T. Watanabe
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THE HARDY-LITTLEWOOD PROPERTY OF FLAG VARIETIES

TAKAO WATANABE

Abstract. We study the asymptotic distribution of rational points on a generalized flag variety which are of bounded height and satisfy some congruence conditions in the formulation analogous to a strongly Hardy-Littlewood variety.

Let X be an affine variety in an affine space V over $\mathbb{Q}$ and B_T the set of $x \in X(\mathbb{R})$ with $\|x\| \leq T$ for a Euclidean norm $\|\cdot\|$ on $V(\mathbb{R})$. The Hardy-Littlewood method allows us to expect that the cardinality of $B_T \cap X(\mathbb{Z})$ is asymptotically equal to the volume of B_T with respect to some measure on $X(\mathbb{R})$. On the basis of such expectation, Borovoi and Rudnick [BR] introduced the notion of a Hardy-Littlewood variety in the adelic manner. Namely, an affine variety X is called a strongly Hardy-Littlewood variety if the asymptotic behavior

$$|(B_T \times B_f) \cap X(\mathbb{Q})| \sim \omega_{X(\mathbb{A}_\mathbb{Q})}(B_T \times B_f) \quad \text{as } T \to \infty$$

holds for any open compact subset B_f of the finite adele $X(\mathbb{A}_{\mathbb{Q},f})$, where $\omega_{X(\mathbb{A}_\mathbb{Q})}$ denotes the measure on $X(\mathbb{A}_\mathbb{Q})$ attached to a gauge form on X. It is known that many affine symmetric spaces have the strongly Hardy-Littlewood property.

In this paper, we study the asymptotic distribution of rational points of bounded height on a generalized flag variety in the formulation analogous to a strongly Hardy-Littlewood variety. Let k be an algebraic number field, G a connected reductive algebraic group defined over k, Q a maximal k-parabolic subgroup of G and $X = Q\backslash G$ a generalized flag variety over k. The adele group $G(\mathbb{A})$ of G has the unimodular subgroup $G(\mathbb{A})^1$ consisting of all elements $g \in G(\mathbb{A})$ that satisfy $|\chi(g)|_\mathbb{A} = 1$ for any k-rational character χ of G. Similarly, the unimodular subgroup $Q(\mathbb{A})^1$ of $Q(\mathbb{A})$ is defined, see Notation below for its precise definition. The homogeneous space $Y = Q(\mathbb{A})^1\backslash G(\mathbb{A})^1$ is appropriate to our purpose by the reason that the set $X(k)$

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of k-rational points of X is naturally regarded as a subset of Y and there is a unique right $G(\mathbb{A})^1$-invariant measure ω_Y on Y matching with Tamagawa measures $\omega_{G(\mathbb{A})^1}$ and $\omega_{Q(\mathbb{A})^1}$ of $G(\mathbb{A})^1$ and $Q(\mathbb{A})^1$, respectively. It is observed that Y is decomposed into the direct product of the infinite part Y_∞ and the finite part Y_f, and Y_f is naturally identified with the homogeneous space $Q(\mathbb{A}_f)\backslash G(\mathbb{A}_f)$. By a strongly k-rational representation π of G, the variety X is embedded into a projective space, and the height H_π is defined on $X(k)$. Since H_π is extended to a positive real valued function on Y, we can define the "ball" B_T of radius T as the set of $y \in Y_\infty$ with $H_\pi(y) \leq T$. Then the main theorem of this paper is stated that the asymptotic behavior

$$(0.1) \qquad |(B_T \times B_f) \cap X(k)| \sim \frac{\tau(Q)}{\tau(G)} \omega_Y(B_T \times B_f) \quad \text{as } T \to \infty$$

holds for any open subset B_f of Y_f. Here $\tau(G)$ and $\tau(Q)$ stand for the Tamagawa numbers of G and Q, respectively. In view of the equality $(B_T \times Y_f) \cap X(k) = \{x \in X(k) : H_\pi(x) \leq T\}$, (0.1) yields the asymptotic distribution of rational points $x \in X(k)$ which satisfy $H_\pi(x) \leq T$ together with congruence conditions provided by B_f. The volume $\omega_Y(B_T \times B_f)$ is explicitly computed in the following sense. If K_f is a good maximal compact subgroup of the finite adele group $G(\mathbb{A}_f)$ and B_f is the image of an open subgroup $D_f \subset K_f$ to $Y_f = Q(\mathbb{A}_f)\backslash G(\mathbb{A}_f)$, then

$$\omega_Y(B_T \times B_f) = \frac{[D_f(K_f \cap Q(\mathbb{A}_f)) : D_f] C_G d_Q}{[K_f : D_f] C_Q d_G e_Q} T^{e_Q[k:\mathbb{Q}]/e_\pi},$$

where d_G, d_Q and e_Q are positive integers depending on G and Q, e_π is a positive rational numbers depending on π and these constants are easily computed. Both C_G and C_Q are also positive real constants depending on G and Q, however the determination of their explicit values is more complicated than other constants. In some particular cases, e.g., the case that G splits over k or G is a special orthogonal group, we can describe C_G/C_Q by using the special values of the Dedekind zeta function of k (cf. Section 7).

Our result gives an affirmative partial answer to a question mentioned in the last paragraph of [MW2, Section 4.3]. The asymptotic formula of rational points of bounded height on any generalized flag variety was first obtained by Franke, Manin and Tschinkel [FMT]. In the case of $B_f = Y_f$, Corollary to Theorem 5 in [FMT] deduces the asymptotic behavior of the form $|(B_T \times Y_f) \cap X(k)| \sim cT^{e_Q[k:\mathbb{Q}]/e_\pi}$, where c is a constant. However, it is not clear in [FMT] that the leading term $cT^{e_Q[k:\mathbb{Q}]/e_\pi}$ is described in terms of the volume of $B_T \times Y_f$. In order to explain it more precisely, we mention the difference between the method of [FMT] and that of this paper. A crucial observation in [FMT] is that the height zeta function can be identified with one of the Langlands-Eisenstein series. Then, by using the analytic properties of Langlands-Eisenstein series and a standard Tauberian argument, Franke, Manin and Tschinkel established their asymptotic formula. Thus the volume $\omega_Y(B_T \times Y_f)$ does not occur in [FMT]. In this paper, we investigate directly the function $F_T(g) = |(B_T \times B_f) \cap X(k)g|\omega_Y(B_T \times B_f)^{-1}$ on $G(k)\backslash G(\mathbb{A})^1$. By using the theory of constant terms of Eisenstein series, we will prove that the inner product $\langle \theta, F_T \rangle$ of any pseudo-Eisenstein series θ on $G(k)\backslash G(\mathbb{A})^1$ and F_T satisfies

$$\langle \theta, F_T \rangle \longrightarrow \frac{\tau(Q)}{\tau(G)} \langle \theta, 1 \rangle \quad \text{as } T \to \infty.$$

This and the argument similar to [DRS] and [MW1] lead us to

$$F_T(g) \longrightarrow \frac{\tau(Q)}{\tau(G)} \quad \text{as } T \to \infty$$

for every $g \in G(k)\backslash G(\mathbb{A})^1$, and hence we immediately obtain (0.1). In view of this, the expression of the main term of $|(B_T \times B_f) \cap X(k)|$ by $\omega_Y(B_T \times B_f)$ is a significant point of our result.

Notation. As usual, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ denote the ring of integers, the field of rational, real and complex numbers, respectively. The group of positive real numbers is denoted by $\mathbb{R}_+^\times$.

