

## ON ELLIPTIC CURVES IN $SL_2(\mathbb{C})/\Gamma$ , SCHANUEL'S CONJECTURE AND GEODESIC LENGTHS

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*Dedicated to Sakiko*

**Abstract.** Let  $\Gamma$  be a discrete cocompact subgroup of  $SL_2(\mathbb{C})$ . We conjecture that the quotient manifold  $X = SL_2(\mathbb{C})/\Gamma$  contains infinitely many non-isogenous elliptic curves and prove this is indeed the case if Schanuel's conjecture holds. We also prove it in the special case where  $\Gamma \cap SL_2(\mathbb{R})$  is cocompact in  $SL_2(\mathbb{R})$ .

Furthermore, we deduce some consequences for the geodesic length spectra of real hyperbolic 2- and 3-folds.

### §1. Introduction

Let  $\Gamma$  be a discrete cocompact subgroup of  $SL_2(\mathbb{C})$ . We are interested in closed complex analytic subspaces of the complex quotient manifold  $X = SL_2(\mathbb{C})/\Gamma$ . It is well-known that  $X$  contains no hypersurfaces and it is easy to show that it contains no curves of genus 0. The existence of curves of genus  $\geq 2$  is an unsolved problem.

On the other hand, it is not hard to show that there do exist curves of genus one (elliptic curves). (For these assertions, see [3], [9].)

Our goal is to investigate how many different curves of genus one can be embedded in one such quotient manifold. There are only countably many abelian varieties which can be embedded into a quotient manifold of a complex semisimple Lie group by a discrete cocompact subgroup ([9, Cor. 4.6.2]). Thus the question is: Is the number of non-isomorphic elliptic curves in such a quotient  $SL_2(\mathbb{C})/\Gamma$  finite or countably infinite?

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Under the additional assumption that  $\Gamma \cap SL_2(\mathbb{R})$  is cocompact in  $SL_2(\mathbb{R})$  we show that there are infinitely many isogeny classes of elliptic curves in  $X$  (Thm. 2). We will see that there do exist discrete cocompact subgroups in  $SL_2(\mathbb{C})$  with this property (Cor. 3). We conjecture that this additional assumption ( $\Gamma \cap SL_2(\mathbb{R})$  being cocompact in  $SL_2(\mathbb{R})$ ) is not needed and show that it can be dropped provided *Schanuel's conjecture* is true (see Cor. 2).

In order to show that there are infinitely many non-isogenous elliptic curves, one first has to construct elliptic curves and then one has to investigate under which conditions they are isogenous. There is a well-known way to construct elliptic curves in  $X = SL_2(\mathbb{C})/\Gamma$ , going back to ideas of Mostow ([4]). In fact every elliptic curve in  $X$  arises in this way ([3]). This method works as follows: If  $\gamma \in \Gamma$  is a semisimple element of infinite order, then the centralizer  $C = \{g \in SL_2(\mathbb{C}) : g\gamma = \gamma g\}$  is isomorphic to  $\mathbb{C}^*$  as a complex Lie group and  $C \cap \Gamma$  is a discrete subgroup containing  $\gamma$  and therefore commensurable with  $\{\gamma^k : k \in \mathbb{Z}\}$ . The quotient of  $\mathbb{C}^*$  by an infinite discrete subgroup is necessarily compact. Hence for every semisimple element  $\gamma \in \Gamma$  of infinite order we obtain an elliptic curve  $E \subset X = SL_2(\mathbb{C})/\Gamma$  which arises as orbit of the centralizer  $C$ . Moreover, this elliptic curve  $E \simeq C/(C \cap \Gamma)$  is isogenous to  $C/\langle \gamma \rangle$  and therefore isogenous to  $\mathbb{C}^*/\langle \lambda \rangle$  where  $\lambda$  and  $\lambda^{-1}$  are the eigenvalues of the matrix  $\gamma \in SL_2(\mathbb{C})$ .

Thus our problem is to investigate how many different eigenvalues occur and under which circumstances different eigenvalues lead to non-isogenous elliptic curves.

First we show that for every Zariski-dense subgroup  $\Gamma \subset SL_2(\mathbb{C})$  there are infinitely many pairwise multiplicatively independent complex numbers occurring as eigenvalues for elements of  $\Gamma$  (Thm. 1).

We conjecture that, if the eigenvalues are algebraic numbers (this is known to be the case if  $\Gamma$  is cocompact), then multiplicatively independent eigenvalues always lead to non-isogenous elliptic curves. We can prove that this conjecture holds if Schanuel's conjecture from transcendental number theory is true.

Even without assuming Schanuel's conjecture to be true we can prove the existence of infinitely many non-isogenous elliptic curves in the case where the eigenvalues are real.

In this way we obtained the desired result in the special case where the intersection  $\Gamma \cap SL_2(\mathbb{R})$  is cocompact in  $SL_2(\mathbb{R})$ .

Using an arithmetic construction one can show that discrete cocompact subgroups  $\Gamma$  for which  $\Gamma \cap SL_2(\mathbb{R})$  is cocompact in  $SL_2(\mathbb{R})$  do indeed exist.

These results on elliptic curves in  $SL_2(\mathbb{C})/\Gamma$  can be related to questions on the length of closed geodesics on real hyperbolic manifolds of dimension 2 or 3. More precisely, let  $M$  be a compact real Riemannian manifold (without boundary) of dimension 2 or 3 which carries a Riemannian metric of constant negative curvature. Let  $\Lambda$  be set of all positive real numbers occuring as length of a closed geodesic on  $M$ . Then  $\Lambda$  contains infinitely many elements which are pairwise linearly independent over  $\mathbb{Q}$  (Thm. 3).

## §2. Multiplicatively independent eigenvalues

### 2.1. Announcement of Theorem 1

DEFINITION. Two non-zero elements  $x, y$  in a field  $k$  are called *multiplicatively dependent* if there exists a pair  $(p, q) \in \mathbb{Z} \times \mathbb{Z} \setminus \{(0, 0)\}$  such that  $x^q = y^p$ .

They are called *multiplicatively independent* if they are not multiplicatively dependent.

By this definition a root of unity is multiplicatively dependent with every other element of  $k^*$ . Thus, if  $x, y \in k^*$  are multiplicatively independent, this implies in particular that neither  $x$  nor  $y$  is a root of unity.

Note that being multiplicatively dependent is an equivalence relation on the set of all elements of  $k^*$  which are not roots of unity.

The purpose to of this section is to prove the following theorem:

THEOREM 1. *Let  $\Gamma$  be a subgroup of  $SL_2(\mathbb{C})$  which is dense in the algebraic Zariski topology.*

*Then there exists infinitely many pairwise multiplicatively independent complex numbers  $\lambda$  which occur as eigenvalues for elements of  $\Gamma$ .*

### 2.2. A fact from combinatorics

As a preparation for the proof of Thm. 1 we need a combinatorial fact.

LEMMA 1. *Let  $S$  be a finite set,  $\phi : \mathbb{N} \rightarrow S$  a map.*

*Then there exists a natural number  $N \leq \#S$  and an element  $s \in S$  such that*

$$A_{s,N} = \{x \in \mathbb{N} : \phi(x) = s = \phi(x + N)\}$$

*is infinite.*

*Proof.* Assume the contrary. Then  $A_{s,N}$  is a finite set for all  $s \in S$ ,  $1 \leq N \leq \#S$ . Hence there is a number  $M \in \mathbb{N}$  such that  $x < M$  for all  $x \in \bigcup_{s \in S} \bigcup_{N \leq \#S} A_{s,N}$ .

But this implies that  $\phi(M+i) \neq \phi(M+j)$  for all  $0 \leq i < j \leq \#S$ , which is impossible by the pigeon-hole principle.  $\square$

### 2.3. Roots in finitely generated fields

We need the following well-known fact on finitely generated fields.

LEMMA 2. *Let  $K$  be a finitely generated field extension of  $\mathbb{Q}$ .*

*Then for every element  $x \in K$  one of the properties hold:*

- $x = 0$ ,
- $x$  is an invertible algebraic integer (i.e. a unit) or
- there exists a discrete valuation  $v : K^* \rightarrow \mathbb{Z}$  with  $v(x) \neq 0$ .

