

HILBERT-KUNZ MULTIPLICITY OF THREE-DIMENSIONAL LOCAL RINGS

KEI-ICHI WATANABE AND KEN-ICHI YOSHIDA

Abstract. In this paper, we investigate the lower bound $s_{\text{HK}}(p, d)$ of Hilbert-Kunz multiplicities for non-regular unmixed local rings of Krull dimension d containing a field of characteristic $p > 0$. Especially, we focus on three-dimensional local rings. In fact, as a main result, we will prove that $s_{\text{HK}}(p, 3) = 4/3$ and that a three-dimensional complete local ring of Hilbert-Kunz multiplicity $4/3$ is isomorphic to the non-degenerate quadric hypersurface $k[[X, Y, Z, W]]/(X^2 + Y^2 + Z^2 + W^2)$ under mild conditions.

Furthermore, we pose a generalization of the main theorem to the case of $\dim A \geq 4$ as a conjecture, and show that it is also true in case $\dim A = 4$ using the similar method as in the proof of the main theorem.

Introduction

Let A be a commutative Noetherian ring containing an infinite field of characteristic $p > 0$ with unity. In [15], Kunz proved the following theorem, which gives a characterization of regular local rings of positive characteristic.

KUNZ' THEOREM. ([15]) *Let (A, \mathfrak{m}, k) be a local ring of characteristic $p > 0$. Then the following conditions are equivalent:*

- (1) *A is a regular local ring.*
- (2) *A is reduced and is flat over the subring $A^p = \{a^p : a \in A\}$. In other words, the Frobenius map $F : A \rightarrow A$ ($a \mapsto a^p$) is flat.*
- (3) *$l_A(A/\mathfrak{m}^{[q]}) = q^d$ for any $q = p^e$, $e \geq 1$, where $\mathfrak{m}^{[q]} = (a^q : a \in \mathfrak{m})$ and $l_A(M)$ denotes the length of an A -module M .*

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Furthermore, in [16], Kunz observed that $l_A(A/\mathfrak{m}^{[q]})/q^d$ ($q = p^e$) is a reasonable measure for the singularity of a local ring. Based on the idea of Kunz, Monsky [18] proved that there exists a constant $c = c(A)$ such that

$$l_A(A/\mathfrak{m}^{[q]}) = cq^d + O(q^{d-1})$$

and defined the notion of *Hilbert-Kunz multiplicity* by $e_{\text{HK}}(A) = c$. In 1990's, Han and Monsky [10] have given an algorithm to compute the Hilbert-Kunz multiplicity for any hypersurface of Briskorn-Fermat type

$$A = k[X_0, \dots, X_n]/(X_0^{d_0} + \dots + X_n^{d_n}).$$

See e.g. [1], [2], [4], [24] about the other examples. Hochster and Huneke [11] have given a ‘‘Length Criterion for Tight Closure’’ in terms of Hilbert-Kunz multiplicity (see Theorem 1.8) and indicated the close relation between tight closure and Hilbert-Kunz multiplicity. In [22], the authors proved a theorem which gives a characterization of regular local rings in terms of Hilbert-Kunz multiplicity:

THEOREM A. ([22, Theorem 1.5]) *Let (A, \mathfrak{m}, k) be an unmixed local ring of positive characteristic. Then A is regular if and only if $e_{\text{HK}}(A) = 1$.*

Many researchers have tried to improve this theorem. For example, Blickle and Enescu [3] recently proved the following theorem:

THEOREM B. (Blickle-Enescu [3]) *Let (A, \mathfrak{m}, k) be an unmixed local ring of characteristic $p > 0$. Then the following statements hold:*

- (1) *If $e_{\text{HK}}(A) < 1 + \frac{1}{d!}$, then A is Cohen-Macaulay and F -rational.*
- (2) *If $e_{\text{HK}}(A) < 1 + \frac{1}{p^a d!}$, then A is regular.*

So it is natural to consider the following problem:

PROBLEM C. *Let $d \geq 2$ be any integer. Determine the lower bound $(s_{\text{HK}}(p, d))$ of Hilbert-Kunz multiplicities for d -dimensional non-regular unmixed local rings of characteristic p . Also, characterize the local rings A for which $e_{\text{HK}}(A) = s_{\text{HK}}(p, d)$ holds.*

In case of one-dimensional local rings, it is easy to answer to this problem. In fact, $s_{\text{HK}}(p, 1) = 2$; $e_{\text{HK}}(A) = 2$ if and only if $e(A) = 2$. In case of two-dimensional Cohen-Macaulay local rings, the authors [23] have given a complete answer to this problem. Namely, we have $s_{\text{HK}}(p, 2) = \frac{3}{2}$ by the theorem below.

THEOREM D. (see also Corollary 2.6) *Let (A, \mathfrak{m}, k) be a two-dimensional Cohen-Macaulay local ring of positive characteristic. Put $e = e(A)$, the multiplicity of A . Then the following statements hold:*

- (1) $e_{\text{HK}}(A) \geq \frac{e+1}{2}$.
- (2) *Suppose that $k = \bar{k}$. Then $e_{\text{HK}}(A) = \frac{e+1}{2}$ holds if and only if the associated graded ring $\text{gr}_{\mathfrak{m}}(A)$ is isomorphic to the Veronese subring $k[X, Y]^{(e)}$.*

In the following, let us explain the organization of this paper. In Section 1, we recall the notions of Hilbert-Kunz multiplicity and tight closure etc. and gather several fundamental properties of them. In particular, Goto-Nakamura's theorem (Theorem 1.9) is important because it plays a central role in the proof of the main result (Theorem 3.1).

In Section 2, we give a key result to estimate Hilbert-Kunz multiplicities for local rings of lower dimension. Indeed, Theorem 2.2 is a refinement of the argument in [23, Section 2]. Also, the point of our proof is to estimate $l_A(\mathfrak{m}^{[q]}/J^{[q]})$ (where J is a minimal reduction of \mathfrak{m}) using volumes in \mathbb{R}^d .

In Section 3, we prove the following theorem as the main result in this paper.

THEOREM 3.1. *Let (A, \mathfrak{m}, k) be a three-dimensional unmixed local ring of characteristic $p > 0$. Then the following statements hold.*

- (1) *If A is not regular, then $e_{\text{HK}}(A) \geq \frac{4}{3}$.*
- (2) *Suppose that $k = \bar{k}$ and $\text{char } k \neq 2$. Then the following conditions are equivalent:*
 - (a) $e_{\text{HK}}(A) = \frac{4}{3}$.
 - (b) $\widehat{A} \cong k[[X, Y, Z, W]]/(X^2 + Y^2 + Z^2 + W^2)$.

Also, we study lower bounds on $e_{\text{HK}}(A)$ for local rings A having a given (small) multiplicity e . In particular, we will prove that any three-dimensional unmixed local ring A with $e_{\text{HK}}(A) < 2$ is F -rational.

In Section 4, we consider a generalization of Theorem 3.1 and pose the following conjecture:

CONJECTURE 4.2. *Let $d \geq 1$ be an integer and $p > 2$ a prime number. Put*

$$A_{p,d} := \overline{\mathbb{F}_p}[[X_0, X_1, \dots, X_d]]/(X_0^2 + \dots + X_d^2).$$

Let (A, \mathfrak{m}, k) be a d -dimensional unmixed local ring with $k = \overline{\mathbb{F}_p}$. Then the following statements hold.

- (1) If A is not regular, then $e_{\text{HK}}(A) \geq e_{\text{HK}}(A_{p,d}) \geq 1 + \frac{cd}{d!}$ (see 4.2 for the definition of c_d). In particular, $s_{\text{HK}}(p, d) = e_{\text{HK}}(A_{p,d})$.
- (2) If $e_{\text{HK}}(A) = e_{\text{HK}}(A_{p,d})$, then the \mathfrak{m} -adic completion \widehat{A} of A is isomorphic to $A_{p,d}$ as local rings.

Also, we prove that this is true in case of $\dim A = 4$. Namely we will prove the following theorem.

THEOREM 4.3. *Let (A, \mathfrak{m}, k) be a four-dimensional unmixed local ring of characteristic $p > 0$. Also, suppose that $k = \overline{k}$ and $\text{char } k \neq 2$. Then $e_{\text{HK}}(A) \geq \frac{5}{4}$ if $e(A) \geq 3$. Also, the following statements hold.*

- (1) If A is not regular, then $e_{\text{HK}}(A) \geq e_{\text{HK}}(A_{p,4}) = \frac{29p^2+15}{24p^2+12}$.
- (2) The following conditions are equivalent:
 - (a) Equality holds in (1).
 - (b) $e_{\text{HK}}(A) < \frac{5}{4}$.
 - (c) \widehat{A} is isomorphic to $A_{p,4}$.

§1. Preliminaries

Throughout this paper, let A be a commutative Noetherian ring with unity. Furthermore, we assume that A has a positive characteristic p , that is, it contains a prime field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$, unless otherwise specified. For every positive integer e , let $q = p^e$. If I is an ideal of A , then $I^{[q]} = (a^q : a \in I)A$. Also, we fix the following notation: $l_A(M)$ (resp. $\mu_A(M)$) denotes the length (resp. the minimal number of generators) of M for any finitely generated A -module M .

First, we recall the notion of Hilbert-Kunz multiplicity (see [15], [16], [18]). Also, see [17] or [20] for usual multiplicity.