Let k be an algebraic number field of finite degree over $\mathbb{Q}$, $\mathfrak{O}$ the ring of integers in k and $\mathfrak{V}$ the set of all places of k. We write $\mathfrak{V}_\infty$ and $\mathfrak{V}_f$ for the sets of all infinite places and all finite places of k, respectively. For $v \in \mathfrak{V}$, k_v denotes the completion of k at v. If v is finite, $\mathfrak{O}_v$ denotes the ring of integers in k_v. We fix, once and for all, a Haar measure μ_v on k_v normalized so that $\mu_v(\mathfrak{O}_v) = 1$ if $v \in \mathfrak{V}_f$, $\mu_v([0,1]) = 1$ if v is a real place and $\mu_v(\{a \in k_v : a\overline{a} \le 1\}) = 2\pi$ if v is an imaginary place. Then the absolute value $|\cdot|_v$ on k_v is defined as $|a|_v = \mu_v(aC)/\mu_v(C)$, where C is an arbitrary compact subset of k_v with nonzero measure. We denote by $\mathbb{A}$ the adele ring of k, by $\mathbb{A}_f$ the finite adele ring of k and by $|\cdot|_{\mathbb{A}} = \prod_{v \in \mathfrak{V}} |\cdot|_v$ the idele norm on the idele group $\mathbb{A}^\times$.

Let G be a connected affine algebraic group defined over k. For any k-algebra R, $G(R)$ stands for the set of R-rational points of G. Let $\mathbf{X}^*(G)$ and $\mathbf{X}_k^*(G)$ be the free $\mathbb{Z}$-modules consisting of all rational characters and all k-rational characters of G, respectively. The absolute Galois group $\mathrm{Gal}(\overline{k}/k)$ acts on $\mathbf{X}^*(G)$. The representation of $\mathrm{Gal}(\overline{k}/k)$ in the space $\mathbf{X}^*(G) \otimes_{\mathbb{Z}} \mathbb{Q}$ is denoted by σ_G and the corresponding Artin L-function is denoted by $L(s, \sigma_G) = \prod_{v \in \mathfrak{V}_f} L_v(s, \sigma_G)$. We set $\sigma_k(G) = \lim_{s \to 1}(s-1)^n L(s, \sigma_G)$, where $n = \operatorname{rank} \mathbf{X}_k^*(G)$. Let ω^G be a nonzero right invariant gauge form on G defined over k. From ω^G and the fixed Haar measure μ_v on k_v, one can construct a right invariant Haar measure ω_v^G on $G(k_v)$. Then, the Tamagawa measure on $G(\mathbb{A})$ is well defined by $\omega_{\mathbb{A}}^G = |D_k|^{-\dim G/2} \omega_\infty^G \omega_f^G$, where $\omega_\infty^G = \prod_{v \in \mathfrak{V}_\infty} \omega_v^G$, $\omega_f^G = \sigma_k(G)^{-1} \prod_{v \in \mathfrak{V}_f} L_v(1, \sigma_G) \omega_v^G$ and $|D_k|$ is the absolute value of the discriminant of k. For $\chi \in \mathbf{X}_k^*(G)$, let $|\chi|_{\mathbb{A}}$ be the continuous homomorphism $G(\mathbb{A}) \to \mathbb{R}_+^\times$ defined by $|\chi|_{\mathbb{A}}(g) = |\chi(g)|_{\mathbb{A}}$. We write $G(\mathbb{A})^1$ for the intersection of kernels of all such $|\chi|_{\mathbb{A}}$'s. If $\chi_1, \ldots, \chi_n$ is a $\mathbb{Z}$-basis of $\mathbf{X}_k^*(G)$, then the mapping

$$g \longmapsto (|\chi_1(g)|_{\mathbb{A}}, \ldots, |\chi_n(g)|_{\mathbb{A}})$$

yields an isomorphism from the quotient group $G(\mathbb{A})^1 \backslash G(\mathbb{A})$ to $(\mathbb{R}_+^\times)^n$. We put the Lebesgue measure dt on $\mathbb{R}$ and the invariant measure dt/t on $\mathbb{R}_+^\times$. Then there exists uniquely a Haar measure $\omega_{G(\mathbb{A})^1}$ of $G(\mathbb{A})^1$ such that the Haar measure on $G(\mathbb{A})^1 \backslash G(\mathbb{A})$ matching with $\omega_{\mathbb{A}}^G$ and $\omega_{G(\mathbb{A})^1}$ is equal to the pull-back of the measure $\prod_{i=1}^n dt_i/t_i$ on $(\mathbb{R}_+^\times)^n$ by the above isomorphism. The measure $\omega_{G(\mathbb{A})^1}$ is independent of the choice of a $\mathbb{Z}$-basis of $\mathbf{X}_k^*(G)$. Since $G(k)$ is a discrete subgroup of $G(\mathbb{A})^1$, we put the counting measure $\omega_{G(k)}$ on $G(k)$. Then the Tamagawa number $\tau(G)$ is defined to be the volume of the quotient space $G(k) \backslash G(\mathbb{A})^1$ with respect to the measure $\omega_G = \omega_{G(k)} \backslash \omega_{G(\mathbb{A})^1}$. Here, in general, if μ_A and μ_B denote Haar measures on a locally compact unimodular group A and its closed unimodular subgroup B, respectively, then $\mu_B \backslash \mu_A$ (resp. μ_A/μ_B) denotes a unique right (resp. left) A-invariant measure on the homogeneous space $B \backslash A$ (resp. A/B) matching with μ_A and μ_B.

If X is an algebraic variety defined over k, then $X(k)$ denotes the set of k-rational points of X. In addition, if X is affine, then $X(\mathbb{A})$ and $X(\mathbb{A}_f)$ stands for the adele and the finite adele of X, respectively. We say that a subset D of $X(\mathbb{A})$ is decomposable if D is of the form $D_\infty \times D_f$, where D_∞ and D_f are subsets of $\prod_{v \in \mathfrak{V}_\infty} X(k_v)$ and $X(\mathbb{A}_f)$, respectively.

If X is a locally compact topological space, $C_0(X)$ denotes the space of all compactly supported continuous functions on X. If X is a finite set, $|X|$ denotes the cardinal number of X. For two non-decreasing functions $F_1(T)$, $F_2(T)$ of real variable T, $F_1(T) \sim F_2(T)$ means $\lim_{T\to\infty} F_1(T)/F_2(T) = 1$ if $F_2(T) \neq 0$ for T large enough, otherwise, $F_1(T) \equiv 0$.

§1. Preliminaries

In the following, let G be a connected reductive group defined over k. We fix a maximally k-split torus S of G, a maximal k-torus S_1 of G containing S, a minimal k-parabolic subgroup P of G containing S and a Borel subgroup B of P containing S_1. Then, we denote by Φ_k the relative root system of G with respect to S and by Δ_k the set of simple roots of Φ_k corresponding to P.

Let M be the centralizer of S in G. Then P has a Levi decomposition $P = MU$, where U is the unipotent radical of P. For every standard k-parabolic subgroup R of G, R has a unique Levi subgroup M_R containing M. We denote by U_R the unipotent radical of R. Throughout this paper, we fix a maximal compact subgroup K of $G(\mathbb{A})$ satisfying the following property; For every standard k-parabolic subgroup R of G, $K \cap M_R(\mathbb{A})$ is a maximal compact subgroup of $M_R(\mathbb{A})$ and $M_R(\mathbb{A})$ possesses an Iwasawa decomposition $(M_R(\mathbb{A}) \cap U(\mathbb{A}))M(\mathbb{A})(K \cap M_R(\mathbb{A}))$. It is known that such maximal compact subgroup of $G(\mathbb{A})$ exists. We set $K^R = K \cap R(\mathbb{A})$, $K^{M_R} = K \cap M_R(\mathbb{A})$, $P^R = M_R \cap P$ and $U^R = M_R \cap U$.

