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ON l-ADIC ITERATED INTEGRALS, I
ANALOG OF ZAGIER CONJECTURE

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Abstract. We are studying some aspects of the action of Galois groups on the torsor of paths connecting two (possibly tangential) points on a projective line minus a finite number of points. We obtain objects which formally behave like classical iterated integrals and polylogarithms. We formulate an analog of Zagier conjecture for these l-adic analogs of iterated integrals and polylogarithms.

§0. Introduction

0.1. The classical complex iterated integrals appear in the study of mixed Hodge structures on fundamental groups and on torsors of paths (see [D], [BD] and [W3]). In this paper we shall study their l-adic analogs.

The notion of a tangential base point (see [D]) is very important in this paper. We use a definition given in [N2].

Let K be a number field and let X be a projective line $\mathbf{P}^1_K$ minus a finite number of K-points. Let z and v be two K-points or tangential base points defined over K of X. Let $\pi_1(X_{\bar K}; v)$ be the l-completion of the étale fundamental group of $X_{\bar K}$ based at v. We denote by $\pi(X_{\bar K}; z, v)$ the $\pi_1(X_{\bar K}; v)$-torsor of (l-adic) paths from v to z. The Galois group $G_K := \mathrm{Gal}(\bar K/K)$ acts on the set $\pi(X_{\bar K}; z, v)$. To describe this action of G_K we shall proceed in the following way.

Let us fix a path p from v to z. Then the map

$$\pi(X_{\bar K}; z, v) \ni q \longrightarrow p^{-1}q \in \pi_1(X_{\bar K}; v) \tag{0.1.1}$$

is a bijection. The action of G_K on the torsor $\pi(X_{\bar K}; z, v)$ transported to an action of G_K on $\pi_1(X_{\bar K}; v)$ by the map (0.1.1) is given by

$$\pi_1(X_{\bar K}; v) \ni S \longrightarrow \mathfrak{f}_p(\sigma) \cdot \sigma(S) \in \pi_1(X_{\bar K}; v),$$

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where $\sigma \in G_K$ and

$$\mathfrak{f}_p(\sigma) := p^{-1} \cdot \sigma(p). \tag{0.1.2}$$

The function $\mathfrak{f}_p : G_K \to \pi_1(X_{\bar{K}}; v)$ has the following important property.

Proposition A. (see Section 1) *The function $\mathfrak{f}_p : G_K \to \pi_1(X_{\bar{K}}; v)$ is a cocycle, i.e.,*

$$\mathfrak{f}_p(\tau \cdot \sigma) = \mathfrak{f}_p(\tau) \cdot \tau(\mathfrak{f}_p(\sigma)). \tag{0.1.3}$$

This (well known) result was the starting point of the paper (see also Theorem A and B in [I1]).

Let $X := \mathbf{P}^1_K \setminus \{a_1, \dots, a_n, \infty\}$. The fundamental group $\pi_1(X_{\bar{K}}; v)$ is a pro-l free group freely generated by n generators, which we denote by $x_1, \dots, x_n$ and which will be constructed below. The element $\mathfrak{f}_p(\sigma) \in \pi_1(X_{\bar{K}}; v)$, hence

$$\mathfrak{f}_p(\sigma) \equiv x_1^{\alpha_1(\sigma)} \cdot x_2^{\alpha_2(\sigma)} \cdots x_n^{\alpha_n(\sigma)} \cdot \prod_{i<j} (x_i, x_j)^{\beta_{i,j}(\sigma)}$$
$$\mod \big((\pi_1(X_{\bar{K}}; v), \pi_1(X_{\bar{K}}; v)), \pi_1(X_{\bar{K}}; v)\big)$$

for some $\alpha_i(\sigma)$ and $\beta_{i,j}(\sigma)$ in $\mathbf{Z}_l$. Let G_K act on $\mathbf{Z}_l$ as a multiplication by the cyclotomic character $\chi : G_K \to \mathbf{Z}_l^*$. It follows from Proposition A that the exponents $\alpha_i : G_K \to \mathbf{Z}_l$ are cocycles (see Corollary 2.2.2). The obvious question is if the exponents $\beta_{i,j} : G_K \to \mathbf{Z}_l$ are also cocycles. This question and its generalization are studied in Sections 6 and 11.

The fundamental group $\pi_1(X_{\bar{K}}; v)$ we embed into the algebra $\mathbf{Q}_l\{\{X_1, \dots, X_n\}\}$ of non-commutative formal power series in n non-commuting variables $X_1, \dots, X_n$ ($n+1$ is a number of points removed from $\mathbf{P}^1_K$) sending a loop around a_i onto e^{X_i} for $i = 1, \dots, n$. The actions of G_K on the fundamental group $\pi_1(X_{\bar{K}}; v)$ and on the torsor $\pi(X_{\bar{K}}; z, v)$ we transport to linear actions of G_K on $\mathbf{Q}_l\{\{X_1, \dots, X_n\}\}$. Hence we get representations

$$\varphi : G_K \longrightarrow \mathrm{Aut}(\mathbf{Q}_l\{\{X_1, \dots, X_n\}\})$$

in a case of the action deduced from the action on $\pi_1(X_{\bar{K}}; v)$ and

$$\psi_p : G_K \longrightarrow \mathrm{GL}(\mathbf{Q}_l\{\{X_1, \dots, X_n\}\})$$

in a case of the action deduced from the action on the torsor $\pi(X_{\bar{K}}; z, v)$.

If $\sigma \in G_{K(\mu_{l^\infty})} := \mathrm{Gal}(\bar{K}/K(\mu_{l^\infty}))$ then $\psi_p(\sigma)$ is a pro-unipotent automorphism of $\mathbf{Q}_l\{\{X_1, \ldots, X_n\}\}$. Hence $\log \psi_p(\sigma)$ is defined and we have the following result.

PROPOSITION B. (see Section 5) *Let $\sigma \in G_{K(\mu_{l^\infty})}$. Then we have*

$$\log \psi_p(\sigma) = L_{(\log \psi_p(\sigma))(1)} + \log \varphi(\sigma),$$

where for $w \in \mathbf{Q}_l\{\{X_1, \ldots, X_n\}\}$, L_w is a left multiplication by w.

The operator $\log \varphi(\sigma)$ is a derivation of the $\mathbf{Q}_l$-algebra $\mathbf{Q}_l\{\{X_1, \ldots, X_n\}\}$. Let us fix a path γ_i from v to a tangential base point at a_i for $i = 1, \ldots, n$. The generator

$$x_i := \gamma_i^{-1} \cdot \text{small loop around } a_i \cdot \gamma_i$$

we send to e^{X_i} for $i = 1, \ldots, n$. Then we show the following result.

PROPOSITION C. (see Section 5) *Let $\sigma \in G_{K(\mu_{l^\infty})}$. Then we have*

$$(\log \varphi(\sigma))(X_i) = [X_i, (\log \psi_{\gamma_i}(\sigma))(1)]$$

for $i = 1, \ldots, n$.

P. Deligne in [D] and Y. Ihara in [I1], [I2] have studied the Galois action on $\mathbf{P}^1 \setminus \{0, 1, \infty\}$. They got results related to our Proposition C. Their results in the case of $\mathbf{P}^1 \setminus \{0, 1, \infty\}$ motivated our study of more general situations.

The power series $(\log \psi_p(\sigma))(1)$ is a Lie element and its coefficients (with $\sigma \in G_{K(\mu_{l^\infty})}$ varing) in a Hall base we shall call l-adic iterated integrals (see Definition 5.3.0). These l-adic iterated integrals are functions from $G_{K(\mu_{l^\infty})}$ to $\mathbf{Q}_l$. They depend on points v and z and also on a choice of a path p from v to z (compare with the classical integral $\int_v^z \frac{dz}{z}$ which depends on v and z and on a choice of a path p from v to z). They have all formal properties of iterated integrals on $X(\mathbf{C})$. In [W1] we studied functional equations of iterated integrals. The l-adic iterated integrals have the same functional equations as classical complex iterated integrals on $X(\mathbf{C})$ (see Section 10). We have an analog of Zagier conjecture for l-adic iterated integrals as in [W3] (see Section 7).

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paper were presented in unofficial talk in Berkeley and in Oberwolfach in autumn 1999, in Luminy in spring 2000 and in summer 2001, in Acquafreda di Maretea in 2001 and in Cambridge in autumn 2002.

The present paper is a rewritten version of the first six sections of [W4].

§1. Torsors of paths

1.0. Let X be a smooth algebraic variety defined over a number field K. We denote by $\hat{X}(K)$ the union of K-points of X and tangential base points of X defined over K.

Let us fix a prime number l. Let $z, v \in \hat{X}(K)$. Let $\pi_1(X_{\bar{K}}; v)$ be the l-completion, i.e., the maximal pro-l quotient of the étale fundamental group of $X_{\bar{K}}$ with a base point at v. We denote by $\pi(X_{\bar{K}}; z, v)$ the profinite set of homotopy classes of (l-adic) paths from v to z. The set $\pi(X_{\bar{K}}; z, v)$ is a $\pi_1(X_{\bar{K}}; v)$-torsor. We set $G_K := \mathrm{Gal}(\bar{K}/K)$. The group G_K acts on $\pi_1(X_{\bar{K}}; v)$ and on $\pi(X_{\bar{K}}; z, v)$ and the action of G_K is compatible with the action of $\pi_1(X_{\bar{K}}; v)$ on $\pi(X_{\bar{K}}; z, v)$, i.e., $\sigma(p \cdot S) = \sigma(p) \cdot \sigma(S)$ for $p \in \pi(X_{\bar{K}}; z, v)$, $S \in \pi_1(X_{\bar{K}}; v)$ and $\sigma \in G_K$.

In this section we shall study elementary properties of the action of the Galois group G_K on the torsor of paths $\pi(X_{\bar{K}}; z, v)$. The set $\pi(X_{\bar{K}}; z, v)$ is difficult to handle. We fix a path from v to z and using this path we identify the set $\pi(X_{\bar{K}}; z, v)$ with the fundamental group $\pi_1(X_{\bar{K}}; v)$. The group $\pi_1(X_{\bar{K}}; v)$ is more familiar and we describe the action of G_K on $\pi(X_{\bar{K}}; z, v)$ in terms of the action of G_K on $\pi_1(X_{\bar{K}}; v)$.

Let us fix a path $p \in \pi(X_{\bar{K}}; z, v)$. Then

$$t_p : \pi(X_{\bar{K}}; z, v) \longrightarrow \pi_1(X_{\bar{K}}; v)$$

given by $t_p(q) := p^{-1} \cdot q$ is a bijection. The map t_p is not G_K-equivariant. However using this map we shall transport the action of G_K on $\pi(X_{\bar{K}}; z, v)$ into the action of G_K on $\pi_1(X_{\bar{K}}; v)$, which is a more familiar object.

Let $\sigma \in G_K$. We set

$$\sigma_p := t_p \circ \sigma \circ t_p^{-1},$$

where $\sigma : \pi(X_{\bar{K}}; z, v) \to \pi(X_{\bar{K}}; z, v)$ is the map induced by σ.

Definition 1.0.1. We define a function $\mathfrak{f}_p : G_K \to \pi_1(X_{\bar{K}}; v)$ setting

$$\mathfrak{f}_p(\sigma) := p^{-1} \cdot \sigma(p) \in \pi_1(X_{\bar{K}}; v)$$

for any $\sigma \in G_K$.

LEMMA 1.0.2. *The action of G_K on $\pi_1(X_{\bar{K}}; v)$ transported by the isomorphism t_p from the action of G_K on $\pi(X_{\bar{K}}; z, v)$ is given by*

$$\sigma_p(S) = \mathfrak{f}_p(\sigma) \cdot \sigma(S),$$

where $S \in \pi_1(X_{\bar{K}}; v)$ and $\sigma \in G_K$.

Proof. We have $\sigma_p(S) = t_p \circ \sigma \circ t_p^{-1}(S) = t_p(\sigma(p \cdot S)) = p^{-1} \cdot \sigma(p) \cdot \sigma(S) = \mathfrak{f}_p(\sigma) \cdot \sigma(S)$.

This action of G_K on $\pi_1(X_{\bar{K}}; v)$ transported by the isomorphism t_p depends on a choice of a path p from v to z. Let $q \in \pi(X_{\bar{K}}; z, v)$ be another path from v to z. One easily verifies that

$$\mathfrak{f}_p(\sigma) = (q^{-1}p)^{-1} \cdot \mathfrak{f}_q(\sigma) \cdot \sigma(q^{-1}p) \tag{1.0.3}$$

and

$$t_p(r) = t_q((qp^{-1}) \cdot r) = (p^{-1}q) \cdot t_q(r) \tag{1.0.4}$$

for any r in $\pi(X_{\bar{K}}; z, v)$.

The relation between actions of σ_p and σ_q is described in the next lemma.

LEMMA 1.0.5. *For any $\sigma \in G_K$ and $S \in \pi_1(X_{\bar{K}}; v)$ we have*

$$\sigma_p(S) = (q^{-1}p)^{-1} \cdot \sigma_q((q^{-1}p) \cdot S).$$

Proof. The lemma follows from Lemma 1.0.2 and from (1.0.3).

We finish this section describing some elementary properties of the element $\mathfrak{f}_p(\sigma)$.

LEMMA 1.0.6. *Let p be a path from v to z and let q be a path from w to v. Then we have*

$$\mathfrak{f}_{pq}(\sigma) = q^{-1} \cdot \mathfrak{f}_p(\sigma) \cdot q \cdot \mathfrak{f}_q(\sigma) \quad \textit{and} \quad \mathfrak{f}_{p^{-1}}(\sigma) = p \cdot (\mathfrak{f}_p(\sigma))^{-1} \cdot p^{-1}$$

for any $\sigma \in G_K$.

Proof. An easy verification we left to the reader.

Proposition 1.0.7. *The function $\mathfrak{f}_p : G_K \to \pi_1(X_{\bar{K}}; v)$ is a cocycle, i.e., for any τ and σ in G_K we have*

$$\mathfrak{f}_p(\tau \cdot \sigma) = \mathfrak{f}_p(\tau) \cdot \tau(\mathfrak{f}_p(\sigma)).$$

Proof. We have $\mathfrak{f}_p(\tau \cdot \sigma) = p^{-1} \cdot \tau(\sigma(p)) = p^{-1} \cdot \tau(p) \cdot \tau(p^{-1}) \cdot \tau(\sigma(p)) = \mathfrak{f}_p(\tau) \cdot \tau(\mathfrak{f}_p(\sigma))$.

Corollary 1.0.8. *We have*

$$\mathfrak{f}_p(\tau^{-1}) = \tau^{-1}(\mathfrak{f}_p(\tau)^{-1}).$$

Remark. Let p be a path from $\overrightarrow{01}$ to $\overrightarrow{10}$ on $\mathbf{P}^1(\mathbf{C}) \setminus \{0, 1, \infty\}$. The element $\mathfrak{f}_p(\sigma)$ was used by Ihara in [I2]. Its Hodge-De Rham incarnation appears in [D] and [Dr].

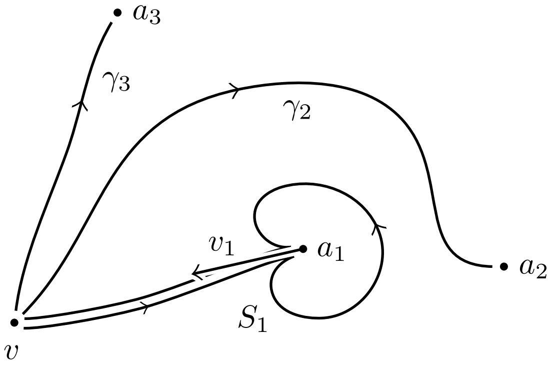
§2. Geometric generators of $\pi_1(X(\mathbf{C}); v)$

2.0. Let $X = \mathbf{P}^1_{\mathbf{C}} \setminus \{a_1, \ldots, a_{n+1}\}$ and let $v \in \hat{X}(\mathbf{C})$. We shall construct a canonical family of generators of $\pi_1(X(\mathbf{C}); v)$. The Galois action on fundamental groups will be described in terms of these generators.

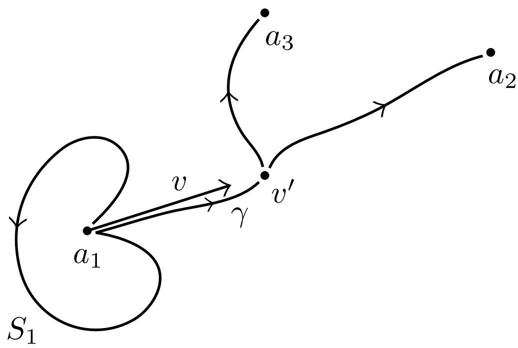
Let us choose a tangential base point v_i (a tangent vector) at a_i for $i = 1, 2, \ldots, n+1$.

2.0.1. Let us assume that $v \in X(\mathbf{C})$. Let $\Gamma = \{\gamma_k\}_{k=1,\ldots,n+1}$ be a family of smooth paths from v to each v_k such that any two paths do not intersect, no path self-intersects and for each k, $\gamma_k([0,1[) \subset X(\mathbf{C})$. The indices are choosen in such a way that when we make a small circle around v in the opposite clockwise direction starting from γ_1, then we meet successively $\gamma_2, \gamma_3, \ldots, \gamma_{n+1}$. The element $S_k \in \pi_1(X(\mathbf{C}); v)$ is defined in the following way: we move along γ_k, near a_k we make a small circle around a_k in the opposite clockwise direction and we return along γ_k to v (see Picture 1).

2.0.2. Without loss of generality we can assume that v is a tangential base point at a_1. Let $v' \in X(\mathbf{C})$ be near a_1 in the direction v. Let $\Gamma = \{\gamma'_k\}_{k=2,\ldots,n+1}$ be a family of smooth paths from v' to each v_k satisfying the conditions from 2.0.1. Let S'_k be defined by the path γ'_k. Let γ be a path $[0,1] \ni t \to a_1 + t(v' - a_1) \in X(\mathbf{C})$. We set $\gamma_k := \gamma'_k \cdot \gamma$ and $S_k := \gamma^{-1} \cdot S'_k \cdot \gamma$ for $k = 2, \ldots, n+1$. S_1 is a small circle around a_1 starting from v in the opposite clockwise direction (see Picture 2).



Picture 1



Picture 2

Lemma 2.0.3. *The elements* $S_1, \ldots, S_{n+1}$ *generate* $\pi_1(X(\mathbf{C}); v)$ *and satisfy the only relation*

$$S_{n+1} \cdots S_1 = 1.$$

Definition 2.0.4. The ordered sequence $(S_1, \ldots, S_{n+1})$ we shall call a sequence of geometric generators of $\pi_1(X(\mathbf{C}); v)$ associated to a family of paths Γ.

2.1. Let $F_{n+1} = F_{n+1}(x_1, \ldots, x_{n+1})$ be a free group on $n+1$ elements $(x_1, \ldots, x_{n+1})$. Let $\mathcal{B}_{n+1}(x_1, \ldots, x_{n+1})$ be a subgroup of $\mathrm{Aut}(F_{n+1})$ consisting of automorphisms f such that $f(x_i) = t_i \cdot x_{\mu(i)} \cdot t_i^{-1}$ $(i = 1, \ldots, n+1)$ and $f(x_{n+1}) \cdots f(x_1) = x_{n+1} \cdots x_1$, where $t_i \in F_{n+1}$ and $\mu \in S_{n+1}$ is a permutation.

