1_K \setminus \{a_1, \dots, a_n, \infty\}$, *let* $z, v \in \hat{X}(K)$ *and let* p *be a path from* v *to* z. *Let* $g_i : X \to \mathbf{P}^1_K \setminus \{0, \infty\}$ *be given by* $g_i(z) = z - a_i$ *for* $i = 1, \dots, n$. *Then*

$$\mathfrak{f}_p \equiv x_1^{l(z-a_1)_{g_1(p)\cdot q_1} - l(v-a_1)_{q_1}} \cdot \dots \cdot x_n^{l(z-a_n)_{g_n(p)\cdot q_n} - l(v-a_n)_{q_n}} \mod \Gamma^2\pi_1(X_{\bar{K}}; v),$$

where q_i *is any path from* $\overrightarrow{01}$ *to* $v - a_i$ *on* $\mathbf{P}^1_{\bar{K}} \setminus \{0, \infty\}$ *for* $i = 1, \dots, n$.

Proof. Without loss of generality we can suppose that $X = \mathbf{P}^1_K \setminus \{a, \infty\}$ and $g : X \to \mathbf{P}^1_K \setminus \{0, \infty\}$ is given by $g(z) = z - a$. Let p be a path from v to z on $X_{\bar{K}}$. Then $g(p)$ is a path from $v - a$ to $z - a$ on $\mathbf{P}^1_K \setminus \{0, \infty\}$. Let q be any path from $\overrightarrow{01}$ to $v - a$ on $\mathbf{P}^1_{\bar{K}} \setminus \{0, \infty\}$. We have

$$\mathfrak{f}_{g(p)\cdot q} = q^{-1} \cdot \mathfrak{f}_{g(p)} \cdot q \cdot \mathfrak{f}_q = q^{-1} \cdot g_*(\mathfrak{f}_p) \cdot q \cdot \mathfrak{f}_q.$$

It follows from Corollary 11.0.7 that

$$x^{l(z-a)_{g(p)\cdot q}} = q^{-1} \cdot g_*(\mathfrak{f}_p) \cdot q \cdot x^{l(v-a)_q},$$

where x is a loop around 0. Hence we get that $\mathfrak{f}_p = x_a^{l(z-a)_{g(p)\cdot q} - l(v-a)_q}$, where $g_*(x_a) = q \cdot x \cdot q^{-1}$.

11.1. In this subsection we shall study functional equations of l-adic polylogarithms. We shall prove the distribution relation and the Abel five term equation for l-adic dilogarithms. We shall show that l-adic dilogarithms satisfy these functional equations without lower degree terms.

We start with the discussion of the l-adic analog of the functional equation

$$\log(x \cdot y) = \log x + \log y$$

of the classical logarithm.

PROPOSITION 11.1.0. *Let $\zeta, y \in \mathbf{P}^1(K) \setminus \{0, \infty\}$. Then there exist paths γ from $\overrightarrow{01}$ to ζ, δ from $\overrightarrow{01}$ to y and φ from $\overrightarrow{01}$ to $y \cdot \zeta$ such that*

$$l(y \cdot \zeta)_\varphi = l(y)_\delta + l(\zeta)_\gamma$$

on G_K.

Proof. Let $g : \mathbf{P}^1_K \setminus \{0, \infty\} \to \mathbf{P}^1_K \setminus \{0, \infty\}$ be given by $g(z) = y \cdot z$. Let p be a path from $\overrightarrow{01}$ to ζ. Then $g(p)$ is a path from $\overrightarrow{0y}$ to $y \cdot \zeta$. We recall that x is a standard generator of $\pi_1(\mathbf{P}^1_{\bar{K}} \setminus \{0, \infty\}, \overrightarrow{01})$. Let us fix a path q from $\overrightarrow{01}$ to $\overrightarrow{0y}$. Let us set $x' = q \cdot x \cdot q^{-1}$. Then x' is a generator of $\pi_1(\mathbf{P}^1_{\bar{K}} \setminus \{0, \infty\}, \overrightarrow{0y})$. Observe that $g_*(x) = x'$. It follows from Corollary 11.0.7 that

$$\mathfrak{f}_p(\sigma) = x^{l(\zeta)_p(\sigma)} \quad \text{and} \quad \mathfrak{f}_{g(p)\cdot q}(\sigma) = x^{l(y\cdot\zeta)_{g(p)\cdot q}(\sigma)}.$$

On the other side we have

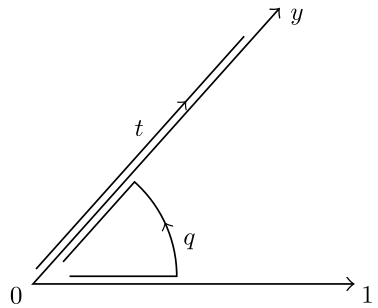
$$\begin{aligned}\mathfrak{f}_{g(p)\cdot q}(\sigma) &= q^{-1} \cdot \mathfrak{f}_{g(p)}(\sigma) \cdot q \cdot \mathfrak{f}_q(\sigma) = q^{-1} \cdot g_*(\mathfrak{f}_p(\sigma)) \cdot q \cdot \mathfrak{f}_q(\sigma) \\ &= x^{l(\zeta)_p(\sigma)} \cdot x^{l(\overrightarrow{0y})_q(\sigma)} = x^{l(\zeta)_p(\sigma) + l(\overrightarrow{0y})_q(\sigma)}.\end{aligned}$$

Comparing exponents we get

$$l(y \cdot \zeta)_{g(p)\cdot q} = l(\zeta)_p + l(\overrightarrow{0y})_q.$$

Let t be the canonical path from $\overrightarrow{0y}$ to y. Then $t \cdot q$ is a path from $\overrightarrow{01}$ to y (see Picture 2).

We have $x^{l(y)_{t\cdot q}(\sigma)} = \mathfrak{f}_{t\cdot q}(\sigma) = q^{-1} \cdot \mathfrak{f}_t(\sigma) \cdot q \cdot \mathfrak{f}_q(\sigma) = q^{-1} \cdot \mathfrak{f}_t(\sigma) \cdot q \cdot x^{l(\overrightarrow{0y})_q(\sigma)}$. It rests to calculate $\mathfrak{f}_t(\sigma)$. Without loss of generality we can suppose that $y = 1$ and t is the canonical path from $\overrightarrow{01}$ to 1. Then it is clear that $\mathfrak{f}_t(\sigma) = 1$. Hence $l(\overrightarrow{0y})_q = l(y)_{t\cdot q}$. This finishes the proof of the proposition.

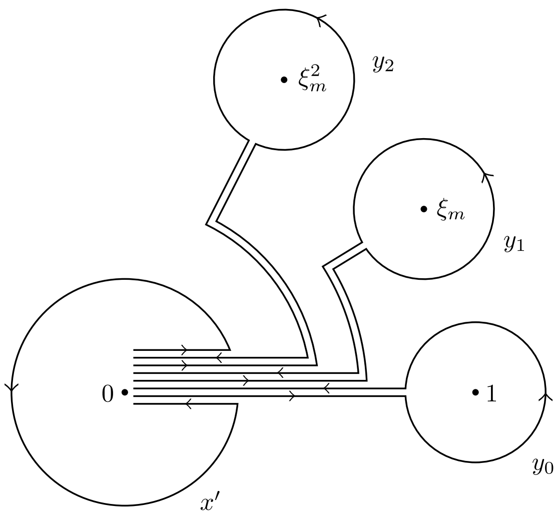


Picture 2

Now we shall discuss the l-adic analog of the functional equation

$$Li_2(z^m) = m\Big(\sum_{i=0}^{m} Li_2(\xi_m^i z)\Big)$$

of the classical dilogarithm. Let $Y = \mathbf{P}^1_{\mathbf{Q}(\mu_m)} \setminus \{0, \mu_m, \infty\}$ and $V = \mathbf{P}^1_{\mathbf{Q}(\mu_m)} \setminus \{0, 1, \infty\}$. We choose generators $x', y_0, \dots, y_{m-1}$ of $\pi_1(Y_{\bar{\mathbf{Q}}}, \overrightarrow{01})$ as on the picture.