For the convenience of the reader we sketch a proof.

*Proof.* Let  $K_0$  denote the algebraic closure of  $\mathbb{Q}$  in  $K$ . Then  $K_0$  is a number field and  $K$  can be regarded as function field of a projective variety  $V$  defined over  $K_0$ . Let  $f \in K$ . If  $f \notin K_0$ , then  $f$  is a non-constant rational function and therefore there is a discrete valuation given by the pole/zero-order along a hypersurface which does not annihilate  $f$ . If  $f \in K_0$ , then either  $f = 0$ , or  $f$  is a unit, i.e. an invertible algebraic integer or an extension of a  $p$ -adic valuation is non-zero for  $f$ .  $\square$

Let  $K$  be a field and  $W_K$  the group of roots of unity contained in  $K$ . Let  $x \in K^*$ . We want to measure up to which degree  $d$  it is possible to find a  $d$ -th root of  $x$  in  $K$  (modulo  $W_K$ ). For this purpose we define

$$\rho_K(x) = \sup\{n \in \mathbb{N} : \exists \alpha \in K : \alpha^n x^{-1} \in W_K\} \in \mathbb{N} \cup \{\infty\}.$$

LEMMA 3. *Let  $K$  be a finitely generated field extension of  $\mathbb{Q}$  and  $x \in K^*$ . Then  $\rho_K(x) < \infty$  unless  $x$  is a root of unity.*

*Proof.* Let  $x$  be an element of  $K^*$  which is not a root of unity. First we discuss the case in which there exists a discrete valuation  $v : K^* \rightarrow \mathbb{Z}$  with  $v(x) \neq 0$ . In this case  $\alpha^n x^{-1} \in W_K$  for  $\alpha \in K$  implies  $v(\alpha) = \frac{1}{n}v(x) \in \mathbb{Z}$ . Therefore  $\rho_K(x) \leq |v(x)|$  in this case.

Now let us discuss the case where every discrete valuation on  $K$  annihilates  $x$ . By Lemma 2, this implies that  $x$  is contained in the algebraic

closure  $K_0$  of  $\mathbb{Q}$  in  $K$  and moreover that  $x \in \mathcal{O}_{K_0}^*$ , i.e.  $x$  is an invertible algebraic integer. Assume that there are elements  $\alpha \in K$ ,  $w \in W_K$  and  $n \in \mathbb{N}$  such that  $\alpha^n = xw$ . Then  $\alpha^{nN} = x^N$  for some  $N \in \mathbb{N}$ . As a consequence,  $\alpha$  is integral over  $\mathcal{O}_{K_0}$ . Similarly,  $\alpha^{-nN} = x^{-N}$  implies that  $\alpha^{-1}$  is integral over  $\mathcal{O}_{K_0}$ . Thus we obtain: *If  $\alpha^n x^{-1} \in W_K$  for some  $\alpha \in K$  and  $n \in \mathbb{N}$ , then  $\alpha \in \mathcal{O}_{K_0}^*$ .*

Therefore

$$\rho_K(x) = \sup\{n \in \mathbb{N} : \exists \alpha \in \mathcal{O}_{K_0}^* : \alpha^n x^{-1} \in W_K\}.$$

A theorem of Dirichlet states that  $\mathcal{O}_{K_0}^*$  is a finitely generated abelian group (with respect to multiplication). Thus  $\mathcal{O}_{K_0}^*/W_K \simeq \mathbb{Z}^d$  for some  $d \in \mathbb{N}$ . This implies  $\rho_K(x) < \infty$ . □

LEMMA 4. *Let  $K$  be a field,  $x \in K^*$  with  $\rho_K(x) < \infty$ . Assume that there are integers  $p \in \mathbb{Z}$ ,  $q \in \mathbb{Z} \setminus \{0\}$  and an element  $\beta \in K^*$  such that  $\beta^q x^{-p} \in W_K$ .*

*Then  $\frac{p}{q}\rho_K(x) \in \mathbb{Z}$ .*

*Proof.* Let  $n = \rho_K(x)$ . Assume that  $\frac{p}{q}n \notin \mathbb{Z}$  and let  $\Gamma$  denote the additive subgroup of  $\mathbb{Q}$  generated by  $1/n$  and  $p/q$ . Now  $\frac{1}{n}\mathbb{Z} \subsetneq \Gamma$ , hence there is a natural number  $N > n$  such that  $\Gamma = \frac{1}{N}\mathbb{Z}$ . Since  $\Gamma$  is generated by  $1/n$  and  $p/q$ , there are integers  $k, m \in \mathbb{Z}$  such that

$$k\frac{1}{n} + m\frac{p}{q} = \frac{kq + nmp}{nq} = \frac{1}{N}.$$

Since  $n = \rho_K(x)$ , there is an element  $\alpha \in K^*$  with  $\alpha^n x^{-1} \in W_K$ . Now we define

$$\gamma = \alpha^k \beta^m.$$

We claim that  $\gamma^N x^{-1} \in W_K$ . Indeed, since  $N = nq/(kq + nmp)$ , this condition is equivalent to  $\gamma^{nq} x^{-kq - nmp} \in W_K$  which can be verified as follows:

$$\gamma^{nq} x^{-kq - nmp} = \alpha^{knq} \beta^{mnq} x^{-kq - nmp} = (\alpha^n x^{-1})^{kq} (\beta^q x^{-p})^{nm} \in W_K.$$

But  $\gamma^N x^{-1} \in W_K$  implies  $\rho_K(x) \geq N$ , contradicting  $N > n = \rho_K(x)$ .

Thus we see that  $p/q$  must be contained in  $\frac{1}{n}\mathbb{Z}$ . □

The statement of the lemma may be reformulated in the following way:

COROLLARY 1. *Let  $K$  be a field,  $x \in K^*$  with  $\rho_K(x) < \infty$ . Let*

$$\Theta_{K,x} = \left\{ \frac{p}{q} \in \mathbb{Q} : \exists \beta \in K^* : \beta^q x^{-p} \in W_K \right\}.$$

*Then  $\Theta_{K,x}$  is a discrete subgroup of  $(\mathbb{Q}, +)$ , generated by  $1/\rho_K(x)$ .*

Next we verify that the behaviour of  $\rho_K(x)$  under finite field extensions is as to be expected.

LEMMA 5. *Let  $L/K$  be a finite field extension of degree  $d$  and  $x \in K^*$  with  $\rho_K(x) < \infty$ .*

*Then there exists a natural number  $s$  which divides  $d$  such that  $\rho_L(x) = s\rho_K(x)$ .*

*Proof.* In the notation of Cor. 1  $\Theta_{K,x}$  is a subgroup of  $\Theta_{L,x}$ .

On the other hand, if there is an element  $\beta \in L^*$  and a natural number  $n$  such that  $\beta^n x^{-1} \in W_L$ , then

$$N_{L/K}(\beta^n x^{-1}) = (N_{L/K}(\beta))^n x^{-d} \in W_K$$

and consequently  $\frac{d}{n}\rho_K(x) \in \mathbb{Z}$  (Lemma 4). Thus  $1/n \in \frac{1}{\rho_L(x)}\mathbb{Z}$  implies  $1/n \in \frac{1}{d\rho_K(x)}\mathbb{Z}$ .

Combined, these facts yield

$$\frac{1}{\rho_K(x)}\mathbb{Z} \subset \frac{1}{\rho_L(x)}\mathbb{Z} \subset \frac{1}{d\rho_K(x)}\mathbb{Z}.$$

This implies the statement of the lemma. □

### 2.4. An auxiliary proposition

PROPOSITION 1. *Let  $K$  be a finitely generated field extension of  $\mathbb{Q}$ ,  $\bar{K}$  an algebraic closure,  $S$  a finite subset of  $K^*$  and  $\Lambda \subset \bar{K}^*$  a subset such that the following properties are fulfilled:*

- (1)  $\deg K(\lambda)/K \leq 2$  for every  $\lambda \in \Lambda$ ,
- (2) for every  $\lambda \in \Lambda$  there exists an element  $\mu \in S$  and integers  $p, q \in \mathbb{Z} \setminus \{0\}$  such that  $\lambda^p = \mu^q$ .