Let us set $F_{n+1}^* := F_{n+1}/\langle x_{n+1} \cdots x_1 \rangle$. The group $\mathcal{B}_{n+1}(x_1, \ldots, x_{n+1})$ acts as an automorphism group on F_{n+1}^*. This automorphism group we denote by $\mathcal{B}^*_{n+1}(x_1,\dots,x_{n+1})$. Let

$$\mathcal{B}^{(1)*}_{n+1}(x_1,\dots,x_{n+1}) := \ker\bigl(\pi : \mathcal{B}^*_{n+1}(x_1,\dots,x_{n+1}) \to \Sigma_{n+1}\bigr),$$

where π is the obvious projection.

The next lemma is well known.

LEMMA 2.1.1. (see [W2]) *Let $(S_1,\dots,S_{n+1})$ be a sequence of geometric generators of $\pi_1(X(\mathbf{C});v)$. Then any other sequence of geometric generators of $\pi_1(X(\mathbf{C});v)$ is of the form $(f(S_1),\dots,f(S_{n+1}))$, where $f \in \mathcal{B}^*_{n+1}(S_1,\dots,S_{n+1})$.*

DEFINITION 2.1.2. Let $s = (S_1,\dots,S_{n+1})$ and $s' = (S'_1,\dots,S'_{n+1})$ be two sequences of geometric generators of $\pi_1(X(\mathbf{C});v)$. We say that s and s' are in the same permutation class if there is $f \in \mathcal{B}^{(1)*}_{n+1}(S_1,\dots,S_{n+1})$ such that $f(S_i) = S'_i$ for each i.

2.2. Let K be a number field. Let $a_1,\dots,a_{n+1}$ be K-points of the projective line $\mathbf{P}^1_K$. Let $X = \mathbf{P}^1_K \setminus \{a_1,\dots,a_{n+1}\}$ and let $v \in \hat{X}(K)$. Let us choose a tangential base point $v_k \in \hat{X}(K)$ at a_k for $k = 1,\dots,n+1$. Let us fix an embedding $K \subset \mathbf{C}$. Let $\Gamma = \{\gamma_k\}_{k=1,\dots,n+1}$ be a family of paths on $X(\mathbf{C})$ from v to each v_k and let $S_1,\dots,S_{n+1}$ be a family of geometric generators of $\pi_1(X(\mathbf{C});v)$ associated to Γ.

The geometric generators of $\pi_1(X(\mathbf{C});v)$ can be interpreted as elements of $\pi_1(X_{\bar{K}};v)$. The path γ_k from v to v_k can be interpreted as an l-adic path, i.e., a natural transformation of fiber functors over v and over v_k from étale coverings of $X_{\bar{K}}$ to sets. A small circle around a_k based at v_k is defined in the proof of Proposition 2.2.1. However it would be very interesting to construct "geometric generators" of $\pi_1(X_{\bar{K}};v)$ in purely algebraic way.

Below we shall describe the action of G_K on $\pi_1(X_{\bar{K}};v)$ in terms of these generators. The result seems to be well known (see [I1, pages 51 and 52] and [AI, page 128]). We give however a sketch of a proof because of the importance of this result in our studies.

Let $\chi : G_K \to \mathbf{Z}^*_l$ be the cyclotomic character.

PROPOSITION 2.2.1. *Let $\sigma \in G_K$. Then*

$$\sigma(S_k) = (\mathfrak{f}_{\gamma_k}(\sigma))^{-1} \cdot S_k^{\chi(\sigma)} \cdot \mathfrak{f}_{\gamma_k}(\sigma)$$

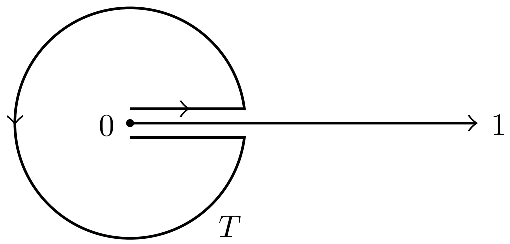
for $k = 1,\dots,n+1$.

Proof. Without loss of generality we can assume that $a_k = 0$, $a_{n+1} = \infty$ and $v_k = \overrightarrow{01}$. Consider the following Galois equivariant map

$$\pi_1(\operatorname{Spec} \bar{K}[[z]][\tfrac{1}{z}], \overrightarrow{01}) \longrightarrow \pi_1(X_{\bar{K}}, v_k),$$

where $\bar{K}[[z]][\frac{1}{z}]$ is the algebra of formal Laurent power series. The fundamental group $\pi_1(\operatorname{Spec} \bar{K}[[z]][\frac{1}{z}], \overrightarrow{01})$ is isomorphic to $\mathbf{Z}_l$. The group G_K acts on $\pi_1(\operatorname{Spec} \bar{K}[[z]][\frac{1}{z}], \overrightarrow{01})$ by the cyclotomic character $\chi : G_K \to \mathbf{Z}_l^*$. (See [I1] and [N1, p. 94].)

Let us fix an embedding of $\bar{K}$ into $\mathbf{C}$. We recall that the elements of $\pi_1(\operatorname{Spec} \bar{K}[[z]][\frac{1}{z}], \overrightarrow{01})$ act on Puiseux elements z^{1/l^n} by analytic continuation. We define a canonical generator T of $\pi_1(\operatorname{Spec} \bar{K}[[z]][\frac{1}{z}], \overrightarrow{01})$ requiring that $T(z^{1/l^n}) = e^{2\pi i/l^n} \cdot z^{1/l^n}$ (see Picture 3).



Picture 3

We denote by T_k the image of T in $\pi_1(X_{\bar{K}}, v_k)$. Clearly we have $\sigma(T_k) = T_k^{\chi(\sigma)}$. Observe that $S_k = \gamma_k^{-1} \cdot T_k \cdot \gamma_k$. Hence we get $\sigma(S_k) = \sigma(\gamma_k^{-1}) \cdot T_k^{\chi(\sigma)} \cdot \sigma(\gamma_k) = \sigma(\gamma_k^{-1}) \cdot \gamma_k \cdot (\gamma_k^{-1} \cdot T_k^{\chi(\sigma)} \cdot \gamma_k) \cdot (\gamma_k^{-1}) \cdot \sigma(\gamma_k) = (\mathfrak{f}_{\gamma_k}(\sigma))^{-1} \cdot S_k^{\chi(\sigma)} \cdot \mathfrak{f}_{\gamma_k}(\sigma)$.

Let $z \in \hat{X}(K)$ and let p be a path from v to z. Let us define functions $\alpha_i : G_K \to \mathbf{Z}_l$ for $i = 1, 2, \ldots, n$ by the following congruence

$$\mathfrak{f}_p(\sigma) \equiv \prod_{i=1}^{n} S_i^{\alpha_i(\sigma)} \mod \big(\pi_1(X_{\bar{K}}; v), \pi_1(X_{\bar{K}}; v)\big).$$

Let G_K act on $\mathbf{Z}_l$ as a multiplication by the cyclotomic character $\chi : G_K \to \mathbf{Z}_l^*$.

Corollary 2.2.2. *The functions $\alpha_i : G_K \to \mathbf{Z}_l$ for $i = 1, 2, \ldots, n$ are cocycles.*

Proof. The corollary follows from Propositions 1.0.7 and 2.2.1.

§3. Filtrations of G_K associated with the lower central series of π_1

3.0. In this section we shall study various filtrations of the group G_K obtained from the action of G_K on fundamental groups and on torsors of paths. The filtrations obtained from the action on fundamental groups were already studied by Ihara (see [I1]), Nakamura and Tsunogai (see [NT]) and others.

These filtrations are associated to the lower central series filtrations. Hence we recall here the definition of the lower central series of a group.

Let π be a group. The subgroups $\Gamma^n\pi$ of the lower central series are defined recursively by

$$\Gamma^1\pi := \pi, \quad \Gamma^{n+1}\pi := (\Gamma^n\pi, \pi), \quad n = 1, 2, \ldots$$

(see [MKS, Section 5.3]).

Let $X = \mathbf{P}^1_K \setminus \{a_1, \ldots, a_{n+1}\}$ and let $z, v \in \hat{X}(K)$. Fix an embedding of $\bar{K}$ into $\mathbf{C}$. Let $x = (x_1, \ldots, x_{n+1})$ be a sequence of geometric generators of $\pi_1(X(\mathbf{C}); v)$ associated with a family of paths $\Gamma = \{\gamma_i\}_{i=1,\ldots,n+1}$. The action of G_K on $\pi_1(X_{\bar{K}}; v)$ preserves $\Gamma^{i+1}\pi_1(X_{\bar{K}}; v)$, hence G_K acts also on the quotient group $\pi_1(X_{\bar{K}}; v)/\Gamma^{i+1}\pi_1(X_{\bar{K}}; v)$.

We set

$$G_i = G_i(X, v) := \ker\big(G_K \to \mathrm{Aut}(\pi_1(X_{\bar{K}}; v)/\Gamma^{i+1}\pi_1(X_{\bar{K}}; v))\big).$$

Observe that $G_1 = \mathrm{Gal}(\bar{K}/K(\mu_{l^\infty}))$. The quotient group G_i/G_{i+1} is isomorphic to a finite direct sum of several copies of $\mathbf{Z}_l$ (see [NT, Theorem (5.11)]). This implies that G_k/G_i are l-adic Lie groups.

The group $G_K/G_1 \subset \mathbf{Z}_l^*$ acts on G_i/G_{i+1} and the G_K/G_1-module G_i/G_{i+1} is isomorphic to $\mathbf{Z}_l(i)^{n_i}$ (see [I1] in the special case, when $X = \mathbf{P}^1_{\mathbf{Q}} \setminus \{0, 1, \infty\}$). Below we shall show that this result is a corollary of a more general statement.

Let us set $G_\infty = G_\infty(X, v) := \bigcap_{i=1}^\infty G_i(X, v)$. Then $G_1/G_\infty = \varprojlim_i G_1/G_i$ is a pro l-adic Lie group.

We say that two paths $p, q \in \pi(X_{\bar{K}}; z, v)$ are Γ^i-equivalent if $p^{-1} \cdot q \in \Gamma^i\pi_1(X_{\bar{K}}; v)$. The set of Γ^i-equivalence classes, which we denote by $\pi(X_{\bar{K}}; z, v)/\Gamma^i$, is a $\pi_1(X_{\bar{K}}; v)/\Gamma^i\pi_1(X_{\bar{K}}; v)$-torsor. The action of G_K on $\pi(X_{\bar{K}}; z, v)$ induces an action of G_K on $\pi(X_{\bar{K}}; z, v)/\Gamma^i$ compatible with the structure of the $\pi_1(X_{\bar{K}}; v)/\Gamma^i\pi_1(X_{\bar{K}}; v)$-torsor.

We introduce a subgroup $H_i = H_i(X; z, v)$ of G_i by

$$H_i = H_i(X; z, v) := \ker\big(G_i(X, v) \to \mathrm{Aut}_{Set}(\pi(X_{\bar{K}}; z, v)/\Gamma^i)\big).$$

PROPOSITION 3.0.1. *The conjugation on H_j by elements of G_K induces an action of $G_K/G_1 \subset \mathbf{Z}_l^*$ on the quotient group H_j/H_{j+1}. Moreover H_j/H_{j+1} is isomorphic to a finite direct sum $\mathbf{Z}_l(j)^{m_j}$ as a G_K/G_1-module.*

Proof. Let us fix a path p from v to z. The map $t_p : \pi(X_{\bar{K}}; z, v) \to \pi_1(X_{\bar{K}}; v)$ is G_K-equivariant, if $\sigma \in G_K$ acts by σ_p on $\pi_1(X_{\bar{K}}; v)$. The map t_p induces a G_K-equivariant map

$$\pi(X_{\bar{K}}; z, v)/\Gamma^j \pi(X_{\bar{K}}; z, v) \to \pi_1(X_{\bar{K}}; v)/\Gamma^j \pi_1(X_{\bar{K}}; v).$$

Hence we get that

$$H_j = \ker\big(G_j \to \mathrm{Aut}_{Set}(\pi_1(X_{\bar{K}}; v)/\Gamma^j \pi_1(X_{\bar{K}}; v))\big).$$

Let $\sigma \in H_j$. Proposition 2.2.1 implies that $\sigma(x_k) = (\mathfrak{f}_{\gamma_k}(\sigma))^{-1} \cdot x_k \cdot \mathfrak{f}_{\gamma_k}(\sigma)$ for $k = 1, \ldots, n, n+1$. Observe that $\mathfrak{f}_{\gamma_k}(\sigma) \in \Gamma^j \pi_1(X_{\bar{K}}; v)$ for $k = 1, \ldots, n, n+1$ and $\mathfrak{f}_p(\sigma) \in \Gamma^j \pi_1(X_{\bar{K}}; v)$. The sequence

$$\big(\mathfrak{f}_p(\sigma), \mathfrak{f}_{\gamma_1}(\sigma), \ldots, \mathfrak{f}_{\gamma_n}(\sigma)\big) \in \Gamma^j \pi_1(X_{\bar{K}}; v) \times (\Gamma^j \pi_1(X_{\bar{K}}; v))^n$$

determines the map σ_p. This implies that the quotient group H_j/H_{j+1} is isomorphic to a closed subgroup of

$$\Gamma^j \pi_1(X_{\bar{K}}; v)/\Gamma^{j+1} \pi_1(X_{\bar{K}}; v) \times \big(\Gamma^j \pi_1(X_{\bar{K}}; v)/\Gamma^{j+1} \pi_1(X_{\bar{K}}; v)\big)^n.$$

Therefore the quotient group H_j/H_{j+1} is isomorphic to a finite direct sum $\mathbf{Z}_l^{m_j}$.

Let $\tau \in G_K$ and $\sigma \in H_j$. We shall show that $\tau \cdot \sigma \cdot \tau^{-1} = \chi(\tau)^j \cdot \sigma$ in H_j/H_{j+1}. It follows from Proposition 1.0.7 and Corollary 1.0.8 that $\mathfrak{f}_p(\tau \cdot \sigma \cdot \tau^{-1}) = \mathfrak{f}_p(\tau) \cdot \tau(\mathfrak{f}_p(\sigma)) \cdot (\tau \cdot \sigma \cdot \tau^{-1})(\mathfrak{f}_p(\tau)^{-1})$. Observe that $\mathfrak{f}_p(\tau) \cdot \tau(\mathfrak{f}_p(\sigma)) \cdot (\tau \cdot \sigma \cdot \tau^{-1})(\mathfrak{f}_p(\tau)^{-1}) = \tau(\mathfrak{f}_p(\sigma)) \mod \Gamma^{j+1}\pi_1(X_{\bar{K}}; v)$ and $\tau(\mathfrak{f}_p(\sigma)) = \chi(\tau)^j \cdot \mathfrak{f}_p(\sigma) \mod \Gamma^{j+1}\pi_1(X_{\bar{K}}; v)$. This implies the proposition because we have also

$$\mathfrak{f}_{\gamma_i}(\tau \cdot \sigma \cdot \tau^{-1}) = \chi(\tau)^j \cdot \mathfrak{f}_{\gamma_i}(\sigma) \mod \Gamma^{j+1}\pi_1(X_{\bar{K}}; v)$$

for $i = 1, \ldots, n, n+1$.

Corollary 3.0.2. *The conjugation on G_j by elements of G_K induces an action of $G_K/G_1 \subset \mathbf{Z}_l^*$ on the quotient group G_j/G_{j+1}. Moreover G_j/G_{j+1} is isomorphic to a finite direct sum $\mathbf{Z}_l(j)^{n_j}$ as a G_K/G_1-module.*

Proof. The corollary is a special case of Proposition 3.0.1 if $z = v$ and p is a constant path.

The class of the element $\sigma \in H_j$ modulo H_{j+1} is completely determined by its coordinates

$$\begin{aligned}&\big(\mathfrak{f}_p(\sigma), \mathfrak{f}_{\gamma_1}(\sigma), \ldots, \mathfrak{f}_{\gamma_n}(\sigma)\big)\\ &\quad \in \big(\Gamma^j\pi_1(X_{\bar{K}}; v)/\Gamma^{j+1}\pi_1(X_{\bar{K}}; v)\big) \times \big(\Gamma^j\pi_1(X_{\bar{K}}; v)/\Gamma^{j+1}\pi_1(X_{\bar{K}}; v)\big)^n.\end{aligned}$$

Apparentely the first coordinate $\mathfrak{f}_p(\sigma) \mod \Gamma^{j+1}\pi_1(X_{\bar{K}}; v)$ depends on a choice of a path p from v to z. However we have the following result.

Lemma 3.0.3. *Let $\sigma \in H_j$ and let p and q be two paths from v to z. Then $\mathfrak{f}_p(\sigma) \equiv \mathfrak{f}_q(\sigma) \mod \Gamma^{j+1}\pi_1(X_{\bar{K}}; v)$.*

Proof. Let us set $S = p^{-1} \cdot q$. Then $\mathfrak{f}_q(\sigma) = \mathfrak{f}_{p\cdot S}(\sigma) = S^{-1} \cdot \mathfrak{f}_p(\sigma) \cdot \sigma(S)$. Observe that $\sigma(S) = S \mod \Gamma^{j+1}\pi_1(X_{\bar{K}}; v)$. Hence we get that $\mathfrak{f}_q(\sigma) = \mathfrak{f}_p(\sigma) \mod \Gamma^{j+1}\pi_1(X_{\bar{K}}; v)$.

It follows from Proposition 3.0.1 that H_k/H_i are l-adic Lie groups. Let us set

$$H_\infty = H_\infty(X; z, v) := \bigcap_{i=1}^{\infty} H_i(X; z, v).$$

Then $H_1/H_\infty = \varprojlim_i H_1/H_i$ is a pro l-adic Lie group.

Definition 3.0.4. Let A and B be nilpotent groups with exponents in $\mathbf{Z}_l$. We say that a homomorphism $h : A \to B$ of groups with exponents in $\mathbf{Z}_l$ is an f-epimorphism if for any $b \in B$ there exists a positive integer n and an element $a \in A$ such that $h(a) = b^{l^n}$.

Remark. If A and B are $\mathbf{Z}_l$-modules and if B is a finitely generated $\mathbf{Z}_l$-module then $h : A \to B$ is an f-epimorphism if and only if $\operatorname{coker}(h)$ is finite.

PROPOSITION 3.0.5. *The natural homomorphisms*

$$H_i/H_k \longrightarrow G_i/G_k$$

are f-epimorphisms for any $i > 0$ and any $k > 0$ such that $k > i$.

Proof. The equality $H_1 = G_1$ implies that the natural homomorphism $g : H_1/H_k \to G_1/G_k$ is an epimorphism for any k. After the Malcev rational completion we obtain an epimorphism $g_0 : H_1/H_k \otimes \mathbf{Q} \to G_1/G_k \otimes \mathbf{Q}$ of nilpotent groups with exponents in $\mathbf{Q}_l$. The category of nilpotent groups with exponent in $\mathbf{Q}_l$ and the category of nilpotent Lie algebras over $\mathbf{Q}_l$ are equivalent. Hence passing to Lie algebras we get an epimorphism $\mathrm{Lie}(g_0) : \mathrm{Lie}(H_1/H_k \otimes \mathbf{Q}) \to \mathrm{Lie}(G_1/G_k \otimes \mathbf{Q})$ of finite dimensional nilpotent Lie algebras over $\mathbf{Q}_l$. The construction of the Malcev rational completion and then passing to Lie algebras are functorial. Therefore the Galois group G_K acts linearly on both Lie algebras and the morphism $\mathrm{Lie}(g_0)$ is G_K-equivariant. Now the standard weight arguments imply that the natural morphism $\mathrm{Lie}(H_i/H_k \otimes \mathbf{Q}) \to \mathrm{Lie}(G_i/G_k \otimes \mathbf{Q})$ is an epimorphism. Hence the homomorphism of nilpotent groups $H_i/H_k \otimes \mathbf{Q} \to G_i/G_k \otimes \mathbf{Q}$ is also an epimorphism. This implies that the natural map $H_i/H_k \to G_i/G_k$ is an f-epimorphism.