Picture 3

Let $f : Y \to V$ be given by $f(z) = z^m$. We have $f_*(x') = x^m$, $f_*(y_0) = y$ and $f_*(y_i) = x^{-i} \cdot y \cdot x^i$. Let $z \in \hat{Y}(\mathbf{Q}(\mu_m))$ and let p be a path from $\overrightarrow{01}$

to z. We define functions $\lambda(z)$, $\mu_0(z),\dots,\mu_{m-1}(z)$, $\nu_0(z),\dots,\nu_{m-1}(z)$ from $G_{\mathbf{Q}(\mu_m)}$ to $\mathbf{Z}_l$ by the following congruence

$$\begin{aligned}(11.1.1)\qquad \mathfrak{f}_p \equiv x'^{\lambda(z)}\cdot y_0^{\mu_0(z)}\cdot y_1^{\mu_1(z)}\cdot\dots\cdot y_{m-1}^{\mu_{m-1}(z)}\\ \cdot (y_0,x')^{\nu_0(z)}\cdot\dots\cdot(y_{m-1},x')^{\nu_{m-1}(z)}\\ \cdot\prod_{i<j}(y_i,y_j)^{\alpha_{ij}(z)} \mod \Gamma^3\pi_1(Y_{\bar{\mathbf{Q}}};\overrightarrow{01}).\end{aligned}$$

Observe that $f(p)$ is a path from $\overrightarrow{01}$ to z^m. Hence we have

$$\mathfrak{f}_{f(p)}\equiv x^{\kappa^0_{z^m}}\cdot y^{\kappa^1_{z^m}}\cdot(y,x)^{\kappa^2_{z^m}} \mod \Gamma^3\pi_1(V_{\bar{\mathbf{Q}}};\overrightarrow{01}),$$

(see Definition 11.0.3).

LEMMA 11.1.2. *We have* $\kappa^0_{z^m}=m\lambda(z)$, $\kappa^1_{z^m}=\mu_0(z)+\mu_1(z)+\dots+\mu_{m-1}(z)$ *and* $\kappa^2_{z^m}=m(\nu_0(z)+\dots+\nu_{m-1}(z))+\mu_1(z)+\dots+i\mu_i(z)+\dots+(m-1)\mu_{m-1}(z)$.

Proof. We have

$$\begin{aligned}f_*\mathfrak{f}_p &\equiv x^{m\lambda(z)}\cdot y^{\mu_0(z)}\cdot x^{-1}\cdot y^{\mu_1(z)}\cdot x\cdot\dots\\ &\quad\cdot x^{-(m-1)}\cdot y^{\mu_{m-1}(z)}\cdot x^{m-1}\cdot(y,x)^{m(\nu_0(z)+\dots+\nu_{m-1}(z))}\\ &\equiv x^{m\lambda(z)}\cdot y^{\mu_0(z)+\dots+\mu_{m-1}(z)}\cdot(y,x)^{m(\nu_0(z)+\dots+\nu_{m-1}(z))+\sum_{i=0}^{m-1}i\mu_i(z)}\\ &\qquad \mod \Gamma^3\pi_1(V_{\bar{\mathbf{Q}}};\overrightarrow{01}).\end{aligned}$$

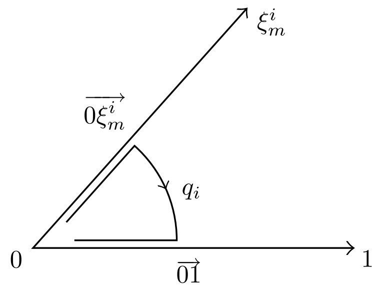
Observe that $f_*\mathfrak{f}_p=\mathfrak{f}_{f(p)}$. Comparing exponents of $f_*\mathfrak{f}_p$ and $\mathfrak{f}_{f(p)}$ we get the equalities of the lemma.

Let q_i be a path from $\overrightarrow{0\xi^i_m}$ to $\overrightarrow{01}$ as on Picture 4.

Let us set $x_i:=q_i^{-1}\cdot x'\cdot q_i$ and $y^{(i)}_k:=q_i^{-1}\cdot y_k\cdot q_i$. Let $f_i:Y\to V$ be given by $f_i(z)=\xi_m^{-i}\cdot z$. Observe that $(f_i)_*\overrightarrow{0\xi^i_m}=\overrightarrow{01}$, $(f_i)_*(x_i)=x$, $(f_i)_*(y^{(i)}_i)=y$ and $(f_i)_*(y^{(i)}_k)=1$ for $k\neq i$.

LEMMA 11.1.3. *We have*

$$\kappa^0_{\xi_m^{-i}z}=\lambda(z)+\frac{i}{m}(1-\chi),\quad \kappa^1_{\xi_m^{-i}z}=\mu_i(z)\ \text{ and }\ \kappa^2_{\xi_m^{-i}z}=\nu_i(z)+\frac{i}{m}(1-\chi)\mu_i(z).$$



Picture 4

Proof. $f_i(pq_i)$ is a path from $\overrightarrow{01}$ to $\xi_m^{-i}z$. Hence we have

$$\mathfrak{f}_{f_i(pq_i)} \equiv x^{\kappa^0_{\xi_m^{-i}z}} \cdot y^{\kappa^1_{\xi_m^{-i}z}} \cdot (y,x)^{\kappa^2_{\xi_m^{-i}z}} \mod \Gamma^3\pi_1(V_{\bar{\mathbf{Q}}};\overrightarrow{01}),$$

by the Definition 11.0.3. On the other side

$$(f_i)_*\mathfrak{f}_{pq_i} = (f_i)_*(q_i^{-1}\cdot \mathfrak{f}_p \cdot q_i)\cdot (f_i)_*(\mathfrak{f}_{q_i}).$$

Hence it follows from (11.1.1) that

$$\begin{aligned}(f_i)_*\mathfrak{f}_{pq_i} &\equiv x^{\lambda(z)}\cdot y^{\mu_i(z)}\cdot (y,x)^{\nu_i(z)}\cdot x^{\frac{i}{m}(1-\chi)}\\ &\equiv x^{\lambda(z)+\frac{i}{m}(1-\chi)}\cdot y^{\mu_i(z)}\cdot (y,x)^{\nu_i(z)+\frac{i}{m}(1-\chi)\mu_i(z)} \mod \Gamma^3\pi_1(V_{\bar{\mathbf{Q}}};\overrightarrow{01}).\end{aligned}$$

We have the identity

$$(f_i)_*\mathfrak{f}_{pq_i} = \mathfrak{f}_{f_i(pq_i)}.$$

Hence comparing exponents of $(f_i)_*\mathfrak{f}_{pq_i}$ and $\mathfrak{f}_{f_i(pq_i)}$ we get the equalities of the lemma.

PROPOSITION 11.1.4. *Let $l_2(z^m)$ be calculated along the path $f(p)$ and let $l_2(\xi_m^{-i}z)$ be calculated along the $\mathbf{Q}_l$-path $f_i(pq_i)\cdot x^{\frac{i}{m}}$ for $i = 0, 1, \dots, m-1$. Then we have*

$$l_2(z^m) = m\Big(\sum_{i=0}^{m-1} l_2(\xi_m^{-i}z)\Big).$$

Proof. It follows from Corollary 11.0.7 that

$$l_2(z)_p(\sigma) = \kappa_z^2(\sigma)_p - \frac{1}{2}\kappa_z^0(\sigma)_p\cdot \kappa_z^1(\sigma)_p.$$

Hence it follows from Lemma 11.1.2 that

$$l_2(z^m)_{f(p)} = m\Big(\sum_{i=0}^{m-1} \nu_i(z)\Big) + \sum_{i=0}^{m-1} i\mu_i(z) - \frac{1}{2}m\lambda(z)\Big(\sum_{i=0}^{m-1} \mu_i(z)\Big).$$

Let us calculate $l_2(\xi_m^{-i}z)_{f_i(pq_i)\cdot x^{\frac{i}{m}}}$. We have

$$\mathfrak{f}_{f_i(pq_i)\cdot x^{\frac{i}{m}}}(\sigma) = x^{-\frac{i}{m}} \cdot \mathfrak{f}_{f_i(pq_i)}(\sigma) \cdot x^{\frac{i}{m}\chi(\sigma)}.$$