*Then there exists a finite subgroup  $W \subset \bar{K}^*$  and a finite subset  $S' \subset \bar{K}^*$  such that for every  $\lambda \in \Lambda$  there exists an element  $\alpha \in S'$ , an integer  $N \in \mathbb{Z}$  and an element  $w \in W$  such that  $\alpha^N w = \lambda$ .*

*Moreover, the set  $S'$  can be chosen in such a way that none of its elements is a root of unity.*

*Proof.* For each element  $\mu \in S$  which is not a root of unity we choose an element  $\alpha_\mu \in \bar{K}^*$  such that

$$(\alpha_\mu)^{2\rho_K(\mu)} = \mu.$$

Let  $S'$  be the set of all these elements  $\alpha_\mu$ . Evidently none of these elements  $\alpha_\mu$  is a root of unity. Let  $L$  denote the field generated by  $K$  and the elements of  $S'$ . Note that  $L$  is a finitely generated field. Let  $L_0$  denote the algebraic closure of  $\mathbb{Q}$  in  $L$ . Then  $L_0$  is a number field. Let  $d_0$  denote its degree (over  $\mathbb{Q}$ ). Recall that for any natural number, in particular for  $2d_0$ , there are only finitely many roots of unity of degree not greater than this number. Let  $W$  be the set of all roots of unity  $w$  in  $\bar{K}^*$  for which  $\deg(L(w)/L) \leq 2$ . Then  $\deg(\mathbb{Q}(w)/\mathbb{Q}) \leq 2d_0$  for every  $w \in W$ . Therefore  $W$  is a finite group. By construction it contains every root of unity which is in  $L(\lambda)$  for some  $\lambda \in \Lambda$ .

Now choose an arbitrary element  $\lambda \in \Lambda$ . If  $\lambda$  is a root of unity, it is contained in  $W$  implying that  $\lambda = \alpha^0 w$  for  $w = \lambda$  and  $\alpha$  arbitrary. Thus we may assume that  $\lambda$  is not a root of unity. There are integers  $p, q \in \mathbb{Z} \setminus \{0\}$  and an element  $\mu \in S$  such that  $\lambda^p = \mu^q$ . Since  $\lambda$  is not a root of unity, this implies that neither  $\mu$  can be a root of unity. Thus  $\rho_K(\mu) < \infty$  (Lemma 3) and there is an element  $\alpha_\mu \in S'$  with  $\alpha_\mu^{2\rho_K(\mu)} = \mu$ .

By Lemma 4 the equality  $\mu^q = \lambda^p$  implies

$$\frac{q}{p} \rho_{K(\lambda)}(\mu) \in \mathbb{Z}.$$

Thanks to Lemma 5 we know that either  $\rho_{K(\lambda)}(\mu) = \rho_K(\mu)$  or  $\rho_{K(\lambda)}(\mu) = 2\rho_K(\mu)$ . In both cases it follows that

$$2\frac{q}{p} \rho_K(\mu) \in \mathbb{Z}.$$

In other words, there is an integer  $N \in \mathbb{Z}$  such that  $2q\rho_K(\mu) = pN$ . Therefore

$$(\alpha_\mu^N)^p = \alpha_\mu^{pN} = \alpha_\mu^{2q\rho_K(\mu)} = \mu^q = \lambda^p.$$

Hence  $(\alpha_\mu^{-N}\lambda)^p = 1$ . Let  $w = \alpha_\mu^{-N}\lambda$ . Then  $w$  is a root of unity which is contained in the field  $L(\lambda)$ . It follows that  $w \in W$ . Thus we have verified that there exist elements  $\alpha \in S'$ ,  $N \in \mathbb{Z}$  and  $w \in W$  such that  $\alpha^N w = \lambda$ .  $\square$

**2.5. Proof of Theorem 1**

*Proof.* If  $\Gamma$  is a Zariski-dense subgroup of  $SL_2(\mathbb{C})$ , then  $\Gamma$  contains a finitely generated torsion-free subgroup  $\Gamma_0$  which is still Zariski-dense (see [9, Lemma 1.7.12 and Prop. 1.7.2]). Fix a finite set  $E$  of generators of  $\Gamma_0$ . Let  $k$  be the field generated by all the matrix coefficients of elements of  $E$ . Then  $k$  is a finitely generated extension field of  $\mathbb{Q}$  and  $\Gamma_0 \subset SL_2(k)$ .

Let  $\Lambda$  be the set of all complex numbers other than 1 and  $-1$  occurring as an eigenvalue for an element  $\gamma \in \Gamma_0$ . We observe that a number  $\lambda \in \mathbb{C}^* \setminus \{1, -1\}$  is contained in  $\Lambda$  if and only if there exists an element  $A \in \Gamma_0$  such that  $Tr(A) = \lambda + \lambda^{-1}$ . Since  $\Gamma_0$  is Zariski dense, the set

$$\{Tr(A) : A \in \Gamma_0\}$$

is Zariski dense in  $\mathbb{C}$ . It follows that  $\Lambda$  is an infinite set.

We claim that  $\Lambda$  contains no root of unity. Indeed, assume that a root of unity  $\omega$  is contained in  $\Lambda$ . Then  $\omega \neq 1, -1$  and consequently  $\omega \neq \omega^{-1}$ . Therefore every element  $A \in SL_2(\mathbb{C})$  with  $\omega$  as an eigenvalue is conjugate to

$$\begin{pmatrix} \omega & \\ & \omega^{-1} \end{pmatrix}.$$

As a consequence, such a matrix  $A$  is of finite order. This contradicts the assumption that  $\Gamma_0$  is torsion-free. Thus  $\Lambda$  can not contain any root of unity.

Let  $\Sigma$  denote the set of all complex numbers which are roots of unity. As remarked before, the notion of “multiplicative dependence” defines an equivalence relation on  $\mathbb{C}^* \setminus \Sigma$ .

Let us assume that the statement of the theorem fails. Since  $\Lambda \subset \mathbb{C}^* \setminus \Sigma$  and since “multiplicative dependence” defines an equivalence relation on  $\mathbb{C}^* \setminus \Sigma$ , it follows that there is a finite set  $S$  and complex numbers  $(\mu_i)_{i \in S} \in \mathbb{C}^* \setminus \Sigma$  such that for every  $\lambda \in \Lambda$  there exists an index  $i \in S$  and non-zero integers  $p, q \in \mathbb{Z} \setminus \{0\}$  with  $\lambda^p = \mu_i^q$ .

Let  $K$  denote the field generated by  $k$  and all the elements  $\mu_i$  ( $i \in S$ ). Recall that every element of  $\Lambda$  is an eigenvalue for a matrix in  $SL_2(k) \subset SL_2(K)$ . Therefore  $\deg(K(\lambda)/K) \leq 2$  for every  $\lambda \in \Lambda$ .

We may now invoke Proposition 1.

Thus we obtain the following statement: *There are finitely many complex numbers  $(\alpha_i)_{i \in S'}$ , none of which is a root of unity, and a finite subgroup  $W$  of the multiplicative group  $\mathbb{C}^*$  such that for every  $\lambda \in \Lambda$  there are  $i \in S'$ ,  $n \in \mathbb{Z}$  and  $w \in W$  such that  $\lambda = \alpha_i^n w$ .*



Evidently we may assume that the numbers  $\alpha_i$  are pairwise multiplicatively independent. Thus we make this assumption.

By adjoining all the elements of  $W$  and all the  $\alpha_i$  ( $i \in S'$ ) to  $K$ , we also may deduce that in this case there exists a finitely generated field  $L$  containing all the  $\alpha_i$  ( $i \in S'$ ) and all  $\lambda \in \Lambda$  and  $w \in W$ .

Let  $\lambda \in \Lambda$ ,  $\zeta \in S'$ ,  $q \in \mathbb{Z} \setminus \{0\}$  and  $w_0 \in W$  such that  $\lambda = w_0 \alpha_\zeta^q$ .

