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LINEAR RELATIONS BETWEEN FOURIER COEFFICIENTS OF SPECIAL SIEGEL MODULAR FORMS

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Abstract. In this paper we give certain linear relations between the Fourier coefficients of Siegel modular forms that are obtained from Ikeda lifts.

§1. Introduction

Let n and k be positive integers with $n \equiv k \pmod 2$. In [5], T. Ikeda constructed a lifting map which associates to a cuspidal Hecke eigenform f of weight $2k$ with respect to $\Gamma_1 := SL_2(\mathbf{Z})$ a cuspidal Hecke eigenform F of weight $k+n$ with respect to the Siegel modular group $\Gamma_{2n} := Sp_{2n}(\mathbf{Z}) \subset GL_{4n}(\mathbf{Z})$ of genus $2n$. By Ikeda's construction, the Fourier coefficients of F are given in terms of (essentially) squarefree Fourier coefficients of modular forms of half-integral weight and products of special values of modified local singular series polynomials.

The existence of this lifting, in terms of a relation between associated zeta functions, was previously conjectured by Duke-Imamoglu and independently by Ibukiyama, in a somewhat different form.

If $n = 1$, the lift comes down to the classical Saito-Kurokawa lift.

In [10], we gave a linear version of Ikeda's lifting map, as a linear map from half-integral weight modular forms to Siegel modular forms of genus $2n$. If $n = 1$, one recovers a formula given by Eichler-Zagier [4] for the Fourier coefficients of the Saito-Kurokawa lifting in terms of the Fourier coefficients of half-integral weight modular forms.

In the classical case $n = 1$, as is well-known the space generated by the lifted forms F has a nice description in terms of certain linear relations between Fourier coefficients ("Maass space"), cf. e.g. [4, sect. 6, formula (9)].

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Let T be a positive definite, half-integral, symmetric matrix of size $2n$ and denote by $D_T := (-1)^n \det(2T)$ its discriminant. The aim of this paper is to show that also in the case $n > 1$ there exist linear relations of a similar kind between certain of the Fourier coefficients $a(T)$ of F, for all F, at least if $n \not\equiv 2 \pmod 4$. Indeed, this follows from the linear description of Ikeda's lifting map in terms of the coefficients $c(m)$ ($m \in \mathbf{N}$) of half-integral weight modular forms given in [10], together with the fact the $c(m)$ often can already be recovered from the $a(T)$ for very special T. However, contrary to the case $n = 1$ it is hard to imagine that for general $n > 1$ these relations can be used to give a linear characterization of the space generated by the F.

In Section 3, we state Ikeda's lifting result and the linear version of it given in [10] in detail, after having recalled several preliminaries in Section 2. In Section 4 we explicitly state the linear relations addressed above in the case $n \equiv 1 \pmod 4$ and give a detailled proof. Section 5 contains some remarks in the other cases $n \not\equiv 1 \pmod 4$.

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§2. Preliminaries

We will recall several facts about local singular series polynomials. As references, the reader may consult [1,6,7]. We will also recall the definition of a certain number-theoretic function which enters into the formulas for the Ikeda lifting given in [10].

Let $T \in M_m(\mathbf{Q})$ be a rational, symmetric, non-degenerate, half-integral matrix of size m.

If m is even, we denote by

$$D_T := (-1)^{\frac{m}{2}} \det(2T)$$

the discriminant of T. Then $D_T \equiv 0, 1 \pmod 4$ and we write $D_T = D_{T,0} f_T^2$ with $D_{T,0}$ the corresponding fundamental discriminant and $f_T \in \mathbf{N}$.