DEFINITION 1.1. (multiplicity, Hilbert-Kunz multiplicity) Let (A, \mathfrak{m}, k) be a local ring of characteristic $p > 0$ with $\dim A = d$. Let I be an \mathfrak{m} -primary ideal of A , and let M be a finitely generated A -module. The (Hilbert-Samuel) multiplicity $e(I, M)$ of I with respect to M is defined by

$$e(I, M) = \lim_{n \rightarrow \infty} \frac{d!}{n^d} l_A(M/I^n M).$$

The *Hilbert-Kunz multiplicity* $e_{\text{HK}}(I, M)$ of I with respect to M is defined by

$$e_{\text{HK}}(I, M) = \lim_{q \rightarrow \infty} \frac{l_A(M/I^{[q]}M)}{q^d}.$$

By definition, we put $e(I) = e(I, A)$ (resp. $e_{\text{HK}}(I) = e_{\text{HK}}(I, A)$) and $e(A) = e(\mathfrak{m})$ (resp. $e_{\text{HK}}(A) = e_{\text{HK}}(\mathfrak{m})$).

We recall several basic results on Hilbert-Kunz multiplicity.

PROPOSITION 1.2. (Fundamental properties (cf. [13], [15], [16], [18], [22])) *Let (A, \mathfrak{m}, k) be a local ring of positive characteristic. Let I, I' be \mathfrak{m} -primary ideals of A , and let M be a finitely generated A -module. Then the following statements hold.*

- (1) *If $I \subseteq I'$, then $e_{\text{HK}}(I) \geq e_{\text{HK}}(I')$.*
- (2) *$e_{\text{HK}}(A) \geq 1$.*
- (3) *$\dim M < d$ if and only if $e_{\text{HK}}(I, M) = 0$.*
- (4) *If $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ is a short exact sequence of finitely generated A -modules, then*

$$e_{\text{HK}}(I, M) = e_{\text{HK}}(I, L) + e_{\text{HK}}(I, N).$$

- (5) (Associative formula)

$$e_{\text{HK}}(I, M) = \sum_{\mathfrak{p} \in \text{Assh}(A)} e_{\text{HK}}(I, A/\mathfrak{p}) \cdot l_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}),$$

where $\text{Assh}(A)$ denotes the set of prime ideals \mathfrak{p} of A with $\dim A/\mathfrak{p} = \dim A$.

- (6) *If J is a parameter ideal of A , then $e_{\text{HK}}(J) = e(J)$. In particular, if J is a minimal reduction of I (i.e., J is a parameter ideal which is contained in I and $I^{r+1} = JI^r$ for some integer $r \geq 0$), then $e_{\text{HK}}(J) = e(I)$.*
- (7) *If A is regular, then $e_{\text{HK}}(I) = l_A(A/I)$.*
- (8) (Localization) *$e_{\text{HK}}(A_{\mathfrak{p}}) \leq e_{\text{HK}}(A)$ holds for any prime ideal \mathfrak{p} such that $\dim A/\mathfrak{p} + \text{height } \mathfrak{p} = \dim A$.*
- (9) *If $x \in I$ is A -regular, then $e_{\text{HK}}(I) \leq e_{\text{HK}}(I/xA)$.*

(10) If $(A, \mathfrak{m}) \rightarrow (B, \mathfrak{n})$ is a flat local ring homomorphism such that $B/\mathfrak{m}B$ is a field, then $e_{\text{HK}}(I) = e_{\text{HK}}(IB)$.

Remark 1. Also, the similar result as above (except (6), (7)) holds for usual multiplicities.

Let (A, \mathfrak{m}, k) be any local ring of positive dimension. The associated graded ring $\text{gr}_{\mathfrak{m}}(A)$ of A with respect to \mathfrak{m} is defined as follows:

$$\text{gr}_{\mathfrak{m}}(A) := \bigoplus_{n \geq 0} \mathfrak{m}^n / \mathfrak{m}^{n+1}.$$

Then $G = \text{gr}_{\mathfrak{m}}(A)$ is a homogeneous k -algebra such that $\mathfrak{M} := G_+$ is the unique homogeneous maximal ideal of G . If $\text{char } A = p > 0$ and $\dim A = d$, then $G_{\mathfrak{M}}$ is also a local ring of characteristic p with $\dim G_{\mathfrak{M}} = d$.

PROPOSITION 1.3. ([22, Theorem (2.15)]) *Let (A, \mathfrak{m}, k) be a local ring of positive characteristic. Let $G = \text{gr}_{\mathfrak{m}}(A)$ the associated graded ring of A with respect \mathfrak{m} as above. Then $e_{\text{HK}}(A) \leq e_{\text{HK}}(G_{\mathfrak{M}}) \leq e(A)$.*

Remark 2. We use the same notation as in the above proposition. Although $e(A) = e(G_{\mathfrak{M}})$ always holds, $e_{\text{HK}}(A) = e_{\text{HK}}(G_{\mathfrak{M}})$ seldom holds.

PROPOSITION 1.4. (cf. [13]) *Let (A, \mathfrak{m}, k) be a local ring of positive characteristic with $d = \dim A$. Let I be an \mathfrak{m} -primary ideal of A . Then*

$$\frac{e(I)}{d!} \leq e_{\text{HK}}(I) \leq e(I).$$

Also, if $d \geq 2$, then the inequality in the left-hand side is strict; see [9].

We say that a local ring A is *unmixed* if $\dim \widehat{A}/\mathfrak{p} = \dim \widehat{A}$ holds for any associated prime ideal \mathfrak{p} of \widehat{A} . The following theorem is an analogy of Nagata's theorem ([20, (40.6)]), which is a starting point in this article.

THEOREM 1.5. ([22, Theorem (1.5)]) *Let (A, \mathfrak{m}, k) be an unmixed local ring of positive characteristic. Then A is regular if and only if $e_{\text{HK}}(A) = 1$.*

It is not so easy to compute Hilbert-Kunz multiplicities in general. However, one has simple formulas for them in case of quotient singularities and in case of binomial hypersurfaces; see below or [4, Theorem 3.1].

THEOREM 1.6. (cf. [22, Theorem (2.7)]) *Let $(A, \mathfrak{m}) \hookrightarrow (B, \mathfrak{n})$ be a module-finite extension of local domains of positive characteristic. Then for every \mathfrak{m} -primary ideal I of A , we have*

$$e_{\text{HK}}(I) = \frac{e_{\text{HK}}(IB)}{[Q(B) : Q(A)]} \cdot [B/\mathfrak{n} : A/\mathfrak{m}],$$

where $Q(A)$ denotes the fraction field of A .

Now let us see some examples of Hilbert-Kunz multiplicities which are given by the above formula. First, we consider the Veronese subring A defined by

$$A = k[[X_1^{i_1} \cdots X_d^{i_d} : i_1, \dots, i_d \geq 0, \sum i_j = r]].$$

Applying Theorem 1.6 to $A \hookrightarrow B = k[[X_1, \dots, X_d]]$, we get

$$(1.1) \quad e_{\text{HK}}(A) = \frac{1}{r} \binom{d+r-1}{r-1}.$$

In particular, if $d = 2$, $r = e(A)$, then $e_{\text{HK}}(A) = \frac{e(A)+1}{2}$.

Next, we consider the homogeneous coordinate rings of quadric hypersurfaces in \mathbb{P}_k^3 . Let k be a field of characteristic $p > 2$, and let R be the homogeneous coordinate ring of the hyperquadric Q defined by $q = q(X, Y, Z, W)$. Put $\mathfrak{M} = R_+$, the unique homogeneous maximal ideal of R , and $A = R_{\mathfrak{M}} \otimes_k \bar{k}$. By suitable coordinate transformation, we may assume that \hat{A} is isomorphic to one of the following rings:

$$(1.2) \quad \begin{cases} k[[X, Y, Z, W]]/(X^2), & \text{if } \text{rank}(q) = 1, \\ k[[X, Y, Z, W]]/(X^2 - YZ), & \text{if } \text{rank}(q) = 2, \\ k[[X, Y, Z, W]]/(XY - ZW), & \text{if } \text{rank}(q) = 3. \end{cases}$$

Then $e_{\text{HK}}(A) = 2, \frac{3}{2}$, or $\frac{4}{3}$, respectively.

In order to state other important properties of Hilbert-Kunz multiplicity, the notion of tight closure is very important. See [11], [12], [13] for definition and the fundamental properties of tight closure. In particular, the notion of F -rational ring is essential in our argument.

DEFINITION 1.7. ([6], [11], [12]) Let (A, \mathfrak{m}, k) be a local ring of positive characteristic. We say that A is *weakly F -regular* (resp. *F -rational*) if every ideal (resp. every parameter ideal) is tightly closed. Also, A is *F -regular* (resp. *F -rational*) if any local ring of A is weakly F -regular (resp. F -rational).

Note that an F -rational local ring is normal and Cohen-Macaulay.

Hochster and Huneke have given the following criterion of tight closure in terms of Hilbert-Kunz multiplicity.

THEOREM 1.8. (Length Criterion for Tight Closure (cf. [11, Theorem 8.17])) *Let $I \subseteq J$ be \mathfrak{m} -primary ideals of a local ring (A, \mathfrak{m}, k) of positive characteristic.*

- (1) *If $I^* = J^*$, then $e_{\text{HK}}(I) = e_{\text{HK}}(J)$.*
- (2) *Suppose that A is excellent, reduced and equidimensional. Then the converse of (1) is also true.*

The following theorem plays an important role in studying Hilbert-Kunz multiplicities for non-Cohen-Macaulay local rings.