Then, after replacing  $\Gamma_0$  by  $g\Gamma_0 g^{-1}$  for an appropriately chosen  $g \in SL_2(\mathbb{C})$ , we obtain

$$\Gamma_0 \ni \gamma = \begin{pmatrix} \lambda & \\ & \lambda^{-1} \end{pmatrix} = \begin{pmatrix} w_0 \alpha_\zeta^q & \\ & w_0^{-1} \alpha_\zeta^{-q} \end{pmatrix}.$$

By the assumption of Zariski density  $\Gamma_0$  must also contain an element  $\delta \in \Gamma_0$  which does not commute with  $\gamma$ . Let

$$\delta = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

be such an element. By the assumption of Zariski density of  $\Gamma_0$  we may and do require that  $a, d \neq 0$ .

Let  $g_n = \gamma^n \delta$  for  $n \in \mathbb{N}$ .

Using Lemma 1, we conclude that there exists a natural number  $N$ , an infinite subset  $A \subset \mathbb{N}$ , an index  $\xi$ , an element  $\tilde{w} \in W$  and sequences of non-zero integers  $m_k, m'_k \in \mathbb{Z} \setminus \{0\}$  such that  $\tilde{w} \alpha_\xi^{m_k}$  resp.  $\tilde{w} \alpha_\xi^{m'_k}$  is an eigenvalue of  $g_k$  resp.  $g_{k+N}$  for all  $k \in A$ . Moreover, we may assume that all the numbers  $m_k$  and  $m'_k$  have the same sign.

Since  $w_0$  is a root of unity, we may invoke the pigeon-hole principle in order to deduce that (by replacing  $A$  with an appropriate smaller set) we may assume that there is an element  $w_1 \in W$  such that  $w_0^k = w_1$  for all  $k \in A$ . Let  $w_2 = w_1 w_0^N$ . Then  $w_2 = w_0^{k+N}$  for all  $k \in \mathbb{N}$ .

Now recall that for an element  $g \in SL_2(\mathbb{C})$  with eigenvalues  $\lambda, \lambda^{-1}$  we have  $Tr(g) = \lambda + \lambda^{-1}$ .

It follows that

$$(1) \quad Tr(\gamma^k \delta) = w_1 \alpha_\zeta^{qk} a + w_1^{-1} \alpha_\zeta^{-qk} d = \tilde{w} \alpha_\xi^{m_k} + \tilde{w}^{-1} \alpha_\xi^{-m_k}$$

and

$$(2) \quad Tr(\gamma^{k+N} \delta) = w_2 \alpha_\zeta^{q(k+N)} a + w_2^{-1} \alpha_\zeta^{-q(k+N)} d = \tilde{w} \alpha_\xi^{m'_k} + \tilde{w}^{-1} \alpha_\xi^{-m'_k}$$

for all  $k \in A$ .

Recall that  $\alpha_\zeta$  is contained in the finitely generated field  $L$  and is not a root of unity. Therefore there exists an absolute value  $|\cdot|$  on  $L$  such that  $|\alpha_\zeta| \neq 1$ . In what follows,  $|\cdot|$  always denotes this (possibly non-archimedean) absolute value on  $L$ .

Using  $|\alpha_\zeta| \neq 1$  and  $a, d, q \neq 0$  we obtain

$$\lim_{k \rightarrow \infty} |w_1 \alpha_\zeta^{qk} a + w_1^{-1} \alpha_\zeta^{-qk} d| = +\infty.$$

Combined with eq. (1), this yields

$$\lim_{k \rightarrow \infty} |\tilde{w} \alpha_\xi^{m_k} + \tilde{w}^{-1} \alpha_\xi^{-m_k}| = +\infty.$$

This is only possible if  $|\alpha_\xi| \neq 1$ .

Without loss of generality we may assume that  $|\alpha_\zeta|, |\alpha_\xi| > 1, q > 0$  and  $m_k, m'_k > 0$  for all  $k \in A$ .

Then

$$\lim_{k \rightarrow \infty} \alpha_\zeta^{-qk} = 0 = \lim_{k \rightarrow \infty} \alpha_\xi^{-m_k} = \lim_{k \rightarrow \infty} \alpha_\xi^{-m'_k}.$$

It follows that the quotient of the respective left hand sides of the equations (2) and (1) converges to  $\alpha_\zeta^{qN} \frac{w_2}{w_1}$ . Evidently the quotient of the respective right hand sides converges to the same value. Hence:

$$\alpha_\zeta^{qN} \frac{w_2}{w_1} = \lim_{k \rightarrow \infty, k \in A} \alpha_\xi^{m'_k - m_k}.$$

The set  $\{\alpha_\xi^n : n \in \mathbb{Z}\}$  is discrete in  $L^*$ , because  $|\alpha_\xi| \neq 1$ . Therefore

$$\alpha_\zeta^{qN} \frac{w_2}{w_1} = \alpha_\xi^{m'_k - m_k}$$

for all sufficiently large  $k$  in  $A$ .

Recall that  $q, N \neq 0$  and  $w_1, w_2 \in W$ . It follows that  $\alpha_\zeta$  and  $\alpha_\xi$  are multiplicatively dependent. But we assumed the numbers  $(\alpha_j)_{j \in S}$  to be multiplicatively independent. Therefore  $\xi = \zeta$ .

By considering the quotient of the right hand side of eq. (1) and its left hand side, we obtain:

$$1 = \lim_{k \rightarrow \infty, k \in A} \frac{\tilde{w}}{w_1 a} \alpha_\xi^{m_k - qk}.$$

Therefore:

$$(3) \quad a = \lim_{k \rightarrow \infty, k \in A} \frac{\tilde{w}}{w_1} \alpha_\xi^{m_k - qk}$$

and consequently

$$a = \frac{\tilde{w}}{w_1} \alpha_\xi^{m_k - qk}$$

for all sufficiently large  $k$  in  $A$ .

Together with eq. (1) this implies that

$$w_1 \alpha_\xi^{qk} a = \tilde{w} \alpha_\xi^{m_k} \quad \text{and} \quad w_1^{-1} \alpha_\xi^{-qk} d = \tilde{w}^{-1} \alpha_\xi^{-m_k}.$$

Combining these two equalities we obtain  $ad = 1$ . Now recall that  $\delta$  was an arbitrarily chosen element in the intersection of  $\Gamma_0$  with the Zariski open subset

$$\Omega = \left\{ A = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in SL_2 : a, d \neq 0, A\gamma \neq \gamma A \right\}.$$

Note that the condition  $A\gamma = \gamma A$  implies that  $A$  is a diagonal matrix and therefore implies that  $ad = 1$ .

Thus we have deduced: *Every element of  $\Gamma_0$  is contained in the algebraic subvariety*

$$\left\{ \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in SL_2 : ad = 1 \text{ or } ad = 0 \right\}.$$

But this contradicts the assumption that  $\Gamma_0$  is Zariski-dense. □

### 2.6. On the absolute values of eigenvalues

For our main goal (i.e. studying elliptic curves in quotients of  $SL_2(\mathbb{C})$ ) we need only to consider the eigenvalues. However, from the point of view of possible applications to the study of geodesic length spectra of real hyperbolic manifolds (see Section 6 below) it might be interesting to deduce a similar result for the absolute values of the eigenvalues. This is the purpose of this subsection.

**PROPOSITION 2.** *Let  $\Gamma$  be a subgroup of  $SL_2(\mathbb{C})$  which is dense in the algebraic Zariski topology.*

*Then there exist infinitely many pairwise multiplicatively independent positive real numbers which occur as the absolute value of an eigenvalue for an element of  $\Gamma$ .*

*Proof.* First we note that  $|z| = \sqrt{z\bar{z}}$  for any complex number. Using this fact, it is clear that for every finitely generated subgroup  $\Gamma$  of  $SL_2(\mathbb{C})$

there is a finitely generated field  $k$  such that every absolute value of an eigenvalue for an element of  $\Gamma$  is contained in a finite extension field of degree at most 4 over  $k$ : We just have to take  $k$  to be the extension field of  $\mathbb{Q}$  generated by all the coefficients and their complex conjugates for all elements in some fixed finite set of generators for  $\Gamma$ .