Let R be a standard k-parabolic subgroup of G. We include the case $R = G$. Let Z_R be the greatest central k-split torus in M_R. The restriction map $\mathbf{X}^*_k(M_R) \to \mathbf{X}^*(Z_R)$ is injective. Since $\mathbf{X}^*_k(M_R)$ has the same rank as $\mathbf{X}^*(Z_R)$, the index

$$d_R = [\mathbf{X}^*(Z_R) : \mathbf{X}^*_k(M_R)] \tag{1.1}$$

is finite. If $\chi_1, \ldots, \chi_r$ is a $\mathbb{Z}$-basis of $\mathbf{X}^*(Z_R)$, then the mapping $z \mapsto (\chi_1(z), \ldots, \chi_r(z))$ yields an isomorphism from $Z_R(\mathbb{A})$ to $(\mathbb{A}^\times)^r$. We regard $\mathbb{R}^\times_+$ as a subgroup of $\mathbb{A}^\times$ by identifying $t \in \mathbb{R}^\times_+$ with the idele $t_{\mathbb{A}} = (t_v)$ such that $t_v = t$ if $v \in \mathfrak{V}_\infty$ and $t_v = 1$ if $v \in \mathfrak{V}_f$. Let A_R denote the inverse image of $(\mathbb{R}^\times_+)^r$ by the isomorphism $Z_R(\mathbb{A}) \to (\mathbb{A}^\times)^r$. Then $M_R(\mathbb{A})$ has the direct product decomposition: $M_R(\mathbb{A}) = A_R M_R(\mathbb{A})^1$. The Haar measure μ_{A_R} on A_R is defined to be the pull-back of the invariant measure $\prod_{i=1}^r dt_i/t_i$ on $(\mathbb{R}^\times_+)^r$ with respect to the isomorphism $z \mapsto (|\chi_1(z)|_{\mathbb{A}}, \ldots, |\chi_r(z)|_{\mathbb{A}})$ from

A_R onto $(\mathbb{R}_+^\times)^r$. It follows from the definition of $\omega_{M_R(\mathbb{A})^1}$ that the Tamagawa measure $\omega_{\mathbb{A}}^{M_R}$ is decomposed into $d_R\mu_{A_R} \cdot \omega_{M_R(\mathbb{A})^1}$. Both A_R and μ_{A_R} are independent of the choice of a basis of $\mathbf{X}^*(Z_R)$. We set $A_R^G = A_R/A_G$.

We define another Haar measure $\nu_{M_R(\mathbb{A})}$ of $M_R(\mathbb{A})$ as follows. Let $\omega_{\mathbb{A}}^M$ and $\omega_{\mathbb{A}}^{U^R}$ be the Tamagawa measures of $M(\mathbb{A})$ and $U^R(\mathbb{A})$, respectively. There is the function δ_{P^R} on $M(\mathbb{A})$ such that the integration formula

$$\int_{U^R(\mathbb{A})} f(mum^{-1})\, d\omega_{\mathbb{A}}^{U^R}(u) = \delta_{P^R}(m)^{-1} \int_{U^R(\mathbb{A})} f(u)\, d\omega_{\mathbb{A}}^{U^R}(u)$$

holds for $f \in C_0(U^R(\mathbb{A}))$. In other words, $\delta_{P^R}^{-1}$ is the modular character of $P^R(\mathbb{A})$. Let $\nu_{K^{M_R}}$ be the Haar measure on K^{M_R} normalized so that the total volume equals one. Then the mapping

$$f \longmapsto \int_{U^R(\mathbb{A})\times M(\mathbb{A})\times K^{M_R}} f(umh)\delta_{P^R}(m)^{-1}\, d\omega_{\mathbb{A}}^{U^R}(u) d\omega_{\mathbb{A}}^{M}(m) d\nu_{K^{M_R}}(h)\,, \qquad (f \in C_0(M_R(\mathbb{A})))$$

defines an invariant measure on $M_R(\mathbb{A})$ and is denoted by $\nu_{M_R(\mathbb{A})}$. There exists a positive constant C_R such that

$$\omega_{\mathbb{A}}^{M_R} = C_R \nu_{M_R(\mathbb{A})}\,. \tag{1.2}$$

We have the following compatibility formula:

(1.3)

$$\int_{G(\mathbb{A})} f(g)\, d\omega_{\mathbb{A}}^G(g) = \frac{C_G}{C_R} \int_{U_R(\mathbb{A})\times M_R(\mathbb{A})\times K} f(umh)\delta_R(m)^{-1}\, d\omega_{\mathbb{A}}^{U_R}(u) d\omega_{\mathbb{A}}^{M_R}(m) d\nu_K(h)$$

for $f \in C_0(G(\mathbb{A}))$, where δ_R^{-1} is the modular character of $R(\mathbb{A})$.

On the homogeneous space $Y_R = R(\mathbb{A})^1\backslash G(\mathbb{A})^1$, we define the right $G(\mathbb{A})^1$-invariant measure ω_{Y_R} by $\omega_{R(\mathbb{A})^1}\backslash\omega_{G(\mathbb{A})^1}$. We note that both $G(\mathbb{A})^1$ and $R(\mathbb{A})^1$ are unimodular. We identify Y_R with $A_G R(\mathbb{A})^1\backslash G(\mathbb{A})$. Then the mapping

$$\iota_R : K/K^R \times A_R^G \longrightarrow Y_R : (\overline{h}, \overline{z}) \longmapsto A_G R(\mathbb{A})^1 z^{-1} h^{-1}$$

is a bijection, where $\overline{h} = hK^R$ and $\overline{z} = zA_G$ for $h \in K$ and $z \in A_R$. Set $\nu_{A_R^G} = \mu_{A_R}/\mu_{A_G}$.

LEMMA 1. *Let D be an open subgroup of K and $\{h_1,\ldots,h_s\}$ be a complete set of coset representatives of K/D. Then, for any right D-invariant function $f\in C_0(Y_R)$, one has*

$$\int_{Y_R} f(y)\,d\omega_{Y_R}(y)=\frac{C_G d_R}{[K:D]C_R d_G}\sum_{i=1}^{s}\int_{A_R^G} f(\iota_R(\overline{h}_i^{-1},\overline{z}))\delta_R(z)\,d\nu_{A_R^G}(\overline{z})\,.$$

Proof. If we set

$$\varphi(y)=\int_K f(yh)\,d\nu_K(h)=\frac{1}{[K:D]}\sum_{i=1}^{s} f(yh_i)\,,$$

then φ is a right K-invariant function on Y_R. By [W, Corollary to Lemma 1],

$$\int_{Y_R}\varphi(y)\,d\omega_{Y_R}(y)=\frac{C_G d_R}{C_R d_G}\int_{A_R^G}\varphi(\iota_R(\overline{e},\overline{z}))\delta_R(z)\,d\nu_{A_R^G}(\overline{z})\,.$$

Since ω_{Y_R} is right $G(\mathbb{A})^1$-invariant, the left hand side equals the integral of $f(y)$ over Y_R. ∎

§2. Heights on flag varieties

Let V_π be a finite dimensional $\overline{k}$-vector space endowed with a k-structure $V_\pi(k)$ and $\pi: G\to GL(V_\pi)$ be an absolutely irreducible k-rational representation. The highest weight space in V_π with respect to B is denoted by x_π. Let Q_π be the stabilizer of x_π in G and λ_π the $\overline{k}$-rational character of Q_π by which Q_π acts on x_π. The representation π is said to be strongly k-rational if x_π is defined over k. Then Q_π is a standard k-parabolic subgroup of G and λ_π is a k-rational character of Q_π. It is known that $\lambda_\pi|_S$ is a non-negative integral linear combination of the fundamental k-weights ([W, Section 1]). We say π is maximal if Q_π is a standard maximal k-parabolic subgroup. This is equivalent to the condition that $\lambda_\pi|_S$ is a positive integer multiple of a single fundamental k-weight.