Hence it follows from Lemma 11.1.3 that

$$l_2(\xi_m^{-i}z)_{f_i(pq_i)x^{\frac{i}{m}}} = \nu_i(z) + \frac{i}{m}\mu_i(z) - \frac{1}{2}\lambda(z)\mu_i(z).$$

Comparing formulas for $l_2(z^m)_{f(p)}$ and $l_2(\xi_m^{-i}z)_{f_i(pq_i)\cdot x^{\frac{i}{m}}}$ we get

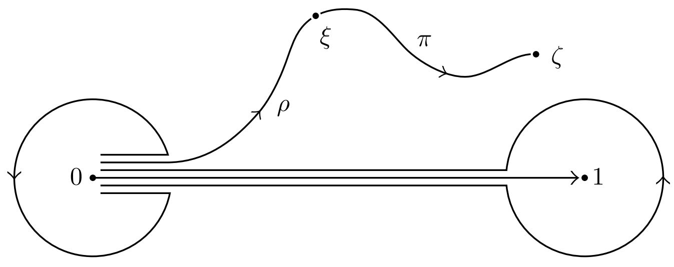
$$l_2(z^m)_{f(p)} = m\Big(\sum_{i=0}^{m-1} l_2(\xi_m^{-i}z)_{f_i(pq_i)\cdot x^{\frac{i}{m}}}\Big).$$

The classical dilogarithm satisfy the functional equation

$$Li_2\Big(\frac{(1-y)z}{z-1}\Big) - Li_2(yz) + Li_2\Big(\frac{(z-1)y}{1-y}\Big) - Li_2\Big(\frac{y}{y-1}\Big) + Li_2(z)$$
$$= \text{lower degree terms.}$$

We shall prove its l-adic analog.

Let $V = \mathbf{P}^1_K \setminus \{0,1,\infty\}$ and let $Y = \mathbf{P}^1_K \setminus \{0,1,\frac{1}{y},\infty\}$, where $y \in K \setminus \{0,1\}$. Let $\xi,\zeta \in \hat{V}(K)$ and let π be a path from ξ to ζ and let ρ be a path from $\overrightarrow{01}$ to ξ (see Picture 5).



Picture 5

Let us set

$$x' = \rho \cdot x \cdot \rho^{-1} \quad \text{and} \quad y' = \rho \cdot y \cdot \rho^{-1}$$

where x, y are generators of $\pi_1(V_{\bar{K}}; \overrightarrow{01})$ as in 11.0. We define functions $\mathfrak{k}(\pi)$, $\mathfrak{k}_1(\pi)$ and $\mathfrak{k}_2(\pi)$ from G_K to $\mathbf{Z}_l$ by the following congruence

$$\mathfrak{f}_\pi \equiv x'^{\mathfrak{k}(\pi)} \cdot y'^{\mathfrak{k}_1(\pi)} \cdot (y', x')^{\mathfrak{k}_2(\pi)} \mod \Gamma^3\pi_1(V_{\bar{K}}; \xi).$$

Lemma 11.1.5. i) *We have*

$$l_2(\zeta)_{\pi\rho} - l_2(\xi)_\rho = \mathfrak{k}_2(\pi) - \frac{1}{2}\mathfrak{k}(\pi)\mathfrak{k}_1(\pi) - \frac{1}{2}\kappa^0_\zeta\kappa^1_\xi + \frac{1}{2}\kappa^0_\xi\kappa^1_\zeta.$$

ii) *If we replace ρ by $\rho_1 = \rho \cdot x^a$ then in terms of new generators $x'' = \rho_1 \cdot x \cdot \rho_1^{-1}$, $y'' = \rho_1 \cdot y \cdot \rho_1^{-1}$ the triple $\mathfrak{k}(\pi)$, $\mathfrak{k}_1(\pi)$, $\mathfrak{k}_2(\pi)$ is replaced by the triple $\mathfrak{k}(\pi)$, $\mathfrak{k}_1(\pi)$, $\mathfrak{k}_2(\pi) + a\mathfrak{k}_1(\pi)$.*

Proof. It follows from the formula $\mathfrak{f}_{\pi\rho} = \rho^{-1}\mathfrak{f}_\pi\rho \cdot \mathfrak{f}_\rho$ (see Lemma 1.0.6) that $\Lambda_{\pi\rho}(\sigma) = \Lambda_\pi(\sigma) \cdot \Lambda_\rho(\sigma)$, where $\Lambda_\pi(\sigma)$ is the image of $\mathfrak{f}_\pi$ by the embedding of $\pi_1(V_{\bar{K}}; \xi)$ into $\mathbf{Q}_l\{\{X, Y\}\}$ sending x' to e^X and y' to e^Y. Applying logarithm we get

$$\log \Lambda_{\pi\rho}(\sigma) \bigcirc (-\log \Lambda_\rho(\sigma)) = \log \Lambda_\pi(\sigma).$$

Comparing coefficient at $[Y, X]$ we get

$$l_2(\zeta)_{\pi\rho} - l_2(\xi)_\rho = \mathfrak{k}_2(\pi) - \frac{1}{2}\mathfrak{k}(\pi)\mathfrak{k}_1(\pi) - \frac{1}{2}\kappa^0_\zeta\kappa^1_\xi + \frac{1}{2}\kappa^0_\xi\kappa^1_\zeta.$$

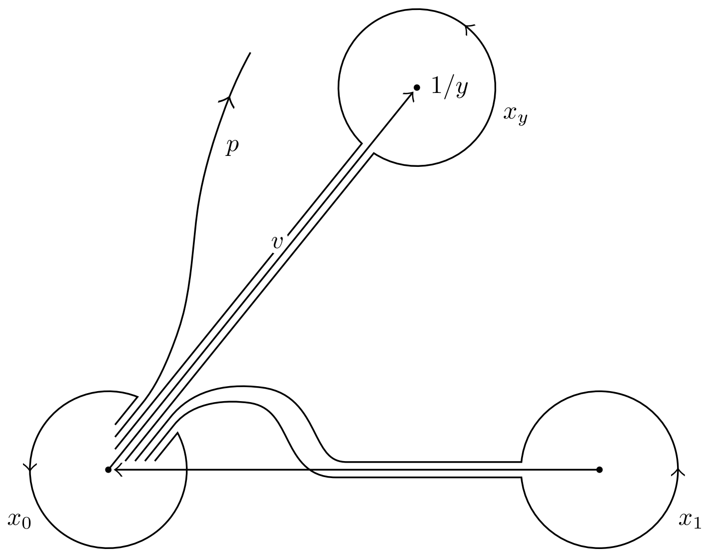
The second part of the lemma follows from the congruence $y'' = x'^a \cdot y' \cdot x'^{-a} \equiv y' \cdot (y', x')^{-a} \mod \Gamma^3\pi_1(V_{\bar{K}}; \xi)$.

Definition 11.1.6. Let us set

$$K(\zeta, \xi) := -\kappa^0_\zeta\kappa^1_\xi + \kappa^0_\xi\kappa^1_\zeta.$$

Observe that $K(\zeta, \xi)$ is a function from G_K to $\mathbf{Q}_l$. After the restriction to $G_{K(\mu_{l^\infty})}$ the function $K(\zeta, \xi)$ does not depend on a choice of paths from $\overrightarrow{01}$ to ξ and ζ.

Now we start to look for l-adic analog of the 5-term functional equation of the classical dilogarithm. Let $f(z) = \frac{(1-y)z}{z-1}$, $g(z) = yz$, $h(z) = \frac{(z-1)y}{1-y}$ and $k(z) = z$. Observe that f, g, h and k define regular maps from Y to V.



Picture 6

Let v be a tangential base point at 0 corresponding to the local parameter yz at 0. Let x_0, x_1, x_y, x_∞ be geometric generators of $\pi_1(Y_{\bar{\mathbf{Q}}}; v)$ – loops around 0, 1, $\frac{1}{y}$ and ∞ respectively (see Picture 6).