Thus the arguments in the proof of the preceding theorem can be applied to deduce the following conclusion:

*Either the statement of the proposition holds,  
or (after conjugation with an appropriate element of  $SL_2(\mathbb{C})$ ) we have*

$$\Gamma \subset \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{C}) : |ad| \in \{0, 1\} \right\}.$$

The condition  $|ad| \in \{0, 1\}$  is equivalent to  $|ad|^2 \in \{0, 1\}$  which is a real algebraic condition.

Hence we have to discuss the *real* algebraic Zariski topology. This is the topology whose closed sets are given as the zero sets of polynomials in the complex coordinates *and* their complex conjugates.

Since  $\Gamma$  is Zariski-dense, the *real Zariski-closure*  $S$  of  $\Gamma$  in  $SL_2(\mathbb{C})$  is either the whole of  $SL_2(\mathbb{C})$  or a real form of  $SL_2(\mathbb{C})$ . Now  $|ad|^2 \in \{0, 1\}$  defines a real algebraic subset of  $SL_2(\mathbb{C})$ . Hence the real Zariski closure  $S$  of  $\Gamma$  cannot be the whole of  $SL_2(\mathbb{C})$ . Furthermore, since  $\Gamma$  is discrete and infinite,  $S$  cannot be compact. Thus  $S$  must be conjugate to  $SL_2(\mathbb{R})$ . However, this leads to a contradiction thanks to the lemma below.  $\square$

LEMMA 6. *There is no element  $A \in SL_2(\mathbb{C})$  such that*

$$A \cdot SL_2(\mathbb{R}) \cdot A^{-1} \subset \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{C}) : |ad| \in \{0, 1\} \right\}.$$

*Proof.* Let  $\rho : SL_2(\mathbb{C}) \rightarrow \mathbb{R}$  denote the function given by

$$\rho \begin{pmatrix} a & b \\ c & d \end{pmatrix} = |ad|.$$

Now let us assume that the assertion of the lemma is wrong. In other words: we assume that there exists an element

$$A = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in SL_2(\mathbb{C})$$

such that  $\rho(g) \in \{0, 1\}$  for every  $g \in A \cdot SL_2(\mathbb{R}) \cdot A^{-1}$ .

Since  $SL_2(\mathbb{R})$  is connected, this implies that  $\rho$  is constant, and its value either 0 or 1.

However,  $\rho$  cannot be constantly zero, because  $|ad| = 0$  is equivalent to  $ad = 0$  and this is a complex algebraic condition. Thus  $\{g \in SL_2(\mathbb{C}) : \rho(g) = 0\}$  is a complex algebraic subvariety and therefore cannot contain the group  $A \cdot SL_2(\mathbb{R}) \cdot A^{-1}$  which is dense in  $SL_2(\mathbb{C})$  with respect to the complex Zariski topology.

This leaves the case where  $\rho$  is constantly 1.

Here explicit calculations show the following:

$$\rho\left(A \cdot \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \cdot A^{-1}\right) = |1 - (txz)^2|,$$

and

$$\rho\left(A \cdot \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \cdot A^{-1}\right) = |1 - (tyw)^2|.$$

Thus  $xz$  and  $yw$  are complex numbers with the property that

$$|1 - (txz)^2| = 1 = |1 - (tyw)^2|$$

for every real number  $t$ . This implies  $xz = yw = 0$ . But now

$$\begin{aligned} \rho\left(A \cdot \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \cdot A^{-1}\right) &= |(xw - yw - xz)(-yz + yw + xz)| \\ &= |-xwyz| = 0 \neq 1 \end{aligned}$$

and we obtain a contradiction to the assumption that  $\rho(ABA^{-1}) = 1$  for all  $B \in SL_2(\mathbb{R})$ . □

### §3. Equivalence of elliptic curves

#### 3.1. Isogeny criteria

An elliptic curve is a one-dimensional abelian variety, or, equivalently a projective smooth algebraic curve of genus 1 (with a basepoint). There are two natural equivalence relations between elliptic curves: isomorphism (as algebraic variety) or isogeny. Two varieties  $V$  and  $W$  are *isogenous* if there exists a variety  $Z$  and unramified coverings  $\pi : Z \rightarrow V$ ,  $\rho : Z \rightarrow W$ .

Over the field of complex numbers, every elliptic curve can be realized as the complex quotient manifold  $\mathbb{C}/\langle 1, \tau \rangle_{\mathbb{Z}}$  where  $\tau \in H^+ = \{z : \Im(z) > 0\}$ . Two elements  $\tau, \tau' \in H^+$  define isomorphic resp. isogenous elliptic curves if

both are contained in the same  $SL_2(\mathbb{Z})$ - resp.  $GL_2^+(\mathbb{Q})$ -orbit for the action on  $H^+$  given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \mapsto \frac{az + b}{cz + d}.$$

Here  $GL_2^+(\mathbb{Q})$  denotes the subgroup of  $GL_2(\mathbb{Q})$  containing all elements with positive determinant.

We need some reformulations of these criteria.

LEMMA 7. *Let  $\Lambda, \Gamma$  be lattices in  $\mathbb{C}$ , and  $\Lambda_{\mathbb{Q}} = \Lambda \otimes \mathbb{Q}$ ,  $\Gamma_{\mathbb{Q}} = \Gamma \otimes \mathbb{Q}$ . Consider the natural map  $\Phi : \Lambda_{\mathbb{Q}} \otimes_{\mathbb{Q}} \Gamma_{\mathbb{Q}} \rightarrow \mathbb{C}$  induced by the inclusion maps  $\Gamma \hookrightarrow \mathbb{C}$ ,  $\Lambda \hookrightarrow \mathbb{C}$ .*

*Then  $\mathbb{C}/\Lambda$  and  $\mathbb{C}/\Gamma$  are isogenous iff  $\dim_{\mathbb{Q}} \ker \Phi > 0$ .*

*Proof.* We may assume  $\Gamma = \langle 1, \tau \rangle_{\mathbb{Z}}$ ,  $\Lambda = \langle 1, \sigma \rangle_{\mathbb{Z}}$ . The kernel  $\ker \Phi$  is positive-dimensional iff there is a linear relation

$$a + b\tau + c\sigma + d\tau\sigma = 0$$

with  $(a, b, c, d) \in \mathbb{Q}^4 \setminus \{(0, 0, 0, 0)\}$ . Using  $\sigma, \tau \in H^+$ , one verifies that

$$-\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{Q}).$$

Thus

$$\sigma = -\frac{a + b\tau}{c + d\tau} = -\begin{pmatrix} a & b \\ c & d \end{pmatrix}(\tau),$$

i.e.  $\dim \ker \Phi > 0$  iff  $\tau$  and  $\sigma$  are contained in the same  $GL_2^+(\mathbb{Q})$ -orbit.  $\square$

LEMMA 8. *For a lattice  $\langle \alpha, \beta \rangle_{\mathbb{Z}} = \Lambda \subset \mathbb{C}$  let  $K_{\Lambda}$  denote the subfield of  $\mathbb{C}$  given by  $K_{\Lambda} = \mathbb{Q}(\alpha/\beta)$ .*

*Then  $K_{\Lambda}$  depends only on  $\Lambda$  and not of the choice of the basis  $(\alpha, \beta)$ .*

*Let  $\Lambda$  and  $\tilde{\Lambda}$  be lattices in  $\mathbb{C}$ .*

*If  $\text{trdeg } K_{\Lambda}/\mathbb{Q} > 0$ , then  $\mathbb{C}/\Lambda$  and  $\mathbb{C}/\tilde{\Lambda}$  are isogenous elliptic curves if and only if  $K_{\Lambda} = K_{\tilde{\Lambda}}$ .*

*Proof.* The independence of the choice of the basis is easily verified.