Let π be a strongly k-rational representation. For convenience, we use a right action of G on V_π defined by $a\cdot g=\pi(g^{-1})a$ for $g\in G$ and $a\in V_\pi$. Then the mapping $g\mapsto x_\pi\cdot g$ gives rise to a k-rational embedding of $Q_\pi\backslash G$ into the projective space $\mathbb{P}V_\pi$.

We write X_{Q_π} for $Q_\pi\backslash G$. Since Q_π is a k-parabolic subgroup, $X_{Q_\pi}(k)$ is naturally identified with $Q_\pi(k)\backslash G(k)$ ([B, Proposition 20.5]). Let us define a height on $X_{Q_\pi}(k)$. We fix a k-basis $\mathbf{e}_1, \dots, \mathbf{e}_n$ of the k-vector space $V_\pi(k)$ and define a local height H_v on $V_\pi(k_v)$ for each $v \in \mathfrak{V}$ as follows:

$$H_v(a_1\mathbf{e}_1 + \cdots + a_n\mathbf{e}_n) = \begin{cases} (|a_1|_v^2 + \cdots + |a_n|_v^2)^{1/(2[k:\mathbb{Q}])} & (\text{if } v \text{ is real}) \\ (|a_1|_v + \cdots + |a_n|_v)^{1/[k:\mathbb{Q}]} & (\text{if } v \text{ is imaginary}) \\ \sup(|a_1|_v, \dots, |a_n|_v)^{1/[k:\mathbb{Q}]} & (\text{if } v \in \mathfrak{V}_f) \end{cases}$$

The global height H_π on $V_\pi(k)$ is defined to be the product of all H_v, that is, $H_\pi(a) = \prod_{v \in \mathfrak{V}} H_v(a)$. By the product formula, H_π is invariant by scalar multiplications. Thus, H_π defines a height on $\mathbb{P}V_\pi(k)$, and on $X_{Q_\pi}(k)$ by restriction. The height H_π is extended to $GL(V_\pi, \mathbb{A})\mathbb{P}V_\pi(k)$ by

$$H_\pi(\xi\overline{a}) = \prod_{v \in \mathfrak{V}} H_v(\xi_v a)$$

for $\xi = (\xi_v) \in GL(V_\pi, \mathbb{A})$ and $\overline{a} = ka \in \mathbb{P}V_\pi(k)$, $a \in V_\pi(k) - \{0\}$. We set

$$\Phi_{\pi,\xi}(g) = H_\pi(\xi(x_\pi \cdot g))/H_\pi(\xi x_\pi)$$

for $g \in G(\mathbb{A})$. Obviously, $\Phi_{\pi,\xi}$ is a continuous function on $G(\mathbb{A})$ and satisfies

$$\Phi_{\pi,\xi}(qg) = |\lambda_\pi(q)^{-1}|_{\mathbb{A}}^{1/[k:\mathbb{Q}]}\Phi_{\pi,\xi}(g)$$

for any $q \in Q_\pi(\mathbb{A})$ and $g \in G(\mathbb{A})$. Thus $\Phi_{\pi,\xi}$ defines a function on $Y_{Q_\pi} = Q_\pi(\mathbb{A})^1\backslash G(\mathbb{A})^1$. It is always possible that one choose an element $\xi \in GL(V_\pi, \mathbb{A})$ so that $\Phi_{\pi,\xi}$ is right K-invariant. In many examples, one can take the identity as such ξ.

§3. The Hardy-Littlewood property of flag varieties

In the following, we assume π is maximal and strongly k-rational. We fix, once and for all, an element $\xi \in GL(V_\pi, \mathbb{A})$ such that $\Phi_{\pi,\xi}$ is right K-invariant. We simply write Q for Q_π and Φ_π for $\Phi_{\pi,\xi}$. Let Δ_Q be the set of nonzero roots $\beta|_{Z_Q}$, $\beta \in \Delta_k$. Since Q is maximal, Δ_Q consists of a single element $\alpha|_{Z_Q}$. Let n_Q be the positive integer such that $n_Q^{-1}\alpha|_{Z_Q}$ is a $\mathbb{Z}$-base of $\mathbf{X}^*(Z_G\backslash Z_Q)$. We set $\alpha_Q = n_Q^{-1}\alpha|_{Z_Q}$. Then the Haar measure ν_{A_Q} equals the pull-back of the measure dt/t by the isomorphism $|\alpha_Q|_{\mathbb{A}} : A_Q^G \to \mathbb{R}_+^\times$. If we set $e_Q = n_Q \dim U_Q$, we have

$$\delta_Q(z) = |\alpha_Q(z)|_{\mathbb{A}}^{e_Q}, \quad (z \in Z_Q(\mathbb{A})). \tag{3.1}$$

The quotient morphism $Z_Q \to Z_G\backslash Z_Q$ induces an isomorphism $\mathbf{X}^*(Z_G\backslash Z_Q) \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathbf{X}^*(Z_Q \cap G^{ss}) \otimes_{\mathbb{Z}} \mathbb{Q}$, where G^{ss} denotes the derived group of G. Under the identification $\mathbf{X}^*(Z_Q \cap G^{ss}) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbf{X}^*(Z_G\backslash Z_Q) \otimes_{\mathbb{Z}} \mathbb{Q}$, there exists the positive rational number e_π such that

$$\lambda_\pi|_{Z_Q \cap G^{ss}} = e_\pi \alpha_Q \,. \tag{3.2}$$

Then $\Phi_\pi(\iota_Q(\overline{h}, \overline{z})) = |\alpha_Q(z)|_{\mathbb{A}}^{e_\pi/[k:\mathbb{Q}]}$ holds for any $(\overline{h}, \overline{z}) \in K/K^Q \times A_Q^G$.

For an open subset D of K and $0 < T$, we set

$$E_\pi(D,T) = \left\{\iota_Q(\overline{h}, \overline{z}) : \overline{h} \in DK^Q/K^Q,\ \overline{z} \in A_Q^G,\ |\alpha_Q(\overline{z})|_{\mathbb{A}} \leq T^{[k:\mathbb{Q}]/e_\pi}\right\}.$$

Obviously, $E_\pi(D,T)$ is contained in $\{y \in Y_Q : \Phi_\pi(y) \leq T\}$, and in particular, the set $E_\pi(K,T) \cap X_Q(k)$ coincides with the set $\{x \in X_Q(k) : H_\pi(\xi x) \leq H_\pi(\xi x_\pi)T\}$. The next is the main theorem of this paper.

THEOREM 1. *Let π and Q be as above and $D = D_\infty \times D_f$ a decomposable open subset of K such that D_∞ equals the infinite part K_∞ of K. Then one has*

$$|E_\pi(D,T) \cap X_Q(k)g| \sim \frac{\tau(Q)}{\tau(G)} \omega_{Y_Q}(E_\pi(D,T)) \quad \textit{as } T \to \infty \tag{3.3}$$

for any $g \in G(\mathbb{A})^1$.