Let us set

$$\mathcal{K}_i(X,v) := \bigcap_{z \in \hat{X}(K)} H_i(X;z,v), \quad \mathcal{K}_i(X) := \bigcap_{(z,v) \in \hat{X}(K)^2} H_i(X;z,v)$$

and

$$\mathcal{K}_\infty(X,v) := \bigcap_{i=1}^{\infty} \mathcal{K}_i(X,v), \quad \mathcal{K}_\infty(X) := \bigcap_{i=1}^{\infty} \mathcal{K}_i(X).$$

3.0.6. Observe that $\mathcal{K}_1(X) = \mathcal{K}_1(X,v) = G_1(X,v) = \mathrm{Gal}(\bar{K}/K(\mu_{l^\infty}))$. We do not know if the maps

$$\mathcal{K}_i(X,v)/\mathcal{K}_k(X,v) \longrightarrow H_i(X;z,v)/H_k(X;z,v)$$

and

$$\mathcal{K}_i(X)/\mathcal{K}_k(X) \longrightarrow H_i(X;z,v)/H_k(X;z,v)$$

are f-epimorphisms for any i and any k. Below we shall show weaker results.

Let T be a nonempty finite subset of $\hat{X}(K)^2$. Let us set

$$\mathcal{K}_i^T(X) := \bigcap_{(z,v)\in T} H_i(X;z,v) \quad \text{and} \quad \mathcal{K}_\infty^T(X) := \bigcap_{i=1}^{\infty} \mathcal{K}_i^T(X).$$

In the same way as Proposition 3.0.5 we show the following result.

Proposition 3.0.7. *Let T and S be nonempty finite subsets of $\hat{X}(K)^2$. Assume that $S \subset T$. Then the maps*

$$\mathcal{K}_i^T(X)/\mathcal{K}_k^T(X) \longrightarrow \mathcal{K}_i^S(X)/\mathcal{K}_k^S(X)$$

are f-epimorphisms for any positive integers k and i such that $k > i$.

Lemma 3.0.8. *The restriction map*

$$H^1(G_K, \mathbf{Q}_l(N)) \longrightarrow H^1(\mathcal{K}_N^T(X), \mathbf{Q}_l(N))$$

is injective.

Proof. Let $\Gamma = \mathrm{Gal}(K(\mu_{l^\infty})/K)$. We recall the reader that $\mathcal{K}_1^T(X) = \mathrm{Gal}(\bar{K}/K(\mu_{l^\infty}))$. The restriction map

$$H^1(G_K, \mathbf{Q}_l(N)) \longrightarrow \mathrm{Hom}_\Gamma(\mathcal{K}_1^T(X)^{ab}, \mathbf{Q}_l(N))$$

is injective. Let $f \in \mathrm{Hom}_\Gamma(\mathcal{K}_1^T(X)^{ab}, \mathbf{Q}_l(N))$. Assume that the composition of f with the natural projection $\mathcal{K}_1^T(X) \to \mathcal{K}_1^T(X)^{ab}$ vanishes on $\mathcal{K}_N^T(X)$. Therefore f induces a Γ-homomorphism $\tilde{f} : (\mathcal{K}_1^T(X)/\mathcal{K}_N^T(X))^{ab} \to \mathbf{Q}_l(N)$. Proposition 3.0.1 implies that the quotient group $\mathcal{K}_1^T(X)/\mathcal{K}_N^T(X)$ is a successive extension of direct sums of $\mathbf{Z}_l(i)$ with $i < N$. Now it follows from weight arguments that $\tilde{f}$ and hence also f are zero maps. This implies the lemma.

Definition 3.0.9. Let $\mathcal{C}$ be the category whose objects are all finite subsets of $\hat{X}(K)^2$ and whose morphisms are inclusions. We set

$$H^1_{\mathcal{C}}(\mathcal{K}_N(X), \mathbf{Q}_l(N)) := \varinjlim_{\mathcal{C}} H^1(\mathcal{K}_N^T(X), \mathbf{Q}_l(N)).$$

Lemma 3.0.10. *The map*

$$H^1(G_K, \mathbf{Q}_l(N)) \longrightarrow H^1_{\mathcal{C}}(\mathcal{K}_N(X), \mathbf{Q}_l(N))$$

is injective.

Proof. The lemma follows from Lemma 3.0.8.

Lemma 3.0.10 will be needed in our formulation of Zagier conjecture in Section 7. We recall also that $H^1(G_K, \mathbf{Q}_l(N))$ for $N > 1$ is a finite dimensional vector space over $\mathbf{Q}_l$. More precisely there is the following result. Let r_1 (resp. r_2) be a number of real (resp. complex) places of K. We assume that l is an odd prime. Let S be a set of maximal ideals of $\mathcal{O}_K$ containing all maximal ideals which divide l and let $\mathcal{O}_{K,S}$ be a ring of S-integers in K. Then

$$\begin{aligned}
&\dim H^1(\operatorname{Spec}\mathcal{O}_{K,S}, \mathbf{Q}_l(N)) = \dim H^1(G_K, \mathbf{Q}_l(N)) = r_2, \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{if } N \text{ is even and greater than 1;} \\
&\dim H^1(\operatorname{Spec}\mathcal{O}_{K,S}, \mathbf{Q}_l(N)) = \dim H^1(G_K, \mathbf{Q}_l(N)) = r_1 + r_2, \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{if } N \text{ is odd and greater than 1.}
\end{aligned}$$

(See [S2, Theorem 1] for $\mathcal{O}_K[\frac{1}{l}]$ and apply Proposition 1 from [S1] for K and $\mathcal{O}_{K,S}$.)

Let us assume that $\mathcal{O}^*_{K,S} \otimes \mathbf{Q}$ is a finite dimensional vector space over $\mathbf{Q}$. Then

$$\dim H^1(\operatorname{Spec}\mathcal{O}_{K,S}, \mathbf{Q}_l(1)) = \dim_{\mathbf{Q}}(\mathcal{O}^*_{K,S} \otimes \mathbf{Q}).$$

The last equality follows from Kummer theory.

3.1. We shall study relations between filtrations $\{G_i\}_{i\in\mathbf{N}}$ and $\{H_i\}_{i\in\mathbf{N}}$ of G_K for different X.

Lemma 3.1.0. *Let $Y = P^1_K \setminus \{b_1, \dots, b_{m+1}\}$ and let $g : Y \to X$ be a non-constant morphism between affine varieties. Let $y, w \in \hat{Y}(K)$ and let $z = g(y)$ and $v = g(w)$. Then we have $G_i(Y, w) \subset G_i(X, v)$ and $H_i(Y; y, w) \subset H_i(X; z, v)$.*

Proof. Observe that the induced map $g_* : \pi_1(Y_{\bar{K}}; w) \to \pi_1(X_{\bar{K}}; v)$ is surjective after passing to the Malcev rational completions and it commutes with the action of G_K. This implies that $G_i(Y, w) \subset G_i(X, v)$. Let p be a path from w to y. Then $\mathfrak{f}_{g(p)}(\sigma) = g_*(\mathfrak{f}_p(\sigma))$. Hence $\mathfrak{f}_p(\sigma) \in \Gamma^i\pi_1(Y_{\bar{K}}; w)$ implies that $\mathfrak{f}_{g(p)}(\sigma) \in \Gamma^i\pi_1(X_{\bar{K}}; v)$. This implies that $H_i(Y; y, w) \subset H_i(X; z, v)$.

As before the weight arguments imply the following result.

PROPOSITION 3.1.1. *The induced maps*

$$G_i(Y,w)/G_{i+k}(Y,w) \longrightarrow G_i(X,v)/G_{i+k}(X,v)$$

and

$$H_i(Y;y,w)/H_{i+k}(Y;y,w) \longrightarrow H_i(X;z,v)/H_{i+k}(X;z,v)$$

are f-epimorphisms for all $i>0$ and all $k>0$.

3.2. We recall that $x=(x_1,x_2,\dots,x_{n+1})$ is a sequence of geometric generators of $\pi_1(X(\mathbf{C});v)$. Then $\pi_1(X_{\bar{K}};v)$ is a free pro-l group on n generators $x_1,\dots,x_n$. Let Lie($\mathbf{X}$) be a free Lie algebra on n generators $X_1,\dots,X_n$. Let us fix a Hall base $\mathcal{B}$ of Lie($\mathbf{X}$). Let $\mathcal{B}_i$ be the set of elements of degree i in $\mathcal{B}$. We introduce a linear order in the set $\mathcal{B}$ in the following way. We fix a linear order in $\mathcal{B}_i$ for every i. We assume that elements of $\mathcal{B}_i$ are smaller than elements of $\mathcal{B}_{i+1}$.

If $e=[\cdots[X_{i_1},X_{i_2}]X_{i_3}\cdots]$, we denote by $e(x)$ the element $(\cdots(x_{i_1},x_{i_2})\,x_{i_3}\cdots)$ of $\pi_1(X_{\bar{K}};v)$. It is well known that any $g\in\pi_1(X_{\bar{K}};v)$ can be written uniquely as an infinite convergent product

$$\prod_{i=1}^{\infty}\prod_{e\in\mathcal{B}_i} e(x)^{\alpha_e}$$

where $\alpha_e\in\mathbf{Z}_l$ and the product is taken in the declared linear order in $\mathcal{B}$.

DEFINITION 3.2.0. Let $z\in\hat{X}(K)$ and let $p\in\pi(X_{\bar{K}};z,v)$. For each $e\in\mathcal{B}_j$ we define maps

$$\kappa_e(p,x): H_j(X;z,v)\longrightarrow \mathbf{Z}_l(j)$$

by the following equations

$$\mathfrak{f}_p(\sigma)\equiv\prod_{e\in\mathcal{B}_j} e(x)^{\kappa_e(p,x)(\sigma)} \mod \Gamma^{j+1}\pi_1(X_{\bar{K}};v).$$

LEMMA 3.2.1. *Let $z\in\hat{X}(K)$ and let $p\in\pi(X_{\bar{K}};z,v)$. Let $e\in\mathcal{B}_j$. The map $\kappa_e(p,x): H_j(X;z,v)\to\mathbf{Z}_l(j)$ is a homomorphism compatible with actions of* $\mathrm{Gal}(K(\mu_{l^\infty})/K)$. *The map $\kappa_e(p,x)$ does not depend on the choice of a path p from v to z and it does not depend on the choice of a sequence of geometric generators $x=(x_1,x_2,\dots,x_{n+1})$ in the same permutation class.*

Proof. Let $x' = (x'_1, \dots, x'_{n+1})$ be another sequence of geometric generators of $\pi_1(X(\mathbf{C}); v)$ associated with a family of paths $\Gamma' = \{\gamma'_i\}_{i=1,\dots,n+1}$. We shall assume that the automorphism of $\pi_1(X(\mathbf{C}); v)$ given by $x_i \to x'_i$ for $i = 1, \dots, n+1$ is in $\mathcal{B}^{(1)*}_{n+1}(x_1, \dots, x_{n+1})$. We have

$$x'_i = f_i(x_1, \dots, x_n)^{-1} \cdot x_i \cdot f_i(x_1, \dots, x_n) \quad (i = 1, \dots, n+1), \tag{3.2.2}$$

where $f_i(x_1, \dots, x_n) := \gamma_i^{-1} \cdot \gamma'_i \in \pi_1(X(\mathbf{C}); v)$. Then

$$\mathfrak{f}_p(\sigma) \equiv \prod_{e \in \mathcal{B}_j} e(x')^{\kappa_e(p,x')(\sigma)} \mod \Gamma^{j+1}\pi_1(X_{\bar{K}}; v).$$

It follows from (3.2.2) that $e(x) \equiv e(x') \mod \Gamma^{j+1}\pi_1(X_{\bar{K}}; v)$. Hence $\kappa_e(p, x) = \kappa_e(p, x')$.

Let $q \in \pi(X_{\bar{K}}; z, v)$ and let $T := p^{-1} \cdot q$. Then it follows from (1.0.4) that

$$\mathfrak{f}_q(\sigma) = T^{-1} \cdot \mathfrak{f}_p(\sigma) \cdot \sigma(T).$$

If $\sigma \in G_j(X, v)$ then $\sigma(T) = T \mod \Gamma^{j+1}\pi_1(X_{\bar{K}}; v)$. Hence we get $\mathfrak{f}_q(\sigma) = T^{-1} \cdot \mathfrak{f}_p(\sigma) \cdot \sigma(T) = \mathfrak{f}_p(\sigma) \mod \Gamma^{j+1}\pi_1(X_{\bar{K}}; v)$. Therefore $\kappa_e(p, x)$ does not depend on the choice of a path p in $\pi(X_{\bar{K}}; z, v)$.

The formula

$$\mathfrak{f}_p(\tau\sigma) = \mathfrak{f}_p(\tau) \cdot \tau(\mathfrak{f}_p(\sigma))$$

(see Proposition 1.0.7) and Proposition 2.2.1 imply that $\kappa_e(p, x)$ is a homomorphism.

Let $\tau \in G_K$ and $\sigma \in H_j(X; z, v)$. Then $\tau\sigma\tau^{-1} \in H_j(X; z, v)$ and

$$\mathfrak{f}_p(\tau\sigma\tau^{-1}) \equiv \prod_{e \in \mathcal{B}_j} e(x)^{\kappa_e(p,x)(\tau\sigma\tau^{-1})} \mod \Gamma^{j+1}\pi_1(X_{\bar{K}}; v).$$

On the other hand

$$\mathfrak{f}_p(\tau\sigma\tau^{-1}) = \mathfrak{f}_p(\tau) \cdot \tau(\mathfrak{f}_p(\sigma)) \cdot \tau\sigma(\mathfrak{f}_p(\tau^{-1})).$$

Working $\mod \Gamma^{j+1}\pi_1(X_{\bar{K}}; v)$ we get

$$\begin{aligned}\mathfrak{f}_p(\tau) \cdot \tau(\mathfrak{f}_p(\sigma)) \cdot \tau\sigma(\mathfrak{f}_p(\tau^{-1})) &\equiv \mathfrak{f}_p(\tau) \cdot \prod_{e \in \mathcal{B}_j} e(x)^{\chi(\tau)^j \kappa_e(p,x)(\sigma)} \cdot \tau(\mathfrak{f}_p(\tau^{-1})) \\ &\equiv \prod_{e \in \mathcal{B}_j} e(x)^{\chi(\tau)^j \kappa_e(p,x)(\sigma)}\end{aligned}$$

because $\sigma(\mathfrak{f}_p(\tau^{-1})) \equiv \mathfrak{f}_p(\tau^{-1}) \bmod \Gamma^{j+1}\pi_1(X_{\bar{K}};v)$ and $\tau(\mathfrak{f}_p(\tau^{-1})) = (\mathfrak{f}_p(\tau))^{-1}$. Hence we get that $\kappa_e(p,x)(\tau\sigma\tau^{-1}) = \chi(\tau)^j\kappa_e(p,x)(\sigma)$.

Observe that the homomorphism $\kappa_e(p,x) : H_j(X;z,v) \to \mathbf{Z}_l(j)$ depends only on $(z,v) \in \hat{X}(K)^2$ and on a linear order $(a_1,\dots,a_{n+1})$ of points removed from $\mathbf{P}^1_K$. Assuming that the linear order $(a_1,\dots,a_{n+1})$ is fixed we set

$$\kappa_e(z,v) := \kappa_e(p,x).$$

§4. Coordinates on the fundamental group and on the torsor

4.0. Let $X = \mathbf{P}^1_K \setminus \{a_1,\dots,a_{n+1}\}$ and let $v \in \hat{X}(K)$. Let $x = (x_1,\dots,x_{n+1})$ be a sequence of geometric generators of $\pi_1(X(\mathbf{C});v)$. Let $\mathbf{Q}_l\{\{X_1,\dots,X_n\}\}$ be an algebra of non-commutative formal power series in n non-commuting variables $X_1,\dots,X_n$. We set $\mathbf{X} := \{X_1,\dots,X_n\}$. To simplify the notation we shall write $\mathbf{Q}_l\{\{\mathbf{X}\}\}$ instead of $\mathbf{Q}_l\{\{X_1,\dots,X_n\}\}$.

We recall that $\mathbf{Q}_l$ is a topological non-archimedian field. Let I be the augmentation ideal of $\mathbf{Q}_l\{\{\mathbf{X}\}\}$. Observe that $\mathbf{Q}_l\{\{\mathbf{X}\}\}/I^m$ is a finite dimensional topological vector space over $\mathbf{Q}_l$ and $\mathbf{Q}_l\{\{\mathbf{X}\}\} = \varprojlim_m \mathbf{Q}_l\{\{\mathbf{X}\}\}/I^m$. We equip $\mathbf{Q}_l\{\{\mathbf{X}\}\}$ with a topology of the projective limit. We recall that $\pi_1(X_{\bar{K}};v)$ is equipped with a pro-finite topology.

We define a continuous embedding

$$k_x : \pi_1(X_{\bar{K}};v) \longrightarrow \mathbf{Q}_l\{\{\mathbf{X}\}\}$$

setting $k_x(x_i) := \exp X_i$ for $i = 1,\dots,n$ and requiring that $k_x(w \cdot w') = k_x(w) \cdot k_x(w')$.

Let $p \in \pi(X_{\bar{K}};z,v)$. Composing t_p (see Section 1) with k_x we get a continuous embedding

$$k_{x,p} : \pi(X_{\bar{K}};z,v) \longrightarrow \mathbf{Q}_l\{\{\mathbf{X}\}\}.$$

Let us set

$$\Lambda_{(p,x)}(\sigma) := k_x(\mathfrak{f}_p(\sigma)).$$

(We shall omit the subscript x if a sequence of geometric generators is fixed and we shall write $\Lambda_p(\sigma)$ instead of $\Lambda_{(p,x)}(\sigma)$.)

Let us denote by $\mathrm{Aut}(\mathbf{Q}_l\{\{\mathbf{X}\}\})$ the group of continuous automorphisms of the $\mathbf{Q}_l$-algebra $\mathbf{Q}_l\{\{\mathbf{X}\}\}$ and by $\mathrm{GL}(\mathbf{Q}_l\{\{\mathbf{X}\}\})$ the group of continuous linear automorphisms of the $\mathbf{Q}_l$-vector space $\mathbf{Q}_l\{\{\mathbf{X}\}\}$.

The action of G_K on $\pi_1(X_{\bar{K}}; v)$ defines a continuous action of G_K on $\mathbf{Q}_l\{\{\mathbf{X}\}\}$,

$$(\ )_x : G_K \longrightarrow \mathrm{Aut}(\mathbf{Q}_l\{\{\mathbf{X}\}\})$$

given by $\sigma_x(\exp X_i) := k_x(\sigma(x_i))$ for $i = 1, \ldots, n$.