Let us fix a prime p. Recall that one defines the local singular series of T at p by

$$b_p(T; s) := \sum_R \nu_p(R)^{-s} \mathrm{e}_p(\mathrm{tr}(TR)) \qquad (s \in \mathbf{C})$$

where R runs over all symmetric (m, m)-matrices with entries in $\mathbf{Q}_p/\mathbf{Z}_p$ and $\nu_p(R)$ is a power of p equal to the product of denominators of elementary divisors of R. Furthermore, for $x \in \mathbf{Q}_p$ we have put $\mathrm{e}_p(x) := e^{2\pi i x'}$ where x' denotes the fractional part of x.

As is well-known, $b_p(T;s)$ is a product of two polynomials in p^{-s} with coefficients in $\mathbf{Z}$. More precisely, one has

$$b_p(T;s) = \gamma_p(T;p^{-s})F_p(T;p^{-s})$$

where

$$\gamma_p(T;X) := \begin{cases} (1-X)(1-(\frac{D_{T,0}}{p})p^{m/2}X)^{-1}\prod_{j=1}^{m/2}(1-p^{2j}X^2), & \text{if } m \text{ is even} \\ (1-X)\prod_{j=1}^{(m-1)/2}(1-p^{2j}X^2), & \text{if } m \text{ is odd} \end{cases}$$

and $F_p(T;X) \in \mathbf{Z}[X]$ has constant term 1.

In the following we will suppose that $m = 2n$ is even. A fundamental result of Katsurada [6] then states that the Laurent polynomial

$$\tilde{F}_p(T;X) := X^{-\operatorname{ord}_p f_T} F_p(T;p^{-n-1/2}X) \tag{1}$$

is symmetric, i.e.

$$\tilde{F}_p(T;X) = \tilde{F}_p(T;X^{-1}).$$

(There is also a corresponding functional equation if T is of odd size, but we won't need it. For the functional equation cf. also [2].)

If p does not divide f_T, then $F_p(T;X) = \tilde{F}_p(T;X) = 1$.

Denote by $V = (\mathbf{F}_p^{2n}, q)$ the quadratic space over $\mathbf{F}_p$ where q is the quadratic form obtained from the quadratic form $x \mapsto T[x]$ ($x \in \mathbf{Z}_p^{2n}$) by reducing modulo p. (For matrices A and B of appropriate sizes over a commutative ring we put $A[B] := B^tAB$ as usual.)

Let $R(V)$ be the radical of V, put $s_p := \dim R(V)$ and denote by W an orthogonal complementary subspace of $R(V)$.

According to [7], one defines a polynomial by

$$H_{n,p}(T;X) := \begin{cases} 1 & \text{if } s_p = 0 \\ \prod_{j=1}^{[\frac{s_p-1}{2}]}(1-p^{2j-1}X^2) & \text{if } s_p > 0,\ s_p \text{ odd} \\ (1+\lambda_p(T)\,p^{\frac{s_p-1}{2}}X)\prod_{j=1}^{[\frac{s_p-1}{2}]}(1-p^{2j-1}X^2) & \text{if } s_p > 0,\ s_p \text{ even,} \end{cases}$$

where for s_p even we have put

$$\lambda_p(T) := \begin{cases} 1 & \text{if } W \text{ is a hyperbolic subspace or } s_p = 2n \\ -1 & \text{otherwise.} \end{cases}$$

For $\mu \in \mathbf{Z}$, $\mu \geq 0$ define $\rho_T(p^\mu)$ by

$$\sum_{\mu \geq 0} \rho_T(p^\mu) X^\mu := \begin{cases} (1-X^2) H_{n,p}(T;X), & \text{if } p|f_T \\ 1, & \text{otherwise.} \end{cases}$$

We extend the function ρ_T multiplicatively to the whole of $\mathbf{N}$ by defining

$$\sum_{a \geq 1} \rho_T(a) a^{-s} := \prod_{p|f_T} ((1-p^{-2s}) H_{n,p}(T;p^{-s})).$$

It follows from the definitions that $\sqrt{a}\,\rho_T(a)$ is an integer.