We assume that

$$x_\infty \cdot x_y \cdot x_1 \cdot x_0 = 1.$$

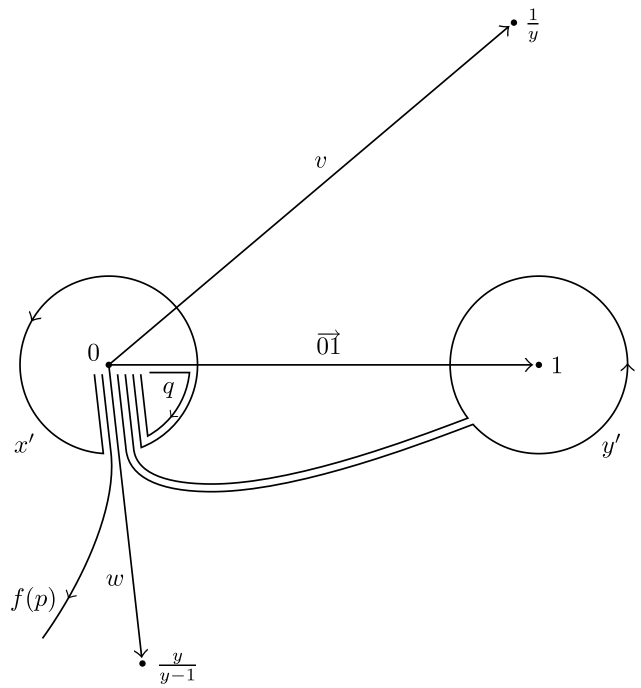
Let $z \in \hat{Y}(K)$ and let $p \in \pi(Y_{\bar{K}}; z, v)$. We introduce functions $\lambda(z)$, $\mu(z)$, $\nu(z)$, $\alpha(z)$, $\beta(z)$ and $\gamma(z)$ from G_K to $\mathbf{Z}_l$ by the following congruence

$$\text{(11.1.7)} \quad \mathfrak{f}_p \equiv x_0^{\lambda(z)} \cdot x_1^{\mu(z)} \cdot x_y^{\nu(z)} \cdot (x_1, x_0)^{\alpha(z)} \cdot (x_y, x_0)^{\beta(z)} \cdot (x_y, x_1)^{\gamma(z)} \mod \Gamma^3 \pi_1(Y_{\bar{K}}; v).$$

We recall that $f : Y \to V$ is given by $f(z) = \frac{(1-y)z}{z-1}$. Observe that $f_*(v) = w$, where w is a tangential base point at 0 corresponding to the local parameter $\frac{y}{y-1} \cdot z$ at 0. Let us set $x' := f_*(x_0)$ and $y' := f_*(x_y)$. Observe that $f(\infty) = 1 - y$. This implies that $f_*(x_\infty) = 1$. Therefore $f_*(x_1) = y'^{-1} \cdot x'^{-1}$. Let q be a path from $\overrightarrow{01}$ to $f_*(v)$ such that $q \cdot x \cdot q^{-1} = x'$ and $q \cdot y \cdot q^{-1} = y'$ (see Picture 7).

By the definition of functions $\mathfrak{k}$ and $\mathfrak{k}_i$ we have

$$\mathfrak{f}_{f(p)} \equiv x'^{\mathfrak{k}(f(p))} \cdot y'^{\mathfrak{k}_1(f(p))} \cdot (y', x')^{\mathfrak{k}_2(f(p))} \mod \Gamma^3 \pi_1(V_{\bar{K}}, f_*(v)).$$



Picture 7

Applying f_* to (11.1.7) we get

$$f_*\mathfrak{f}_p \equiv x'^{\lambda(z)-\mu(z)} \cdot y'^{\nu(z)-\mu(z)} \cdot (y', x')^{-\alpha(z)+\beta(z)-\gamma(z)+\frac{1}{2}\mu(z)^2+\frac{1}{2}\mu(z)} \mod \Gamma^3\pi_1(V_{\bar{K}}, f_*(v)).$$

The equality $f_*\mathfrak{f}_p = \mathfrak{f}_{f(p)}$ implies

$$\text{(11.1.8)} \qquad \mathfrak{k}(f(p)) = \lambda(z) - \mu(z), \quad \mathfrak{k}_1(f(p)) = \nu(z) - \mu(z)$$

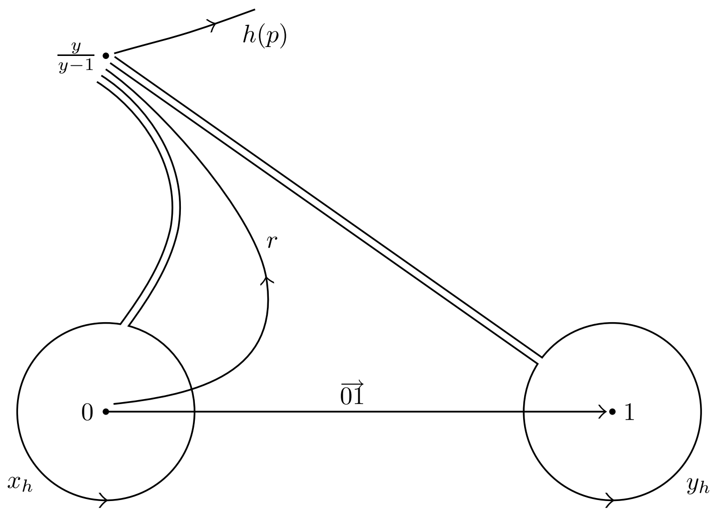
and

$$\text{(11.1.8)} \qquad \mathfrak{k}_2(f(p)) = -\alpha(z) + \beta(z) - \gamma(z) + \frac{1}{2}\mu(z)^2 + \frac{1}{2}\mu(z).$$

We recall that $g : Y \to V$ is given by $g(z) = yz$. Observe that $g_*(v) = \overrightarrow{01}$, $g_*(x_0) = x$, $g_*(x_1) = 1$ and $g_*(x_y) = y$. Comparing coefficients of $\mathfrak{f}_{g(p)}$ and $g_*\mathfrak{f}_p$ we get

$$\text{(11.1.9)} \qquad \mathfrak{k}(g(p)) = \lambda(z), \quad \mathfrak{k}_1(g(p)) = \nu(z), \quad \mathfrak{k}_2(g(p)) = \beta(z).$$

We recall that $h : Y \to V$ is given by $h(z) = \frac{(z-1)y}{1-y}$. Observe that $h_*(v) = \frac{y}{y-1}$. Let us set $x_h := h_*(x_1)$ and $y_h := h_*(x_y)$. Notice that $h_*(x_0) = 1$. Let r be a path from $\overrightarrow{01}$ to $\frac{y}{y-1}$ such that $r \cdot x \cdot r^{-1} = x_h$ and $r \cdot y \cdot r^{-1} = y_h$ (see Picture 8).



Picture 8

Comparing coefficients of $\mathfrak{f}_{h(p)}$ and $h_*\mathfrak{f}_p$ we get

$$\mathfrak{k}(h(p)) = \mu(z), \quad \mathfrak{k}_1(h(p)) = \nu(z), \quad \mathfrak{k}_2(h(p)) = \gamma(z). \tag{11.1.10}$$

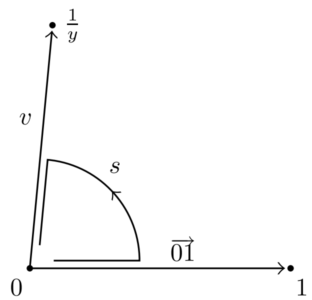
We recall that $k : Y \to V$ is given by $k(z) = z$. Observe that $k_*(v) = v$. Let us set $x_k := k_*(x_0)$ and $y_k := k_*(x_1)$. We have $k_*(x_y) = 1$. Let s be a path from $\overrightarrow{01}$ to v such that

$$s \cdot x \cdot s^{-1} = x_k \quad \text{and} \quad s \cdot y \cdot s^{-1} = y_k$$

(see Picture 9).