We fix a decomposable open subset D of K with $D_\infty = K_\infty$. Since the finite part of K is totally disconnected, there is a decomposable open normal subgroup D_1 of K and $b_0 \in D$ such that $D_1 b_0^{-1} D = b_0^{-1} D$ and $D_{1,\infty} = K_\infty$. If $b_1, \ldots, b_s \in D$ is a complete set of coset representatives of $D_1 K^Q \backslash b_0^{-1} D K^Q$, then $E_\pi(b_0^{-1}D, T) = E_\pi(D,T)b_0$ decomposes into a disjoint union of $E_\pi(D_1,T)b_i$, $i = 1, 2, \ldots, s$. It is easy to see that the truth of (3.3) for D_1 implies the truth of (3.3) for D. Hence, we may assume without loss of generality that D is an open normal subgroup of K to begin with. Then, by Lemma 1, $\omega_{Y_Q}(E_\pi(D,T))$ equals

$$\frac{[DK^Q : D]C_G d_Q}{[K:D]C_Q d_G} \int_0^{T^{[k:\mathbb{Q}]/e_\pi}} t^{e_Q} \frac{dt}{t} = \frac{[DK^Q : D]C_G d_Q}{[K:D]C_Q d_G e_Q} T^{e_Q[k:\mathbb{Q}]/e_\pi}.$$

Let χ_T be the characteristic function of $E_\pi(D,T)$. Define the function F_T on $G(k)\backslash G(\mathbb{A})^1$ as

$$F_T(g) = \frac{1}{\omega_{Y_Q}(E_\pi(D,T))} \sum_{x \in X_Q(k)} \chi_T(xg) = \frac{|E_\pi(D,T) \cap X_Q(k)g|}{\omega_{Y_Q}(E_\pi(D,T))}.$$

(3.3) is equivalent to the assertion that

$$\lim_{T\to\infty} F_T(g) = \frac{\tau(Q)}{\tau(G)}$$

holds for every $g \in G(\mathbb{A})^1$. For a pair of functions ψ_1, ψ_2 on $G(k)\backslash G(\mathbb{A})^1$, we set

$$\langle \psi_1, \psi_2 \rangle = \int_{G(k)\backslash G(\mathbb{A})^1} \psi_1(g)\overline{\psi_2(g)}\, d\omega_G(g)$$

if the integral has the meaning.

Proposition 1. *If*

$$\lim_{T\to\infty} \langle \psi, F_T \rangle = \frac{\tau(Q)}{\tau(G)} \langle \psi, 1 \rangle$$

holds for any $\psi \in C_0(G(k)\backslash G(\mathbb{A})^1)$, *then*

$$\lim_{T\to\infty} F_T(g) = \frac{\tau(Q)}{\tau(G)}$$

for every $g \in G(\mathbb{A})^1$.

Proof. Let $\{U_m\}_{m=1,2,3,\dots}$ be a descending family of neighborhoods of the identity e in $G(\mathbb{A})^1$ such that U_m is decomposable, i.e., $U_m = (U_m)_\infty \times (U_m)_f$, $U_m^{-1} = U_m$, $(U_m)_f = D_f$, $(U_m)_\infty$ is compact and $\bigcap_{m=1}^\infty (U_m)_\infty = \{e\}$. Since Φ_π is continuous and KU_m is compact, there exists the maximum

$$\beta_m = \max_{g\in KU_m} \Phi_\pi(g) = \max_{g_\infty \in K_\infty (U_m)_\infty} \Phi_\pi(g_\infty)\,.$$

From the right K-invariance of Φ_π and $\Phi_\pi(e) = 1$, it follows that $\beta_m \downarrow 1$ as $m \to \infty$. By $D_\infty = K_\infty$ and the definition of $E_\pi(D, T)$, it is evident that

$$E_\pi(D, T)U_m \subset E_\pi(D, \beta_m T)$$

for every m. Therefore,

$$E_\pi(D, \beta_m^{-1}T)g^{-1}g_0^{-1} \subset E_\pi(D, T)g_0^{-1} \subset E_\pi(D, \beta_m T)g^{-1}g_0^{-1}$$

holds for every $g \in U_m = U_m^{-1}$ and a fixed $g_0 \in G(\mathbb{A})^1$. This implies the inequality

$$\begin{aligned}\omega_{Y_Q}(E_\pi(D, \beta_m^{-1}T))F_{\beta_m^{-1}T}(g_0 g) &\le \omega_{Y_Q}(E_\pi(D, T))F_T(g_0)\\ &\le \omega_{Y_Q}(E_\pi(D, \beta_m T))F_{\beta_m T}(g_0 g)\end{aligned}$$

for $g \in U_m$. Let U'_m be the image of $g_0 U_m$ to the quotient $G(k)\backslash G(\mathbb{A})^1$. We choose a real-valued and non-negative function $\psi_m \in C_0(G(k)\backslash G(\mathbb{A})^1)$ such that the support of ψ_m is contained in U'_m and $\langle \psi_m, 1\rangle = 1$. Then the above inequality yields

$$\begin{aligned}\frac{\omega_{Y_Q}(E_\pi(D,\beta_m^{-1}T))}{\omega_{Y_Q}(E_\pi(D,T))}\langle \psi_m, F_{\beta_m^{-1}T}\rangle &\le F_T(g_0)\\ &\le \frac{\omega_{Y_Q}(E_\pi(D,\beta_m T))}{\omega_{Y_Q}(E_\pi(D,T))}\langle \psi_m, F_{\beta_m T}\rangle\,.\end{aligned}$$

By $\omega_{Y_Q}(E_\pi(D,\beta_m T))/\omega_{Y_Q}(E_\pi(D,T)) = \beta_m^{e_Q[k:\mathbb{Q}]/e_\pi}$ and the assumption on F_T, one has

$$\beta_m^{-e_Q[k:\mathbb{Q}]/e_\pi}\frac{\tau(Q)}{\tau(G)} \le \liminf_{T\to\infty} F_T(g_0) \le \limsup_{T\to\infty} F_T(g_0) \le \beta_m^{e_Q[k:\mathbb{Q}]/e_\pi}\frac{\tau(Q)}{\tau(G)}\,.$$

Hence, letting $m \to \infty$, we get the assertion. □

For every function ψ on $G(k)\backslash G(\mathbb{A})^1$, we set

$$\begin{aligned}\Pi_Q^1(\psi)(g) &= \int_{U_Q(k)\backslash U_Q(\mathbb{A})} \psi(ug)\, d\omega_{U_Q}(u)\,,\\ \Pi_Q(\psi)(g) &= \int_{Q(k)\backslash Q(\mathbb{A})^1} \psi(qg)\, d\omega_Q(q)\\ &= \int_{M_Q(k)\backslash M_Q(\mathbb{A})^1} \Pi_Q^1(\psi)(mg)\, d\omega_{M_Q}(m)\end{aligned}$$

when the integrals have the meaning. By the unfolding argument and Lemma 1, we have

$$(3.4)\qquad \begin{aligned}\langle \psi, F_T\rangle &= \int_{G(k)\backslash G(\mathbb{A})^1} \psi(g) F_T(g)\, d\omega_G(g)\\ &= \frac{1}{\omega_{Y_Q}(E_\pi(D,T))}\int_{Y_Q} \Pi_Q(\psi)(y)\chi_T(y)\, d\omega_{Y_Q}(y)\\ &= \frac{e_Q}{T^{e_Q[k:\mathbb{Q}]/e_\pi}}\int_0^{T^{[k:\mathbb{Q}]/e_\pi}} t^{e_Q}\Pi_Q(\psi)(\iota_Q(\overline{e}, |\alpha_Q|_{\mathbb{A}}^{-1}(t)))\frac{dt}{t}\end{aligned}$$

for every right D-invariant $\psi \in C_0(G(k)\backslash G(\mathbb{A})^1)$, where $|\alpha_Q|_{\mathbb{A}}^{-1}$ stands for the inverse map of $|\alpha_Q|_{\mathbb{A}} : A_Q^G \to \mathbb{R}_+^\times$.