Furthermore, without loss of generality we may assume  $\Lambda = \langle 1, \tau \rangle$  and  $\tilde{\Lambda} = \langle 1, \sigma \rangle$  for some  $\tau, \sigma \in H^+$ . Now the statement follows from the fact that for transcendental complex numbers  $\tau, \sigma$  we have  $\mathbb{Q}(\tau) = \mathbb{Q}(\sigma)$  iff there are rational numbers  $a, b, c, d$  such that  $\tau = (a + b\sigma)/(c + d\sigma)$ .

Thus  $\mathbb{Q}(\tau) = \mathbb{Q}(\sigma)$  iff  $\sigma$  and  $\tau$  are in the same  $GL_2^+(\mathbb{Q})$ -orbit in  $H^+$ .  $\square$

### 3.2. Conjectures

We now formulate a conjecture about an isogeny criterion for certain elliptic curves:

CONJECTURE A. *Let  $\alpha_1, \alpha_2 \in \mathbb{C}$  be algebraic numbers with  $|\alpha_i| > 1$ . Let  $E_i$  be the quotient manifold  $\mathbb{C}^*/\{\alpha_i^k : k \in \mathbb{Z}\}$ . Then the following conditions are equivalent:*

- (i) *The numbers  $\alpha_1, \alpha_2$  are multiplicatively dependent (in the sense of Def. 2.1).*
- (ii) *The elliptic curves  $E_1$  and  $E_2$  are isogenous.*

We reformulate the above conjecture into terms of group actions on the upper half plane  $H^+$ .

CONJECTURE A'. *Let*

$$B^+(\mathbb{Q}) = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \in \mathbb{Q}^+, b \in \mathbb{Q} \right\}.$$

*Let  $\beta_1, \beta_2 \in H^+$  and assume that both  $e^{2\pi i\beta_1}$  and  $e^{2\pi i\beta_2}$  are algebraic. Then the following conditions are equivalent:*

- (i') *The number  $\beta_1, \beta_2 \in H^+$  are contained in the same  $B^+(\mathbb{Q})$ -orbit.*
- (ii') *The number  $\beta_1, \beta_2 \in H^+$  are contained in the same  $GL_2^+(\mathbb{Q})$ -orbit.*

We may further reformulate this conjecture in a number-theoretic fashion:

CONJECTURE A''. *Let  $\beta_1, \beta_2 \in H^+$  and assume that both  $e^{2\pi i\beta_1}$  and  $e^{2\pi i\beta_2}$  are algebraic. Then the following conditions are equivalent:*

- (i'')  *$\dim_{\mathbb{Q}}\langle 1, \beta_1, \beta_2 \rangle < 3$ .*
- (ii'')  *$\dim_{\mathbb{Q}}\langle 1, \beta_1, \beta_2, \beta_1\beta_2 \rangle < 4$ .*

Let us verify that these conjectures are equivalent:

PROPOSITION 3. *Let  $\beta_1, \beta_2 \in H^+$  and define  $\alpha_j = e^{2\pi i\beta_j}$  ( $j = 1, 2$ ). Assume that  $\alpha_1, \alpha_2 \in \bar{\mathbb{Q}}$ .*

*The conditions (i), (i') and (i'') in the preceding conjectures are all equivalent. Similarly the conditions (ii), (ii') and (ii'') are equivalent.*

*Therefore the Conjectures A, A' and A'' are equivalent.*

*Proof.* Since  $\Im(\beta_j) > 0$ , the condition  $\dim_{\mathbb{Q}}\langle 1, \beta_1, \beta_2 \rangle < 3$  implies that there exists  $a \in \mathbb{Q}^+$ ,  $d \in \mathbb{Q}$  such that  $\beta_1 = a\beta_2 + d$ . Furthermore, there are numbers  $(a, d) \in \mathbb{Q}^+ \times \mathbb{Q}$  with  $\beta_1 = a\beta_2 + d$  if and only if there are integers  $n, m \in \mathbb{Z} \setminus \{0\}$  such that  $2\pi i(n\beta_1 - m\beta_2) \in 2\pi i\mathbb{Z}$ . Using these observations, it is easy to verify  $(i) \iff (i') \iff (i'')$ .

The equivalence  $(ii) \iff (ii')$  is standard, while the equivalence  $(ii) \iff (ii'')$  is a consequence of Lemma 7.  $\square$

Next we prove that these three equivalent conjectures of ours are true provided the famous Schanuel's conjecture is right.

**PROPOSITION 4.** *The three equivalent conjectures above hold, if Schanuel's conjecture is true.*

Schanuel's Conjecture is the far-reaching conjecture from transcendental number theory which encompasses many important conjecture in this area. It states the following:

**SCHANUEL'S CONJECTURE.** *If  $x_1, \dots, x_n$  are  $\mathbb{Q}$ -linearly independent complex numbers, then the transcendence degree of  $\mathbb{Q}(x_1, \dots, x_n, e^{x_1}, \dots, e^{x_n})$  over  $\mathbb{Q}$  is at least  $n$ .*

Now we prove the proposition.

*Proof.* Indeed, let  $x_1 = 2\pi i$ ,  $x_2 = \log \alpha_1$ ,  $x_3 = \log \alpha_2$ . Schanuel's conjecture then implies that either

- (1)  $\dim_{\mathbb{Q}}\langle 2\pi i, \log \alpha_1, \log \alpha_2 \rangle \leq 2$ , or
- (2)  $2\pi i, \log \alpha_1, \log \alpha_2$  are all three algebraically independent.

Since  $\Re \log \alpha_i = \log |\alpha_i| > 0$  (recall that we assumed  $|\alpha_i| > 1$ ) for  $i = 1, 2$ , in the first case there exist integers  $n, m \in \mathbb{Z} \setminus \{0\}$  such that  $\alpha_1^n = \alpha_2^m$ , i.e.  $\alpha_1$  and  $\alpha_2$  are multiplicatively dependent.

In the second case we can conclude that  $\log \alpha_1/2\pi i$  and  $\log \alpha_2/2\pi i$  are both transcendental and  $\mathbb{Q}(\log \alpha_1/2\pi i) \neq \mathbb{Q}(\log \alpha_2/2\pi i)$ . Hence  $\mathbb{C}^*/\langle \alpha_1 \rangle$  is not isogenous to  $\mathbb{C}^*/\langle \alpha_2 \rangle$  in this case.

Thus we have shown that either  $\alpha_1$  and  $\alpha_2$  are multiplicatively dependent, or  $\mathbb{C}^*/\langle \alpha_1 \rangle$  must be isogenous to  $\mathbb{C}^*/\langle \alpha_2 \rangle$ .  $\square$

*Remark.* Actually we do not use Schanuel's conjecture in its full strength, but only a special case of it. However, even the special statement we need is not yet proven.



**§4. Elliptic curves in  $SL_2(\mathbb{C})/\Gamma$**

Let  $\Gamma$  be a discrete cocompact subgroup of  $SL_2(\mathbb{C})$  and  $X = SL_2(\mathbb{C})/\Gamma$  the quotient manifold. We are interested in elliptic curves embedded into  $X$ . Every elliptic curve embedded into  $X$  is an orbit of a reductive Lie subgroup  $H$  of  $SL_2(\mathbb{C})$  with  $H \simeq \mathbb{C}^*$  (see [9]). Conversely, if  $H$  is a Lie subgroup of  $SL_2(\mathbb{C})$  with  $H \simeq \mathbb{C}^*$  and  $\#(H \cap \Gamma) = \infty$ , then  $H/(H \cap \Gamma)$  is an elliptic curve embedded into  $X$  as an  $H$ -orbit. If  $\gamma \in \Gamma$  is an element of infinite order in a discrete cocompact subgroup  $\Gamma$ , then  $\gamma$  is a semisimple element of  $SL_2(\mathbb{C})$ , and the connected component of the centralizer

$$C(\gamma) = \{g \in SL_2(\mathbb{C}) : g\gamma = \gamma g\}$$

is such a Lie subgroup of  $SL_2(\mathbb{C})$  which has an elliptic curve as a closed orbit in  $X$ . Moreover this elliptic curve is isogenous to the quotient manifold of  $\mathbb{C}^*$  by the infinite cyclic subgroup generated by  $\lambda$  where  $\lambda$  is an eigenvalue of  $\gamma \in SL_2(\mathbb{C})$ .