Comparing coefficients of $\mathfrak{f}_{k(p)}$ and $k_*\mathfrak{f}_p$ we get

$$\mathfrak{k}(k(p)) = \lambda(z), \quad \mathfrak{k}_1(k(p)) = \mu(z), \quad \mathfrak{k}_2(k(p)) = \alpha(z). \tag{11.1.11}$$



Picture 9

It follows from the equalities (11.1.8)–(11.1.11) that

(11.1.12)
$$\mathfrak{k}_2(f(p))-\frac{1}{2}\mathfrak{k}(f(p))\mathfrak{k}_1(f(p))-\mathfrak{k}_2(g(p))+\frac{1}{2}\mathfrak{k}(g(p))\mathfrak{k}_1(g(p))$$
$$+\mathfrak{k}_2(h(p))-\frac{1}{2}\mathfrak{k}(h(p))\mathfrak{k}_1(h(p))+\mathfrak{k}_2(k(p))-\frac{1}{2}\mathfrak{k}(k(p))\mathfrak{k}_1(k(p))=\frac{1}{2}\mu(z).$$

LEMMA 11.1.13. *On $G_{K(\mu_{l^\infty})}$ we have the following equality*

$$K(f(z),f_*(v))-K(g(z),g_*(v))+K(h(z),h_*(v))+K(k(z),k_*(v))=0.$$

Proof. We recall that $\kappa_0(z)=\kappa_z^0$ and $\kappa_1(z)=\kappa_z^1$. Hence we have

$$\begin{aligned}
&K(f(z),f_*(v))-K(g(z),g_*(v))+K(h(z),h_*(v))+K(k(z),k_*(v))\\
&\quad=K(f(z),w)-K(g(z),\overrightarrow{01})+K\Big(h(z),\frac{y}{y-1}\Big)+K(k(z),v)\\
&\quad=-\kappa_0\Big(\frac{(1-y)z}{z-1}\Big)\kappa_1(w)+\kappa_0(w)\kappa_1\Big(\frac{(1-y)z}{z-1}\Big)+\kappa_0(yz)\kappa_1(\overrightarrow{01})\\
&\qquad-\kappa_0(\overrightarrow{01})\kappa_1(yz)-\kappa_0\Big(\frac{(z-1)y}{1-y}\Big)\kappa_1\Big(\frac{y}{y-1}\Big)\\
&\qquad+\kappa_0\Big(\frac{y}{y-1}\Big)\kappa_1\Big(\frac{(z-1)y}{1-y}\Big)-\kappa_0(z)\kappa_1(v)+\kappa_0(v)\kappa_1(z).
\end{aligned}$$

One checks that $\kappa_0(\overrightarrow{0a})=\kappa_0(a)$ and $\kappa_1(\overrightarrow{0a})=0$. The lemma follows from the fact that $\kappa_0(x\cdot y)=\kappa_0(x)+\kappa_0(y)$ and $\kappa_1(z)=\kappa_0(1-z)$ on $G_{K(\mu_{l^\infty})}$.

THEOREM 11.1.14. *There are paths* ($\mathbf{Q}_2$*-paths if* $l=2$) *from* $\overrightarrow{01}$ *to points* $\frac{(1-y)z}{z-1}$, yz, $\frac{(z-1)y}{1-y}$, $\frac{y}{y-1}$ *and* z *such that on* $G_{K(\mu_{l^\infty})}$ *for* l*-adic dilogarithms calculated along these paths we have*

$$l_2\Big(\frac{(1-y)z}{z-1}\Big)-l_2(yz)+l_2\Big(\frac{(z-1)y}{1-y}\Big)-l_2\Big(\frac{y}{y-1}\Big)+l_2(z)=0.$$

Proof. It follows from Lemma 11.1.5, the equality (11.1.12) and Lemma 11.1.13 that

$$\begin{aligned}l_2(f(z))-l_2(f_*(v))-l_2(g(z))+l_2(g_*(v))\\ +l_2(h(z))-l_2(h_*(v))+l_2(k(z))-l_2(k_*(v))=\frac{1}{2}\mu(z).\end{aligned}$$

To eliminate $\frac{1}{2}\mu(z)$ we replace the path s by $s'=s\cdot x^{-1/2}$. Then $x_k'=s'\cdot x\cdot s'^{-1}=x_k$ and $y_k'=(s\cdot x^{-1/2})\cdot y\cdot(s\cdot x^{-1/2})^{-1}=s\cdot y\cdot(y,x)^{1/2}\cdot s^{-1}=y_k\cdot(y_k,x_k)^{1/2}$. In terms of generators x_k' and y_k' of $\pi_1(V_{\bar K};v)$ we have

$$\mathfrak{k}_2(k(p))=\alpha(z)-\frac{1}{2}\mu(z).$$

Observe that $l_2(\overrightarrow{0a})=0$. Hence we get

$$l_2\Big(\frac{(1-y)z}{z-1}\Big)-l_2(yz)+l_2\Big(\frac{(z-1)y}{1-y}\Big)-l_2\Big(\frac{y}{y-1}\Big)+l_2(z)=0$$

for l-adic dilogarithms calculated along the paths $f(p)\cdot q$, $g(p)$, $h(r)\cdot r$, r and $k(p)\cdot s\cdot x^{-1/2}$ respectively.

It would be interesting to choose paths in such a way that we get the Abel equation on G_K without lower degree terms.

11.2. Now we shall discuss functional equations of arbitrary l-adic polylogarithms. The next result is a corollary of Theorem 10.0.7. We recall that a subgroup G_{n+1} of $\pi_1(V_{\bar{\mathbf{Q}}};\overrightarrow{01})$ was defined at the end of Subsection 11.0.

We are not able to show that after a suitable choice of paths l-adic polylogarithms satisfy functional equations without lower degree terms. We have only the following result.

THEOREM 11.2.1. *Let* K *be a number field and let* $V=\mathbf{P}^1_K\setminus\{0,1,\infty\}$. *Let* $a_1,\dots,a_{m+1}$ *be* K*-points of* $\mathbf{P}^1_K$ *and let* $Y=\mathbf{P}^1_K\setminus\{a_1,\dots,a_{m+1}\}$. *Let*

$n_i \in \mathbf{Z}$ for $i = 1, \dots, N$ and let $f_i : Y \to V$ be regular maps defined over K for $i = 1, \dots, N$. Let $z, v \in \hat{Y}(K)$. Let us assume that $\sum_{i=1}^{N} n_i (f_i)_* = 0$ in

$$\mathrm{Hom}(\Gamma^n \pi_1(Y_{\bar{K}}; v)/\Gamma^{n+1}\pi_1(Y_{\bar{K}}; v); \Gamma^n \pi_1(V_{\bar{\mathbf{Q}}}; \overrightarrow{01})/G_{n+1}).$$

Then we have a functional equation

$$\sum_{i=1}^{N} n_i(\mathcal{L}_n(f_i(z)) - \mathcal{L}_n(f_i(v))) = 0$$

on the subgroup $H_n(Y; z, v)$ of G_K.

Proof. The theorem follows from Theorem 10.0.7 and Proposition 11.0.15.

COROLLARY 11.2.2. *Let ξ_m be a primitive m-th root of 1. Then we have*

$$m^{n-1}\Big(\sum_{k=0}^{m-1} \mathcal{L}_n(\xi_m^k z)\Big) = \mathcal{L}_n(z^m)$$

on the subgroup $H_n(\mathbf{P}^1_{\mathbf{Q}(\mu_m)} \setminus \{0, \mu_m, \infty\}; z, \overrightarrow{01})$ of $G_{\mathbf{Q}(\mu_m)}$.

In Part III we shall need a special case of the equality from Corollary 11.2.2.

COROLLARY 11.2.3. *Let ξ_m be a primitive m-th root of 1. Then we have*

$$m^{n-1}\Big(\sum_{k=0}^{m-1} \mathcal{L}_n(\xi_m^k)\Big) = \mathcal{L}_n(1)$$

on the subgroup $H_n(\mathbf{P}_{\mathbf{Q}(\mu_m)} \setminus \{0, \mu_m, \infty\}; \overrightarrow{10}, \overrightarrow{01})$ of $G_{\mathbf{Q}(\mu_m)}$, where $\mathcal{L}_n(1) := \mathcal{L}_n(\overrightarrow{10})$.