THEOREM 1.9. (Goto-Nakamura [8]) *Let (A, \mathfrak{m}, k) be an equidimensional local ring which is a homomorphic image of a Cohen-Macaulay local ring of characteristic $p > 0$. Then*

- (1) *If J is a parameter ideal of A , then $e(J) \geq l_A(A/J^*)$.*
- (2) *Suppose that A is unmixed. If $e(J) = l_A(A/J^*)$ for some parameter ideal J , then A is F -rational (hence is Cohen-Macaulay).*

The next corollary is well-known in case of Cohen-Macaulay local rings (e.g. see [13]).

COROLLARY 1.10. *Let (A, \mathfrak{m}, k) be an unmixed local ring of characteristic $p > 0$. Suppose that $e(A) = 2$. Then \widehat{A} is F -rational if and only if $e_{\text{HK}}(A) < 2$. When this is the case, A is an F -rational hypersurface.*

Proof. Since any Cohen-Macaulay local ring of multiplicity 2 is a hypersurface, it suffices to prove the first statement.

We may assume that A is complete and k is infinite. We can take a minimal reduction J of \mathfrak{m} . First, suppose that $e_{\text{HK}}(A) < 2$. Then we show that A is Cohen-Macaulay, F -rational. By Goto-Nakamura's theorem, we have $2 = e(J) \geq l_A(A/J^*)$. If equality does not hold, then $l_A(A/J^*) = 1$, that is, $J^* = \mathfrak{m}$. Then $e_{\text{HK}}(A) = e_{\text{HK}}(J^*) = e_{\text{HK}}(J) = e(J) = 2$ by Proposition 1.2. This is a contradiction. Hence $e(J) = l_A(A/J^*)$. By Goto-Nakamura's theorem again, we obtain that A is Cohen-Macaulay, F -rational.

Conversely, suppose that A is a complete F -rational local ring. Then since A is Cohen-Macaulay and $J^* = J \neq \mathfrak{m}$, we have $e_{\text{HK}}(A) < e_{\text{HK}}(J) = e(J) = 2$ by the Length Criterion for Tight Closure. \square

The next question is open in general. However, we will show that it is true for $\dim A \leq 3$; see Section 3.

QUESTION 1.11. *If A is an unmixed local ring with $e_{\text{HK}}(A) < 2$, then is it F -rational?*

§2. Estimate of Hilbert-Kunz multiplicities

In this section, we will prove the key result to find a lower bound on Hilbert-Kunz multiplicities. Actually, it is a refinement of the argument which appeared in [22, Section 5] or in [23, Section 2]. The point is to use the tight closure J^* instead of “a parameter ideal J itself”. This enables us to investigate Hilbert-Kunz multiplicities of non-Cohen-Macaulay local rings. In Sections 3, 4, we will apply our method to unmixed local rings with $\dim A = 3, 4$.

Before stating our theorem, we introduce the following notation: Fix $d > 0$. For any positive real number s , we put

$$v_s := \text{vol} \left\{ (x_1, \dots, x_d) \in [0, 1]^d : \sum_{i=1}^d x_i \leq s \right\}, \quad v'_s := 1 - v_s,$$

where $\text{vol}(W)$ denotes the volume of $W \subseteq \mathbb{R}^d$. Then it is easy to see the following fact.

FACT 2.1. *Let s be a positive real number. Using the same notation as above, we have*

- (1) $v_s + v'_s = 1$.
- (2) $v'_{d-s} = v_s$.
- (3) $v_{d/2} = v'_{d/2} = \frac{1}{2}$.
- (4) *If $0 \leq s \leq 1$, then $v_s = \frac{s^d}{d!}$.*

Using the above notation, the key result in this paper can be written as follows:

THEOREM 2.2. *Let (A, \mathfrak{m}, k) be an unmixed local ring of characteristic $p > 0$. Put $d = \dim A \geq 1$. Let J be a minimal reduction of \mathfrak{m} , and let r be an integer with $r \geq \mu_A(\mathfrak{m}/J^*)$, where J^* denotes the tight closure of J . Also, let $s \geq 1$ be a rational number. Then we have*

$$(2.1) \quad e_{\text{HK}}(A) \geq e(A) \left\{ v_s - r \cdot \frac{(s-1)^d}{d!} \right\}.$$

Remark 3. When $1 \leq s \leq 2$, the right-hand side in Equation (2.1) is equal to $e(A)(v_s - r \cdot v_{s-1})$.

Before proving the theorem, we need the following lemma. In what follows, for any positive real number α , we define $I^\alpha := I^n$, where n is the minimum integer which does not exceed α .

LEMMA 2.3. *Let (A, \mathfrak{m}, k) be an unmixed local ring of characteristic $p > 0$ with $\dim A = d \geq 1$. Let J be a parameter ideal of A . Using the same notation as above, we have*

$$\lim_{q \rightarrow \infty} \frac{l_A(A/J^{sq})}{q^d} = \frac{e(J)s^d}{d!}, \quad \lim_{q \rightarrow \infty} l_A \left(\frac{J^{sq} + J^{[q]}}{J^{[q]}} \right) = e(J) \cdot v'_s.$$

Proof. First, note that our assertion holds if A is regular and $J = \mathfrak{m}$. We may assume that A is complete. Let x_1, \dots, x_d be a system of parameters which generates J , and put $R := k[[x_1, \dots, x_d]]$, $\mathfrak{n} = (x_1, \dots, x_d)R$. Then R is a complete regular local ring and A is a finitely generated R -module with $A/\mathfrak{m} = R/\mathfrak{n}$. Since the assertion is clear in case of regular local rings, it suffices to show the following claim.

CLAIM. *Let $\mathcal{I} = \{I_q\}_{q=p^e}$ be a set of ideals of A which satisfies the following conditions:*

- (1) *For each $q = p^e$, $I_q = J_q A$ holds for some ideal $J_q \subseteq R$.*
- (2) *There exists a positive integer t such that $\mathfrak{n}^{tq} \subseteq J_q$ for all $q = p^e$.*
- (3) *$\lim_{q \rightarrow \infty} l_R(R/J_q)/q^d$ exists.*

Then

$$\lim_{q \rightarrow \infty} \frac{l_A(A/I_q)}{q^d} = e(J) \cdot \lim_{q \rightarrow \infty} \frac{l_R(R/J_q)}{q^d}.$$

In fact, since A is unmixed, it is a torsion-free R -module of rank $e := e(J)$. Take a free R -module F of rank e such that $A_W \cong F_W$, where $W = R \setminus \{0\}$. Since F and A are both torsion-free, there exist the following short exact sequences of finitely generated R -modules:

$$0 \rightarrow F \rightarrow A \rightarrow C_1 \rightarrow 0, \quad 0 \rightarrow A \rightarrow F \rightarrow C_2 \rightarrow 0,$$

where $(C_1)_W = (C_2)_W = 0$. In particular, $\dim C_1 < d$ and $\dim C_2 < d$.

Applying the tensor product $- \otimes_R R/J_q$ to the above two exact sequences, respectively, we get

$$\begin{aligned} l_A(A/I_q) &\leq l_R(F/J_q F) + l_R(C_1/J_q C_1), \\ l_R(F/J_q F) &\leq l_A(A/I_q) + l_R(C_2/J_q C_2). \end{aligned}$$

In general, if $\dim_R C < d$, then

$$\frac{l_R(C/J_q C)}{q^d} \leq \frac{l_R(C/\mathfrak{n}^{tq} C)}{q^d} \rightarrow 0 \quad (q \rightarrow \infty).$$

Thus the required assertion easily follows from the above observation. \square

Proof of Theorem 2.2. For simplicity, we put $L = J^*$ and $e = e(A)$. We will give an upper bound of $l_A(\mathfrak{m}^{[q]}/J^{[q]})$. First, we have the following inequality:

$$\begin{aligned} l_A(\mathfrak{m}^{[q]}/J^{[q]}) &\leq l_A\left(\frac{\mathfrak{m}^{[q]} + \mathfrak{m}^{sq}}{J^{[q]}}\right) \\ &= l_A\left(\frac{\mathfrak{m}^{[q]} + \mathfrak{m}^{sq}}{L^{[q]} + \mathfrak{m}^{sq}}\right) + l_A\left(\frac{L^{[q]} + \mathfrak{m}^{sq}}{L^{[q]} + J^{sq}}\right) \\ &\quad + l_A\left(\frac{L^{[q]} + J^{sq}}{J^{[q]} + J^{sq}}\right) + l_A\left(\frac{J^{[q]} + J^{sq}}{J^{[q]}}\right) \\ &=: \ell_1 + \ell_2 + \ell_3 + \ell_4. \end{aligned}$$

Next, we see that $\ell_1 \leq r \cdot l_A(A/J^{(s-1)q}) + O(q^{d-1})$. By our assumption, we can write $\mathfrak{m} = L + Aa_1 + \cdots + Aa_r$. Since $\mathfrak{m}^{(s-1)q} a_i^q \subseteq \mathfrak{m}^{sq} \subseteq \mathfrak{m}^{sq} + L^{[q]}$, we have

$$\begin{aligned} \ell_1 &= l_A\left(\frac{\mathfrak{m}^{[q]} + \mathfrak{m}^{sq}}{L^{[q]} + \mathfrak{m}^{sq}}\right) \leq \sum_{i=1}^r l_A\left(\frac{Aa_i^q + L^{[q]} + \mathfrak{m}^{sq}}{L^{[q]} + \mathfrak{m}^{sq}}\right) \\ &= \sum_{i=1}^r l_A\left(A/(L^{[q]} + \mathfrak{m}^{sq}) : a_i^q\right) \\ &\leq r \cdot l_A(A/\mathfrak{m}^{(s-1)q}). \end{aligned}$$