These facts (for which we refer to [9]) establish the relationship between isogeny classes of elliptic curves embedded in  $X$  on one side and eigenvalues of elements of  $\Gamma$  on the other side.

**PROPOSITION 5.** *If Conjecture A holds, then for every discrete cocompact subgroup  $\Gamma \subset SL_2(\mathbb{C})$  there exist infinitely many isogeny classes of elliptic curves embedded in  $X = SL_2(\mathbb{C})/\Gamma$ .*

*Proof.* If  $\Gamma$  is discrete and cocompact in  $SL_2(\mathbb{C})$ , then it must be Zariski-dense. Hence by Thm. 1 there are infinitely many complex numbers  $\lambda_1, \lambda_2, \dots$  which are pairwise multiplicatively independent and which occur as eigenvalue for elements  $\gamma_1, \gamma_2, \dots$  in  $\Gamma$ .

Being multiplicatively independent implies in particular that none of these numbers  $\lambda_i$  is a root of unity.

Furthermore,  $\Gamma$  is conjugate to a subgroup of  $SL_2(k)$  for some number field  $k$  (see [5, Thm. 7.67]), hence all the numbers  $\lambda_i$  are algebraic numbers.

Let  $H_i$  be the centralizer of  $\gamma_i$  in  $SL_2(\mathbb{C})$ . An element of  $SL_2(\mathbb{C})$  with an eigenvalue different from 1 and  $-1$  is semisimple. Hence  $H_i \simeq \mathbb{C}^*$ . Now  $H_i \cap \Gamma$  is discrete and contains the element  $\gamma_i$ . Because  $\lambda_i$  is not a root of unity,  $\gamma_i$  is of infinite order. It follows that  $\langle \gamma_i \rangle \simeq \mathbb{Z}$  and that  $H_i/(\Gamma \cap H_i)$  is an elliptic curve which is isogenous to  $\mathbb{C}^*/\langle \lambda_i \rangle$ .

Thus the quotients  $H_i/(\Gamma \cap H_i)$  are elliptic curves embedded in  $X = SL_2(\mathbb{C})/\Gamma$  and, provided Conj. A holds, these elliptic curves are pairwise non-isogenous since the  $\lambda_i$  are pairwise multiplicatively independent.  $\square$

In particular:

**COROLLARY 2.** *If Schanuel's conjecture holds, then for every discrete cocompact subgroup  $\Gamma \subset SL_2(\mathbb{C})$  there exists infinitely many isogeny classes of elliptic curves embedded in  $X = SL_2(\mathbb{C})/\Gamma$ .*

#### 4.1. The case where $\Gamma \cap SL_2(\mathbb{R})$ is Zariski dense

**THEOREM 2.** *Let  $\Gamma$  be a discrete subgroup of  $SL_2(\mathbb{C})$  and assume that  $\Gamma \cap SL_2(\mathbb{R})$  is Zariski-dense in  $SL_2$ .*

*Then there exists infinitely many isogeny classes of elliptic curves embedded in  $X = SL_2(\mathbb{C})/\Gamma$ .*

*Proof.* By Thm. 1 there are infinitely many pairwise multiplicatively independent complex numbers  $\lambda_i$  occurring as eigenvalues for elements  $\gamma \in \Gamma \cap SL_2(\mathbb{R})$ . None of these  $\lambda_i$  is a root of unity.

If  $\lambda$  is an eigenvalue for a matrix  $SL_2(\mathbb{R})$ , then either  $\lambda$  is real or  $|\lambda| = 1$ . If  $\lambda$  is an eigenvalue for an element of a discrete subgroup of  $SL_2(\mathbb{R})$  with  $|\lambda| = 1$ , then  $\lambda$  must be a root of unity.

Since none of the  $\lambda_i$  is a root of unity, it follows that all the numbers  $\lambda_i$  are real.

Thus there are infinitely many elliptic curves  $E_i$  in  $X = SL_2(\mathbb{C})/\Gamma$  which are isogenous to  $\mathbb{C}^*/\langle \lambda_i \rangle$  where the numbers  $\lambda_i$  are all real and pairwise multiplicatively independent.

We claim that at most two of these  $E_i$  can be isogenous. Assume the converse, i.e., let  $\lambda_i, \lambda_j$  and  $\lambda_k$  be pairwise multiplicatively independent real numbers larger than 1 such that the three elliptic curves  $E_i, E_j$  and  $E_k$  are all isogenous.

Note that  $E_i = \mathbb{C}/\langle 2\pi i, \log \lambda_i \rangle$  and similarly for  $E_j$  and  $E_k$ . Isogeny of  $E_i$  and  $E_j$  implies that there is a  $\mathbb{Q}$ -linear relation between  $4\pi^2, \log \lambda_i \log \lambda_j, 2\pi i \log \lambda_i$  and  $2\pi i \log \lambda_j$  (see Lemma 7). Now  $4\pi^2 \in \mathbb{R}$  and  $\log \lambda_i \log \lambda_j \in \mathbb{R}$ , while  $2\pi i \log \lambda_i$  and  $2\pi i \log \lambda_j$  are  $\mathbb{Q}$ -linearly independent elements of  $i\mathbb{R}$ . Therefore a  $\mathbb{Q}$ -linear relation can only exist if  $4\pi^2/(\log \lambda_i \log \lambda_j) \in \mathbb{Q}$ .

Similarly the existence of an isogeny of between  $E_j$  and  $E_k$  implies  $4\pi^2/(\log \lambda_j \log \lambda_k) \in \mathbb{Q}$ .

Combined, this yields  $(\log \lambda_i \log \lambda_j)/(\log \lambda_j \log \lambda_k) = \log \lambda_i / \log \lambda_k \in \mathbb{Q}$  which contradicts the assumption of  $\lambda_i$  and  $\lambda_k$  being multiplicatively independent.

This proves the claim.

Thus we obtain an infinite family of elliptic curves in  $SL_2(\mathbb{C})/\Gamma$  such that for each of these curves there is at most one other curve in this family

to which it is isogenous. It follows that there are infinitely many isogeny classes.  $\square$

**§5. Existence of  $\Gamma$  for which  $\Gamma \cap SL_2(\mathbb{R})$  is cocompact in  $SL_2(\mathbb{R})$**

From a differential geometric point of view the torsion-free discrete cocompact subgroups of  $SL_2(\mathbb{C})$  are precisely those groups which occur as fundamental group of compact real hyperbolic threefolds  $M$ . The condition that  $\Gamma \cap SL_2(\mathbb{R})$  is cocompact in  $SL_2(\mathbb{R})$  translates into the condition that there is a real hyperbolic surface geodesically embedded into  $M$ .

However, we use a different point of view to show the existence of such  $\Gamma$ . There is an arithmetic way to produce discrete cocompact subgroups in  $SL_2(\mathbb{C})$  which we employ.

This arithmetic construction (see e.g. [8]) is the following: Let  $K$  be either  $\mathbb{Q}$  or a totally imaginary quadratic extension of  $\mathbb{Q}$ ,  $\bar{K}$  the unique archimedean completion of  $K$ ,  $L/K$  a quadratic extension,  $\lambda \in K^*$  such that  $\lambda \notin N_{L/K}(L^*)$ . Then a central simple  $K$ -algebra can be defined by  $A = \{a + bt : a, b \in L\}$  with multiplication given by  $at = ta^\sigma$  (for  $\text{Gal}(L/K) = \{id, \sigma\}$ ) and  $t^2 = \lambda$ . The elements of norm one constitute a  $K$ -anisotropic simple  $K$ -group  $S$ . Now  $S(\mathcal{O}_K)$  becomes a discrete cocompact subgroup of  $S(\bar{K})$ . If  $\bar{K} = \mathbb{R}$ , then  $S(\bar{K}) = SL_2(\mathbb{R})$  if  $A \otimes \mathbb{R} \simeq Mat(2, \mathbb{R})$  and  $S(\bar{K}) = SU(2)$  if  $A \otimes \mathbb{R}$  is isomorphic to the algebra of quaternions.