Both corollaries follow immediately from Theorem 11.2.1. We give however a detailed proof of Corollary 11.2.3 because of it importance in Part III.

Proof of Corollary 11.2.3. We shall use the notation of Subsection 11.1, where we discussed the l-adic analog of the functional equation $Li_2(z^m) = m\big(\sum_{i=0}^{m} Li_2(\xi_m^i z)\big)$. We shall use also the following notation. If a and b are elements of a group then $(a, b^1) := (a, b) = a \cdot b \cdot a^{-1} \cdot b^{-1}$ and $(a, b^n) := ((a, b^{n-1}), b)$ for $n > 1$.

We recall that $Y = \mathbf{P}^1_{\mathbf{Q}(\mu_m)} \setminus \{0, \mu_m, \infty\}$ and $f : Y \to V$ is given by $f(z) = z^m$. Let p be a path from $\overrightarrow{01}$ to $\overrightarrow{10}$, the interval $[0, 1]$. Let $\sigma \in H_n(Y; \overrightarrow{10}, \overrightarrow{01})$. Then we have

$$\mathfrak{f}_p(\sigma) \equiv (y_0, x'^{n-1})^{\nu_0^n(\overrightarrow{10})(\sigma)} \cdot \dots \cdot (y_{m-1}, x'^{n-1})^{\nu_{m-1}^n(\overrightarrow{10})(\sigma)}$$

modulo a subgroup generated by $\Gamma^{n+1}\pi_1(Y_{\bar{K}}; \overrightarrow{01})$ and commutators which contain at least two y's. Observe that $f(p)$ is a path from $\overrightarrow{01}$ to $m \cdot \overrightarrow{10}$. Then for any $\sigma \in H_n(V; \overrightarrow{10}, \overrightarrow{01})$, and therefore also for any $\sigma \in H_n(Y; \overrightarrow{10}, \overrightarrow{01})$ we have

$$\mathfrak{f}_{f(p)}(\sigma) \equiv (y, x^{n-1})^{\kappa^n_{\overrightarrow{10}}(\sigma)} \mod G_{n+1}.$$

It follows from the equality $f_* \mathfrak{f}_p = \mathfrak{f}_{f(p)}$ that

$$m^{n-1}(\nu_0^n(\overrightarrow{10}) + \dots + \nu_{m-1}^n(\overrightarrow{10})) = \kappa^n_{\overrightarrow{10}} \tag{11.2.4}$$

on $H_n(Y; \overrightarrow{10}, \overrightarrow{01})$. We recall that $f_i : Y \to V$ is given by $f_i(z) = \xi_m^{-i} \cdot z$. Observe that $(f_i)_* \mathfrak{f}_{pq_i}(\sigma) \equiv (y, x^{n-1})^{\nu_i(\overrightarrow{10})(\sigma)} \mod G_{n+1}$ for $\sigma \in H_n(Y; \overrightarrow{10}, \overrightarrow{0\xi_m^i}) = H_n(Y; \overrightarrow{10}, \overrightarrow{01})$ and $\mathfrak{f}_{f_i(pq_i)}(\sigma) \equiv (y, x^{n-1})^{\kappa^n_{\xi_m^{-i}}(\sigma)} \mod G_{n+1}$ for $\sigma \in H_n(Y; \overrightarrow{\xi_n^{-i}0}, \overrightarrow{01}) = H_n(Y; \overrightarrow{10}, \overrightarrow{01})$, where q_i is a path from $\overrightarrow{0\xi_m^i}$ to $\overrightarrow{01}$ as on Picture 4. Hence we get

$$\nu_i^n(\overrightarrow{10}) = \kappa^n_{\xi_m^{-i}} \tag{11.2.5}$$

on $H_n(Y; \overrightarrow{\xi_n^{-i}0}, \overrightarrow{01}) = H_n(Y; \overrightarrow{10}, \overrightarrow{01})$. It follows from (11.2.4) and (11.2.5) that

$$m^{n-1}\left(\sum_{i=0}^{m-1} \kappa^n_{\xi_m^{-i}}\right) = \kappa^n_{\overrightarrow{10}}$$

on $H_n(Y; \overrightarrow{\xi_n^{-i}0}, \overrightarrow{01}) = H_n(Y; \overrightarrow{10}, \overrightarrow{01})$. For $\sigma \in H_n(Y; \overrightarrow{10}, \overrightarrow{01})$ we have $\kappa^n_{\xi_m^{-i}}(\sigma) = \mathcal{L}_n(\xi_m^{-i})(\sigma)$ and $\kappa^n_{\overrightarrow{10}}(\sigma) = \mathcal{L}_n(\overrightarrow{10})(\sigma)$. This finishes the proof of Corollary 11.2.3.

One of the most useful functional equations of classical polylogarithms is the relation between $Li_n(z)$ and $Li_n(\frac{1}{z})$. For l-adic polylogarithms we have the following result.

Corollary 11.2.6. *For any $z \in V(K)$, we have*

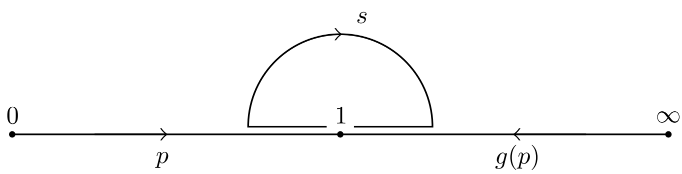
$$\mathcal{L}_n(z) + (-1)^n \mathcal{L}_n\Big(\frac{1}{z}\Big) = 0$$

on the subgroup $H_n(V_{\bar{\mathbf{Q}}}; z, \overrightarrow{01})$.

Proof. It follows from Theorem 11.2.1 that

$$\mathcal{L}_n(z) - \mathcal{L}_n(\overrightarrow{01}) + (-1)^n \Big(\mathcal{L}_n\Big(\frac{1}{z}\Big) - \mathcal{L}_n(\overrightarrow{\infty 1})\Big) = 0.$$

$\mathcal{L}_n(\overrightarrow{01})$ vanishes. Hence we have to calculate $\mathcal{L}_n(\overrightarrow{\infty 1})$. Let p a path from $\overrightarrow{01}$ to $\overrightarrow{10}$ and let s a path from $\overrightarrow{10}$ to $\overrightarrow{1\infty}$ as on the picture.



Picture 10

Let $g : V \to V$ be given by $g(z) = \frac{1}{z}$. Let us set $q := g(p)^{-1} \cdot s \cdot p$. We denote by π'' the subgroup $[\Gamma^2\pi_1(V_{\bar{K}}; \overrightarrow{01}), \Gamma^2\pi_1(V_{\bar{K}}; \overrightarrow{01})]$ of $\pi_1(V_{\bar{K}}; \overrightarrow{01})$. Let $(\Gamma^{n+1}\pi_1(V_{\bar{K}}; \overrightarrow{01}), \pi'')$ be a normal subgroup of $\pi_1(V_{\bar{K}}; \overrightarrow{01})$ generated by $\Gamma^{n+1}\pi_1(V_{\bar{K}}; \overrightarrow{01})$ and π''.