Finally, let

$$\mathcal{D}(T) := GL_{2n}(\mathbf{Z}) \backslash \{ G \in M_{2n}(\mathbf{Z}) \cap GL_{2n}(\mathbf{Q}) \mid T[G^{-1}] \text{ half-integral} \}$$

where $GL_{2n}(\mathbf{Z})$ operates by left-multiplication. Then $\mathcal{D}(T)$ is finite as is easy to see. For $a \in \mathbf{N}$ with $a|f_T$ put

$$\phi(a;T) := \sqrt{a} \sum_{d^2|a} \sum_{G \in \mathcal{D}(T), |\det(G)|=d} \rho_{T[G^{-1}]}(\frac{a}{d^2}). \tag{2}$$

Note that on the right hand side of (2) we have $\frac{a}{d^2}|f_{T[G^{-1}]}$ and that $\phi(a;T) \in \mathbf{Z}$ for all a.

§3. Lifting maps

Let f be a normalized cuspidal Hecke eigenform of even integral weight $2k$ with respect to Γ_1. For a prime p, let $\lambda(p)$ and α_p be the p-th Fourier coefficient and the Satake p-parameter of f, respectively. Thus

$$1 - \lambda(p) X + p^{2k-1} X^2 = (1 - p^{k-1/2} \alpha_p X)(1 - p^{k-1/2} \alpha_p^{-1} X).$$

Note that α_p is determined only up to inversion.

Let

$$g = \sum_{m \geq 1, (-1)^k m \equiv 0,1 \pmod 4} c(m) e^{2\pi i m z} \qquad (z \in \mathcal{H} = \text{ upper half-plane})$$

be a cuspidal Hecke eigenform of weight $k + \frac{1}{2}$ and level 4 contained in the "plus" space which corresponds to f under the Shimura correspondence [9,12].

Let $n \in \mathbf{N}$ with $n \equiv k \pmod 2$. For T a positive definite, symmetric, half-integral matrix of size $2n$ define

$$a_f(T) := c(|D_{T,0}|)\, f_T^{k-1/2} \prod_{p|f_T} \tilde{F}_p(T;\alpha_p) \tag{3}$$

where we have used the notation explained in Section 2. Note that for n and k of the same parity $(-1)^k D_{T,0} > 0$.

THEOREM [5]. *The function*

$$F(Z) := \sum_{T>0} a_f(T) e^{2\pi i\, tr(TZ)}$$
$$(Z \in \mathcal{H}_{2n} = \textit{Siegel upper half-space of genus } 2n),$$

where T runs over all positive definite, symmetric, half-integral matrices of size $2n$, is a cuspidal Siegel-Hecke eigenform of weight $k+n$ with respect to Γ_{2n}.

THEOREM [10]. *With the notation of Section 2, one has*

$$a_f(T) = \sum_{a|f_T} a^{k-1}\phi(a;T)c(\frac{|D_T|}{a^2}). \tag{4}$$

§4. Linear relations

We keep all notations of the preceding sections.

If T_1 and T_2 are quadratic matrices over a commutative ring, we write $T_1 \oplus T_2$ for the diagonal block matrix $\begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}$.

Let $n \in \mathbf{N}$ with $n \equiv 1 \pmod 4$. Let T_0 be a positive definite, integral, even, symmetric, unimodular matrix of size $2n-2$. (Note that $2n-2 \equiv 0 \pmod 8$; for example, one can take for T_0 the matrix of $\frac{n-1}{4}$ copies of the standard E_8-lattice.)

For $m \in \mathbf{N}$ with $m \equiv 0, 3 \pmod 4$, let $\mathcal{T}_m$ be any positive definite, half-integral, symmetric (2,2)-matrix of discriminant $-m$ whose associated quadratic form represents the number 1. Note that for given m all such forms are equivalent; for $\mathcal{T}_m$ one can take for example $\begin{pmatrix} \frac{m}{4} & 0 \\ 0 & 1 \end{pmatrix}$ or

$\begin{pmatrix} \frac{m+1}{4} & 1/2 \\ 1/2 & 1 \end{pmatrix}$ according as $m \equiv 0 \pmod 4$ or $m \equiv 3 \pmod 4$, respectively.