Since J is a minimal reduction of \mathfrak{m} , we have $l_A(\mathfrak{m}^{(s-1)q}/J^{(s-1)q}) = O(q^{d-1})$. Thus we have the required inequality. Similarly, we get

$$\ell_2 = l_A\left(\frac{L^{[q]} + \mathfrak{m}^{sq}}{L^{[q]} + J^{sq}}\right) \leq l_A(\mathfrak{m}^{sq}/J^{sq}) = O(q^{d-1}).$$

Also, we have $l_A(L^{[q]}/J^{[q]}) = O(q^{d-1})$ by Length Criterion for Tight Closure. Hence $\ell_3 = O(q^{d-1})$ and thus

$$l_A(\mathfrak{m}^{[q]}/J^{[q]}) \leq r \cdot l_A(A/J^{(s-1)q}) + l_A\left(\frac{J^{[q]} + J^{sq}}{J^{[q]}}\right) + O(q^{d-1}).$$

It follows from the above argument that

$$\begin{aligned} e_{\text{HK}}(J) - e_{\text{HK}}(\mathfrak{m}) &\leq r \cdot \lim_{q \rightarrow \infty} \frac{l_A(A/J^{(s-1)q})}{q^d} + \lim_{q \rightarrow \infty} \frac{1}{q^d} l_A\left(\frac{J^{[q]} + J^{sq}}{J^{[q]}}\right) \\ &= r \cdot e \cdot \frac{(s-1)^d}{d!} + e \cdot v'_s. \end{aligned}$$

Since $e_{\text{HK}}(J) = e(J) = e$, $e_{\text{HK}}(A) = e_{\text{HK}}(\mathfrak{m})$ and $v'_s = 1 - v_s$, we get the required inequality. \square

The following fact is known, which gives a lower bound on Hilbert-Kunz multiplicities for hypersurface local rings.

FACT 2.4. (cf. [1], [2], [22]) *Let (A, \mathfrak{m}, k) be a hypersurface local ring of characteristic $p > 0$ with $d = \dim A \geq 1$. Then*

$$e_{\text{HK}}(A) \geq \beta_{d+1} \cdot e(A),$$

where β_{d+1} is given by the following equivalent formulas:

- (a) $\frac{1}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin \theta}{\theta}\right)^{d+1} d\theta;$
- (b) $\frac{1}{2^d d!} \sum_{\ell=0}^{\lfloor \frac{d}{2} \rfloor} (-1)^\ell (d+1-2\ell)^d \binom{d+1}{\ell};$
- (c) $\text{vol}\left\{\underline{x} \in [0, 1]^d : \frac{d-1}{2} \leq \sum x_i \leq \frac{d+1}{2}\right\} = 1 - v_{\frac{d-1}{2}} - v'_{\frac{d+1}{2}}.$

TABLE 1.

d	0	1	2	3	4	5	6
β_{d+1}	1	1	$\frac{3}{4}$	$\frac{2}{3}$	$\frac{115}{192}$	$\frac{11}{20}$	$\frac{5633}{11520}$

Remark 4. The above inequality is *not* best possible in general. In case of $d \geq 4$, one cannot prove the formula in the above fact as a corollary of our theorem. See also Proposition 3.9 and Theorem 4.3.

The following lemma is an analogy of Sally's theorem: If A is a Cohen-Macaulay local ring, then $\mu_A(\mathfrak{m}/J) = \mu_A(\mathfrak{m}) - \dim A \leq e(A) - 1$.

LEMMA 2.5. *Let (A, \mathfrak{m}, k) be an unmixed local ring of positive characteristic, and let J be a minimal reduction of \mathfrak{m} .*

- (1) $\mu_A(\mathfrak{m}/J^*) \leq e(A) - 1$.
- (2) *If A is not F -rational, then $\mu_A(\mathfrak{m}/J^*) \leq e(A) - 2$.*

Proof. We may assume that A is complete and thus is a homomorphic image of a Cohen-Macaulay local ring.

(1) By Goto-Nakamura's Theorem, we have that $\mu_A(\mathfrak{m}/J^*) \leq l_A(\mathfrak{m}/J^*) \leq e(J) - 1 = e - 1$.

(2) If A is not F -rational, then $l_A(A/J^*) \leq e(J) - 1 = e - 1$. Thus $\mu_A(\mathfrak{m}/J^*) \leq e - 2$, as required. \square

Using Theorem 2.2 and Lemma 2.5, one can prove the following corollary, which has been already proved in [23] in the case of Cohen-Macaulay local rings.

COROLLARY 2.6. (cf. [23]) *Let (A, \mathfrak{m}, k) be a two-dimensional unmixed local ring of characteristic $p > 0$. Put $e = e(A)$. Then*

$$(2.2) \quad e_{\text{HK}}(A) \geq \frac{e+1}{2}.$$

Also, suppose $k = \bar{k}$. Then the equality holds if and only if $\text{gr}_{\mathfrak{m}}(A)$ is isomorphic to the Veronese subring $k[X, Y]^{(e)} = k[X^e, X^{e-1}Y, \dots, XY^{e-1}, Y^e]$.

Moreover, if A is not F -rational, then we have

$$e_{\text{HK}}(A) \geq \frac{e^2}{2(e-1)}.$$

EXAMPLE 2.7. (Fakhruddin-Trivedi [7, Corollary 3.19]) Let E be an elliptic curve over a field $k = \bar{k}$ of characteristic $p > 0$, and let \mathcal{L} be a very ample line bundle on E of degree $e \geq 2$. Let R be the homogeneous coordinate ring (the section ring of \mathcal{L}) defined by

$$R = \bigoplus_{n \geq 0} H^0(E, \mathcal{L}^{\otimes n}).$$

Also, put $A = R_{\mathfrak{M}}$, where \mathfrak{M} be the unique homogeneous maximal ideal of R . Then we have $e_{\text{HK}}(A) = \frac{e^2}{2(e-1)}$.

§3. Lower bounds in the case of three-dimensional local rings

In this section, we prove the main theorem in this paper, which gives the lower bound of Hilbert-Kunz multiplicities for non-regular unmixed local rings of dimension 3.

THEOREM 3.1. *Let (A, \mathfrak{m}, k) be a three-dimensional unmixed local ring of characteristic $p > 0$. Then*

- (1) *If A is not regular, then $e_{\text{HK}}(A) \geq \frac{4}{3}$.*
- (2) *Suppose that $k = \bar{k}$ and $\text{char } k \neq 2$. Then the following conditions are equivalent:*
 - (a) $e_{\text{HK}}(A) = \frac{4}{3}$.
 - (b) $\widehat{A} \cong k[[X, Y, Z, W]]/(X^2 + Y^2 + Z^2 + W^2)$.
 - (c) $\text{gr}_{\mathfrak{m}}(A) \cong k[X, Y, Z, W]/(X^2 + Y^2 + Z^2 + W^2)$. *That is, $\text{gr}_{\mathfrak{m}}(A) \cong k[X, Y, Z, W]/(XY - ZW)$.*

PROPOSITION 3.2. *Let (A, \mathfrak{m}, k) be a three-dimensional unmixed local ring of characteristic $p > 0$. If $e_{\text{HK}}(A) < 2$, then A is F -rational.*

From now on, we divide the proof of Theorem 3.1 and Proposition 3.2 into several steps. In the following, we assume the following condition.

(#): Let (A, \mathfrak{m}, k) be a three-dimensional unmixed local ring of characteristic $p > 0$, and $e(A) = e$, the multiplicity of A . Also, suppose that \mathfrak{m} has a minimal reduction J .

Suppose that A is not regular under the assumption (#). Then $e = e(A)$ is an integer with $e \geq 2$. Thus the first assertion of Theorem 3.1 follows from the following lemma. Also, this implies that if $e_{\text{HK}}(A) = \frac{4}{3}$ then $e(A) = 2$ without extra assumptions.

LEMMA 3.3. *Under the assumption (#), we have*

- (1) *If $e \geq 5$, then $e_{\text{HK}}(A) > 2$.*
- (2) *If $e = 4$, then $e_{\text{HK}}(A) \geq \frac{7}{4} > \frac{4}{3}$.*
- (3) *If $e = 3$, then $e_{\text{HK}}(A) \geq \frac{13}{8} > \frac{4}{3}$.*
- (4) *If $e = 2$, then $e_{\text{HK}}(A) \geq \frac{4}{3}$.*

Remark 5. The lower bounds of $e_{\text{HK}}(A)$ in Lemma 3.3 are not best possible.