We use this in the following way: Let  $F_1 = \mathbb{Q}[\sqrt{2}]$ ,  $F_2 = \mathbb{Q}[i]$ ,  $F_3 = \mathbb{Q}[i, \sqrt{2}]$  and  $p = 5$ . We observe that the prime ideal  $(5)$  splits in  $F_2$ :  $5 = (2 + i)(2 - i)$ . Now  $(2 + i)$  is prime in  $\mathbb{Z}[i]$  and both residue class fields for  $5$  in  $\mathbb{Z}$  resp.  $2 + i$  (or  $2 - i$ ) in  $\mathbb{Z}[i]$  are isomorphic to the finite field  $\mathbb{F}_5 = \mathbb{Z}/5\mathbb{Z}$ . Note that  $2$  is not a square in  $\mathbb{F}_5$ . As a consequence the prime ideals  $(5)$  and  $(2 + i)$  (and similarly for  $(2 - i)$ ) are totally inert with respect to the the field extensions  $\mathbb{Q}[\sqrt{2}]/\mathbb{Q}$  resp.  $\mathbb{Q}[i, \sqrt{2}]/\mathbb{Q}[i]$ . It follows that  $5$  is not contained in the image of the *norm* for either the field extension  $\mathbb{Q}[i, \sqrt{2}]/\mathbb{Q}[i]$  or the field extension  $\mathbb{Q}[\sqrt{2}]/\mathbb{Q}$ .

Thus we may use the above construction with

$$(K, L, \lambda) = (\mathbb{Q}[i], \mathbb{Q}[i, \sqrt{2}], 5)$$

resp.  $(\mathbb{Q}, \mathbb{Q}[\sqrt{2}], 5)$  to obtain a discrete cocompact subgroup  $\Gamma$  resp.  $\Gamma_1$  in  $S(\mathbb{C}) \simeq SL_2(\mathbb{C})$  resp.  $S(\mathbb{R})$ . Evidently  $\Gamma_1 = \Gamma \cap S(\mathbb{R})$ . Now observe that  $\mathbb{Q}[\sqrt{2}] \subset \mathbb{R}$  implies  $A \otimes \mathbb{R} \simeq Mat(2, \mathbb{R})$ . Thus  $S(\mathbb{R}) \simeq SL_2(\mathbb{R})$ .

We have thus established:

PROPOSITION 6. *There exists a discrete subgroup  $\Gamma$  in  $SL_2(\mathbb{C})$  such that both  $SL_2(\mathbb{C})/\Gamma$  and  $SL_2(\mathbb{R})/(SL_2(\mathbb{R}) \cap \Gamma)$  are compact.*

In combination with Thm. 2 this implies the following:

COROLLARY 3. *There exists a discrete cocompact subgroup  $\Gamma$  in  $SL_2(\mathbb{C})$  such that the complex quotient manifold  $X = SL_2(\mathbb{C})/\Gamma$  contains infinitely many pairwise non-isogenous elliptic curves.*

## §6. Geodesic length spectra for hyperbolic manifolds

Here we want to relate our results on eigenvalues of elements of discrete groups to the study of closed geodesics on real hyperbolic manifolds (As standard references for hyperbolic manifolds, see [2], [6]).

A *real hyperbolic manifold* is a complete Riemannian manifold with constant curvature  $-1$ . In each dimension  $n$  there is a unique simply-connected real hyperbolic manifold  $H^n$ .

Let  $\mathbb{H} = \{z + wj : z, w \in \mathbb{C}\}$  denote the division algebra of *quaternions*, i.e., the algebra given by  $j^2 = -1$  and  $zj = j\bar{z}$  for all  $z \in \mathbb{C}$ .

Now  $H^2$  can be realized as  $H^2 \simeq \{z + tj \in \mathbb{H} : z \in \mathbb{R}, t \in \mathbb{R}^+\}$  and  $H^3$  as  $H^3 \simeq \{z + tj \in \mathbb{H}, z \in \mathbb{C}, t \in \mathbb{R}^+\}$ . In both cases the hyperbolic metric is obtained from the euclidean metric by multiplication with  $1/t$ . Let  $\rho$  denote the induced distance function.

The isometry group  $G$  of  $H^2$  resp.  $H^3$  is  $PSL_2(\mathbb{R})$  resp.  $PSL_2(\mathbb{C})$  with the action given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \zeta \longmapsto (a\zeta + b)(c\zeta + d)^{-1}$$

where the calculations take place in the algebra of quaternions.

Explicit calculations show that for any  $A \in G$  we have

$$\inf_{x \in H} \rho(x, Ax) = \log(\max\{|\lambda|^2, |\lambda^{-2}|\})$$

where the infimum is taken over all points of  $H^2$  resp.  $H^3$  and  $(\lambda, \lambda^{-1})$  are the roots of the characteristic polynomial of  $\tilde{A}$  where  $\tilde{A}$  is an element of  $SL_2(\mathbb{C})$  which projects onto  $A \in G \subset PSL_2(\mathbb{C}) = SL_2(\mathbb{C})/\{I, -I\}$ .

For a complete Riemannian manifold with strictly negative curvature there is a unique closed geodesic for every element of the fundamental group.

Therefore: If  $\Gamma$  is a torsion-free discrete subgroup of  $G$  then the set of lengths of closed geodesics of  $H/\Gamma$  coincides with the set of logarithms of absolute values of squares of eigenvalues of elements of  $\Gamma$ .

Moreover, if  $H = H^3$ , one can show that the logarithm of the eigenvalue of an element  $g \in \Gamma$  is the “complex length” of the corresponding closed geodesic in the following sense: Let  $\gamma$  be a closed geodesic in a compact hyperbolic 3-fold  $M$ . Let  $s$  be the length of  $\gamma$  in the usual sense. If we fix a point  $p \in \gamma$ , then the holonomy along  $\gamma$  defines an orthogonal transformation of the normal space  $T_p(M)/T_p(\gamma)$ . This normal space is isomorphic to  $\mathbb{R}^2$ , thus an orthogonal transformation is simply a rotation by an angle  $\theta$ . Now the “complex length” of  $\gamma$  is defined to be  $s + i\theta$  ([7]).

The set of all real resp. complex numbers occuring as (complex) length for a closed geodesic is denoted as (complex) geodesic length spectrum. (In the literature, usually multiplicities are taken into account, and sometimes only simple closed geodesics are considered. For our point of interest (the  $\mathbb{Q}$ -linear independence of geodesic lengths) these distinctions are of no relevance.)

Therefore we obtain:

**PROPOSITION 7.** *Assume that  $M$  is a compact real hyperbolic 3-manifold. Then there exist infinitely many closed geodesics on  $M$  such that their complex lengths are pairwise  $\mathbb{Q}$ -linearly independent.*

Using the results of Subsection 2.4. on the absolute values of the eigenvalues we also obtain:

**THEOREM 3.** *Let  $M$  be a compact real hyperbolic manifold of dimension two or three and  $\Lambda$  its geodesic length spectrum.*

*Then  $\Lambda$  contains infinitely many pairwise  $\mathbb{Q}$ -linearly independent elements.*

Another consequence is the following:

**COROLLARY 4.** *Let  $\Gamma$  be a Zariski-dense subgroup in  $SL_2(\mathbb{C})$ .*

*Then there exist two elements  $\gamma_1, \gamma_2 \in \Gamma$  with respective eigenvalues  $\lambda_1, \lambda_2 \in \mathbb{R}$  such that the numbers  $|\log \lambda_1|, |\log \lambda_2|$  generate a dense subgroup of the additive group  $(\mathbb{R}, +)$ .*

There is a related result of Benoist ([1]) which implies that the subgroup of  $(\mathbb{R}, +)$  generated by *all* the logarithms of the absolute values of eigenvalues of elements of  $\Gamma$  is dense. Thus, for  $SL_2(\mathbb{R})$  and  $SL_2(\mathbb{C})$  we can improve this result of Benoist. However, Benoist's work applies to other semisimple Lie groups as well where our results concern only  $SL_2(\mathbb{R})$  and  $SL_2(\mathbb{C})$ .

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