§4. Preliminaries on Eisenstein series

We recall the theory of Eisenstein series following [H], [MW]. Let R be a standard k-parabolic subgroup of G. We set

$$\operatorname{Re}\mathfrak{a}_R = X^*(Z_G\backslash Z_R)\otimes_{\mathbb{Z}}\mathbb{R}\,,\quad \mathfrak{a}_R = \operatorname{Re}\mathfrak{a}_R\otimes_{\mathbb{R}}\mathbb{C} = \operatorname{Re}\mathfrak{a}_R + \sqrt{-1}\operatorname{Re}\mathfrak{a}_R\,.$$

Every $\Lambda\in\mathfrak{a}_R$ of the form $\chi_1\otimes s_1+\cdots+\chi_r\otimes s_r$, $\chi_i\in X^*(Z_G\backslash Z_R)$, $s_i\in\mathbb{C}$ gives rise to a quasi-character of A_R^G by

$$z\longmapsto z^\Lambda = |\chi_1(z)|_{\mathbb{A}}^{s_1}\cdots|\chi_r(z)|_{\mathbb{A}}^{s_r}$$

for $z\in A_R^G$. By this way, $\mathfrak{a}_R$ is identified with the group of quasi-characters of A_R^G. There is a unique $\rho_R\in\operatorname{Re}\mathfrak{a}_R$ such that $z^{2\rho_R}=\delta_R(z)$. If R' is a standard k-parabolic subgroup of G such that $R'\subset R$, then $Z_G\backslash Z_R$ (resp. A_R^G) is a subgroup of $Z_G\backslash Z_{R'}$ (resp. $A_{R'}^G$) and hence there is a natural surjection from $\mathfrak{a}_{R'}$ onto $\mathfrak{a}_R$. The kernel of this surjection is denoted by $\mathfrak{a}_{R'}^R$. Since the quasi-characters of $M_R(\mathbb{A})^1\backslash M_R(\mathbb{A})$ is restricted to $M_{R'}(\mathbb{A})^1\backslash M_{R'}(\mathbb{A})$ ([MW, I.1.4.(2)]), there is a splitting $\mathfrak{a}_R\to\mathfrak{a}_{R'}$, and hence a direct product decomposition: $\mathfrak{a}_{R'}=\mathfrak{a}_R\oplus\mathfrak{a}_{R'}^R$. The subspace $\mathfrak{a}_{R'}^R$ is identified with the group of quasi-characters of $A_{R'}^R=A_{R'}/A_R$ by the similar way as above. If $(\delta_{R'}^R)^{-1}$ denotes the modular character of $(M_R\cap R')(\mathbb{A})$, there is a unique $\rho_{R'}^R\in\operatorname{Re}\mathfrak{a}_{R'}^R$ such that $z^{2\rho_{R'}^R}=\delta_{R'}^R(z)$ for $z\in A_{R'}^R$. One has $\rho_{R'}=\rho_R+\rho_{R'}^R$. We always consider $\mathfrak{a}_R$ as a subspace of $\mathfrak{a}_P$ and fix an admissible inner product $(\,\cdot\,,\,\cdot\,)$ on $\operatorname{Re}\mathfrak{a}_P$. Then $\operatorname{Re}\mathfrak{a}_{R'}=\operatorname{Re}\mathfrak{a}_R\oplus\operatorname{Re}\mathfrak{a}_{R'}^R$ is an orthogonal decomposition. For each root $\beta\in\Phi_k$, $\beta^\vee$ denotes the coroot $2(\beta,\beta)^{-1}\beta$. Let Δ_R denote the set consisting of nonzero roots $\beta|_{Z_R}$, $\beta\in\Delta_k$. It is obvious that Δ_R is contained in $\operatorname{Re}\mathfrak{a}_R$ and spans $\mathfrak{a}_R$ as a $\mathbb{C}$-vector space. We set

$$\mathfrak{c}_R=\{\Lambda\in\mathfrak{a}_R : (\operatorname{Re}\Lambda-\rho_R,\beta^\vee|_{Z_R})>0 \text{ for all } \beta|_{Z_R}\in\Delta_R\}$$

and

$$\mathfrak{c}_{R'}^R=\big\{\Lambda\in\mathfrak{a}_{R'}^R : (\operatorname{Re}\Lambda-\rho_{R'}^R,\beta^\vee|_{Z_{R'}})>0 \text{ for all } \beta|_{Z_{R'}}\in\Delta_{R'} \text{ with } \beta|_{Z_R}=0\big\}\,.$$

A map $z_R : G(\mathbb{A})\to A_R^G=A_GM_R(\mathbb{A})^1\backslash M_R(\mathbb{A})$ is defined by $z_R(g)=A_GM_R(\mathbb{A})^1m$ if $g=umh$, $u\in U_R(\mathbb{A})$, $m\in M_R(\mathbb{A})$ and $h\in K$.

For a smooth function $\eta\in C_0^\infty(A_R^G)$, its Mellin transform is defined to be

$$\widehat{\eta}(\Lambda)=\int_{A_R^G}\eta(z)z^{-(\Lambda+\rho_R)}\,d\nu_{A_R^G}(z)\,.$$

We choose the measure $d\Lambda$ on $\mathfrak{a}_R$ so that the following inversion formula holds for any $\eta \in C_0^\infty(A_R^G)$:

$$\eta(z) = \int_{\Lambda \in \Lambda_0 + \sqrt{-1}\,\mathrm{Re}\,\mathfrak{a}_R} \widehat{\eta}(\Lambda) z^{\Lambda+\rho_R}\, d\Lambda\,,$$

where $\Lambda_0 \in \mathrm{Re}\,\mathfrak{a}_R$ is a base point.

Let $\mathcal{A}_{0,R} = \mathcal{A}_0(A_R^G U_R(\mathbb{A}) M_R(k)\backslash G(\mathbb{A})^1)$ be the space of cuspidal automorphic forms on $A_R^G U_R(\mathbb{A}) M_R(k)\backslash G(\mathbb{A})^1$. For an open subgroup $D \subset K$, $\mathcal{A}_{0,R}^D$ denotes the set of right D-invariant cusp forms in $\mathcal{A}_{0,R}$. For $\varphi \in \mathcal{A}_{0,R}$, $\eta \in C_0^\infty(A_R^G)$ and $\Lambda \in \mathfrak{c}_R$, the pseudo-Eisenstein series $\theta_{\varphi,\eta}$ and the Eisenstein series $E(\varphi,\Lambda)$ on $G(k)\backslash G(\mathbb{A})^1$ are defined as follows:

$$\theta_{\varphi,\eta}(g) = \sum_{\gamma \in R(k)\backslash G(k)} \varphi(\gamma g)\eta(z_R(\gamma g))\,,$$
$$E(\varphi,\Lambda)(g) = \sum_{\gamma \in R(k)\backslash G(k)} z_R(\gamma g)^{\Lambda+\rho_R}\varphi(\gamma g)\,.$$

It is known that both series are absolutely convergent, $\theta_{\varphi,\eta}$ is a rapidly decreasing function on $G(k)\backslash G(\mathbb{A})^1$ and $E(\varphi,\Lambda)$ is meromorphically continued on the whole $\mathfrak{a}_R$. If $\Lambda_0 \in \mathrm{Re}\,\mathfrak{a}_R \cap \mathfrak{c}_R$ is fixed, then $\theta_{\varphi,\eta}$ is expressed as

$$\theta_{\varphi,\eta}(g) = \int_{\Lambda \in \Lambda_0 + \sqrt{-1}\,\mathrm{Re}\,\mathfrak{a}_R} \widehat{\eta}(\Lambda) E(\varphi,\Lambda)(g)\, d\Lambda\,.$$

We need intertwining operators to describe constant terms of pseudo-Eisenstein series. Let W_G be the relative Weyl groups of (G,S). We take a pair of a standard k-parabolic subgroup R' and an element $w \in W_G$ such that $wM_Rw^{-1} = M_{R'}$. Then, for $\Lambda \in \mathfrak{c}_R$ and $\varphi \in \mathcal{A}_{0,R}$, we consider

$$(M(w,\Lambda)\varphi)(g) = z_{R'}(g)^{-(w\Lambda+\rho_{R'})}$$
$$\times \int_{(U_{R'}(\mathbb{A})\cap wU_R(\mathbb{A})w^{-1})\backslash U_{R'}(\mathbb{A})} \varphi(w^{-1}ug) z_R(w^{-1}ug)^{\Lambda+\rho_R}\, d\omega_{\mathbb{A}}^{U_{R'}}(u)\,.$$

The integral of the right-hand side converges absolutely and $M(w,\Lambda)\varphi$ is contained in $\mathcal{A}_{0,R'}$. Moreover, the operator $M(w,\Lambda)$ is meromorphically continued to the whole $\mathfrak{a}_R$. The adjoint operator $M(w,\Lambda)^*$ of $M(w,\Lambda)$ with respect to the L^2-inner product on $\mathcal{A}_{0,R}$ equals $M(w^{-1}, -w\overline{\Lambda})$.