Let $\sigma \in H_n(V_{\bar{\mathbf{Q}}}; z, \overrightarrow{01})$. Then we have

$$\text{(11.2.7)} \quad \mathfrak{f}_q(\sigma) = \prod_{i+j=n,\, i\geq 1,\, j\geq 1} (((y,x)x^{i-1})y^{j-1})^{\kappa_{i,j}(\overrightarrow{\infty 1})(\sigma)} \mod (\Gamma^{n+1}\pi_1(V_{\bar{K}}, \overrightarrow{01}), \pi'').$$

for some $\kappa_{i,j}(\overrightarrow{\infty 1})(\sigma) \in \mathbf{Z}_l$. It follows from Lemma 1.0.6 and from equality (10.0.1) that

$$\text{(11.2.8)} \qquad \mathfrak{f}_q = q^{-1} \cdot g_*(\mathfrak{f}_p)^{-1} \cdot q \cdot p^{-1} \cdot \mathfrak{f}_s \cdot p \cdot \mathfrak{f}_p.$$

Observe that

$$\text{(11.2.9)} \qquad q^{-1} \cdot g_*(y) \cdot q = y \quad \text{and} \quad q^{-1} \cdot g_*(x) \cdot q = x^{-1} \cdot y^{-1}.$$

Let $\sigma \in H_n(V_{\bar{\mathbf{Q}}}; z, \overrightarrow{01})$. Then we have

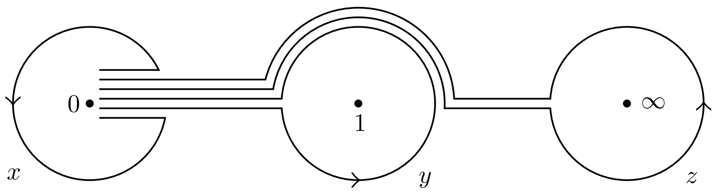
$$(11.2.10)\quad \mathfrak{f}_p(\sigma) = \prod_{i+j=n,\, i\geq 1,\, j\geq 1} (((y,x)x^{i-1})y^{j-1})^{\kappa_{i,j}(\overrightarrow{10})(\sigma)} \mod (\Gamma^{n+1}\pi_1(V_{\bar{K}}, \overrightarrow{01}), \pi'')$$

for some $\kappa_{i,j}(\overrightarrow{10})(\sigma) \in \mathbf{Z}_l$. It follows from (11.2.7)–(11.2.10) that

$$\kappa_{n-1,1}(\overrightarrow{\infty 1}) = (-1)^n \kappa_{n-1,1}(\overrightarrow{10})(\sigma) + \kappa_{n-1,1}(\overrightarrow{10})(\sigma).$$

Hence $\kappa_{n-1,1}(\overrightarrow{\infty 1}) = 0$ if n is odd.

We shall show that $\kappa_{n-1,1}(\overrightarrow{10})$ vanishes for n even. Let x, y and z be generators of $\pi_1(V_{\bar{K}}; \overrightarrow{01})$ as on the picture.



Picture 11

Then we have $z \cdot y \cdot x = 1$. It follows from Proposition 2.2.1 that

$$(\mathfrak{f}_q(\sigma)(x,y))^{-1} \cdot z^{\chi(\sigma)} \cdot (\mathfrak{f}_q(\sigma)(x,y)) \cdot (\mathfrak{f}_p(\sigma)(x,y))^{-1} \cdot y^{\chi(\sigma)} \cdot (\mathfrak{f}_p(\sigma)(x,y)) \cdot x^{\chi(\sigma)} = 1.$$

Let $\sigma \in H_n(V_{\bar{K}}; z, \overrightarrow{01})$. It follows from (11.2.8) and (11.2.9) that

$$\mathfrak{f}_q(\sigma)(x,y) = (\mathfrak{f}_p(\sigma)(x^{-1}y^{-1}, y))^{-1} \cdot (\mathfrak{f}_p(\sigma)(x,y)).$$

Hence we get

$$(\mathfrak{f}_p(\sigma)(x,y))^{-1} \cdot (\mathfrak{f}_p(\sigma)(x^{-1}y^{-1}, y)) \cdot x^{-1} \cdot y^{-1} \cdot (\mathfrak{f}_p(\sigma)(x^{-1}y^{-1}, y))^{-1} \cdot (\mathfrak{f}_p(\sigma)(x,y)) \cdot (\mathfrak{f}_p(\sigma)(x,y))^{-1} \cdot y \cdot (\mathfrak{f}_p(\sigma)(x,y)) \cdot x = 1.$$

Comparing exponents at (y, x^n) we get $(1 + (-1)^n)\kappa_{n-1,1}(\overrightarrow{10}) = 0$. Hence $\kappa_{n-1,1}(\overrightarrow{10}) = 0$ for n even (see also [I1], [I2] and [D], where the element $\mathfrak{f}_p(\sigma)$ is studied). Therefore $\kappa_{n-1,1}(\overrightarrow{\infty 1}) = 0$ for any n. The equality $\kappa_{n-1,1}(\overrightarrow{\infty 1}) = \mathcal{L}_n(\overrightarrow{\infty 1})$ implies the corollary.

The fact that $\kappa_{n-1,1}(\overrightarrow{10})$ vanishes for n even implies the following well known result.

COROLLARY 11.2.11.

$$\mathcal{L}_{2n}(\overrightarrow{10}) = 0.$$

§12. Monodromy of l-adic iterated integrals and l-adic polylogarithms

12.0. We shall show here that suitable defined l-adic polylogarithms form a local system with the similar shape of the monodromy representation as the local system of classical polylogarithms given in [BD]. We start with the discussion of the monodromy of arbitrary l-adic iterated integrals. The notation is the same as in Section 10.

Let p be a path from v to z on $X_{\bar{K}}$ and let $S \in \pi_1(X_{\bar{K}}; v)$. Then we have

$$\mathfrak{f}_{pS}(\sigma) = S^{-1} \cdot \mathfrak{f}_p(\sigma) \cdot S \cdot \mathfrak{f}_S(\sigma). \tag{12.0.0}$$

Let $\mathrm{Map}(G_K; \pi_1(X_{\bar{K}}; v))$ be the set of all maps from G_K to $\pi_1(X_{\bar{K}}; v)$. We define a map

$$r_{z,v;p} : \pi_1(X_{\bar{K}}; v) \longrightarrow \mathrm{Aut}_{set}(\mathrm{Map}(G_K; \pi_1(X_{\bar{K}}; v)))$$

setting

$$r_{z,v;p}(S)(w)(\sigma) := S^{-1} \cdot w(\sigma) \cdot S \cdot \mathfrak{f}_S(\sigma),$$

for $S \in \pi_1(X_{\bar{K}}; v)$, $w \in \mathrm{Map}(G_K; \pi_1(X_{\bar{K}}; v))$ and $\sigma \in G_K$.

Further we drop the indices $_{z,v;p}$ to simplify the notation.

LEMMA 12.0.1. *The map $r_{z,v;p}$ is a representation of $\pi_1(X_{\bar{K}}; v)$.*

Proof. Let $S, T \in \pi_1(X_{\bar{K}}; v)$. We have $r(T)(r(S)w)(\sigma) = T^{-1}(S^{-1} \cdot w(\sigma) \cdot S \cdot \mathfrak{f}_S(\sigma)) \cdot T \cdot \mathfrak{f}_T(\sigma) = (S \cdot T)^{-1} \cdot w(\sigma) \cdot (S \cdot T) \cdot (T^{-1} \cdot \mathfrak{f}_S(\sigma) \cdot T \cdot \mathfrak{f}_T(\sigma)) = (S \cdot T)^{-1} \cdot w(\sigma) \cdot (S \cdot T) \cdot \mathfrak{f}_{ST}(\sigma) = r(S \cdot T)(w)(\sigma)$. We recall that in our notation $S \cdot T$ means that first we go along T and then along S. Therefore r is a representation of $\pi_1(X_{\bar{K}}; v)$.

We recall that $k_x : \pi_1(X_{\bar{K}}; v) \to \mathbf{Q}_l\{\{\mathbf{X}\}\}$ is a continous multiplicative embedding given by $k_x(x_i) = e^{X_i}$ for $i = 1, \ldots, n$ and that for a path p from v to z we set $\Lambda_p(\sigma) := k_x(\mathfrak{f}_p(\sigma))$.

Let $\mathrm{Map}(G_K; \mathbf{Q}_l\{\{\mathbf{X}\}\})$ be the set of all maps from G_K to $\mathbf{Q}_l\{\{\mathbf{X}\}\}$. Observe that $\mathrm{Map}(G_K; \mathbf{Q}_l\{\{\mathbf{X}\}\})$ is a vector space over $\mathbf{Q}_l$. We denote by $GL(\mathrm{Map}(G_K; \mathbf{Q}_l\{\{\mathbf{X}\}\}))$ the group of linear automorphisms of the vector space $\mathrm{Map}(G_K; \mathbf{Q}_l\{\{\mathbf{X}\}\})$.