Proof. We may assume that A is complete. By Lemma 2.5(1), we can apply Theorem 2.2 with $r = e - 1$. Namely, if $1 \leq s \leq 2$, then

$$(3.1) \quad e_{\text{HK}}(A) \geq e(v_s - (e-1)v_{s-1}) = e \left(\frac{s^3}{6} - (e+2) \frac{(s-1)^3}{6} \right).$$

Define the real-valued function $f_e(s)$ by the right-hand side of Eq. (3.1). Then one can easily calculate $\max_{1 \leq s \leq 2} f_e(s)$. In fact, if $e \geq 2$, then

$$\max_{1 \leq s \leq 2} f_e(s) = f \left(\frac{e+2+\sqrt{e+2}}{e+1} \right) = \frac{e}{6} \left(\frac{e+2+\sqrt{e+2}}{e+1} \right)^2.$$

But, in order to prove the lemma, it is enough to use the following values only:

s	$\frac{3}{2}$	$\frac{7}{4}$	2
$f_e(s)$	$\frac{e(25-e)}{48}$	$\frac{e(289-27e)}{384}$	$\frac{e(6-e)}{6}$

(1) We show that $e_{\text{HK}}(A) > 2$ if $e \geq 5$. If $e \geq 13$, then by Proposition 1.4,

$$e_{\text{HK}}(A) \geq \frac{e}{3!} \geq \frac{13}{6} > 2.$$

So we may assume that $5 \leq e \leq 12$. Applying Eq. (3.1) for $s = \frac{3}{2}$, we get

$$e_{\text{HK}}(A) \geq \frac{e(25-e)}{48} \geq \frac{5(25-5)}{48} = \frac{25}{12} > 2.$$

(2) Suppose that $e = 4$. Applying Eq. (3.1) for $s = \frac{3}{2}$, we get

$$e_{\text{HK}}(A) \geq \frac{e(25-e)}{48} = \frac{7}{4}.$$

(3) Suppose that $e = 3$. Applying Eq. (3.1) for $s = \frac{7}{4}$, we get

$$e_{\text{HK}}(A) \geq \frac{e(289 - 27e)}{384} = \frac{13}{8}.$$

(4) Suppose that $e = 2$. Applying Eq. (3.1) for $s = 2$,

$$e_{\text{HK}}(A) \geq \frac{e(6 - e)}{6} = \frac{4}{3},$$

as required. □

Before proving the second assertion of Theorem 3.1, we prove Proposition 3.2. For that purpose, we now focus non- F -rational local rings.

Now suppose that A is not F -rational. If $e = 2$, then $e_{\text{HK}}(A) = 2$ by Lemma 1.10. On the other hand, if $e \geq 5$, then $e_{\text{HK}}(A) > 2$ by Lemma 3.3. Thus in order to prove Proposition 3.2, it is enough to investigate the cases of $e = 3, 4$. Namely, Proposition 3.2 follows from the following lemma.

LEMMA 3.4. *Suppose that A is not F -rational under the assumption (#). Then*

- (1) *If $e = 3$, then $e_{\text{HK}}(A) \geq 2$.*
- (2) *If $e = 4$, then $e_{\text{HK}}(A) > 2$.*

Proof. By Lemma 2.5(2), we can apply Theorem 2.2 for $r = e - 2$. Thus if $1 \leq s \leq 2$, then

$$(3.2) \quad e_{\text{HK}}(A) \geq e \left(\frac{s^3}{6} - (e + 1) \frac{(s - 1)^3}{6} \right).$$

(1) Suppose that $e = 3$. Applying Eq. (3.2) for $s = 2$, we get

$$e_{\text{HK}}(A) \geq \frac{e(7 - e)}{6} = 2.$$

(2) Suppose that $e = 4$. Applying Eq. (3.2) for $s = \frac{7}{4}$, we get

$$e_{\text{HK}}(A) \geq \frac{e(316 - 27e)}{384} = \frac{13}{6} > 2,$$

as required. □

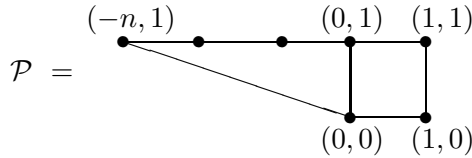
EXAMPLE 3.5. Let $R = k[T, xT, xyT, yT, x^{-1}yT, x^{-2}yT, \dots, x^{-n}yT]$ be a rational normal scroll and put $\mathfrak{m} = (T, xT, xyT, yT, x^{-1}yT, \dots, x^{-n}yT)$. Then $A = R_{\mathfrak{m}}$ is a three-dimensional Cohen-Macaulay F -rational local domain with $e(A) = n + 2$, and

$$e_{\text{HK}}(A) = \frac{e(A)}{2} + \frac{e(A)}{6(n+1)}.$$

Proof. Let $\mathcal{P} \subseteq \mathbb{R}$ be a convex polytope with vertex set

$$\Gamma = \{(0, 0), (1, 0), (1, 1), (0, 1), (-1, 1), \dots, (-n, 1)\},$$

and put $\tilde{\mathcal{P}} := \{(\alpha, 1) \in \mathbb{R}^3 : \alpha \in \mathcal{P}\}$ and $d\mathcal{P} := \{d \cdot \alpha : \alpha \in \mathcal{P}\}$ for every integer $d \geq 0$. Also, if we define a cone $\mathcal{C} = \mathcal{C}(\tilde{\mathcal{P}}) := \{r\beta : \beta \in \tilde{\mathcal{P}}, 0 \leq r \in \mathbb{Q}\}$ and regard R as a homogeneous k -algebra with $\deg x = \deg y = 0$ and $\deg T = 1$, then the basis of R_d corresponds to the set $\{(\alpha, d) \in \mathbb{Z}^3 : \alpha \in \mathbb{Z}^2 \cap d\mathcal{P}\} = \{(\alpha, d) \in \mathbb{Z}^3 : \alpha \in \mathbb{Z}^2\} \cap \mathcal{C}$.



If we put $\Gamma_q = \{(0, 0), (q, 0), (q, q), (0, q), (-q, q), \dots, (-nq, q)\}$, then $\mathfrak{m}^{[q]} = (x^a y^b T^q : (a, b) \in \Gamma_q)$. Since $[\mathfrak{m}^{[q]}]_d = \sum_{(a,b) \in \Gamma_q} R_{d-q} x^a y^b T^q$, we have

$$\begin{aligned} e_{\text{HK}}(A) &= \lim_{q \rightarrow \infty} \frac{1}{q^3} l_A(A/\mathfrak{m}^{[q]}) \\ &= \lim_{q \rightarrow \infty} \frac{1}{q^3} \# \left\{ \mathbb{Z}^3 \cap \left(\mathcal{C} \setminus \bigcup_{(a,b) \in \Gamma_q} (a, b, q) + \mathcal{C} \right) \right\}, \end{aligned}$$

that is,

$$e_{\text{HK}}(A) = \lim_{q \rightarrow \infty} \frac{1}{q^3} \left[\sum_{d=0}^{\infty} \# \left\{ \mathbb{Z}^2 \cap \left(d\mathcal{P} \setminus \bigcup_{(a,b) \in \Gamma_q} (a, b) + \max\{0, d - q\}\mathcal{P} \right) \right\} \right].$$

Also, if we define a real continuous function $f : [0, \infty) \rightarrow \mathbb{R}$ by

$$f(t) = \text{the volume of } \left[t\mathcal{P} \setminus \bigcup_{(a,b) \in \Gamma} (a, b) + \max\{0, t - 1\}\mathcal{P} \right] \text{ in } \mathbb{R}^2,$$

then $e_{\text{HK}}(A) = \int_0^\infty f(t) dt$. Let us denote the volume of $M \subseteq \mathbb{R}^2$ by $\text{vol}(M)$. To calculate $e_{\text{HK}}(A)$, we need to determine $f(t)$. Namely, we need to show the following claim.

CLAIM.

$$f(t) = \begin{cases} \text{vol}(t\mathcal{P}), & 0 \leq t < 1; \\ \text{vol}(t\mathcal{P}) - (n+4) \text{vol}((t-1)\mathcal{P}), & 1 \leq t < \frac{n+2}{n+1}; \\ \frac{(n+2)t(2-t)}{2} + (n+2) \frac{(2-t)^2}{2n}, & \frac{n+2}{n+1} \leq t < 2; \\ 0, & t \geq 2. \end{cases}$$

To prove the claim, we may assume that $t \geq 1$. For simplicity, we put $M_{a,b} = (a,b) + (t-1)\mathcal{P}$ for every $(a,b) \in \Gamma$. First suppose that $1 \leq t < \frac{n+2}{n+1}$. Then since $1 - n(t-1) > t-1$, $M_{0,0} \cap M_{1,0} = \emptyset$. Similarly, one can easily see that any two $M_{a,b}$ do not intersect each other; see Figure 1. Thus $f(t) = \text{vol}(t\mathcal{P}) - (n+4) \text{vol}((t-1)\mathcal{P})$.

Next suppose that $\frac{n+2}{n+1} \leq t < 2$. Then $\mathcal{P} \cap \{(x,y) \in \mathbb{R}^2 : 0 \leq y \leq t-1\} = M_{0,0} \cup M_{1,0} \cup T_0$, where T_0 is a triangle with vertex $(t-1, 0)$, $(1, 0)$ and $(t-1, \frac{2-t}{n})$. Similarly, there exist $(n+1)$ -triangles T_1, \dots, T_{n+1} having the same volumes as T_0 such that

$$\mathcal{P} \cap \{(x,y) \in \mathbb{R}^2 : 1 \leq y \leq t\} = M_{-n,1} \cup \dots \cup M_{1,1} \cup M_{0,1} \cup M_{1,1} \cup T_1 \cup \dots \cup T_{n+1}$$

and any two T_i 's do not intersect each other; see Figure 2. Thus

$$\begin{aligned} f(t) &= \text{vol}(\mathcal{P} \cap \{(x,y) \in \mathbb{R}^2 : t-1 \leq y \leq 1\}) + (n+2) \text{vol}(T_0) \\ &= \frac{(n+2)t(2-t)}{2} + (n+2) \frac{(2-t)^2}{2n}. \end{aligned}$$

Finally, suppose that $t \geq 2$. Then since \mathcal{P} is covered by $M_{a,b}$'s, we have $f(t) = 0$, as required.