§5. Proof of Theorem 1

Let π, Q, D and F_T be the same as in Section 3. On account of Proposition 1, we must prove

$$\lim_{T\to\infty}\langle\psi,F_T\rangle=\frac{\tau(Q)}{\tau(G)}\langle\psi,1\rangle$$

for every $\psi\in C_0(G(k)\backslash G(\mathbb{A}))$. By [DRS, Lemma 2.4], it is enough to prove

$$\lim_{T\to\infty}\langle\theta_{\varphi,\eta},F_T\rangle=\frac{\tau(Q)}{\tau(G)}\langle\theta_{\varphi,\eta},1\rangle$$

for all pseudo-Eisenstein series $\theta_{\varphi,\eta}$.

Proposition 2. *Let R be a standard k-parabolic subgroup of G, $\varphi\in\mathcal{A}_{0,R}$ and $\eta\in C_0^\infty(A_R^G)$. If $R\neq P$, i.e., R is not a minimal k-parabolic subgroup, then*

$$\langle\theta_{\varphi,\eta},F_T\rangle=\langle\theta_{\varphi,\eta},1\rangle=0\,.$$

Proof. First, by (1.3) and $\omega_{G(\mathbb{A})^1}=(d_G\mu_{A_G})\backslash\omega_{\mathbb{A}}^G$, one has

(5.1)

$$\begin{aligned}\langle\theta_{\varphi,\eta},1\rangle&=\int_{R(k)\backslash G(\mathbb{A})^1}\varphi(g)\eta(z_R(g))\,d(\omega_{R(k)}\backslash\omega_{G(\mathbb{A})^1})(g)\\&=\frac{C_G}{C_Rd_G}\int_{U_R(k)\backslash U_R(\mathbb{A})\times A_GM_R(k)\backslash M_R(\mathbb{A})\times K}\varphi(mh)\eta(z_R(m))\\&\qquad\times\delta_R(m)^{-1}\,d\omega_{U_R}(u)d(\mu_{A_G}\omega_{G(k)}\backslash\omega_{\mathbb{A}}^{M_R})(m)d\nu_K(h)\\&=\frac{C_Gd_R}{C_Rd_G}\int_{M_R(k)\backslash M_R(\mathbb{A})^1\times K}\varphi(mh)\left\{\int_{A_R^G}\eta(z)z^{-2\rho_R}\,d\nu_{A_R^G}(z)\right\}\\&\qquad\times d\omega_{M_R}(m)d\nu_K(h)\\&=\frac{C_Gd_R}{C_Rd_G}\widehat{\eta}(\rho_R)\langle\varphi,1\rangle_R\,,\end{aligned}$$

where we set

$$\langle\varphi,1\rangle_R=\int_{M_R(k)\backslash M_R(\mathbb{A})^1\times K}\varphi(mh)\,d\omega_{M_R}(m)d\nu_K(h)\,.$$

From the cuspidality of φ, it follows $\langle\varphi,1\rangle_R=0$, and hence $\langle\theta_{\varphi,\eta},1\rangle=0$.

Next we compute $\Pi_Q(\theta_{\varphi,\eta})$. Since Q is maximal, there is an only one simple root $\alpha \in \Delta_k$ such that $\alpha|_{Z_Q} \neq 0$. We define a subset $W(M_R, M_Q)$ of the Weyl group W_G by

$$W(M_R, M_Q) = \{w \in W_G : w^{-1}(\beta) > 0 \text{ for all } \beta \in \Delta_k - \{\alpha\} \\ \text{and } wRw^{-1} \subset Q\}\,.$$

Then the constant term of the Eisenstein series $E(\varphi, \Lambda)$ along U_Q is given by the formula

$$\Pi^1_Q(E(\varphi,\Lambda))(g) \\ = \sum_{w\in W(M_R,M_Q)} \sum_{\gamma\in M_Q(k)\cap R^w(k)\backslash M_Q(k)} (M(w,\Lambda)\varphi)(\gamma g) z_{R^w}(\gamma g)^{w\Lambda+\rho_{R^w}}\,,$$

where R^w denotes wRw^{-1} ([MW, Proposition II.1.7]). If $W(M_R, M_Q)$ is empty, this constant term is zero. Thus $\Pi^1_Q(\theta_{\varphi,\eta})(g)$ equals

(5.2)

$$\sum_{w\in W(M_R,M_Q)} \int_{\Lambda\in\Lambda_0+\sqrt{-1}\,\mathrm{Re}\,\mathfrak{a}_R} \widehat{\eta}(\Lambda) \\ \times \sum_{\gamma\in M_Q(k)\cap R^w(k)\backslash M_Q(k)} (M(w,\Lambda)\varphi)(\gamma g) z_{R^w}(\gamma g)^{w\Lambda+\rho_{R^w}}\, d\Lambda \\ = \sum_{w\in W(M_R,M_Q)} \int_{\Lambda\in w\Lambda_0+\sqrt{-1}\,\mathrm{Re}\,\mathfrak{a}_{R^w}} \widehat{\eta}(w^{-1}\Lambda) \\ \times \sum_{\gamma\in M_Q(k)\cap R^w(k)\backslash M_Q(k)} (M(w,w^{-1}\Lambda)\varphi)(\gamma g) z_{R^w}(\gamma g)^{\Lambda+\rho_{R^w}}\, d\Lambda\,.$$

We take $m \in A_G\backslash M_Q(\mathbb{A})$ and $m_1 \in M_Q(\mathbb{A})^1$ so that $m = m_1 z_Q(m)$. Then one has $z_{R^w}(\gamma m) = z_Q(m) z_{R^w}(\gamma m_1)$ and $z_{R^w}(\gamma m)^\Lambda = z_Q(m)^{\Lambda_1} z_{R^w}(\gamma m_1)^{\Lambda_2}$ for $\Lambda = \Lambda_1 + \Lambda_2$, $\Lambda_1 \in \mathfrak{a}_Q$ and $\Lambda_2 \in \mathfrak{a}^Q_{R^w}$ because of $\gamma m_1 \in M_Q(\mathbb{A})^1$. We choose a base point $\Lambda_{1,0} \in \mathrm{Re}\,\mathfrak{a}_Q$ and $\Lambda_{w,0} \in \mathrm{Re}\,\mathfrak{a}^Q_{R^w}$ as follows: $(-\Lambda_{1,0}, \alpha^\vee|_{Z_Q})$ is sufficiently large, and $(\Lambda_{w,0} - \rho^Q_{R^w}, \beta^\vee|_{Z_{R^w}}) > 0$ for all $\beta|_{Z_{R^w}} \in \Delta_{R^w}$ with $\beta|_{Z_Q} = 0$. Then we can shift the integral domain of (5.2) from $w\Lambda_0 + \sqrt{-1}\,\mathrm{Re}\,\mathfrak{a}_{R^w}$ to $\Lambda_{1,0} + \Lambda_{w,0} + \sqrt{-1}\,\mathrm{Re}\,\mathfrak{a}_{R^w}$ ([MW, Lemma II.2.2]).