The action of G_K on $\pi(X_{\bar{K}}; z, v)$ defines a continuous action of G_K on $\mathbf{Q}_l\{\{\mathbf{X}\}\}$,

$$(\ )_{x,p} : G_K \longrightarrow \mathrm{GL}(\mathbf{Q}_l\{\{\mathbf{X}\}\})$$

given by $\sigma_{x,p}(w) := \Lambda_{(p,x)}(\sigma) \cdot \sigma_x(w)$.
(We shall omit the subscript x if a sequence of geometric generators is fixed and we shall write σ instead of σ_x and σ_p instead of $\sigma_{x,p}$. We hope that these notations will not cause confusions with notations used in Section 1. There σ (resp. σ_p) denotes an automorphism of $\pi_1(X(\mathbf{C}); v)$ (resp. a bijection of $\pi_1(X(\mathbf{C}); v)$) induced from the action of G_K on $\pi_1(X(\mathbf{C}); v)$ (resp. on the $\pi_1(X(\mathbf{C}); v)$-torsor $\pi(X_{\bar{K}}; z, v)$).)

4.1. The subgroups $G_i(X, v)$ and $H_i(X; z, v)$ of G_K can be described in terms of the action of G_K on $\mathbf{Q}_l\{\{\mathbf{X}\}\}$ in the following way.

LEMMA 4.1.1. *Let $X = \mathbf{P}^1_K \setminus \{a_1, \ldots, a_{n+1}\}$ and let $z, v \in \hat{X}(K)$. We have*

$$G_i(X; v) = \ker(G_K \to \mathrm{Aut}(\mathbf{Q}_l\{\{\mathbf{X}\}\}/I^{i+1}))$$

and

$$H_i(X; z, v) = \ker(G_i(X; v) \to \mathrm{GL}(\mathbf{Q}_l\{\{\mathbf{X}\}\}/I^i)).$$

We shall omit an easy proof.

4.2. Let $\lambda \in \mathbf{Q}_l^*$. We define a continuous automorphism of $\mathbf{Q}_l$-algebras

$$\rho(\lambda) : \mathbf{Q}_l\{\{\mathbf{X}\}\} \longrightarrow \mathbf{Q}_l\{\{\mathbf{X}\}\}$$

setting $\rho(\lambda)(w) := \lambda^i w$ if w is homogenous of degree i.

Let $\sigma \in G_K$. We set

$$\varphi_x(\sigma) := \sigma_x \circ \rho(\chi(\sigma)^{-1})$$

and

$$\psi_{x,p}(\sigma) := \sigma_{x,p} \circ \rho(\chi(\sigma)^{-1}).$$

Observe that $\varphi_x(\sigma)$ (resp. $\psi_{x,p}(\sigma)$) is a pro-unipotent automorphism of $\mathbf{Q}_l$-algebra (resp. pro-unipotent $\mathbf{Q}_l$-linear automorphism of) $\mathbf{Q}_l\{\{\mathbf{X}\}\}$.

Remark. If $\sigma \in G_1$ then $\varphi_x(\sigma) = \sigma_x$ and $\psi_{x,p}(\sigma) = \sigma_{x,p}$.

Lemma 4.2.1. *We have*

$$\varphi_x(\tau \cdot \sigma) = \varphi_x(\tau) \circ \big(\rho(\chi(\tau)) \circ \varphi_x(\sigma) \circ \rho(\chi(\tau)^{-1})\big)$$

and

$$\psi_{x,p}(\tau \cdot \sigma) = \psi_{x,p}(\tau) \circ \big(\rho(\chi(\tau)) \circ \psi_{x,p}(\sigma) \circ \rho(\chi(\tau)^{-1})\big).$$

We can interpret the equalities from Lemma 4.2.1 in the following way.

Corollary 4.2.2. *Let* G_K *acts on* $\mathrm{Aut}(\mathbf{Q}_l\{\{\mathbf{X}\}\})$ (*resp.* $\mathrm{GL}(\mathbf{Q}_l\{\{\mathbf{X}\}\})$) *by* $\sigma(a) := \rho(\chi(\sigma))\circ a\circ\rho(\chi(\sigma)^{-1})$. *Then the maps* $\varphi_x : G_K \to \mathrm{Aut}(\mathbf{Q}_l\{\{\mathbf{X}\}\})$ *and* $\psi_{x,p} : G_K \to \mathrm{Aut}(\mathbf{Q}_l\{\{\mathbf{X}\}\})$ *are* 1-*cocycles.*

Let $\lambda \in \mathbf{Q}_l^*$. We shall denote by a^λ the automorphism $\rho(\lambda^{-1}) \circ a \circ \rho(\lambda)$

§5. l-adic iterated integrals

5.0. The purpose of this section is to introduce objects called by us l-adic iterated integrals (see Definition 5.3.0). These l-adic iterated integrals evaluated at z are functions from the Galois group G_K to $\mathbf{Q}_l$, which to $\sigma \in G_K$ associate coefficients of the power series $(\log \psi_{x,p}(\sigma))(1)$ ($(\log \sigma_{x,p})(1)$ if $\sigma \in G_{K(\mu_{l^\infty})}$). These l-adic iterated integrals correspond to suitably normalized classical complex iterated integrals.

Let $a_1, \dots, a_{n+1}$ be K-points of the projective line $\mathbf{P}^1_K$. Let $X = \mathbf{P}^1_K \setminus \{a_1, \dots, a_{n+1}\}$ and let $v \in \hat{X}(K)$ be a base point. Let us choose a tangential base point v_i at a_i for $i = 1, 2, \dots, n+1$. Let $x = (x_1, \dots, x_{n+1})$ be a sequence of geometric generators of $\pi_1(X(\mathbf{C}); v)$ associated with a family of paths $\Gamma = \{\gamma_i\}_{i=1,\dots,n+1}$ from v to each v_i. It follows from Section 3 that G_1/G_∞ is a pro-unipotent l-adic Lie group. Hence $(G_1/G_\infty) \otimes \mathbf{Q}$ - the rational completion of G_1/G_∞ - is a pro-unipotent $\mathbf{Q}_l$-Lie group. Let us set $\mathfrak{g} = \mathfrak{g}(X, v) := T_{id}((G_1/G_\infty) \otimes \mathbf{Q}) = \mathrm{Lie}((G_1/G_\infty) \otimes \mathbf{Q})$ - the tangent space of $(G_1/G_\infty) \otimes \mathbf{Q}$ at the identity.

We shall denote by $\mathrm{Der}(\mathbf{Q}_l\{\{\mathbf{X}\}\})$ the Lie algebra of continuous derivations of the $\mathbf{Q}_l$-algebra $\mathbf{Q}_l\{\{\mathbf{X}\}\}$ and by $\mathrm{End}(\mathbf{Q}_l\{\{\mathbf{X}\}\})$ the Lie algebra of continuous automorphisms of the $\mathbf{Q}_l$-vector space $\mathbf{Q}_l\{\{\mathbf{X}\}\}$.

We have the following commutative diagram:

$$\begin{array}{ccc} G_1/G_\infty & \xrightarrow{(\ )_x} & \mathrm{Aut}(\mathbf{Q}_l\{\{\mathbf{X}\}\}) \\ \Big\downarrow \log & & \Big\downarrow \log \\ \mathfrak{g} & \xrightarrow{\mathrm{Lie}(\ )_x} & \mathrm{Der}(\mathbf{Q}_l\{\{\mathbf{X}\}\}). \end{array}$$

(The upper horizontal arrow is induced by the action of G_K on $\mathbf{Q}_l\{\{\mathbf{X}\}\}$, the lower horizontal arrow is the induced map on tangent spaces, log on the right side is defined only on pro-unipotent automorphisms.)

Let $z \in \hat{X}(K)$ and let $p \in \pi(X_{\bar{K}}; z, v)$. It follows from Section 3 that $H_1/H_\infty = H_1(X; z, v)/H_\infty(X; z, v)$ is a pro-unipotent l-adic Lie group. Hence $(H_1/H_\infty) \otimes \mathbf{Q}$ is a pro-unipotent $\mathbf{Q}_l$-Lie group. Let us set $\mathfrak{h} = \mathfrak{h}(X; z, v) := T_{id}((H_1/H_\infty) \otimes \mathbf{Q}) = \mathrm{Lie}((H_1/H_\infty) \otimes \mathbf{Q})$ - the tangent space of $(H_1/H_\infty) \otimes \mathbf{Q}$ at the identity. We have the following commutative diagram:

$$\begin{array}{ccc} H_1/H_\infty & \xrightarrow{(\ )_{x,p}} & \mathrm{GL}(\mathbf{Q}_l\{\{\mathbf{X}\}\}) \\ \Big\downarrow \log & & \Big\downarrow \log \\ \mathfrak{h} & \xrightarrow{\mathrm{Lie}(\ )_{x,p}} & \mathrm{End}(\mathbf{Q}_l\{\{\mathbf{X}\}\}). \end{array}$$

Let $T \subset \hat{X}(K)^2$ be a finite subset containing a pair (z, v). We have epimorphisms $\mathcal{K}_1^T(X)/\mathcal{K}_\infty^T(X) \to G_1/G_\infty$ and $\mathcal{K}_1^T(X)/\mathcal{K}_\infty^T(X) \to H_1/H_\infty$ and the induced epimorphisms of Lie algebras

$$\mathrm{Lie}(\mathcal{K}_1^T(X)/\mathcal{K}_\infty^T(X) \otimes \mathbf{Q}) \longrightarrow \mathfrak{g} \quad \text{and} \quad \mathrm{Lie}(\mathcal{K}_1^T(X)/\mathcal{K}_\infty^T(X) \otimes \mathbf{Q}) \longrightarrow \mathfrak{h}.$$

Hence we can consider that the homomorphisms $(\ )_x$ and $(\ )_{x,p}$ are defined on $\mathcal{K}_1^T(X)/\mathcal{K}_\infty^T(X)$ and that the morphisms of Lie algebras $\mathrm{Lie}(\ )_x$ and $\mathrm{Lie}(\ )_{x,p}$ are defined on $\mathrm{Lie}(\mathcal{K}_1^T(X)/\mathcal{K}_\infty^T(X) \otimes \mathbf{Q})$.

The image of the morphism $(\ )_x$ (resp. $\mathrm{Lie}(\ )_x$) is contained in the "braid-like" subgroup of $\mathrm{Aut}(\mathbf{Q}_l\{\{\mathbf{X}\}\})$ (resp. Lie subalgebra of Der $(\mathbf{Q}_l\{\{\mathbf{X}\}\})$). We recall their definitions. We also describe subgroups and subalgebras containing images of morphisms $(\ )_{x,p}$ and $\mathrm{Lie}(\ )_{x,p}$ respectively.

5.1. We recall that $\mathbf{X} = \{X_1, \dots, X_2\}$. Let $\mathrm{Lie}(\mathbf{X})$ be a free Lie algebra over $\mathbf{Q}_l$ on the set $\mathbf{X}$. Let us set

$$L(\mathbf{X}) := \varprojlim_i \mathrm{Lie}(\mathbf{X})/\Gamma^i \,\mathrm{Lie}(\mathbf{X}).$$

We identify $L(\mathbf{X})$ with Lie elements in $\mathbf{Q}_l\{\{\mathbf{X}\}\}$.

We introduce the following notation. If A and B belong to a Lie algebra then we define $[[A,B]B^0] := [A,B]$, $[[A,B]B^1] := [[A,B],B]$ and $[[A,B]B^m] := [[[A,B]B^{m-1}],B]$ for $m > 1$.

Definition 5.1.0. Let us define subgroups

$$\begin{aligned}\mathrm{Aut}^*(\mathbf{Q}_l\{\{\mathbf{X}\}\}) := \{f \in \mathrm{Aut}(\mathbf{Q}_l\{\{\mathbf{X}\}\}) \mid \\ \forall X_i \in \mathbf{X}\ \exists l_i \in L(\mathbf{X}),\ f(X_i) = e^{-l_i}\cdot X_i \cdot e^{l_i}\};\end{aligned}$$

$$\mathrm{Aut}^* L(\mathbf{X}) := \left\{ f \in \mathrm{Aut}\, L(\mathbf{X}) \ \middle| \ \forall X_i \in \mathbf{X}\ \exists l_i \in L(\mathbf{X}),\ f(X_i) = X_i + \sum_{m=1}^{\infty} \frac{1}{m!}[[X_i, l_i]l_i^{m-1}] \right\}$$

and Lie subalgebras

$$\begin{aligned}\mathrm{Der}^*(\mathbf{Q}_l\{\{\mathbf{X}\}\}) := \{D \in \mathrm{Der}(\mathbf{Q}_l\{\{\mathbf{X}\}\}) \mid \\ \forall X_i \in \mathbf{X}\ \exists A_i \in L(\mathbf{X}), D(X_i) = X_i\cdot A_i - A_i \cdot X_i\};\end{aligned}$$

$$\mathrm{Der}^* L(\mathbf{X}) := \{D \in \mathrm{Der}\, L(\mathbf{X}) \mid \forall X_i \in \mathbf{X}\ \exists A_i \in L(\mathbf{X}), D(X_i) = [X_i, A_i]\}$$

and

$$\begin{aligned}\mathrm{Der}^* \mathrm{Lie}(\mathbf{X}) := \{D \in \mathrm{Der}\,\mathrm{Lie}(\mathbf{X}) \mid \\ \forall X_i \in \mathbf{X}\ \exists A_i \in \mathrm{Lie}(\mathbf{X}), D(X_i) = [X_i, A_i]\}.\end{aligned}$$

Lemma 5.1.1. *We have*

i) $\mathrm{Aut}^*(\mathbf{Q}_l\{\{\mathbf{X}\}\}) = \mathrm{Aut}^* L(\mathbf{X})$;

ii) $\mathrm{Der}^*(\mathbf{Q}_l\{\{\mathbf{X}\}\}) = \mathrm{Der}^* L(\mathbf{X})$;

iii) *The Lie algebra of* $\mathrm{Aut}^*(\mathbf{Q}_l\{\{\mathbf{X}\}\})$ (*resp.* $\mathrm{Aut}^* L(\mathbf{X})$) *is* $\mathrm{Der}^*(\mathbf{Q}_l\{\{\mathbf{X}\}\})$ (*resp.* $\mathrm{Der}^* L(\mathbf{X})$).

Proof. The first part follows from the well known formula

$$e^{-l_i} \cdot X_i \cdot e^{l_i} = X_i + \sum_{m=1}^{\infty} \frac{1}{m!}[[X_i, l_i]l_i^{m-1}]. \tag{5.1.2}$$

The second part is obvious, so it rests to show the last statement of the lemma. It is well known that the Lie algebra of the group of automorphisms of a $\mathbf{Q}_l$-algebra is the Lie algebra of derivations of this $\mathbf{Q}_l$-algebra. Let D be a derivation of the $\mathbf{Q}_l$-algebra $\mathbf{Q}_l\{\{\mathbf{X}\}\}$. Suppose that $\exp tD \in \mathrm{Aut}^*(\mathbf{Q}_l\{\{\mathbf{X}\}\})$. Then $(\exp tD)(X_i) = e^{-l_i(t)} \cdot X_i \cdot e^{l_i(t)}$ for $i = 1, \ldots, n$. The elements $l_i(t)$ are in $L(\mathbf{X})$. We can suppose that the coefficient of $l_i(t)$ at X_i vanishes. Then we have $l_i(0) = 0$ and $l_i(t)$ depends smoothly on t. Hence $A_i := \lim_{t\to 0} \frac{1}{t} l_i(t)$ exists and belongs to $L(\mathbf{X})$. Comparing Taylor developments of $(\exp tD)(X_i)$ and $e^{-l_i(t)} \cdot X_i \cdot e^{l_i(t)}$ we get $D(X_i) = [X_i, A_i]$ for $i = 1, \ldots, n$. Therefore D belongs to $\mathrm{Der}^*(\mathbf{Q}_l\{\{\mathbf{X}\}\})$.

Proposition 5.1.3. *Let $\sigma \in G_K$. Then $\varphi_x(\sigma) \in \mathrm{Aut}^*(\mathbf{Q}_l\{\{\mathbf{X}\}\})$ and $\log \varphi_x(\sigma) \in \mathrm{Der}^*(\mathbf{Q}_l\{\{\mathbf{X}\}\})$.*

Proof. It follows from Proposition 2.2.1 that

$$\sigma_x(X_i) = (\Lambda_{(\gamma_i,x)}(\sigma))^{-1} \cdot \chi(\sigma) X_i \cdot \Lambda_{(\gamma_i,x)}(\sigma)$$

for $i = 1, \ldots, n$. Hence $\varphi_x(\sigma) \in \mathrm{Aut}^*(\mathbf{Q}_l\{\{\mathbf{X}\}\})$. It follows from Lemma 5.1.1 that $\log \varphi_x(\sigma) \in \mathrm{Der}^*(\mathbf{Q}_l\{\{\mathbf{X}\}\})$.

To give an explicit formula for $\log \varphi_x(X_i)$ we need to study Galois actions on torsors of paths. The action of Galois groups on torsors of paths requires to introduce semi-direct products of Lie algebras. Below we give the necessary definitions.

Let L be a Lie algebra and let $\mathcal{D}$ be a Lie subalgebra of the algebra of Lie derivations of L. We equip the direct product $L \times \mathcal{D}$ with a Lie bracket

$$[(l, D), (l_1, D_1)] := ([l, l_1] + D(l_1) - D_1(l), [D, D_1]).$$

The resulting Lie algebra we denote by $L \tilde{\times} \mathcal{D}$ and we call it a semi-direct product of L and $\mathcal{D}$.

If $g \in \mathbf{Q}_l\{\{\mathbf{X}\}\}$ then L_g denotes left multiplication by g. $L_{\exp(L(\mathbf{X}))}$ is the set of left multiplications by elements of $\exp(L(\mathbf{X}))$ and $L_{L(\mathbf{X})}$ is the set of left multiplications by elements of $L(\mathbf{X})$.

Lemma 5.1.4. *Let $\mathcal{G}$ be a subgroup of* $\mathrm{GL}(\mathbf{Q}_l\{\{\mathbf{X}\}\})$ *generated by* $L_{\exp(L(\mathbf{X}))}$ *and* $\mathrm{Aut}^*(\mathbf{Q}_l\{\{\mathbf{X}\}\})$. *Then $\mathcal{G}$ is a semi-direct product of* $L_{\exp(L(\mathbf{X}))}$ *and* $\mathrm{Aut}^*(\mathbf{Q}_l\{\{\mathbf{X}\}\})$, *which we denote by* $L_{\exp(L(\mathbf{X}))} \tilde{\times} \mathrm{Aut}^*(\mathbf{Q}_l\{\{\mathbf{X}\}\})$. *The Lie algebra of* $L_{\exp(L(\mathbf{X}))} \tilde{\times} \mathrm{Aut}^*(\mathbf{Q}_l\{\{\mathbf{X}\}\})$ *is equal to a semi-direct product of Lie algebras* $L_{L(\mathbf{X})} \tilde{\times} \mathrm{Der}^* L(\mathbf{X}) \approx L(\mathbf{X}) \tilde{\times} \mathrm{Der}^* L(\mathbf{X})$.

Proof. Let $f, f_1 \in \exp(L(\mathbf{X}))$ and $\phi, \phi_1 \in \mathrm{Aut}^*(\mathbf{Q}_l\{\{\mathbf{X}\}\})$. Then we have

$$(L_f \circ \phi) \circ (L_{f_1} \circ \phi_1) = L_{f\cdot\phi(f_1)} \circ (\phi \circ \phi_1).$$

This implies that $\mathcal{G}$ is a semi-direct product of $L_{\exp(L(\mathbf{X}))}$ and $\mathrm{Aut}^*(\mathbf{Q}_l\{\{\mathbf{X}\}\})$. It follows from Lemma 5.1.1 that the Lie algebra of $\mathrm{Aut}^*(\mathbf{Q}_l\{\{\mathbf{X}\}\})$ is $\mathrm{Der}^* L(\mathbf{X})$. The Lie algebra of $L_{\exp(L(\mathbf{X}))}$ is $L_{L(\mathbf{X})}$. Hence the Lie algebra of $\mathcal{G}$ is equal to $L_{L(\mathbf{X})} \times \mathrm{Der}^* L(\mathbf{X})$ as a vector space.