Using the above claim, let us calculate $e_{\text{HK}}(A)$. Note that $\text{vol}(t\mathcal{P}) = \frac{(n+2)t^2}{2}$.

$$\begin{aligned} e_{\text{HK}}(A) &= \int_0^{\frac{n+2}{n+1}} \frac{(n+2)t^2}{2} dt - (n+4) \int_1^{\frac{n+2}{n+1}} \frac{(n+2)(t-1)^2}{2} dt \\ &\quad + \int_{\frac{n+2}{n+1}}^2 \frac{(n+2)t(2-t)}{2} dt + (n+2) \int_{\frac{n+2}{n+1}}^2 \frac{(2-t)^2}{2n} dt \\ &= (n+2) \left[\frac{1}{2} + \frac{1}{6(n+1)} \right], \end{aligned}$$

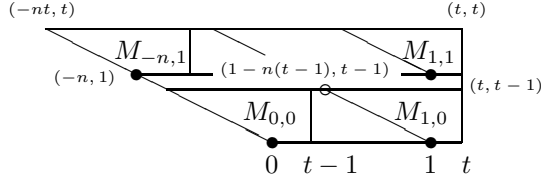


FIGURE 1. The case where $1 \leq t < \frac{n+2}{n+1}$

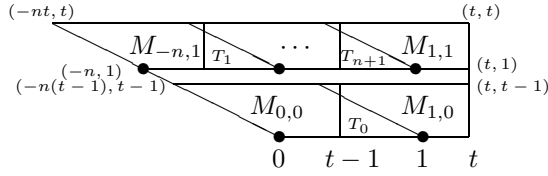


FIGURE 2. The case where $\frac{n+2}{n+1} \leq t < 2$

as required. □

DISCUSSION 3.6. Let A be a complete local ring which satisfies (#). Also, suppose that $e = 3$. What is the smallest value of $e_{\text{HK}}(A)$ among such rings?

The function $f_e(s) = 3\left(\frac{s^3}{6} - 5\frac{(s-1)^3}{6}\right)$, which appeared in Eq. (3.1), takes the maximal value

$$f\left(\frac{5 + \sqrt{5}}{4}\right) = \frac{15 + 5\sqrt{5}}{16} = 1.636\dots$$

in $s \in [1, 2]$. Hence $e_{\text{HK}}(A) \geq 1.636\dots$. But we believe that this is not best possible.

Suppose that $e_{\text{HK}}(A) < 2$. Then A is F -rational by Lemma 3.4. Thus it is Cohen-Macaulay and $3+1 \leq v = \text{emb}(A) \leq d+e-1 = 3+3-1 = 5$. If $v \neq 5$, then A is a hypersurface and $e_{\text{HK}}(A) \geq \frac{2}{3} \cdot e = 2$ by Fact 2.4. Hence we may assume that $v = 5$, that is, A has maximal embedding dimension. If we write as $A = R/I$, where R is a complete regular local ring with $\dim R = 5$, then $\text{height } I = 2$. By Hilbert-Burch's theorem, there exists a 2×3 -matrix \mathbb{M} such that $I = I_2(\mathbb{M})$, the ideal generated by all 2-minors of \mathbb{M} . In particular, A can be written as $A = B/aB$, where $B = k[X]/I_2(X)$,

X is a generic 2×3 -matrix and a is a prime element of B . This implies that

$$e_{\text{HK}}(A) = e_{\text{HK}}(B/aB) \geq e_{\text{HK}}(B) = 3 \left\{ \frac{1}{2} + \frac{1}{4!} \right\} = \frac{13}{8} = 1.625;$$

see [5, Section 3].

For example, if $A = k[[T, xT, xyt, yT, x^{-1}yT]]$ is a rational normal scroll, then $e_{\text{HK}}(A) = \frac{7}{4} = 1.75$ by Example 3.5. Is this the smallest value?

DISCUSSION 3.7. Let A be a complete local ring which satisfies (#). Also, suppose that $e = 4$. What is the smallest value of $e_{\text{HK}}(A)$ among such rings?

As in Discussion 3.6, it suffices to consider F -rational local rings only. For example, let $A = k[[x, y, z]]^{(2)}$ be the Veronese subring. Then A is an F -rational local domain with $e(A) = 4$ and $e_{\text{HK}}(A) = 2$. Also, let A be the completion of the Rees algebra $R(\mathfrak{n})$ over an F -rational double point (R, \mathfrak{n}) of dimension 2. Then A is an F -rational local domain with $e(A) = 4$ and $e_{\text{HK}}(A) \geq 2$ (we believe that this inequality is strict).

On the other hand, the function $f_e(s)$ which appeared in Eq. (3.1), takes the maximal value

$$f\left(\frac{6 + \sqrt{6}}{5}\right) = \frac{28 + 8\sqrt{6}}{25} = 1.903\dots$$

in $s \in [1, 2]$. Hence the fact that we can prove now is “ $e_{\text{HK}}(A) \geq 1.903\dots$ ” only.

Based on Corollary 2.6 and Discussion 3.7, we pose the following conjecture.

CONJECTURE 3.8. Let A be a complete local ring which satisfies (#), and let $r \geq 2$ be an integer. If $e(A) = r^2$, then

$$e_{\text{HK}}(A) \geq \frac{(r+1)(r+2)}{6}.$$

Also, the equality holds if and only if A is isomorphic to $k[[x, y, z]]^{(r)}$.

In the rest of this section, we prove the second statement of Theorem 3.1. Let (A, \mathfrak{m}, k) be a complete local ring which satisfies (#). If

$e_{\text{HK}}(A) = \frac{4}{3}$, then A is an F -rational hypersurface with $e(A) = 2$ by the above observation. Furthermore, suppose that $k = \bar{k}$ and $\text{char } k \neq 2$. Then we may assume that A can be written as the form $k[[X, Y, Z, W]]/(X^2 - \varphi(Y, Z, W))$. To study Hilbert-Kunz multiplicities for these rings, we prove the improved version of Theorem 2.2.

PROPOSITION 3.9. *Let k be an algebraically closed field of $\text{char } k \neq 2$, and let $A = k[[X, Y, Z, W]]/(X^2 - \varphi(Y, Z, W))$ be an F -rational hypersurface local ring. Let a, b, c be integers with $2 \leq a \leq b \leq c$.*

Suppose that there exists a function $\text{ord} : A \rightarrow \mathbb{Q} \cup \{\infty\}$ which satisfies the following conditions:

- (1) $\text{ord}(\alpha) \geq 0$; and $\text{ord}(\alpha) = \infty \iff \alpha = 0$.
- (2) $\text{ord}(x) = 1/2$, $\text{ord } y = 1/a$, $\text{ord } z = 1/b$, and $\text{ord } w = 1/c$.
- (3) $\text{ord}(\varphi) \geq 1$.
- (4) $\text{ord}(\alpha + \beta) \geq \min\{\text{ord}(\alpha), \text{ord}(\beta)\}$.
- (5) $\text{ord}(\alpha\beta) \geq \text{ord}(\alpha) + \text{ord}(\beta)$.

Then we have

$$e_{\text{HK}}(A) \geq 2 - \frac{abc}{12}(N^3 - n^3),$$

where

$$N = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} - \frac{1}{2}, \quad n = \max\left\{0, N - \frac{2}{c}\right\}.$$

In particular, if $(a, b, c) \neq (2, 2, 2)$, then $e_{\text{HK}}(A) > \frac{4}{3}$.

Remark 6. The third condition $\text{ord}(\varphi) \geq 1$ is important. For example, if $\varphi \equiv y^2 \pmod{(z, w)^3}$, then one can take $(a, b, c) = (2, 3, 3)$, but $(a, b, c) = (2, 3, 4)$.

Proof. First, we define a filtration $\{F_n\}_{n \in \mathbb{Q}}$ as follows:

$$F_n := \{\alpha \in A : \text{ord}(\alpha) \geq n\}.$$

Then every F_n is an ideal and $F_m F_n \subseteq F_{m+n}$ holds for all $m, n \in \mathbb{Q}$. Using F_n instead of \mathfrak{m}^n , we shall estimate $l_A(\mathfrak{m}^{[q]}/J^{[q]})$.

Set $J = (y, z, w)A$ and fix a sufficiently large power $q = p^e$. Put

$$s = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}, \quad N = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} - \frac{1}{2}.$$

Since J is a minimal reduction of \mathfrak{m} and $xy^{q-1}z^{q-1}w^{q-1}$ generates the socle of $A/J^{[q]}$, we have that $F_{sq} \subseteq J^{[q]}$. Also, since $B = A/J^{[q]}$ is an Artinian Gorenstein local ring, we get

$$F_{\frac{(N+1)q}{2}} B \subseteq 0 :_B F_{\frac{Nq}{2}} B \cong K_{B/F_{\frac{Nq}{2}} B},$$

where K_C denotes a canonical module of a local ring C . Hence, by the Matlis duality theorem, we get

$$l_A \left(\frac{F_{\frac{(N+1)q}{2}} + J^{[q]}}{J^{[q]}} \right) \leq l_B(F_{\frac{(N+1)q}{2}}) \leq l_B(K_{B/F_{\frac{Nq}{2}} B}) = l_B(B/F_{\frac{Nq}{2}} B).$$

On the other hand, since $x^q \in F_{\frac{q}{2}}$ by the assumption, we have

$$x^q F_{\frac{Nq}{2}} \subseteq F_{\frac{(N+1)q}{2}}.$$

Therefore by a similar argument as in the proof of Theorem 2.2, we get

$$\begin{aligned} l_A(\mathfrak{m}^{[q]}/J^{[q]}) &\leq l_A \left(\frac{Ax^q + J^{[q]} + F_{\frac{(N+1)q}{2}}}{F_{\frac{(N+1)q}{2}} + J^{[q]}} \right) + l_A \left(\frac{F_{\frac{(N+1)q}{2}} + J^{[q]}}{J^{[q]}} \right) \\ &\leq l_A \left(A/(J^{[q]} + F_{\frac{(N+1)q}{2}}) : x^q \right) + l_B(B/F_{\frac{Nq}{2}} B) \\ &\leq 2 \cdot l_A \left(A/J^{[q]} + F_{\frac{Nq}{2}} \right). \end{aligned}$$