Let us define a map

$$R_{z,v;p} : \pi_1(X_{\bar{K}}; v) \longrightarrow GL(\mathrm{Map}(G_K; \mathbf{Q}_l\{\{\mathbf{X}\}\}))$$

setting

$$R_{z,v;p}(S)(W)(\sigma) := k_x(S)^{-1} \cdot W(\sigma) \cdot k_x(S) \cdot \Lambda_S(\sigma).$$

Proposition 12.0.2. *The map $R_{z,v;p}$ is a representation of $\pi_1(X_{\bar{K}}; v)$.*

Proof. To simplify the notation let us set $R = R_{z,v;p}$. Let $S, T \in \pi_1(X_{\bar{K}}; v)$. We have $R(T)(R(S)(W))(\sigma) = k_x(T)^{-1} \cdot (R(S)(W)(\sigma)) \cdot k_x(T) \cdot \Lambda_T(\sigma) = k_x(T)^{-1} \cdot (k_x(S)^{-1} \cdot W(\sigma) \cdot k_x(S) \cdot \Lambda_S(\sigma)) \cdot k_x(T) \cdot \Lambda_T(\sigma) = k_x(S \cdot T)^{-1} \cdot W(\sigma) \cdot k_x(S \cdot T) \cdot k_x(T)^{-1} \cdot \Lambda_S(\sigma) \cdot k_x(T) \cdot \Lambda_T(\sigma) = R(S \cdot T)(W)(\sigma)$.

It follows from Lemma 10.3.1 that the embedding $k_x : \pi_1(X_{\bar{K}}; v) \to \mathbf{Q}_l\{\{\mathbf{X}\}\}$ extends uniquely to a continous multiplicative embedding $\bar{k}_x : \pi_1(X_{\bar{K}}; v) \otimes \mathbf{Q} \to \mathbf{Q}_l\{\{\mathbf{X}\}\}$.

Proposition 12.0.3. *The representation $R_{z,v;p}$ extends to the representation*

$$\bar{R}_{z,v;p} : \pi_1(X_{\bar{K}}; v) \otimes \mathbf{Q} \longrightarrow GL(\mathrm{Map}(G_K; \mathbf{Q}_l\{\{\mathbf{X}\}\})).$$

Let $S \in \pi_1(X_{\bar{K}}; v) \otimes \mathbf{Q}$. Then we have

$$\bar{R}_{z,v;p}(S)(W)(\sigma) = \bar{k}_x(S)^{-1} \cdot W(\sigma) \cdot \bar{k}_x(S) \cdot \bar{R}_{z,v;p}(S)(1)(\sigma).$$

Proof. We define an increasing filtration $\{\mathcal{W}_{-i}\}_{i \in \mathbf{N}}$ of the $\mathbf{Q}_l$-vector space $\mathrm{Map}(G_K; \mathbf{Q}_l\{\{\mathbf{X}\}\})$ setting

$$\mathcal{W}_{-2k} = \mathcal{W}_{-2k-1} \text{ to be a set of all maps from } G_K \text{ to } I^k,$$

where I^k is a k-th power of the augmentation ideal of $\mathbf{Q}_l\{\{\mathbf{X}\}\}$. Let $S \in \pi_1(X_{\bar{K}}; v)$ and let $W \in \mathcal{W}_{-2k}$. Then we have

$$R_{z,v;p}(S)(W) \equiv W \mod \mathcal{W}_{-2(k+1)}.$$

Hence the image of $R_{z,v;p}$ is in the subgroup of pro-unipotent automorphisms of the vector space $\mathrm{Map}(G_K; \mathbf{Q}_l\{\{\mathbf{X}\}\})$. This implies that the representation $R_{z,v;p}$ extends to the representation

$$\bar{R}_{z,v;p} : \pi_1(X_{\bar{K}}; v) \otimes \mathbf{Q} \to GL(\mathrm{Map}(G_K; \mathbf{Q}_l\{\{\mathbf{X}\}\})).$$

Let $S \in \pi_1(X_{\bar{K}}; v) \otimes \mathbf{Q}$ be such that $S^{l^m} \in \pi_1(X_{\bar{K}}; v)$. Then we have

$$R_{z,v;p}(S^{l^m})(W)(\sigma) = k_x(S^{l^m})^{-1} \cdot W(\sigma) \cdot k_x(S^{l^m}) \cdot \Lambda_{S^{l^m}}(\sigma),$$

where $\Lambda_{S^{l^m}}(\sigma) = R_{z,v;p}(S^{l^m})(1)(\sigma)$. This implies that

$$\bar{R}_{z,v;p}(S)(W)(\sigma) = \bar{k}_x(S)^{-1} \cdot W(\sigma) \cdot \bar{k}_x(S) \cdot \bar{R}_{z,v;p}(S)(1)(\sigma).$$

The elements $S \in \pi_1(X_{\bar{K}}; v) \otimes \mathbf{Q}$ such that $S^{l^m} \in \pi_1(X_{\bar{K}}; v)$ for some m are dense in $\pi_1(X_{\bar{K}}; v) \otimes \mathbf{Q}$ hence the last formula holds for any $S \in \pi_1(X_{\bar{K}}; v) \otimes \mathbf{Q}$. This finishes the proof of the proposition.

12.1. Now we shall study monodromy of l-adic polylogarithms, more exactly, we shall study monodromy of coefficients at $X^{n-1}Y$ of the power series $\Lambda_p(\sigma)$. Let $V = \mathbf{P}^1_K \setminus \{0, 1, \infty\}$ and let p be a path from $\overrightarrow{01}$ to z. From now on the notation is the same as in Subsection 11.0.

We define functions $\lambda_i(z)_p$, $\mu_j(z)_p$ and $\nu_{i,j}(z)_p$ from G_K to $\mathbf{Q}_l$ by the congruence

$$\Lambda_p(\sigma) \equiv 1 + \sum_{k=1}^{\infty} \frac{(l(z)_p(\sigma))^k}{k!} X^k + \sum_{i=1}^{\infty} \lambda_i(z)_p(\sigma) X^{i-1} Y + \sum_{j=2}^{\infty} \mu_j(z)_p(\sigma) Y X^{j-1} + \sum_{i,j=1}^{\infty} \nu_{i,j}(z)_p(\sigma) X^i Y X^j$$

modulo the ideal generated by monomials with at least two Y's.

The function $\lambda_1(z)_p = l_1(z)_p$ and the l-adic polylogarithms $l_k(z)_p$ can be expressed by the function $\lambda_k(z)_p$ and the functions $l(z)_p$ and $\lambda_i(z)_p$ with $i < k$.

Proposition 12.1.1. *The monodromy transformation of functions $l(z)_p$ and $\lambda_n(z)_p$ is as follows*:

$$x : l(z)_p \longrightarrow l(z)_p + (\chi - 1), \quad \lambda_n(z)_p \longrightarrow \lambda_n(z)_p + \sum_{i=1}^{n-1} \frac{(-1)^{n-i}}{(n-i)!} \lambda_i(z)_p,$$

$$\mu_n(z)_p \longrightarrow \mu_n(z)_p + \sum_{i=2}^{n-1} \frac{\chi^{n-i}}{(n-i)!} \mu_i(z)_p + \frac{\chi^{n-1}}{(n-1)!} \lambda_1(z)_p$$

and

$$y : l(z)_p \longrightarrow l(z)_p, \quad \lambda_1(z)_p \longrightarrow \lambda_1(z)_p + (\chi - 1),$$
$$\lambda_n(z)_p \longrightarrow \lambda_n(z)_p + \chi \frac{(l(z)_p)^{n-1}}{(n-1)!}$$

for $n > 1$ *and* $\mu_n(z)_p \to \mu_n(z)_p - \frac{(l(z)_p)^{n-1}}{(n-1)!}$.

Proof. The proposition follows from the formula

$$\Lambda_{p \cdot S}(\sigma) = k(S)^{-1} \cdot \Lambda_p(\sigma) \cdot k(S) \cdot \Lambda_S(\sigma),$$

which for $S = x$ gives

$$\Lambda_{p \cdot x}(\sigma) = e^{-X} \cdot \Lambda_p(\sigma) \cdot e^{\chi(\sigma)X}.$$

For $S = y$ the formula is more complicated, however when we restrict our attention to coefficients with only one Y then the formula have the same simple form.

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