Let $f, g \in L(\mathbf{X})$ and let $D, E \in \mathrm{Der}^* L(\mathbf{X})$. Observe that $L_f + D$ is the tangent vector at $t = 1$ to the curve $t \to L_{\exp tf} \circ \exp tD$. To calculate a Lie bracket of the Lie algebra of $\mathcal{G}$ we need to calculate the coefficient at t^2 of the commutator

$$(L_{\exp tf} \circ \exp tD, L_{\exp tg} \circ \exp tE).$$

This coefficient is equal $L_{[f,g]+D(g)-E(f)} + [D, E]$. This shows that the Lie algebra of $\mathcal{G}$ is the semi-direct product of Lie algebras $L_{L(\mathbf{X})} \tilde{\times} \mathrm{Der}^* L(\mathbf{X}) \approx L(\mathbf{X}) \tilde{\times} \mathrm{Der}^* L(\mathbf{X})$.

Proposition 5.1.5. *Let $\sigma \in G_K$. Then* $\psi_{x,p}(\sigma) \in L_{\exp(L(\mathbf{X}))} \tilde{\times} \mathrm{Aut}^*(\mathbf{Q}_l\{\{\mathbf{X}\}\})$ *and* $\log \psi_{x,p}(\sigma) \in L_{L(\mathbf{X})} \tilde{\times} \mathrm{Der}^* L(\mathbf{X})$.

Proof. Let $\sigma \in G_K$ and $w \in \mathbf{Q}_l\{\{\mathbf{X}\}\}$. We have

$$\psi_{x,p}(\sigma)(w) = \Lambda_{(p,x)}(\sigma) \cdot \varphi_x(\sigma)(w). \tag{5.1.6}$$

It follows from (5.1.6) that $\psi_{x,p}(\sigma)$ belongs to the semi-direct product

$$L_{\exp(L(\mathbf{X}))} \tilde{\times} \mathrm{Aut}^*(\mathbf{Q}_l\{\{\mathbf{X}\}\}).$$

The Lie algebra of the semi-direct product of groups $L_{\exp(L(\mathbf{X}))} \tilde{\times} \mathrm{Aut}^*(\mathbf{Q}_l\{\{\mathbf{X}\}\})$ is equal to a semi-direct product of Lie algebras $L_{L(\mathbf{X})} \tilde{\times} \mathrm{Der}^* L(\mathbf{X}) \approx L(\mathbf{X}) \tilde{\times} \mathrm{Der}^* L(\mathbf{X})$ by Lemma 5.1.4. Therefore $\log \psi_{x,p}(\sigma) \in L_{L(\mathbf{X})} \tilde{\times} \mathrm{Der}^* L(\mathbf{X})$. This finishes the proof of the proposition.

Below we shall calculate both components of $\log \psi_{x,p}(\sigma)$.

We denote by $\bigcirc$ a product given by the Baker-Campbell-Hausdorff formula (BCH formula) (see [MKS, Theorem 5.19]).

PROPOSITION 5.1.7. *The element* $\log \psi_{x,p}(\sigma)(1) \in L(\mathbf{X})$ *and we have*

$$\log \psi_{x,p}(\sigma) = L_{(\log \psi_{x,p}(\sigma))(1)} + \log \varphi_x(\sigma).$$

Proof. Let $g, h \in L(\mathbf{X})$ and $D \in \mathrm{Der}^*(\mathbf{Q}_l\{\{\mathbf{X}\}\})$. Then $[L_g, L_h] = L_{[g,h]}$ and $[D, L_g] = L_{D(g)}$. Hence all terms of $\log \psi_{x,p}(\sigma) - \log \varphi_x(\sigma) = L_{\log \Lambda_{(p,x)}(\sigma)} \bigcirc \log \varphi_x(\sigma) - \log \varphi_x(\sigma)$ are of the form L_g for some $g \in L(\mathbf{X})$. Therefore $\log \psi_{x,p}(\sigma) = L_g + \log \varphi_x(\sigma)$ for some $g \in L(\mathbf{X})$. Evaluating both sides of the equality at 1 we get that $g = (\psi_{x,p}(\sigma))(1)$. This finishes the proof of the proposition.

PROPOSITION 5.1.8. *Let* $\sigma \in G_K$. *Then we have*

$$\log \varphi_x(\sigma)(X_k) = [X_k; (\log \psi_{x,\gamma_k}(\sigma))(1)]$$

for $k = 1, \ldots, n$.

Proof. One computes easily that

$$\begin{aligned}(\log \psi_{x,\gamma_k}(\sigma))(1) &= (\Lambda_{(\gamma_k,x)}(\sigma) - 1)\\ &\quad - \tfrac{1}{2}\big(\Lambda_{(\gamma_k,x)}(\sigma) \cdot \varphi_x(\sigma)(\Lambda_{(\gamma_k,x)}(\sigma)) - 2\Lambda_{(\gamma_k,x)}(\sigma) + 1\big)\\ &\quad + \tfrac{1}{3}\big(\Lambda_{(\gamma_k,x)}(\sigma) \cdot \varphi_x(\sigma)(\Lambda_{(\gamma_k,x)}(\sigma)) \cdot \varphi_x(\sigma)^2(\Lambda_{(\gamma_k,x)}(\sigma))\\ &\qquad - 3\Lambda_{(\gamma_k,x)}(\sigma) \cdot \varphi_x(\sigma)(\Lambda_{(\gamma_k,x)}(\sigma)) + 3\Lambda_{(\gamma_k,x)}(\sigma) - 1\big) \cdots .\end{aligned}$$

This implies that $(\log \psi_{x,\gamma_k}(\sigma))(X_k) = X_k \cdot ((\log \psi_{x,\gamma_k}(\sigma))(1))$. Now it follows from Proposition 5.1.7 that $\log \varphi_x(\sigma)(X_k) = [X_k; (\log \psi_{x,\gamma_k}(\sigma))(1)]$.

5.2. The main object of our study are coefficients of the operator $\log \psi_{x,p}(\sigma)$ for varing σ (see Definition 5.3.0). The element $\log \psi_{x,p}(\sigma) \in L_{L(\mathbf{X})} \tilde{\times} \mathrm{Der}^* L(\mathbf{X})$. Hence to study these coefficients we need to study linear forms on the Lie algebra $L_{L(\mathbf{X})} \tilde{\times} \mathrm{Der}^* L(\mathbf{X})$ and on various Lie subalgebras of this Lie algebra. First we define suitable linear forms which evaluated on the element $\log \psi_{x,p}(\sigma)$ gives coefficients. Next we are studying properties of the operators induced by the Lie brackets on these linear forms.

The free Lie algebra $\mathrm{Lie}(\mathbf{X})$ (resp. the completed free Lie algebra $L(\mathbf{X})$) has an obvious $\mathbf{Q}$-structure - a free Lie algebra over $\mathbf{Q}$ on the set $\mathbf{X}$ (resp. a completed free Lie algebra over $\mathbf{Q}$ on the set $\mathbf{X}$). Therefore the Lie algebras of derivations $\operatorname{Der}\operatorname{Lie}(\mathbf{X})$ and $\operatorname{Der}^*\operatorname{Lie}(\mathbf{X})$ (resp. $\operatorname{Der} L(\mathbf{X})$ and $\operatorname{Der}^* L(\mathbf{X})$) have also $\mathbf{Q}$-structures.

Let $\operatorname{Lie}(\mathbf{X})_m$ be a vector subspace of $\operatorname{Lie}(\mathbf{X})$ of homogenous elements of degree m. Let $(\operatorname{Lie}(\mathbf{X})_m)^*$ be the dual vector space of the finite dimentional vector space $\operatorname{Lie}(\mathbf{X})_m$. We define the graded dual $\operatorname{Lie}(\mathbf{X})^\diamond$ of the free Lie algebra by

$$\operatorname{Lie}(\mathbf{X})^\diamond := \bigoplus_{m=1}^{\infty}(\operatorname{Lie}(\mathbf{X})_m)^*.$$

Let $\langle X_i\rangle$ be a vector subspace of $\operatorname{Lie}(\mathbf{X})$ generated by X_i and let $\langle X_i\rangle^*$ be the dual vector space. We define the subspace of $\operatorname{Lie}(\mathbf{X})^\diamond$ of linear forms killing $\langle X_i\rangle$ by

$$(\operatorname{Lie}(\mathbf{X})/\langle X_i\rangle)^\diamond := \ker(\operatorname{Lie}(\mathbf{X})^\diamond \to \langle X_i\rangle^*).$$

We shall define the graded dual of the semi-direct product $\operatorname{Lie}(\mathbf{X}) \tilde{\times} \operatorname{Der}^* \operatorname{Lie}(\mathbf{X})$. We start with the following observation. Let $D \in \operatorname{Der}^*\operatorname{Lie}(\mathbf{X})$ be such that $D(X_i) = [X_i, A_i]$ for $i = 1, \dots, n$. The map

$$f : \operatorname{Der}^*\operatorname{Lie}(\mathbf{X}) \longrightarrow \bigoplus_{i=1}^{n}(\operatorname{Lie}(\mathbf{X})/\langle X_i\rangle)$$

given by $f(D) = (A_1, \dots, A_n)$ is an isomorphism of vector spaces. The isomorphism f is compatible with $\mathbf{Q}$-structures on both vector spaces. The isomorphism f identifies $\operatorname{Der}^*\operatorname{Lie}(\mathbf{X})$ with $\bigoplus_{i=1}^{n}(\operatorname{Lie}(\mathbf{X})/\langle X_i\rangle)$. We define the graded dual of the Lie algebra $\operatorname{Der}^*\operatorname{Lie}(\mathbf{X})$ by

$$(\operatorname{Der}^*\operatorname{Lie}(\mathbf{X}))^\diamond := \bigoplus_{i=1}^{n}(\operatorname{Lie}(\mathbf{X})/\langle X_i\rangle)^\diamond.$$

The dual of a semi-direct product of two Lie algebras is a direct sum of duals of these two Lie algebras. Hence we set

$$(\operatorname{Lie}(\mathbf{X}) \tilde{\times} \operatorname{Der}^*\operatorname{Lie}(\mathbf{X}))^\diamond := \operatorname{Lie}(\mathbf{X})^\diamond \oplus (\operatorname{Der}^*\operatorname{Lie}(\mathbf{X}))^\diamond.$$

Definition 5.2.0. Let V be a vector space. We say that V is a Lie coalgebra if V is equipped with a linear map $d : V \to V \otimes V$ satisfying

i) $\tau \circ d + d = 0$, where $\tau(a \otimes b) = b \otimes a$;

ii) $\sum_{i=0}^{2} \sigma^i \circ (d \otimes id_V) \circ d = 0$, where $\sigma(a \otimes b \otimes c) = b \otimes c \otimes a$.

It follows from i) that d factors through $d : V \to V \wedge V$, where

$$V \wedge V = \Big\{ \sum_{i \in I} n_i (a_i \otimes b_i - b_i \otimes a_i) \in V \otimes V \Big\}.$$

Farther we shall also denote $V \wedge V$ by $\bigwedge^2 V$.

LEMMA 5.2.1. i) *If V is a Lie coalgebra then the dual vector space V^* equipped with $[\ \ ] := d^* : V^* \otimes V^* \to V^*$ is a Lie algebra.*

ii) *If L is a Lie algebra then L^* equipped with $d := [\ \ ]^* : L^* \to L^* \otimes L^*$ is a Lie coalgebra.*

COROLLARY 5.2.2. *The vector spaces* $\mathrm{Lie}(\mathbf{X})^\diamond$, $(\mathrm{Der}^* \mathrm{Lie}(\mathbf{X}))^\diamond$ *and* $(\mathrm{Lie}(\mathbf{X}) \tilde{\times} \mathrm{Der}^* \mathrm{Lie}(\mathbf{X}))^\diamond$ *equipped with $d := [\ \ ]^*$ are Lie coalgebras.*

Proof. The dual vector spaces $\mathrm{Lie}(\mathbf{X})^*$, $(\mathrm{Der}^* \mathrm{Lie}(\mathbf{X}))^*$ and $(\mathrm{Lie}(\mathbf{X}) \tilde{\times} \mathrm{Der}^* \mathrm{Lie}(\mathbf{X}))^*$ equipped with $d := [\ \ ]^*$ are Lie coalgebras. Observe that d preserves $\mathrm{Lie}(\mathbf{X})^\diamond$, $(\mathrm{Der}^* \mathrm{Lie}(\mathbf{X}))^\diamond$ and $(\mathrm{Lie}(\mathbf{X}) \tilde{\times} \mathrm{Der}^* \mathrm{Lie}(\mathbf{X}))^\diamond$. Hence these vector spaces are also Lie coalgebras.

5.2.3. The vector spaces $\mathrm{Lie}(\mathbf{X})^\diamond$, $(\mathrm{Der}^* \mathrm{Lie}(\mathbf{X}))^\diamond$ and $(\mathrm{Lie}(\mathbf{X}) \tilde{\times} \mathrm{Der}^* \mathrm{Lie}(\mathbf{X}))^\diamond$ are canonically embedded as Lie coalgebras into $L(\mathbf{X})^*$, $(\mathrm{Der}^* L(\mathbf{X}))^*$ and $(L(\mathbf{X}) \tilde{\times} \mathrm{Der}^* L(\mathbf{X}))^*$ respectively. When we view these vector spaces as vector subspaces of $L(\mathbf{X})^*$, $(\mathrm{Der}^* L(\mathbf{X}))^*$ and $(L(\mathbf{X}) \tilde{\times} \mathrm{Der}^* L(\mathbf{X}))^*$ then we denote them by $L(\mathbf{X})^\diamond$, $(\mathrm{Der}^* L(\mathbf{X}))^\diamond$ and $(L(\mathbf{X}) \tilde{\times} \mathrm{Der}^* L(\mathbf{X}))^\diamond$ respectively.

5.3. Below we shall give the very definition of the l-adic iterated integrals. Observe that the element $(\log \psi_{x,p}(\sigma))(1)$ is a Lie element in $\mathbf{Q}_l\{\{\mathbf{X}\}\}$ by Proposition 5.1.7.

DEFINITION 5.3.0. Let us fix a Hall base $\mathcal{B}$ of $\mathrm{Lie}(\mathbf{X})$. Let $\sigma \in G_K$. We set

$$a_{x,p}(\sigma) := (\log \psi_{x,p}(\sigma))(1) = \sum_{e \in \mathcal{B}} a^e_{x,p}(\sigma) \cdot e.$$

Let $\phi \in L(\mathbf{X})^\diamond$ be a linear form defined over $\mathbf{Q}$. We set

$$a^\phi_{x,p}(\sigma) := \phi((\log \psi_{x,p}(\sigma))(1)).$$

The functions $a^e_{x,p} : G_K \to \mathbf{Q}_l$ we shall call l-adic iterated integrals.

If $e \in \mathcal{B}$ then we denote by e^* the dual vector with respect to this base $\mathcal{B}$. Observe that the set $\{e^*\}_{e\in\mathcal{B}}$ is a linear base of $\mathrm{Lie}(\mathbf{X})^\diamond$. Hence any $a^\phi_{x,p}$ is a linear combination of a finite number of $a^e_{x,p}$.

THEOREM 5.3.1. *Let $e \in \mathcal{B}$ be an element of degree i. We have*:

i) $a^e_{x,p}(\sigma) = 0$ *for* $\sigma \in H_{i+1}$.

ii) $a^e_{x,p}(\tau \cdot \sigma) = a^e_{x,p}(\tau) + a^e_{x,p}(\sigma)$ *for any* $\tau, \sigma \in H_i$.

iii) *The homomorphism* $a^e_{x,p|H_i} : H_i \to \mathbf{Q}_l(i)$ *is compatible with the action of* $\mathrm{Gal}(K(\mu_{l^\infty})/K)$ *on* H_i *and* $\mathbf{Q}_l(i)$.

iv) *The homomorphism* $a^e_{x,p|H_i} : H_i \to \mathbf{Q}_l(i)$ *depends only on z and v. It does not depend on the choice of geometric generators x* (*in a given permutation class*) *and on a choice of a path p from v to z.*

Proof. The point i) follows from the definition of the group H_i and from Lemma 4.1.1. Let $\tau, \sigma \in H_i$. Then $\psi_{x,p}(\sigma) = \sigma_{x,p}$, $\psi_{x,p}(\tau) = \tau_{x,p}$ and $\psi_{x,p}(\tau\cdot\sigma) = (\tau\cdot\sigma)_{x,p}$ It follows from the point i) that

$$(\log \sigma_{x,p})(1) = \sum_{e\in\mathcal{B}^i} a^e_{x,p}(\sigma)\cdot e + \sum_{j\geq i+1}\sum_{e\in\mathcal{B}^j} a^e_{x,p}(\sigma)\cdot e. \tag{5.3.2}$$

We have $(\tau\cdot\sigma)_{x,p} = \tau_{x,p}\circ\sigma_{x,p}$. The BCH formula implies

$$\log(\tau\cdot\sigma)_{x,p} = \log\tau_{x,p} + \log\sigma_{x,p} + \frac{1}{2}[\log\tau_{x,p}, \log\sigma_{x,p}] + \cdots .$$

Evaluating both sides of the equality at 1 we get

$$(\log(\tau\cdot\sigma)_{x,p})(1) = (\log\tau_{x,p})(1) + (\log\sigma_{x,p})(1) + A(\tau,\sigma)(1),$$

where $A(\tau,\sigma) = \frac{1}{2}[\log\tau_{x,p}, \log\sigma_{x,p}] + \cdots$. It follows from the point i) that terms of degree i of $A(\tau,\sigma)(1)$ vanish. Hence $a^e_{x,p}(\tau\cdot\sigma) = a^e_{x,p}(\tau) + a^e_{x,p}(\sigma)$ for $e \in \mathcal{B}^i$ and $\tau, \sigma \in H_i$. The points iii) and iv) follow from Lemma 3.2.1.

We recall from Proposition 5.1.7 that

$$\log\psi_{x,p}(\sigma) = L_{(\log\psi_{x,p}(\sigma))(1)} + \log\varphi_x(\sigma).$$

The l-adic iterated integrals introduced in Definition 5.3.0 are coefficients of the element $(\log\psi_{x,p}(\sigma))(1)$. We must also study coefficients of the operator $\log\varphi_x(\sigma)$. We recall that $\varphi_x(\sigma)$ is an automorphism of $\mathbf{Q}_l\{\{\mathbf{X}\}\}$ induced by the action of σ on $\pi_1(X_{\bar{K}}; v)$ twisted by the cyclotomic character (see Section 4). Hence the operator $\varphi_x(\sigma)$ depends only on a choice of geometric generators $x = (x_1, \ldots, x_{n+1})$ and on a choice of a base point v.

DEFINITION 5.3.3. Let $\varepsilon \in (\mathrm{Der}^* L(\mathbf{X}))^\diamond$ be a linear form of degree m and let $\sigma \in G_K$. We set

$$\varepsilon(v)(\sigma) := \varepsilon(\log \varphi_x(\sigma)).$$

Observe that $\varepsilon(v)$ is a function from G_K to $\mathbf{Q}_l$. We shall use functions $\varepsilon(v)$ to express the action of the operator d on l-adic iterated integrals. Any function $\varepsilon(v)$ is in fact a linear combination of l-adic iterated integrals defined in Definition 5.3.0. However it is still very useful to have a separated notation for these functions.