In fact, since

$$\begin{aligned} &\lim_{q \rightarrow \infty} \frac{1}{q^3} l_A \left(A/J^{[q]} + F_{\frac{Nq}{2}} \right) \\ &= e(A) \cdot \lim_{q \rightarrow \infty} \frac{1}{q^3} \text{vol} \left\{ (x, y, z) \in [0, q]^3 : \frac{y}{a} + \frac{z}{b} + \frac{w}{c} \leq \frac{Nq}{2} \right\} \\ &= 2 \cdot \text{vol} \left\{ (x, y, z) \in [0, 1]^3 : \frac{y}{a} + \frac{z}{b} + \frac{w}{c} \leq \frac{N}{2} \right\} \\ &= \frac{abc}{24} (N^3 - n^3), \end{aligned}$$

we get

$$e_{\text{HK}}(A) \geq 2 - 2 \cdot \frac{abc}{24} (N^3 - n^3) = 2 - \frac{abc}{12} (N^3 - n^3),$$

as required. □

EXAMPLE 3.10. Let k be an algebraically closed field of char $k \neq 2$, and let (A, \mathfrak{m}, k) be a hypersurface. Put $\text{gr}_{\mathfrak{m}}(A) = k[X, Y, Z, W]/(g(X, Y, Z, W))$.

$$g(X, Y, Z, W) = X^2 + Y^3 + Z^3 + W^3 \implies e_{\text{HK}}(A) \geq \frac{55}{32};$$

$$g(X, Y, Z, W) = X^2 + Y^2 + Z^3 + W^3 \implies e_{\text{HK}}(A) \geq \frac{14}{9};$$

$$g(X, Y, Z, W) = X^2 + Y^2 + Z^2 + W^c \implies e_{\text{HK}}(A) \geq \frac{3}{2} - \frac{2}{3c^2}.$$

Proof of Theorem 3.1(2). Put $G = \text{gr}_{\mathfrak{m}}(A)$ and $\mathfrak{M} = \text{gr}_{\mathfrak{m}}(A)_+$. The implication (a) \Rightarrow (b) follows from Proposition 3.9. (b) \Rightarrow (c) is clear. Suppose (c). Then $e_{\text{HK}}(G\mathfrak{M}) = \frac{4}{3}$. Also, by Proposition 1.3 and Theorem 3.1(1), we have that $\frac{4}{3} \leq e_{\text{HK}}(A) \leq e_{\text{HK}}(G\mathfrak{M}) = \frac{4}{3}$. Thus $e_{\text{HK}}(A) = \frac{4}{3}$, as required. \square

Also, the following corollary follows from the proof of Proposition 3.9 and Example 3.10.

COROLLARY 3.11. *Let A be a local ring which satisfies (#). Also, assume that $k = \bar{k}$ and $p \neq 2$. Then the following conditions are equivalent:*

- (1) $\frac{4}{3} < e_{\text{HK}}(A) \leq \frac{3}{2}$.
- (2) $\text{gr}_{\mathfrak{m}}(A) \cong k[X, Y, Z]/(X^2 + Y^2 + Z^2)$.
- (3) *A is isomorphic to a hypersurface $k[[X, Y, Z, W]]/(X^2 + Y^2 + Z^2 + W^c)$ for some integer $c \geq 3$.*

When this is the case, $e_{\text{HK}}(A) \geq \frac{3}{2} - \frac{2}{3c^2}$.

§4. A generalization of the main result to higher dimensional case

In this section, we want to generalize Theorem 3.1 to the case of $\dim A \geq 4$. Let $d \geq 1$ be an integer and $p > 2$ a prime number. If we put

$$A_{p,d} := \overline{\mathbb{F}}_p[[X_0, X_1, \dots, X_d]]/(X_0^2 + \dots + X_d^2),$$

then we can guess that $e_{\text{HK}}(A_{p,d}) = s_{\text{HK}}(p, d)$ holds according to the observations until the previous section. In the following, let us formulate this as a conjecture and prove that it is also true in case of $\dim A = 4$.

In [10], Han and Monsky gave an algorithm to calculate $e_{\text{HK}}(A_{p,d})$, but it is not so easy to represent $e_{\text{HK}}(A_{p,d})$ as a quotient of two polynomials of p for any fixed $d \geq 1$.

d	1	2	3	4
$e_{\text{HK}}(A_{p,d})$	2	$\frac{3}{2}$	$\frac{4}{3}$	$\frac{29p^2+15}{24p^2+12}$

On the other hand, surprisingly, Monsky proved the following theorem:

THEOREM 4.1. (Monsky [19]) *Under the above notation, we have*

$$(4.1) \quad \lim_{p \rightarrow \infty} e_{\text{HK}}(A_{p,d}) = 1 + \frac{c_d}{d!},$$

where

$$(4.2) \quad \sec x + \tan x = \sum_{d=0}^{\infty} \frac{c_d}{d!} x^d \quad \left(|x| < \frac{\pi}{2} \right).$$

Remark 7. It is known that the Taylor expansion of $\sec x$ (resp. $\tan x$) at origin can be written as follows:

$$\begin{aligned} \sec x &= \sum_{i=0}^{\infty} \frac{E_{2i}}{(2i)!} x^{2i}, \\ \tan x &= \sum_{i=1}^{\infty} (-1)^{i-1} \frac{2^{2i}(2^{2i}-1)B_{2i}}{(2i)!} x^{2i-1}, \end{aligned}$$

where E_{2i} (resp. B_{2i}) is said to be Euler number (resp. Bernoulli number).

Also, c_d appeared in Eq. (4.1) is a positive integer since $\cos t$ is an unit element in a ring $\mathcal{H} = \left\{ \sum_{n=0}^{\infty} a_n \frac{t^n}{n!} : a_n \in \mathbb{Z} \text{ for all } n \geq 0 \right\}$.

Based on the above observation, we pose the following conjecture.

CONJECTURE 4.2. *Let $d \geq 1$ be an integer and $p > 2$ a prime number. Put*

$$A_{p,d} := \overline{\mathbb{F}_p}[[X_0, X_1, \dots, X_d]] / (X_0^2 + \dots + X_d^2).$$

Let (A, \mathfrak{m}, k) be a d -dimensional unmixed local ring with $k = \overline{\mathbb{F}_p}$. Then the following statements hold.

- (1) *If A is not regular, then $e_{\text{HK}}(A) \geq e_{\text{HK}}(A_{p,d}) \geq 1 + \frac{c_d}{d!}$. In particular, $s_{\text{HK}}(p, d) = e_{\text{HK}}(A_{p,d})$.*

(2) If $e_{\text{HK}}(A) = e_{\text{HK}}(A_{p,d})$, then $\widehat{A} \cong A_{p,d}$ as local rings.

In the following, we prove that this is true in case of $\dim A = 4$. Note that

$$\lim_{p \rightarrow \infty} e_{\text{HK}}(A_{p,4}) = \lim_{p \rightarrow \infty} \frac{29p^2 + 15}{24p^2 + 12} = \frac{29}{24} = 1 + \frac{c_4}{4!}.$$

THEOREM 4.3. *Let (A, \mathfrak{m}, k) be an unmixed local ring of characteristic $p > 0$ with $\dim A = 4$. If $e(A) \geq 3$, then $e_{\text{HK}}(A) \geq \frac{5}{4} = \frac{30}{24}$.*

Suppose that $k = \overline{k}$ and $\text{char } k \neq 2$. Put

$$A_{p,4} = \overline{\mathbb{F}}_p[[X_0, X_1, \dots, X_4]]/(X_0^2 + \dots + X_4^2).$$

Then the following statement holds.

(1) *If A is not regular, then*

$$e_{\text{HK}}(A) \geq e_{\text{HK}}(A_{p,4}) = \frac{29p^2 + 15}{24p^2 + 12}.$$

(2) *The following conditions are equivalent:*

(a) *Equality holds in (1).*

(b) $e_{\text{HK}}(A) < \frac{5}{4}$.

(c) *The completion of A is isomorphic to $A_{p,4}$.*

Proof. Put $e = e(A)$, the multiplicity of A . We may assume that A is complete with $e \geq 2$ and k is infinite. In particular, A is a homomorphic image of a Cohen-Macaulay local ring, and there exists a minimal reduction J of \mathfrak{m} . Then $\mu_A(\mathfrak{m}/J^*) \leq e - 1$ by Lemma 2.5. We first show that $e_{\text{HK}}(A) \geq \frac{5}{4}$ if $e \geq 3$.

CLAIM 1. *If $3 \leq e \leq 10$, then $e_{\text{HK}}(A) \geq \frac{5}{4}$.*

Putting $r = e - 1$ and $s = 2$ in Theorem 2.2, since $v_2 = \frac{1}{2}$, we have

$$e_{\text{HK}}(A) \geq e \left\{ v_2 - \frac{(e-1)1^4}{4!} \right\} = \frac{(13-e)e}{24} \geq \frac{(13-3) \cdot 3}{24} = \frac{30}{24},$$

as required.