THEOREM. *Let $n, k \in \mathbf{N}$ with $n \equiv k \pmod 2$. Suppose that $n \equiv 1 \pmod 4$. Let f be a normalized cuspidal Hecke eigenform of weight $2k$ with respect to Γ_1. Then with the above notation, for each positive definite, symmetric, half-integral matrix T of size $2n$ the Fourier coefficients of the Ikeda lift F of f given by (3) satisfy the linear relation*

$$a_f(T) = \sum_{a | f_T} a^{k-1} \phi(a; T)\, a_f(\mathcal{T}_{|D_T|/a^2} \oplus \frac{1}{2} T_0).$$

Proof. Note that by our assumption $D_T < 0$. In view of (4), it is sufficient to prove that

$$a_f(\mathcal{T}_m \oplus \frac{1}{2} T_0) = c(m) \tag{5}$$

for all $m \in \mathbf{N}$ with $m \equiv 0, 3 \pmod 4$.

We first claim that

$$\tilde{F}_p(\mathcal{T} \oplus \frac{1}{2} T_0; X) = \tilde{F}_p(\mathcal{T}; X) \tag{6}$$

for any rational, symmetric, non-degenerate, half-integral matrix $\mathcal{T}$ and all p, where for our purposes it is sufficient to prove (6) only for $\mathcal{T}$ of even rank, say $2r$.

Indeed, for fixed p let L and U be the lattices over $\mathbf{Z}_p$ corresponding to $\mathcal{T}$ and $\frac{1}{2} T_0$, respectively. Then U is an even unimodular hyperbolic lattice. The set $\mathcal{D}_p(\mathcal{T})$ (defined in the same way as $\mathcal{D}(\mathcal{T})$ in Section 3, but with $\mathbf{Z}$ replaced by $\mathbf{Z}_p$) can be identified with the set of isomorphism classes of $\mathbf{Z}_p$-integral lattices $\tilde{L} \subset L \otimes \mathbf{Q}_p$ containing L, and as is well-known the map from $\mathcal{D}_p(\mathcal{T})$ to $\mathcal{D}_p(\mathcal{T} \oplus \frac{1}{2} T_0)$ induced by $\tilde{L} \mapsto \tilde{L} \oplus U$ is a bijection. In fact, the surjectivity is a consequence of Propos. 5.2.2 in [8] and the injectivity follows from Lemma 5.3.1 in [8] (cf. also [11; 82:15, 92:3 and 93.14a]).

On the other hand, by [7, Thm. 2] (compare also [9, Propos. 1]) one has the identity

$$\tilde{F}_p(\mathcal{T}; X) = X^{-\operatorname{ord}_p f_{\mathcal{T}}} \sum_{G \in \mathcal{D}_p(\mathcal{T})} X^{2 \operatorname{ord}_p |\det G|} \cdot (1 - (\frac{D_{\mathcal{T},0}}{p}) p^{-\frac{1}{2}} X) \cdot H_{r,p}(\mathcal{T}[G^{-1}]; X).$$

Hence (6) follows.

From (6), in particular we obtain that

$$\tilde{F}_p(\mathcal{T}_m \oplus \frac{1}{2}T_0; X) = \tilde{F}_p(\mathcal{T}_m; X).$$

Thus by (3), the proof of (5) is reduced to the case $n = 1$, i.e. to showing that

$$a_f(\mathcal{T}_m) = c(m) \tag{7}$$

for all m. This, however, is the situation of the Maass space and (7), of course, is well-known (cf. [4; 5, sect. 16; 9, sect. 6]). Note that (6) also follows easily from certain recursion formulas for the local singular series polynomials given in [6], cf. in particular [6; Thm. 2.6 (1), proof of Thm. 4.1 and p. 418].