PROPOSITION 5.3.4. *There are* $e_1, \ldots, e_r \in \mathcal{B}_m$ *and* $\alpha_{k,i} \in \mathbf{Q}_l$ *for* $0 < k < n+1$ *and* $0 < i < r+1$ *such that*

$$\varepsilon(v) = \sum_{k=1}^{n} \sum_{i=1}^{r} \alpha_{k,i} a_{x,\gamma_k}^{e_i}.$$

If ε *is defined over* $\mathbf{Q}$ *then* $\alpha_{k,i}$ *are in* $\mathbf{Q}$.

Proof. The proposition follows from Proposition 5.1.8.

We shall see later that the function $a_{x,p}^e : G_K \to \mathbf{Q}_l$ depends on a choice of a path p from v to z. Assume that e is of degree m. It follows from Theorem 5.3.1 iv) that the restriction of $a_{x,p}^e$ to the subgroup $H_m(X; z, v)$ depends only on z and v. It does not depend on a choice of a path p. This motivate the following definition.

DEFINITION 5.3.5. Let $e \in \mathcal{B}$ be an element of degree m and let $\varphi \in L(\mathbf{X})^\diamond$ be a linear form of degree m. We set

$$\mathcal{L}^e(z,v) := a^e_{x,p|H_m(X;z,v)} \quad \text{and} \quad \mathcal{L}^\varphi(z,v) := a^\varphi_{x,p|H_m(X;z,v)}.$$

Let $\varepsilon \in (\mathrm{Der}^* L(\mathbf{X}))^\diamond$ be a linear form of degree m. We set

$$\mathcal{L}^\varepsilon(v) := \varepsilon(v)_{|H_m(X;z,v)}.$$

It follows from Proposition 5.3.4 that

$$\mathcal{L}^\varepsilon(v) = \sum_{k=1}^{n} \sum_{i=1}^{r} \alpha_{k,i} \mathcal{L}^{e_i}(v_k, v).$$

§6. Cocycle conditions

6.0. It follows from Proposition 1.0.7 that the function $\mathfrak{f}_p : G_K \to \pi_1(X_{\bar{K}}; v)$ is a cocycle. Similarly Lemma 4.2.1 implies that the functions $\varphi_x : G_K \to \mathrm{Aut}(\mathbf{Q}_l\{\{\mathbf{X}\}\})$ and $\psi_{x,p} : G_K \to \mathrm{GL}(\mathbf{Q}_l\{\{\mathbf{X}\}\})$ are cocycles. The map $(\ \ )_{x,p} : G_{K(\mu_{l^\infty})} \to \mathrm{GL}(\mathbf{Q}_l\{\{\mathbf{X}\}\})$ is a homomorphism. However coefficients of these matrix valued functions usually are not cocycles or homomorphisms.

Let $\varphi \in L(\mathbf{X})^\diamond$ be a linear form of degree m. The function $a^\varphi_{x,p} : G_{K(\mu_{l^\infty})} \to \mathbf{Q}_l(m)$ (resp. $a^\varphi_{x,p} : G_K \to \mathbf{Q}_l(m)$) usually is not a homomorphism (resp. a cocycle). We are looking for conditions when a linear combination of various $a^\varphi_{x,p}$ with $\mathbf{Q}_l$ coefficients is a homomorphism (resp. a cocycle).

Let T be a finite subset of $\hat{X}(K)^2$ containing a pair (z, v). It follows from Section 5.0 that $a^\varphi_{x,p}$ and $\varepsilon(v)$ can be also considered as functions from the Lie algebra $\mathrm{Lie}(\mathcal{K}^T_1(X)/\mathcal{K}^T_\infty(X))$ to $\mathbf{Q}_l$.

Lemma 6.0.1. *Let $\varphi \in L(\mathbf{X})^\diamond$ be a linear form of degree m and let T be a finite subset of $\hat{X}(K)^2$ containing a pair (z, v). Assume that*

$$d(\varphi) = \sum_{k+j=m} \Bigg(\sum_{e\in\mathcal{B}_k,\, e'\in\mathcal{B}_j} c_{e,e'} e^* \wedge e'^* + \sum_{e\in\mathcal{B}_k,\, \varepsilon\in(\mathrm{Der}^*\, L(\mathbf{X}))^j} b_{e,\varepsilon} e^* \wedge \varepsilon \Bigg)$$

in $\bigwedge^2 (L(\mathbf{X}) \tilde{\times} \mathrm{Der}^*\, L(\mathbf{X}))^\diamond$. Then we have

$$d(a^\varphi_{x,p}) = \sum_{k+j=m} \Bigg(\sum_{e\in\mathcal{B}_k,\, e'\in\mathcal{B}_j} c_{e,e'} a^e_{x,p} \wedge a^{e'}_{x,p} + \sum_{e\in\mathcal{B}_k,\, \varepsilon\in(\mathrm{Der}^*\, L(\mathbf{X}))^j} b_{e,\varepsilon} a^e_{x,p} \wedge \varepsilon(v) \Bigg)$$

in $\bigwedge^2 \big(\mathrm{Lie}(\mathcal{K}^T_1(X)/\mathcal{K}^T_\infty(X))\big)^*$, where

$$\big(\mathrm{Lie}(\mathcal{K}^T_1(X)/\mathcal{K}^T_\infty(X))\big)^* := \mathrm{Hom}_{\mathbf{Z}_l}\big(\mathrm{Lie}(\mathcal{K}^T_1(X)/\mathcal{K}^T_\infty(X)); \mathbf{Q}_l\big).$$

Proof. The lemma is an obvious consequence of the fact that the map

$$\mathrm{Lie}(\ \ )_{x,p} : \mathrm{Lie}(\mathcal{K}^T_1(X)/\mathcal{K}^T_\infty(X)) \longrightarrow L_{L(\mathbf{X})} \tilde{\times} \mathrm{Der}^* L(\mathbf{X})$$

is a morphism of Lie algebras.

Proposition 6.0.2. *Let $(z_i, v_i) \in \hat{X}(K)^2$, let $\varphi_i \in L(\mathbf{X})^\diamond$ be a linear form of degree m, let p_i be a path from v_i to z_i and let x_i be a sequence of geometric generators of $\pi_1(X(\mathbf{C}); v_i)$ for $i = 1, \dots, N$. Let $n_1, \dots, n_N$ be in $\mathbf{Q}_l$. Let T be a finite subset of $\hat{X}(K)^2$ containing pairs (z_i, v_i) for $i = 1, \dots, N$.*

i) *Assume that* $d\big(\sum_{i=1}^N n_i a^{\varphi_i}_{x_i,p_i}\big) = 0$ *in* $\bigwedge^2\big(\mathrm{Lie}(\mathcal{K}^T_1(X)/\mathcal{K}^T_\infty(X))\big)^*$. *Then* $\sum_{i=1}^N n_i a^{\varphi_i}_{x_i,p_i}$ *is a homomorphism from* $\mathcal{K}^T_1(X)/\mathcal{K}^T_\infty(X)$ *to* $\mathbf{Q}_l(m)$.

ii) *Assume that for any* τ *and* σ *in* G_K

$$\sum_{i=1}^N n_i\varphi_i\big([\cdots[\cdots[\log\psi_{x_i,p_i}(\tau),\log\psi_{x_i,p_i}(\sigma)^{\chi(\tau)^{-1}}],\log\psi_{x_i,p_i}(\tau)]\cdots](1)\big)=0$$

for all Lie brackets of lengths $2,3,\ldots,m$. *Then* $\sum_{i=1}^N n_i a^{\varphi_i}_{x_i,p_i}$ *is a cocycle on* G_K *with values in* $\mathbf{Q}_l(m)$.

Proof. We start with the proof of the first part of the proposition. Let $\tau,\sigma\in\mathcal{K}^T_1(X)$. Let us set $T_i=\log\tau_{x_i,p_i}$ and $S_i=\log\sigma_{x_i,p_i}$. The equality $\tau_{x_i,p_i}\circ\sigma_{x_i,p_i}=(\tau\sigma)_{x_i,p_i}$ implies

$$\log(\tau\sigma)_{x_i,p_i}=T_i+S_i+\frac{1}{2}[T_i,S_i]-\frac{1}{12}[[T_i,S_i]T_i]+\cdots .$$

Evaluating φ_i on the last equality we get

$$a^{\varphi_i}_{x_i,p_i}(\tau\sigma)=a^{\varphi_i}_{x_i,p_i}(\tau)+a^{\varphi_i}_{x_i,p_i}(\sigma)+\frac{1}{2}\varphi_i([T_i,S_i])-\frac{1}{12}\varphi_i([[T_i,S_i]T_i])+\cdots .$$

Observe that $\varphi_i([T_i,S_i])=d\varphi_i(T_i\otimes S_i)=da^{\varphi_i}_{x_i,p_i}(\log\tau\otimes\log\sigma)$. Hence $\sum_{i=1}^N n_i\varphi_i([T_i,S_i])=0$. Observe that $\varphi([[T,S]R])=((d\otimes id)\circ d)(\varphi)(T\otimes S\otimes R)$. This implies $\sum_{i=1}^N n_i\varphi_i([[T_i,S_i]T_i])=0$. We apply the same arguments to others brackets and finally we get

$$\sum_{i=1}^N n_i a^{\varphi_i}_{x_i,p_i}(\tau\sigma)=\sum_{i=1}^N n_i a^{\varphi_i}_{x_i,p_i}(\tau)+\sum_{i=1}^N n_i a^{\varphi_i}_{x_i,p_i}(\sigma).$$

Now we assume that $\tau,\sigma\in G_K$. The equality

$$\psi_{x_i,p_i}(\tau\sigma)=\psi_{x_i,p_i}(\tau)\circ\psi_{x_i,p_i}(\sigma)^{\chi(\tau)^{-1}}$$

(see Lemma 4.2.1) implies that

$$\begin{aligned}\log\psi_{x_i,p_i}(\tau\sigma)&=\log\psi_{x_i,p_i}(\tau)\bigcirc\log\psi_{x_i,p_i}(\sigma)^{\chi(\tau)^{-1}}\\&=\log\psi_{x_i,p_i}(\tau)+\log\psi_{x_i,p_i}(\sigma)^{\chi(\tau)^{-1}}\\&\qquad+\frac{1}{2}[\log\psi_{x_i,p_i}(\tau),\log\psi_{x_i,p_i}(\sigma)^{\chi(\tau)^{-1}}]+\cdots .\end{aligned}$$

Now the second part of the proposition follows immediately from the assumptions ii).

6.1. We shall define filtrations of the Lie algebras $\mathrm{Der}^* L(\mathbf{X})$ and $L(\mathbf{X}) \tilde{\times} \mathrm{Der}^* L(\mathbf{X})$ associated with the lower central series of $L(\mathbf{X})$.
Let us set

$$\mathrm{Der}^*_k L(\mathbf{X}) := \{D \in \mathrm{Der}^* L(\mathbf{X}) \mid \forall X_i \in \mathbf{X}\ \exists A_i \in \Gamma^k L(\mathbf{X}), D(X_i) = [X_i, A_i]\}$$

and

$$\gamma_k(L(\mathbf{X}) \tilde{\times} \mathrm{Der}^* L(\mathbf{X})) := \Gamma^k L(\mathbf{X}) \tilde{\times} \mathrm{Der}^*_k L(\mathbf{X}).$$

LEMMA 6.1.0. *$\mathrm{Der}^*_k L(\mathbf{X})$ (resp. $\gamma_k(L(\mathbf{X}) \tilde{\times} \mathrm{Der}^* L(\mathbf{X}))$) is a Lie ideal of $\mathrm{Der}^* L(\mathbf{X})$ (resp. $L(\mathbf{X}) \tilde{\times} \mathrm{Der}^* L(\mathbf{X})$). We have isomorphisms of Lie algebras*

$$\bigoplus_{k=1}^{\infty} \mathrm{Der}^*_k L(\mathbf{X}) / \mathrm{Der}^*_{k+1} L(\mathbf{X}) = \mathrm{Der}^* \mathrm{Lie}(\mathbf{X})$$

and

$$\bigoplus_{k=1}^{\infty} \gamma_k(L(\mathbf{X}) \tilde{\times} \mathrm{Der}^* L(\mathbf{X})) / \gamma_{k+1}(L(\mathbf{X}) \tilde{\times} \mathrm{Der}^* L(\mathbf{X})) = \mathrm{Lie}(\mathbf{X}) \tilde{\times} \mathrm{Der}^* \mathrm{Lie}(\mathbf{X}).$$

Proof. The lemma follows from the fact that the graded associated Lie algebra $\bigoplus_{k=1}^{\infty} \Gamma^k L(\mathbf{X}) / \Gamma^{k+1} L(\mathbf{X})$ is canonically isomorphic to $\mathrm{Lie}(\mathbf{X})$.

Let T be a finite subset of $\hat{X}(K)^2$. We set

$$\mathfrak{k}^T(X) := gr\, \mathrm{Lie}(\mathcal{K}^T_1(X)/\mathcal{K}^T_\infty(X)) := \bigoplus_{i=1}^{\infty} \mathrm{Lie}(\mathcal{K}^T_i(X)/\mathcal{K}^T_{i+1}(X)) \otimes \mathbf{Q}.$$

The homomorphism of Lie algebras

$$\mathrm{Lie}(\ \ )_{x,p} : \mathrm{Lie}(\mathcal{K}^T_1(X)/\mathcal{K}^T_\infty(X)) \longrightarrow L(\mathbf{X}) \tilde{\times} \mathrm{Der}^* L(\mathbf{X})$$

is compatible with filtrations $\{\mathrm{Lie}(\mathcal{K}^T_i(X)/\mathcal{K}^T_\infty(X))\}_{i=1}^{\infty}$ of $\mathrm{Lie}(\mathcal{K}^T_1(X)/\mathcal{K}^T_\infty(X))$ and $\{\gamma_i(L(\mathbf{X}) \tilde{\times} \mathrm{Der}^* L(\mathbf{X}))\}_{i=1}^{\infty}$ of $L(\mathbf{X}) \tilde{\times} \mathrm{Der}^* L(\mathbf{X})$. Therefore it induces a homomorphism of associated graded Lie algebras

$$\pi^T_{z,v} : \mathfrak{k}^T(X) \longrightarrow \mathrm{Lie}(\mathbf{X}) \tilde{\times} \mathrm{Der}^* \mathrm{Lie}(\mathbf{X}).$$

Let us set

$$\mathfrak{k}^T(X)^\diamond := \bigoplus_{i=1}^{\infty} \big(\mathrm{Lie}(\mathcal{K}^T_i(X)/\mathcal{K}^T_{i+1}(X)) \otimes \mathbf{Q}\big)^*.$$

Then $\mathfrak{k}^T(X)^\diamond$ is a Lie coalgebra and we have a homomorphism of Lie coalgebras

$$(\pi^T_{z,v})^\diamond : (\mathrm{Lie}(\mathbf{X}) \tilde{\times} \mathrm{Der}^* \mathrm{Lie}(\mathbf{X}))^\diamond \longrightarrow \mathfrak{k}^T(X)^\diamond.$$

Moreover an inclusion $S \subset T$ of finite subsets of $\hat{X}(K)^2$ induces a morphism of Lie coalgebras

$$\mathfrak{k}^S(X)^\diamond \longrightarrow \mathfrak{k}^T(X)^\diamond.$$

DEFINITION 6.1.1. Let $\mathcal{C}$ be the category whose objects are all finite subsets of $\hat{X}(K)^2$ and whose morphisms are inclusions. We set

$$\mathfrak{k}(X)^\diamond := \varinjlim_{\mathcal{C}} \mathfrak{k}^T(X)^\diamond.$$

The $\mathbf{Q}_l$-vector space $\mathfrak{k}(X)^\diamond$ is a Lie coalgebra and morphisms $(\pi^T_{z,v})^\diamond$: $(\mathrm{Lie}(\mathbf{X}) \tilde{\times} \mathrm{Der}^* \mathrm{Lie}(\mathbf{X}))^\diamond \to \mathfrak{k}^T(X)^\diamond$ induce a morphism of Lie coalgebras

$$\pi^\diamond_{z,v} : (\mathrm{Lie}(\mathbf{X}) \tilde{\times} \mathrm{Der}^* \mathrm{Lie}(\mathbf{X}))^\diamond \longrightarrow \mathfrak{k}(X)^\diamond.$$

Observe that

$$\mathcal{L}^e(z,v) = e^* \circ \pi_{z,v} = \pi^\diamond_{z,v}(e^*)$$

and

$$\mathcal{L}^\varepsilon(v) = \varepsilon \circ \pi_{v,v} = \varepsilon \circ \pi_{z,v} = \pi^\diamond_{v,v}(\varepsilon) = \pi^\diamond_{z,v}(\varepsilon)$$

for any $e \in \mathcal{B}$ and for any $\varepsilon \in (\mathrm{Der}^* \mathrm{Lie}(\mathbf{X}))^\diamond$ of degree n.

Hence we get

$$d\mathcal{L}^e(z,v) = d(\pi^\diamond_{z,v}(e^*)) = (\pi^\diamond_{z,v} \wedge \pi^\diamond_{z,v})(d(e^*)).$$

Warning: $\pi^\diamond_{z,v}$ is not injective, hence we can have $d(e^*) \neq 0$ but $d(\mathcal{L}^e(z,v)) = 0$.

PROPOSITION 6.1.2. *Let $\varphi \in L(\mathbf{X})^\diamond$ be a linear form of degree m. If*

$$d(\varphi) = \sum_{k+j=m} \Big(\sum_{e \in \mathcal{B}_k,\, e' \in \mathcal{B}_j} c_{e,e'} e^* \wedge e'^* + \sum_{e \in \mathcal{B}_k,\, \varepsilon \in (\mathrm{Der}^* L(\mathbf{X}))^j} b_{e,\varepsilon} e^* \wedge \varepsilon \Big)$$

in $\bigwedge^2 (L(\mathbf{X}) \tilde{\times} \mathrm{Der}^* L(\mathbf{X}))^\diamond$ then

$$d(\mathcal{L}^\varphi(z,v)) = \sum_{k+j=m} \Big(\sum_{e \in \mathcal{B}_k,\, e' \in \mathcal{B}_j} c_{e,e'} \mathcal{L}^e(z,v) \wedge \mathcal{L}^{e'}(z,v) + \sum_{e \in \mathcal{B}_k,\, \varepsilon \in (\mathrm{Der}^* L(\mathbf{X}))^j} b_{e,\varepsilon} \mathcal{L}^e(z,v) \wedge \mathcal{L}^\varepsilon(v) \Big)$$

in $\mathfrak{k}(X)^\diamond$.

§7. Analog of Zagier conjecture

7.0. We shall present here a conjecture which is an l-adic analog of conjectures concerning iterated integrals from [W3]. These conjectures are generalizations of the Zagier conjecture for classical complex polylogarithms. The main ideas come from the Deligne-Beilinson paper.

We assume that there exists a category of mixed Tate motives over $\operatorname{Spec} K$ such as in [BD]. (We do not know if recent constructions of Voevodsky and others are sufficient for our purpose.) We shall denote this category by $\mathcal{MM}_K$. The category $\mathcal{MM}_K$ is a tannakian category and it is equivalent to a category of representations of a pro-algebraic group Π_K defined over $\mathbf{Q}$. Let $U_K := \ker(\Pi_K \to \mathbf{G_m})$. The group U_K is a pro-algebraic pro-unipotent group defined over $\mathbf{Q}$. We denote by $\operatorname{Lie} U_K$ its Lie algebra. This Lie algebra is equipped with the weight filtration. Let

$$\mathcal{L}ie\, U_K = \bigoplus_{n=1}^{\infty} (\mathcal{L}ie\, U_K)_n$$

be the associated graded Lie algebra. We set

$$(\mathcal{L}ie\, U_K)^{\diamond} := \bigoplus_{n=1}^{\infty} (\mathcal{L}ie\, U_K)_n^{\diamond}.$$

$(\mathcal{L}ie\, U_K)^{\diamond}$ equipped with d - the dual of the Lie bracket - is a Lie coalgebra.