CLAIM 2. *If $11 \leq e \leq 29$, then $e_{\text{HK}}(A) \geq \frac{737}{384} (> \frac{5}{4})$.*

By Fact 2.4, we have $v_{3/2} = \frac{1-\beta_{4+1}}{2} = \frac{77}{384}$. Putting $r = e - 1$ and $s = \frac{3}{2}$ in Theorem 2.2, we have

$$e_{\text{HK}}(A) \geq e \left\{ v_{3/2} - \frac{e-1}{24} \cdot \left(\frac{1}{2}\right)^4 \right\} = \frac{(78-e)e}{384} \geq \frac{11(78-11)}{384} = \frac{737}{384},$$

as required.

CLAIM 3. *If $e \geq 30$, then $e_{\text{HK}}(A) \geq \frac{5}{4}$.*

By Proposition 1.4, we have $e_{\text{HK}}(A) \geq \frac{e}{4!} \geq \frac{30}{24}$.

In the following, we assume that $k = \bar{k}$, $\text{char } k \neq 2$ and $e \geq 2$. To see (1), (2), we may assume that $e = 2$ by the above argument. Then since $e_{\text{HK}}(A) = 2$ if A is not F -rational, we may also assume that A is F -rational and thus is a hypersurface. Thus A can be written as the following form:

$$A = k[[X_0, X_1, \dots, X_4]]/(X_0^2 - \varphi(X_1, X_2, X_3, X_4)).$$

If A is isomorphic to $A_{p,4}$, then by [10], it is known that $e_{\text{HK}}(A) = \frac{29p^2+15}{24p^2+12}$. Suppose that A is not isomorphic to $A_{p,4}$. Then one can take a minimal numbers of generators x, y, z, w, u of \mathfrak{m} and one can define a function $\text{ord} : A \rightarrow \mathbb{Q} \cup \{\infty\}$ such that

$$\text{ord}(x) = \text{ord}(y) = \text{ord}(z) = \text{ord}(w) = \frac{1}{2}, \quad \text{ord}(u) = \frac{1}{3}.$$

If we put $J = (y, z, w, u)A$ and $F_n = \{\alpha \in A : \text{ord}(\alpha) \geq n\}$, then by a similar argument as in the proof of Proposition 3.9, we have

$$l_A(\mathfrak{m}^{[q]}/J^{[q]}) \leq 2 \cdot l_A(A/J^{[q]} + F_{2q/3}).$$

Divided the both-side by q^d and taking a limit $q \rightarrow \infty$, we get

$$e(A) - e_{\text{HK}}(A) \leq 2 \cdot e(A) \cdot \text{vol} \left\{ (y, z, w, u) \in [0, 1]^4 : \frac{y}{2} + \frac{z}{2} + \frac{w}{2} + \frac{u}{3} \leq \frac{2}{3} \right\}.$$

To calculate the volume in the right-hand side, we put

$$F_u = \begin{cases} \frac{1}{6} \left(\frac{4}{3} - \frac{2}{3}u\right)^3 - 3 \cdot \frac{1}{6} \left(\frac{1}{3} - \frac{2}{3}u\right)^3 & (0 \leq u \leq \frac{1}{2}) \\ \frac{1}{6} \left(\frac{4}{3} - \frac{2}{3}u\right)^3 & (\frac{1}{2} \leq u \leq 1) \end{cases}$$

Then one can easily calculate

$$\text{the above volume} = \int_0^1 F_u \, du = \frac{237}{2^4 3^4}.$$

It follows that

$$e_{\text{HK}}(A) \geq 2 - 4 \times \frac{237}{2^4 3^4} = \frac{411}{324} > \frac{5}{4}.$$

□

The following conjecture also holds if $\dim A \leq 4$.

CONJECTURE 4.4. *Under the same notation as in Conjecture 4.2, if $e(A) \geq 3$, then*

$$e_{\text{HK}}(A) \geq 1 + \frac{c_d + 1}{d!}.$$

DISCUSSION 4.5. Let $d \geq 2$ be an integer and fix a prime number $p \gg d$. Assume that Conjectures 4.2 and 4.4 are true. Also, assume that $s_{\text{HK}}(p, d) < s_{\text{HK}}(p, d - 1)$ for all $d \geq 3$. Let $A = k[X_0, \dots, X_v]/I$ be a d -dimensional homogeneous unmixed k -algebra with $\deg X_i = 1$, and let \mathfrak{m} be the unique homogeneous maximal ideal of A . Suppose that k is an algebraically closed field of characteristic $p > 0$. Then $e_{\text{HK}}(A) = s_{\text{HK}}(p, d)$ implies that $\widehat{A}_{\mathfrak{m}} \cong A_{p,d}$.

In fact, if $e_{\text{HK}}(A) = s_{\text{HK}}(p, d)$, then we may assume that $e_{\text{HK}}(A) < 1 + \frac{c_d + 1}{d!}$. Thus $e(A_{\mathfrak{m}}) = 2$ if Conjecture 4.4 is true. For any prime ideal $PA_{\mathfrak{m}}$ of $A_{\mathfrak{m}}$ such that $P \neq \mathfrak{m}$, we have $e_{\text{HK}}(A_P) \leq e_{\text{HK}}(A_{\mathfrak{m}}) = s_{\text{HK}}(p, d) < s_{\text{HK}}(p, n)$, where $n = \dim A_P < d$. Since A_P is also unmixed, it is regular. Thus $A_{\mathfrak{m}}$ has an isolated singularity. Hence A is a non-degenerate quadric hypersurface. In other words, $\widehat{A}_{\mathfrak{m}}$ is isomorphic to $A_{p,d}$.

REFERENCES

- [1] R. O. Buchweitz and Q. Chen, *Hilbert-Kunz functions of cubic curves and surfaces*, J. Algebra, **197** (1997), 246–267.
- [2] R. O. Buchweitz, Q. Chen and K. Pardue, *Hilbert-Kunz functions*, preprint.
- [3] M. Blickle and F. Enescu, *On rings with small Hilbert-Kunz multiplicity*, Proc. Amer. Math. Soc., **132** (2004), 2505–2509.
- [4] A. Conca, *Hilbert-Kunz functions of monomials and binomial hypersurfaces*, Manuscripta Math., **90** (1996), 287–300.

- [5] K. Eto and K. Yoshida, *Notes on Hilbert-Kunz multiplicity of Rees algebras*, Comm. Algebra, **31** (2003), 5943–5976.
- [6] R. Fedder and K.-i. Watanabe, *A characterization of F -regularity in terms of F -purity*, Commutative algebra (Berkeley, CA, 1987), Math. Sci. Research Inst. Publ., vol. 15, Springer-Verlag, New York (1989), pp. 227–245.
- [7] N. Fakhruddin and V. Trivedi, *Hilbert-Kunz functions and multiplicities for full flag varieties and elliptic curves*, J. Pure Appl. Algebra, **181** (2003), 23–52.
- [8] S. Goto and Y. Nakamura, *Multiplicity and tight closures of parameters*, J. Algebra, **244** (2001), 302–311.
- [9] D. Hanes, *Notes on the Hilbert-Kunz function*, Comm. Algebra, **30** (2002), 3789–3812.
- [10] C. Han and P. Monsky, *Some surprising Hilbert-Kunz functions*, Math. Z., **214** (1993), 119–135.
- [11] M. Hochster and C. Huneke, *Tight closure, invariant theory, and the Briançon-Skoda theorem*, J. Amer. Math. Soc., **3** (1990), 31–116.
- [12] M. Hochster and C. Huneke, *F -regularity, test elements, and smooth base change*, Trans. Amer. Math. Soc., **346** (1994), 1–62.
- [13] C. Huneke, *Tight closure and its applications*, American Mathematical Society, 1996.
- [14] C. Huneke and Y. Yao, *Unmixed local rings with minimal Hilbert-Kunz multiplicity are regular*, Proc. Amer. Math. Soc., **130** (2002), 661–665.
- [15] E. Kunz, *Characterizations of regular local rings of characteristic p* , Amer. J. Math., **41** (1969), 772–784.
- [16] E. Kunz, *On Noetherian rings of characteristic p* , Amer. J. Math., **88** (1976), 999–1013.
- [17] H. Matsumura, *Commutative ring theory*, Cambridge University Press, 1986.
- [18] P. Monsky, *The Hilbert-Kunz function*, Math. Ann., **263** (1983), 43–49.
- [19] P. Monsky, *A personal letter from Monsky to K.-i. Watanabe*.
- [20] M. Nagata, *Local rings*, Interscience, 1962.
- [21] D. Rees, *A note on analytically unramified local rings*, J. London Math. Soc., **36** (1961), 24–28.
- [22] K.-i. Watanabe and K. Yoshida, *Hilbert-Kunz multiplicity and an inequality between multiplicity and colength*, J. Algebra., **230** (2000), 295–317.
- [23] K.-i. Watanabe and K. Yoshida, *Hilbert-Kunz multiplicity of two-dimensional local rings*, Nagoya Math. J., **162** (2001), 87–110.
- [24] K.-i. Watanabe and K. Yoshida, *Hilbert-Kunz multiplicity, McKay correspondence and good ideals in two-dimensional rational singularities*, Manuscripta Math., **104** (2001), 275–294.

Kei-ichi Watanabe
Department of Mathematics
College of Humanities and Sciences
Nihon University
Setagaya-ku
Tokyo 156-0045
Japan
watanabe@math.chs.nihon-u.ac.jp

Ken-ichi Yoshida
Graduate School of Mathematics
Nagoya University
Chikusa-ku
Nagoya 464-8602
Japan
yoshida@math.nagoya-u.ac.jp