§5. Complements

Suppose that $n \equiv 0 \pmod 4$ and let T_0 now be a positive definite, integral, even, symmetric, unimodular matrix of size $2n-8$. Then one can show in a similar way as above that

$$a_f(\mathcal{S}_m \oplus \frac{1}{2}T_0) = c(m) \qquad (m \equiv 0, 1 \pmod 4)$$

where

$$2\mathcal{S}_m = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{m}{2} \end{pmatrix}$$

if $m \equiv 0 \pmod 4$ and

$$2\mathcal{S}_m = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & \frac{m+3}{2} \end{pmatrix}$$

if $m \equiv 1 \pmod 4$, respectively. We leave the details to the reader.

Note that the lattice attached to $2\mathcal{S}_4$ is just the E_8-lattice and that the matrices $\mathcal{S}_m$ are simple analogues in the case of rank 8 of the special matrices $\begin{pmatrix} \frac{m'}{4} & 0 \\ 0 & 1 \end{pmatrix}$ etc. ($m' \equiv 0, 3 \pmod 4$) of Section 4.

Thus from (4) we again obtain certain linear relations among the Fourier coefficients $a_f(T)$.

To proceed in the general case in a similar way, for each $m \in \mathbf{N}$ with $(-1)^n m \equiv 0, 1 \pmod 4$ one would like to find a positive definite, symmetric, half-integral matrix R_m of size $2n$ which satisfies the following condition:

i) if $m \equiv 0 \pmod 4$, then

$$R_m \sim (-1)^{n-1} u_p \begin{pmatrix} (-1)^{n-1} m/4 & 0 \\ 0 & 1 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}^{\oplus(n-1)}$$

with some $u_p \in \mathbf{Z}_p^*$ for all primes p;

ii) if $(-1)^n m \equiv 1 \pmod 4$, then

$$R_m \sim (-1)^{n-1} u_p \begin{pmatrix} ((-1)^{n-1} m + 1)/4 & 1/2 \\ 1/2 & 1 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}^{\oplus(n-1)}$$

with some $u_p \in \mathbf{Z}_p^*$ for all primes p. Here $\sim$ means equivalence over $\mathbf{Z}_p$.

One can construct such an R_m at least unless $n \equiv 2 \pmod 4$ and m is a perfect square. Indeed, denote by $(\ ,\ )_p$ the Hilbert symbol relative to $\mathbf{Q}_p$. Clearly the Hasse invariant of the quadratic form $Q_{m,p}$ over $\mathbf{Z}_p$ defined by the right-hand side of i) resp. ii) is equal to

$$\begin{aligned} c_p(Q_{m,p}) &= (-1,-1)_p^{n(n-1)/2} (u_p, (-1)^n m)_p \\ &= \begin{cases} (u_p, (-1)^n m)_p, & \text{if } p > 2 \\ (-1)^{n(n-1)/2} (u_p, (-1)^n m)_p & \text{if } p = 2. \end{cases} \end{aligned}$$

Suppose that m is not a square. Then we can choose a prime ℓ such that $\operatorname{ord}_\ell m$ is odd and $u_\ell \in \mathbf{Z}_\ell^*$ such that

$$(u_\ell, (-1)^n m)_\ell = (-1)^{n(n-1)/2}.$$

We put $u_p = 1$ for $p \neq \ell$. Then $c_p(Q_{m,p}) = 1$ for almost all p and

$$\prod_p c_p(Q_{m,p}) = 1.$$

Hence the existence of R_m follows from [3; chap. 6, Thm. 1.3 and chap. 9, Thm. 1.2].

Similarly, if m is a square and n is odd, then we can find $u_2 \in \mathbf{Z}_2^*$ such that

$$(u_2, -1)_2 = (-1)^{(n-1)/2}$$

and put $u_p = 1$ for $p > 2$. Finally, if m is a square and $n \equiv 0 \pmod 4$, we put $u_p = 1$ for all p.

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