Let X be a projective line over K minus a finite number of K-points. We shall construct a Lie subcoalgebra of the Lie coalgebra $(\mathcal{L}ie\, U_K)^{\diamond}$ corresponding to the pro-unipotent part of the fundamental group of the tannakian category generated by mixed motives of torsors of paths from v to z on X for all pairs $(z, v) \in \hat{X}(K)^2$.

We shall construct this Lie subcoalgebra of $(\mathcal{L}ie\, U_K)^{\diamond}$ in the inductive way. This Lie subcoalgebra will be a graded Lie coalgebra. The construction in degree 1 will be clear. We shall assume that we have constructed our Lie coalgebra up to degree N, i.e., we have $\bigoplus_{i=1}^{N-1} \mathcal{L}_i$ and $d : \bigoplus_{i=1}^{N-1} \mathcal{L}_i \to \bigwedge^2 \big(\bigoplus_{i=1}^{N-2} \mathcal{L}_i\big)$.

We construct a candidate $\mathcal{L}'_N$ in degree N and $d'_N : \mathcal{L}'_N \to \bigwedge^2 \big(\bigoplus_{i=1}^{N-1} \mathcal{L}_i\big)$. Our construction should be motivic hence we should have the following commutative diagram

$$\begin{array}{ccc}
\ker d'_N & \xrightarrow{\Phi'_N} & \ker d_N \\
\downarrow & & \downarrow \\
\mathcal{L}'_N & \xrightarrow{\bar{\Phi}_N} & (\mathcal{L}ie\, U_K)^{\diamond}_N \\
\downarrow{\scriptstyle d'_N} & & \downarrow{\scriptstyle d_N} \\
\bigwedge^2(\bigoplus_{i=1}^{N-1} \mathcal{L}_i) & \xrightarrow{\bigwedge^2(\bigoplus_{i=1}^{N-1} \Phi_i)} & \bigwedge^2(\bigoplus_{i=1}^{N-1}(\mathcal{L}ie\, U_K)^{\diamond}_i)
\end{array}$$

Then it is clear that $\mathcal{L}_N = \mathcal{L}'_N/\ker\bar{\Phi}_N$. Observe that $\mathcal{L}'_N/\ker\bar{\Phi}_N = \mathcal{L}'_N/\ker\Phi'_N$. In fact we shall conjecture that we have a map

$$\Phi'_N : \ker d'_N \longrightarrow \ker d_N = \mathrm{Ext}^1_{\mathcal{MM}_K}(\mathbf{Q}(0), \mathbf{Q}(N)) \otimes \mathbf{Q}$$

and we shall set $\mathcal{L}_N = \mathcal{L}'_N/\ker\Phi'_N$. This is a short motivic justification of our next steps.

We recall that in the category $\mathcal{MM}_K$

$$\mathrm{Ext}^1_{\mathcal{MM}_K}(\mathbf{Q}(0), \mathbf{Q}(1)) \otimes \mathbf{Q} = K^* \otimes \mathbf{Q}.$$

7.1. Let $X = \mathbf{P}^1_K \setminus \{a_1, \ldots, a_{n+1}\}$. We assume for simplicity that $a_{n+1} = \infty$. Let us choose a tangential base point v_i (a tangent vector) at a_i for $i = 1, 2, \ldots, n+1$. Let $\mathcal{B}$ be a Hall base of $\mathrm{Lie}(\mathbf{X})$ and let $\mathcal{B}_m$ be elements of degree m in $\mathcal{B}$.

For $k = 1$ we set $\mathcal{L}_1 := K^* \otimes \mathbf{Q}$, $d_1 = 0 : \mathcal{L}_1 \to 0$. We define symbols $\{z, v\}_{X_i} \in \mathcal{L}_1$ in the following way. If $(z, v) \in X(K)^2$ then $\{z, v\}_{X_i} := \frac{z-a_i}{v-a_i} \otimes 1 \in \mathcal{L}_1$, if $z \in X(K)$ and $v = \overrightarrow{a_i x}$ then $\{z, v\}_{X_i} := \frac{z-a_i}{x-a_i} \otimes 1 \in \mathcal{L}_1$ and $\{z, v\}_{X_j} := \frac{z-a_j}{a_i-a_j} \otimes 1 \in \mathcal{L}_1$, if $z = \overrightarrow{a_k x'}$ and $v = \overrightarrow{a_l x}$ then $\{z, v\}_{X_i} := \frac{a_k-a_i}{a_l-a_i} \otimes 1 \in \mathcal{L}_1$ for $i \neq k, l$, $\{z, v\}_{X_k} := \frac{x'-a_k}{a_l-a_k} \otimes 1 \in \mathcal{L}_1$ and $\{z, v\}_{X_l} := \frac{a_k-a_l}{x-a_l} \otimes 1 \in \mathcal{L}_1$.

We define a map

$$\varphi_1 : \mathcal{L}_1 \longrightarrow \mathrm{Ext}^1_{\mathcal{MM}_K}(\mathbf{Q}(0), \mathbf{Q}(1)) \otimes \mathbf{Q} = K^* \otimes \mathbf{Q}$$

by $\varphi_1(z \otimes 1) := z \otimes 1$. We define

$$\psi_1 : \mathcal{L}_1 \longrightarrow H^1(\mathcal{K}_1(X), \mathbf{Q}_l(1))$$

by $\psi_1(\{z, v\}_{X_i}) := \mathcal{L}^{X_i}(z, v)$. (We recall that $\mathcal{K}_1(X) = \mathrm{Gal}(\bar{K}/K(\mu_{l^\infty}))$.)

Proposition 7.1.0. *The diagram*

$$\begin{tikzcd}
\mathcal{L}_1 \arrow[r, "\varphi_1"] \arrow[d, "\psi_1"] & \mathrm{Ext}^1_{\mathcal{MM}_K}(\mathbf{Q}(0),\mathbf{Q}(1))\otimes\mathbf{Q} \arrow[d, "realization"] \\
H^1(\mathcal{K}_1(X),\mathbf{Q}_l(1)) & H^1(G_K,\mathbf{Q}_l(1)), \arrow[l, hook']
\end{tikzcd}$$

commutes, where realization associates to $z\otimes 1\in K^*\otimes\mathbf{Q}=\mathrm{Ext}^1_{\mathcal{MM}_K}(\mathbf{Q}(0),\mathbf{Q}(1))\otimes\mathbf{Q}$ *the Kummer character corresponding to* z.

Proof. Let $(z,v)\in\hat{X}(K)^2$ and let p be a path from v to z. First we consider the case when z and v are K-points of X. Let us take $\sigma\in G_{K(\mu_{l^\infty})}$. We shall calculate the coefficient of $(\log\sigma_{x,p})(1)$ at X_i. This coefficient is equal to the exponent of $\mathfrak{f}_p(\sigma)$ at x_i. Let ζ be a coordinate on $\mathbf{P}^1_K$. The loop $\mathfrak{f}_p(\sigma)=p^{-1}\cdot\sigma\cdot p\cdot\sigma^{-1}$ transforms $(\zeta-a_i)^{1/l^n}$ into

$$\frac{\sigma^{-1}((v-a_i)^{1/l^n})}{(v-a_i)^{1/l^n}}\cdot\frac{\sigma((z-a_i)^{1/l^n})}{(z-a_i)^{1/l^n}}\cdot(\zeta-a_i)^{1/l^n}.$$

This finishes the proof of the proposition when $(z,v)\in X(K)^2$. Now we assume that $v=\overrightarrow{a_ix}$ is a tangential base point and z is a K-point. The isomorphism of $\mathbf{P}^1_K$ given by $y\to\frac{y-a_i}{x-a_i}$ is defined over K. Hence we can assume that $a_i=0$, $v=\overrightarrow{01}$ and p is a path from $\overrightarrow{01}$ to $z_1=\frac{z-a_i}{x-a_i}$. Let ζ be a local parameter corresponding to the tangential base point $\overrightarrow{01}$. The loop $\mathfrak{f}_p(\sigma)$ transforms ζ^{1/l^n} into $\sigma\big((\frac{z-a_i}{x-a_i})^{1/l^n}\big)\cdot\big((\frac{z-a_i}{x-a_i})^{1/l^n}\big)^{-1}\cdot\zeta^{1/l^n}$. Hence the exponent of $\mathfrak{f}_p(\sigma)$ at x_i is equal to the Kummer character of $\frac{z-a_i}{x-a_i}$ evaluated at σ. The other cases we left to the readers.

Let $N>1$. We assume that the groups $\mathcal{L}_k$, the symbols $\{z,v\}_e\in\mathcal{L}_k$ for $e\in\mathcal{B}_k$, the homomorphisms $d_k:\mathcal{L}_k\to\bigoplus_{i+j=k}\mathcal{L}_i\wedge\mathcal{L}_j$, $\varphi_k:\ker d_k\to\mathrm{Ext}^1_{\mathcal{MM}_K}(\mathbf{Q}(0),\mathbf{Q}(k))\otimes\mathbf{Q}$ and $\psi_k:\mathcal{L}_k\to H^1_{\mathcal{C}}(\mathcal{K}_k(X),\mathbf{Q}_l(k))$ are defined for $k<N$. We assume that for $k<N$ the diagram

$$\begin{tikzcd}
\ker d_k \arrow[r, "\varphi_k"] \arrow[d, "\psi_k"] & \mathrm{Ext}^1_{\mathcal{MM}_K}(\mathbf{Q}(0),\mathbf{Q}(k))\otimes\mathbf{Q} \arrow[d, "realization"] \\
H^1_{\mathcal{C}}(\mathcal{K}_k(X),\mathbf{Q}_l(k)) & H^1(G_K,\mathbf{Q}_l(k)) \arrow[l, hook']
\end{tikzcd}$$

commutes. We recall that the lower horizontal morphism is injective by Lemma 3.0.10.

Let $\varphi \in \mathrm{Lie}(\mathbf{X})^\diamond$ be a linear form of degree k defined over $\mathbf{Q}$. If $\varphi = \sum_i a_i (e_i^k)^*$ then we set $\{z,v\}_\varphi := \sum_i a_i \{z,v\}_{e_i^k}$.

Let $\varepsilon \in (\mathrm{Der}^* \mathrm{Lie}(\mathbf{X}))^\diamond = \bigoplus_{i=1}^n (\mathrm{Lie}(\mathbf{X})/\langle X_i \rangle)^\diamond$ be a linear form of degree k defined over $\mathbf{Q}$. Assume that $\varepsilon = (\varphi_1, \dots, \varphi_n)$, where $\varphi_i \in (L(\mathbf{X})/\langle X_i \rangle)^\diamond$. Then we set

$$\{v\}_\varepsilon := \sum_{i=1}^n \{v_i, v\}_{\varphi_i}.$$

Observe that $\mathcal{L}^\varepsilon(v) = \sum_{i=1}^n \mathcal{L}^{\varphi_i}(v_i, v)$.

Let $\mathcal{B}_N = \{e_i^N\}_{i \in I}$. For each e_i^N we set

$$\mathcal{L}^{e_i^N} := \bigoplus_{\{z,v\} \in \hat{X}(K)^2} \mathbf{Q}\{z,v\}'_{e_i^N} \text{ -- a vector space over } \mathbf{Q} \text{ on symbols } \{z,v\}'_{e_i^N}$$

and

$$\mathcal{L}'_N := \bigoplus_{i \in I} \mathcal{L}^{e_i^N}.$$

Let $e \in \mathcal{B}_N$. We define

$$d'_N : \mathcal{L}'_N \longrightarrow \bigoplus_{i+j=N} \mathcal{L}_i \wedge \mathcal{L}_j$$

setting

$$d'_N(\{z,v\}'_e) = \sum_{k+j=N} \Bigg(\sum_{e_1 \in \mathcal{B}_k,\, e_2 \in \mathcal{B}_j} c_{e_1,e_2} \{z,v\}_{e_1} \wedge \{z,v\}_{e_2} + \sum_{e' \in \mathcal{B}_k,\, \varepsilon \in (\mathrm{Der}^* L(\mathbf{X}))^j} b_{e',\varepsilon} \{z,v\}_{e'} \wedge \{v\}_\varepsilon \Bigg)$$

if

$$d(e^*) = \sum_{k+j=N} \Bigg(\sum_{e_1 \in \mathcal{B}_k,\, e_2 \in \mathcal{B}_j} c_{e_1,e_2} e_1^* \wedge e_2^* + \sum_{e' \in \mathcal{B}_k,\, \varepsilon \in (\mathrm{Der}^* L(\mathbf{X}))^j} b_{e',\varepsilon} e^* \wedge \varepsilon \Bigg)$$

in $\bigwedge^2 (\mathrm{Lie}(\mathbf{X}) \tilde{\times} \mathrm{Der}^* \mathrm{Lie}(\mathbf{X}))^\diamond$.

Conjecture D_N. There is a homomorphism

$$\varphi'_N : \ker d'_N \longrightarrow \mathrm{Ext}^1_{\mathcal{MM}_K}(\mathbf{Q}(0), \mathbf{Q}(N)) \otimes \mathbf{Q}$$

such that the diagram

$$\begin{array}{ccc} \ker d'_N & \xrightarrow{\varphi'_N} & \mathrm{Ext}^1_{\mathcal{MM}_K}(\mathbf{Q}(0), \mathbf{Q}(N)) \otimes \mathbf{Q} \\ \downarrow{\scriptstyle \psi'_N} & & \downarrow{\scriptstyle realization} \\ H^1_{\mathcal{C}}(\mathcal{K}_N(X), \mathbf{Q}_l(N)) & \hookleftarrow & H^1(G_K, \mathbf{Q}_l(N)), \end{array}$$

commutes, where the map ψ'_N is given by $\psi'_N(\{z,v\}'_{e_i^N}) := \mathcal{L}^{e_i^N}(z,v)$.

If the conjecture is true then we set $\mathcal{L}_N := \mathcal{L}'_N / \ker \varphi'_N$. The maps d_N, ψ_N and φ_N are defined by passing to quotient. The symbol $\{z,v\}_{e_i^N}$ is the image of $\{z,v\}'_{e_i^N}$ in $\mathcal{L}_N$.

Definition 7.1.1. We set

$$\mathcal{L}^K(X) := \bigoplus_{N=1}^{\infty} \mathcal{L}_N.$$

We define $d : \mathcal{L}^K(X) \to \mathcal{L}^K(X) \wedge \mathcal{L}^K(X)$ by setting $d|_{\mathcal{L}_N} := d_N$.

Lemma 7.1.2. *Let $\varepsilon \in (\mathrm{Der}^* \mathrm{Lie}(\mathbf{X}))^\diamond$ be a linear form of degree N defined over $\mathbf{Q}$. Assume that*

$$d\varepsilon = \sum_{p+q=N} \sum_{\varepsilon_1 \in (\mathrm{Der}^* \mathrm{Lie}(\mathbf{X}))^p,\, \varepsilon_2 \in (\mathrm{Der}^* \mathrm{Lie}(\mathbf{X}))^q} a_{\varepsilon_1,\varepsilon_2} \varepsilon_1 \wedge \varepsilon_2$$

in $\bigwedge^2 (\mathrm{Der}^* \mathrm{Lie}(\mathbf{X}))^\diamond$. Then

$$d(\{v\}_\varepsilon) = \sum_{p+q=N} \sum_{\varepsilon_1 \in (\mathrm{Der}^* \mathrm{Lie}(\mathbf{X}))^p,\, \varepsilon_2 \in (\mathrm{Der}^* \mathrm{Lie}(\mathbf{X}))^q} a_{\varepsilon_1,\varepsilon_2} \{v\}_{\varepsilon_1} \wedge \{v\}_{\varepsilon_2}.$$

Proof. We recall that

$$\mathrm{Der}^* \mathrm{Lie}(\mathbf{X}) = \{D \in \mathrm{Der}\,\mathrm{Lie}(\mathbf{X}) \mid \forall X_i \in \mathbf{X}\ \exists A_i \in \mathrm{Lie}(\mathbf{X}), D(X_i) = [X_i, A_i]\}.$$

The derivation $D \in \mathrm{Der}^*(\mathrm{Lie}(\mathbf{X}))$ such that $D(X_i) = [X_i, A_i]$ we shall denote by $D_{(A_1,\dots,A_n)} = D_{(A_i)_{i=1,\dots,n}}$. We have an identification

$$\mathrm{Der}^* \mathrm{Lie}(\mathbf{X}) = \bigoplus_{i=1}^{n} \mathrm{Lie}(\mathbf{X})/\langle X_i \rangle$$

sending $D_{(A_1,\dots,A_n)}$ to a sequence $(A_1, \dots, A_n)$. One easily checks that (7.1.3)

$$[D_{(V_k)_{k=1,\dots,n}}, D_{(W_k)_{k=1,\dots,n}}] = D_{([V_k,W_k]+D_{(V_j)_j}(W_k)-D_{(W_j)_j}(V_k))_{k=1,\dots,n}}.$$

If $e \in \mathcal{B}$ then we set $(e)^i = (a_1, \dots, a_n) \in \bigoplus_{k=1}^{n} \mathrm{Lie}(\mathbf{X})/\langle X_k \rangle$, where $a_i = e$ and $a_j = 0$ for $j \neq i$.

Let $\varepsilon \in (\mathrm{Der}^* \mathrm{Lie}(\mathbf{X}))^\diamond$. Then $\varepsilon = \sum_{i=1}^{n} \big(\sum_{e\in\mathcal{B}} n_{e,i}(e)^{i*}\big)$, where $(e)^{i*}$ is a composition of e^* with the projection $\bigoplus_{k=1}^{n} \mathrm{Lie}(\mathbf{X})/\langle X_k \rangle \to \mathrm{Lie}(\mathbf{X})/\langle X_i \rangle$. We shall compare $d(e^*)$ with $d((e)^{i*})$ in $(\mathrm{Lie}(\mathbf{X}) \tilde{\times} \mathrm{Der}^* \mathrm{Lie}(\mathbf{X}))^\diamond$. Observe that $e^* \in \mathrm{Lie}(\mathbf{X})^\diamond$ and $(e)^{i*} \in (\mathrm{Der}^* \mathrm{Lie}(\mathbf{X}))^\diamond$. It follows from the definition of the Lie bracket in the semi-direct product $\mathrm{Lie}(\mathbf{X}) \tilde{\times} \mathrm{Der}^*(\mathrm{Lie}(\mathbf{X}))$ that

(7.1.4)
$$d(e^*) = \sum_{e_1,e_2\in\mathcal{B}} e^*([e_1,e_2])e_1^* \wedge e_2^* + \sum_{k=1}^{n} \sum_{e_3,e_4\in\mathcal{B}} e^*(D_{(e_4)^k}(e_3))e_3^* \wedge (e_4)^{k*}.$$

Hence we get

$$(7.1.5)\quad d(\{v_i,v\}_e) = \sum_{e_1,e_2\in\mathcal{B}} e^*([e_1,e_2])\{v_i,v\}_{e_1} \wedge \{v_i,v\}_{e_2} + \sum_{k=1}^{n} \sum_{e_3,e_4\in\mathcal{B}} e^*(D_{(e_4)^k}(e_3))\{v_i,v\}_{e_3} \wedge \{v\}_{(e_4)^{k*}}.$$

On the other side it follows from (7.1.3) that

$$(7.1.6)\qquad d((e)^{i*}) = \sum_{e_1,e_2\in\mathcal{B}} e^*([e_1,e_2])(e_1)^{i*} \wedge (e_2)^{i*} + \sum_{k=1}^{n} \sum_{e_3,e_4\in\mathcal{B}} e^*(D_{(e_4)^k}(e_3))(e_3)^{i*} \wedge (e_4)^{k*}.$$

We recall that we have defined $\{v\}_{(e)^{i*}} := \{v_i, v\}_e$. Hence if in the right hand side of the equality (7.1.6) we replace $(e_\alpha)^{j*}$ by $\{v\}_{(e_\alpha)^{j*}}$ then we get the right hand side of the equality (7.1.5). Therefore the lemma is proved for $\varepsilon = (e)^{i*}$. Any $\varepsilon \in (\mathrm{Der}^* \mathrm{Lie}(\mathbf{X}))^\diamond$ is a linear combination of $(e)^{i*}$, hence the lemma is proved for any $\varepsilon \in (\mathrm{Der}^* \mathrm{Lie}(\mathbf{X}))^\diamond$.

PROPOSITION 7.1.7. *The* $\mathbf{Q}$*-vector space* $\mathcal{L}^K(X)$ *equipped with the homomorphism* $d : \mathcal{L}^K(X) \to \mathcal{L}^K(X) \wedge \mathcal{L}^K(X)$ *is a Lie coalgebra.*

Proof. It is enough to show that

$$\sum_{i=0}^{2} \sigma^i \circ (d \otimes id_{\mathcal{L}^K(X)}) \circ d = 0, \tag{7.1.8}$$

where $\sigma(a \otimes b \otimes c) = b \otimes c \otimes a$. In the Lie coalgebra $(\mathrm{Lie}(\mathbf{X}) \tilde{\times} \mathrm{Der}^* \mathrm{Lie}(\mathbf{X}))^\diamond$ we obviously have

$$\sum_{i=0}^{2} \sigma^i \circ (d \otimes id_{(\mathrm{Lie}(\mathbf{X}) \tilde{\times} \mathrm{Der}^* \mathrm{Lie}(\mathbf{X}))^\diamond}) \circ d = 0. \tag{7.1.9}$$

The calculation of $d(\{z, v\}_e)$ (corresponding to $d(e^*)$ in $(\mathrm{Lie}(\mathbf{X}) \tilde{\times} \mathrm{Der}^* \mathrm{Lie}(\mathbf{X}))^\diamond$) involves only symbols $\{z, v\}_{e_1}$ (corresponding to e_1^* in $(\mathrm{Lie}(\mathbf{X}) \tilde{\times} \mathrm{Der}^* \mathrm{Lie}(\mathbf{X}))^\diamond$) and $\{v_i, v\}_{e_2} = \{v\}_{(e_2)^{i*}}$ (corresponding to $(e_2)^{i*}$ in $(\mathrm{Lie}(\mathbf{X}) \tilde{\times} \mathrm{Der}^* \mathrm{Lie}(\mathbf{X}))^\diamond$). Hence the proposition follows from Lemma 7.1.2.

PROPOSITION 7.1.10. *Assume that Conjectures* D_N *are true for all* N *and that for all* N *the maps realization* : $\mathrm{Ext}^1_{\mathcal{MM}_K}(\mathbf{Q}(0), \mathbf{Q}(N)) \otimes \mathbf{Q} \to H^1(G_K, \mathbf{Q}_l(N))$ *are injective. Let* $(z_i, v_i) \in \hat{X}(K)^2$ *and let* $e_i^N \in \mathcal{B}_N$ *for* $i = 1, \ldots, m$. *Let* $n_i \in \mathbf{Q}_l$ *for* $i = 1, \ldots, m$. *Then* $\sum_{i=1}^m n_i \mathcal{L}^{e_i^N}(z_i, v_i) = 0$ *if and only if* $\sum_{i=1}^m n_i \{z_i, v_i\}_{e_i^N} = 0$ *in* $\mathcal{L}_N \otimes \mathbf{Q}_l$.

Proof. It is well known that the restriction map $H^1(G_K, \mathbf{Q}_l(1)) \to H^1(G_{K(\mu_{l^\infty})}, \mathbf{Q}_l(1))$ is injective. Hence it follows from Proposition 7.1.0 that the proposition is true for $k = 1$. Let us assume that it is true for $k < N$. Let $\sum_{i=1}^m n_i \mathcal{L}^{e_i^N}(z_i, v_i) = 0$. This implies that $d(\sum_{i=1}^m n_i \mathcal{L}^{e_i^N}(z_i, v_i)) = \sum_{k+l=N} \sum_{\alpha,\beta} c_{\alpha,\beta}^{k,l} \mathcal{L}^{e_\alpha^k}(z_\alpha, v_\alpha) \cdot \mathcal{L}^{e_\beta^l}(z_\beta, v_\beta) = 0$ in $\mathfrak{t}(X)^\diamond \wedge \mathfrak{t}(X)^\diamond$. Hence for any $\sigma \in \mathcal{K}_k^T(X)/\mathcal{K}_{k+1}^T(X)$ we have $\sum_{\alpha,\beta} c_{\alpha,\beta}^{k,l} \mathcal{L}^{e_\alpha^k}(z_\alpha, v_\alpha)(\sigma) \cdot \mathcal{L}^{e_\beta^l}(z_\beta, v_\beta) = 0$ for T sufficiently big. Hence by the induction hypothesis we have $\sum_{\alpha,\beta} c_{\alpha,\beta}^{k,l} \mathcal{L}^{e_\alpha^k}(z_\alpha, v_\alpha)(\sigma) \cdot \{z_\beta, v_\beta\}_{e_\beta^l} = 0$. Let $f : \mathcal{L} \to \mathbf{Q}_l$ be a homomorphism. We get that for all $\sigma \in \mathcal{K}_k^T(X)/\mathcal{K}_{k+1}^T(X)$, $\sum_{\alpha,\beta} c_{\alpha,\beta}^{k,l} \mathcal{L}^{e_\alpha^k}(z_\alpha, v_\alpha)(\sigma) \cdot f(\{z_\beta, v_\beta\}_{e_\beta^l}) =$

0. The induction hypothesis implies that for any homomorphism $f:\mathcal{L}\to\mathbf{Q}_l$ we have $\sum_{\alpha,\beta} c^{k,l}_{\alpha,\beta}\{z_\alpha,v_\alpha\}_{e^k_\alpha}\cdot f(\{z_\beta,v_\beta\}_{e^l_\beta}) = 0$. This implies that $d(\sum_{i=1}^m n_i\{z_i,v_i\}_{e_i^N}) = 0$. The assumption that the realization and the restriction are injective implies that $\sum_{i=1}^m n_i\{z_i,v_i\}_{e_i^N} = 0$ in $\mathcal{L}^K(X)\otimes\mathbf{Q}_l$.

Corollary 7.1.11. *Assume that Conjectures D_N are true for all N. Assume that for all N the maps realization* : $\mathrm{Ext}^1_{\mathcal{MM}_K}(\mathbf{Q}(0),\mathbf{Q}(N))\otimes\mathbf{Q}\to H^1(G_K,\mathbf{Q}_l(N))$ *are injective. Let $q_i\in\mathbf{Q}$ for $i=1,\dots,m$.*

i) *We have a relation $\sum_{i=1}^m q_i\mathcal{L}^{e_i}(z_i,v_i)=0$ if and only if $\sum_{i=1}^m q_i\{z_i,v_i\}_{e_i} = 0$ in $\mathcal{L}^K(X)$.*

ii) *The vector space of linear relations between functions $\mathcal{L}^e(z,v)$ is defined over $\mathbf{Q}$.*

Proof. The first part follows immediately from Proposition 7.1.10. Observe that a vector space of linear relations between elements $\{z,v\}_e$ is generated by relations with $\mathbf{Q}$-coefficients. This implies the second part of the corollary.

Proposition 7.1.12. *Assume that Conjectures D_N are true for all N. Assume that for all N the maps realization* : $\mathrm{Ext}^1_{\mathcal{MM}_K}(\mathbf{Q}(0),\mathbf{Q}(N))\otimes\mathbf{Q}\to H^1(G_K,\mathbf{Q}_l(N))$ *are injective. Then the Lie coalgebras $(\mathcal{L}^K(X)\otimes\mathbf{Q}_l,d)$ and $(\mathfrak{k}(X)^\diamond,d)$ are isomorphic.*

Proof. Let us define a map

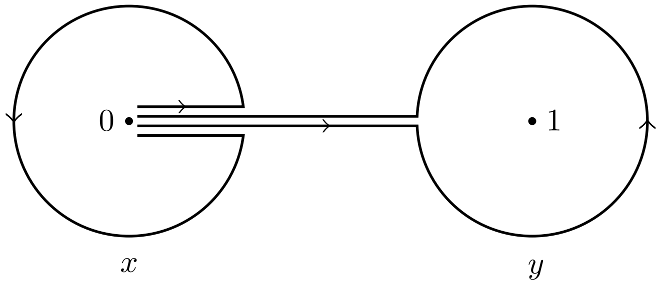
$$r_l:\mathcal{L}^K(X)\otimes\mathbf{Q}_l\longrightarrow\mathfrak{k}(X)^\diamond$$

by $r_l(\{z,v\}_e\otimes 1):=\mathcal{L}^e(z,v)$. The vector space $\mathfrak{k}(X)^\diamond$ is generated over $\mathbf{Q}_l$ by linear forms $\mathcal{L}^e(z,v)$ ($(z,v)\in\hat{X}(K)\times\hat{X}(K)$, $e\in\mathcal{B}$). Corollary 7.1.11 implies that the map r_l is an isomorphism of vector spaces over $\mathbf{Q}_l$. It follows from the definition of d in $\mathcal{L}^K(X)$ that r_l is an isomorphism of Lie coalgebras over $\mathbf{Q}_l$.

§8. Primitive example in the case $\mathbf{P}^1\setminus\{0,1,\infty\}$

8.0. We shall show here that the functions $a^{\varphi}_{x,p}$ are generalizations of characters considered by Soulé, Deligne, Ihara (see [S1], [S2], [D] and [I1]).

Let $V=P^1_{\mathbf{Q}}\setminus\{0,1,\infty\}$. Let us fix a path p from $\overrightarrow{01}$ to $\overrightarrow{10}$. We recall that $\pi_1(V_{\bar{\mathbf{Q}}},\overrightarrow{01})$ is a free group on x - a small loop around 0, and y - a loop



Picture 4

around 1. (One goes from $\overrightarrow{01}$ to $\overrightarrow{10}$ along p, makes a small loop around 1 and returns to $\overrightarrow{01}$ along p (see Picture 4).)

The action of $\sigma \in G_{Q(\mu_{l^\infty})}$ is given by

$$\sigma(x) = x \quad \text{and} \quad \sigma(y) = \mathfrak{f}_p(\sigma)^{-1} \cdot y \cdot \mathfrak{f}_p(\sigma).$$

Let us set $\pi_1' := [\pi_1(V_{\bar{\mathbf{Q}}}, \overrightarrow{01}), \pi_1(V_{\bar{\mathbf{Q}}}, \overrightarrow{01})]$ and $\pi_1'' := [\pi_1', \pi_1']$. The element $\mathfrak{f}_p(\sigma)$ belongs to π_1'. Assume that

$$\mathfrak{f}_p(\sigma) = \prod_{i,j\geq 1} (y^{j-1}(x^{i-1}(x,y)\cdots)\cdots)^{\alpha_{i,j}(\sigma)} \mod \pi_1''. \tag{8.0.1}$$

It implies that

$$\sigma((x,y)) = (x,y) \prod_{i,j\geq 1} (y^{j}(x^{i}(x,y)\cdots)\cdots)^{\alpha_{i,j}(\sigma)} \mod \pi_1''. \tag{8.0.2}$$

Ihara shows that π_1'/π_1'' is a free $\mathbf{Z}_l[[u,v]]$-module generated by (x,y), where $(u+1)\cdot z = x \cdot z \cdot x^{-1}$ and $(v+1)\cdot z = y\cdot z\cdot y^{-1}$ for any $z \in \pi_1'/\pi_1''$ (see [I1, Theorem 2]). It follows from (8.0.2) that $\sigma((x,y)) = h_\sigma(u,v)\cdot(x,y)$, where $h_\sigma(u,v) := 1 + \sum_{i,j\geq 1} \alpha_{i,j}(\sigma)u^i v^j$. Coefficients $\beta_{i,j} : G_{\mathbf{Q}(\mu_{l^\infty})} \to \mathbf{Q}_l(i+j)$ are defined by the equality

$$\log h_\sigma(e^U - 1, e^V - 1) = \sum_{i,j\geq 1} \frac{\beta_{i,j}(\sigma)}{i!j!} U^i V^j \tag{8.0.3}$$

(see [I1, pages 96 and 105]). We shall compare these coefficients with l-adic iterated integrals defined by us.

The inclusion k of $\pi_1(X_{\bar{\mathbf{Q}}}, \overrightarrow{01})$ into $\mathbf{Q}_l\{\{X,Y\}\}$ given by $k(x) = e^X$ and $k(y) = e^Y$ induces an action of σ on $\mathbf{Q}_l\{\{X,Y\}\}$ given by

$$\sigma(X) = X \quad \text{and} \quad \sigma(Y) = \Lambda_p(\sigma)^{-1} \cdot Y \cdot \Lambda_p(\sigma).$$

The logarithm of σ, $\log \sigma \in \mathrm{Der}^*(\mathbf{Q}_l\{\{X,Y\}\})$ and

$$(\log \sigma)(X) = 0, \quad (\log \sigma)(Y) = [Y, \mathcal{L}(X,Y)(\sigma)]$$

for some element $\mathcal{L}(X,Y)(\sigma) \in [L(X,Y), L(X,Y)]$. Let $L' := [L(X,Y), L(X,Y)]$ and $L'' := [L', L']$. Then

$$\mathcal{L}(X,Y)(\sigma) = \sum_{n=2}^{\infty} \sum_{i+j=n,\, i>0,\, j>0} a_{ij}(\sigma)[\cdots[\cdots[Y,X]X^{i-1}]Y^{j-1}] \mod L'',$$

where $a_{ij} : G_{\mathbf{Q}(\mu_{l^\infty})} \to \mathbf{Q}_l(i+j)$. Hence

(8.0.4)
$$(\log \sigma)(Y) = \sum_{n=2}^{\infty} \sum_{i+j=n,\, i>0,\, j>0} a_{ij}(\sigma)[\cdots[\cdots[X,Y]X^{i-1}]Y^{j}] \mod L''.$$

We shall calculate the coefficients $a_{i,j}$.

LEMMA 8.0.5. *We have* $k((y^{b-1}(x^{a-1}(x,y)\cdots)\cdots) = e^{r_{a,b}(X,Y)}$, *where*

$$r_{a,b}(X,Y) = \sum_{i_a,\dots,i_1,j_b,\dots,j_1 \ge 1} \frac{(-1)^{i_a+\cdots+i_1+j_b+\cdots+j_1-1}}{i_a!\cdots i_1! \cdot j_b \cdots j_1!} \times [\cdots[\cdots[Y,X]X^{i_a+\cdots+i_1-1}]Y^{j_b+\cdots+j_1-1}] \mod L''.$$

Proof. First one calculates $r_{1,1}(X,Y)$ and next by induction $r_{a,b}(X,Y)$ for any pair (a,b).

LEMMA 8.0.6. *There is a continuous bijection of vector spaces*

$$L'/L'' \approx \mathbf{Q}_l[[s,t]]$$

given by $[\cdots[\cdots[Y,X]X^{i-1}]Y^{j-1}] \to s^i t^j$. *The element* $r_{a,b}(X,Y) \in L'/L''$ *corresponds to a power series* $-(e^{-s}-1)^a(e^{-t}-1)^b$.

Observe that $\Lambda_p(\sigma) = e^{\varphi_\sigma(X,Y)}$, where $\varphi_\sigma(X,Y) \in L'$. The action of σ on $\mathbf{Q}_l\{\{X,Y\}\}$ induces

$$\sigma : L(X,Y)/L'' \longrightarrow L(X,Y)/L''$$

given by $\sigma(X) = X$ and $\sigma(Y) = Y + [Y, \varphi_\sigma(X,Y)] \mod L''$. It follows from (8.0.1) that

$$\varphi_\sigma(X,Y) = \sum_{i,j\geq 1} \alpha_{i,j}(\sigma) r_{i,j}(X,Y) \mod L''. \tag{8.0.7}$$

We shall calculate $(\log\sigma)(Y)$, where $\sigma : L(X,Y)/L'' \to L(X,Y)/L''$.

PROPOSITION 8.0.8. *The element* $(\log\sigma)(Y) \in L'/L''$ *corresponds to the power series*

$$t\log\Big(1 + \sum_{i,j\geq 1} \alpha_{i,j}(\sigma)(e^{-s}-1)^i(e^{-t}-1)^j\Big) \in \mathbf{Q}_l[[s,t]].$$

Proof. Let $F_\sigma(s,t) \in \mathbf{Q}_l[[s,t]]$ corresponds to $\varphi_\sigma(X,Y) \in L'/L''$. Then the series $-tF_\sigma(s,t)$ corresponds to $(\sigma - Id)(Y)$, the series $tF_\sigma(s,t)^2$ corresponds to $(\sigma - Id)^2(Y)$, the series $t(-F_\sigma(s,t))^n$ corresponds to $(\sigma - Id)^n(Y)$. Hence $(\log\sigma)(Y)$ corresponds to the series $t\log(1 - F_\sigma(s,t))$. It follows from Lemma 8.0.6 that $F_\sigma(s,t) = -\sum_{i,j\geq 1}\alpha_{i,j}(\sigma)(e^{-s}-1)^i(e^{-t}-1)^j$.

It follows from (8.0.4) and Proposition 8.0.8 that the coefficient $a_{i,j}(\sigma)$ is equal to the coefficient of the power series $-\log\big(1 + \sum_{i,j\geq 1}\alpha_{i,j}(\sigma)(e^{-s} - 1)^i(e^{-t}-1)^j\big)$ at s^it^j. It follows from (8.0.3) that $\frac{\beta_{i,j}(\sigma)}{i!j!}$ is the coefficient of the series $\log\big(1 + \sum_{i,j\geq 1}\alpha_{i,j}(\sigma)(e^U-1)^i(e^V-1)^j\big)$ at U^iV^j. Hence we get that $\frac{\beta_{i,j}(\sigma)}{i!j!} = (-1)^{i+j-1}a_{i,j}(\sigma)$. It follows from Proposition 5.1.8 that $(\log\sigma)(Y) = [Y, (\log\sigma_p)(1)]$. We recall that $(\log\sigma_p)(1) = \sum_{e\in\mathcal{B}} a_p^e(\sigma)e$, where $\mathcal{B}$ is a Hall base of $\mathrm{Lie}(X,Y)$. Hence we get that

$$a_{i,j}(\sigma) = a_p^{[\cdots[\cdots[Y,X]X^{i-1}]Y^{j-1}]}(\sigma).$$

Therefore we have proved the following result.

PROPOSITION 8.0.9. *We have*

$$\frac{\beta_{i,j}(\sigma)}{i!j!} = (-1)^{i+j-1}a_p^{[[[Y,X]X^{i-1}]Y^{j-1}]}(\sigma).$$

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