Summing up, (5.2) at $g = m$ is equal to

$$\sum_{w\in W(M_R,M_Q)} \int_{\Lambda_1\in\Lambda_{1,0}+\sqrt{-1}\,\mathrm{Re}\,\mathfrak{a}_Q} z_Q(m)^{\Lambda_1+\rho_Q} \times \sum_{\gamma\in M_Q(k)\cap R^w(k)\backslash M_Q(k)} \Psi_w(\Lambda_1,\gamma m_1)\,d\Lambda_1\,,$$

where

$$\Psi_w(\Lambda_1,m_1) = \int_{\Lambda_2\in\Lambda_{w,0}+\sqrt{-1}\,\mathrm{Re}\,\mathfrak{a}_{R^w}^Q} \widehat{\eta}(w^{-1}(\Lambda_1+\Lambda_2)) \times (M(w,w^{-1}(\Lambda_1+\Lambda_2))\varphi)(m_1) z_{R^w}(m_1)^{\Lambda_2+\rho_{R^w}^Q}\,d\Lambda_2\,.$$

Therefore, for $z\in A_Q^G$,

$$\begin{aligned}
&\Pi_Q(\theta_{\varphi,\eta})(z)\\
&= \int_{M_Q(k)\backslash M_Q(\mathbb{A})^1} \Pi_Q^1(\theta_{\varphi,\eta})(m_1 z)\,d\omega_{M_Q}(m_1)\\
&= \sum_{w\in W(M_R,M_Q)} \int_{\Lambda_1\in\Lambda_{1,0}+\sqrt{-1}\,\mathrm{Re}\,\mathfrak{a}_Q} z^{\Lambda_1+\rho_Q}\\
&\quad\times\left\{\int_{M_Q(k)\backslash M_Q(\mathbb{A})^1} \sum_{\gamma\in M_Q(k)\cap R^w(k)\backslash M_Q(k)} \Psi_w(\Lambda_1,\gamma m_1) d\omega_{M_Q}(m_1)\right\} d\Lambda_1\,.
\end{aligned}$$

By the calculation similar to (5.1), the inner integral equals

$$\begin{aligned}
&\frac{C_Q d_{R^w}}{C_{R^w} d_Q} \int_{A_{R^w}^Q} \left\{\int_{M_{R^w}(k)\backslash M_{R^w}(\mathbb{A})^1\times K^{M_Q}} \Psi_w(\Lambda_1, z_2 m_2 h)\right.\\
&\qquad\left.\times\, d\omega_{M_{R^w}}(m_2) d\nu_{K^{M_Q}}(h)\right\} (\delta_{R^w}^Q)^{-1}(z_2)\,d(\mu_{A_Q}\backslash\mu_{A_{R^w}})(z_2)\\
&= \frac{C_Q d_{R^w}}{C_{R^w} d_Q} \int_{A_{R^w}^Q} \int_{\Lambda_2\in\Lambda_{w,0}+\sqrt{-1}\,\mathrm{Re}\,\mathfrak{a}_{R^w}^Q} \widehat{\eta}(w^{-1}(\Lambda_1+\Lambda_2))\\
&\quad\times\left\{\int_{M_{R^w}(k)\backslash M_{R^w}(\mathbb{A})^1\times K^{M_Q}} (M(w,w^{-1}(\Lambda_1+\Lambda_2))\varphi)(m_2 h)\right.\\
&\qquad\left.\times\, d\omega_{M_{R^w}}(m_2) d\nu_{K^{M_Q}}(h)\right\} z_2^{\Lambda_2-\rho_{R^w}^Q}\, d\Lambda_2 d(\mu_{A_Q}\backslash\mu_{A_{R^w}})(z_2)
\end{aligned}$$

The cuspidality of $M(w,w^{-1}\Lambda)\varphi$ implies

$$\int_{M_{R^w}(k)\backslash M_{R^w}(\mathbb{A})^1\times K^{M_Q}} (M(w,w^{-1}\Lambda)\varphi)(m_2h)\,d\omega_{M_{R^w}}(m_2)d\nu_{K^{M_Q}}(h)=0\,.$$

Hence $\Pi_Q(\theta_{\varphi,\eta})|_{M_Q(\mathbb{A})}\equiv 0$. This implies $\langle\theta_{\varphi,\eta},F_T\rangle=0$ by (3.4). ∎

Next, we consider the case $R=P$. Since P is a minimal k-parabolic subgroup, the constant function $\varphi_0\equiv 1$ is contained in $\mathcal{A}_{0,P}$. We define the inner product on $\mathcal{A}_{0,P}^K=\mathcal{A}_0(M(k)\backslash M(\mathbb{A})^1)^{K^M}$ by

$$\langle\psi_1,\psi_2\rangle_M=\int_{M(k)\backslash M(\mathbb{A})^1}\psi_1(m)\overline{\psi_2(m)}\,d\omega_M(m)\quad(\psi_1,\psi_2\in\mathcal{A}_{0,P}^K).$$

Let W_{M_Q} be the relative Weyl group of (M_Q,S). As a subgroup of W_G, W_{M_Q} is identified with the point wise stabilizer of $\mathfrak{a}_Q$ in W_G. For $w\in W_G$ and a generic $\Lambda\in\mathfrak{a}_P$, the operator $M(w,\Lambda)$ maps $\mathcal{A}_{0,P}^{DK^Q}$ into itself. If $w\in W_{M_Q}$, then the equality $M(w,\Lambda_1+\Lambda_2)=M(w,\Lambda_2)$ holds for $\Lambda_1\in\mathfrak{a}_Q$, $\Lambda_2\in\mathfrak{a}_P^Q$, and $M(w,\Lambda_2)$ is regarded as an operator on $\mathcal{A}_0(A_P^QU(\mathbb{A})M(k)\backslash Q(\mathbb{A})^1)$. We denote by w_0 (resp. w_1) the longest element of W_G (resp. W_{M_Q}). It is known from the theory of local intertwining operators and the Langlands classification theorem that the residue

$$M(w_0)=\lim_{\substack{\Lambda\in\mathfrak{c}_P\\ \Lambda\to\rho_P}}\left(\prod_{\beta\in\Delta_k}(\Lambda-\rho_P,\beta^\vee)\right)M(w_0,\Lambda)$$

exists and yields a projection from $\mathcal{A}_{0,P}$ onto the trivial representation $\mathbb{C}\varphi_0$ of $G(\mathbb{A})^1$ ([FMT, Section 10 (b)]). By the argument of [L] or [Lai], one has

$$M(w_0)\varphi_0=\frac{C_Gd_P\tau(P)}{d_G\tau(G)}\varphi_0\,.$$

In a similar fashion, the residue

$$M(w_1)=\lim_{\substack{\Lambda_2\in\mathfrak{c}_P^Q\\ \Lambda_2\to\rho_P^Q}}\left(\prod_{\beta\in\Delta_k-\{\alpha\}}(\Lambda_2-\rho_P^Q,\beta^\vee)\right)M(w_1,\Lambda_2)$$

yields a projection from $\mathcal{A}_0(A_P^QU(\mathbb{A})M(k)\backslash Q(\mathbb{A})^1)$ onto $\mathbb{C}\varphi_0$ and one has

$$M(w_1)\varphi_0=\frac{C_Qd_P\tau(P)}{d_Q\tau(Q)}\varphi_0\,.$$

LEMMA 2. *For any $\varphi \in \mathcal{A}_{0,P}$,*

$$M(w_0)\varphi = \frac{C_G d_P}{d_G \tau(G)} \langle \varphi, 1\rangle_P \varphi_0 \,.$$

Proof. If $M(w_0)\varphi = c\varphi_0$, then

$$c = \frac{1}{\tau(P)} \langle M(w_0)\varphi, \varphi_0\rangle_P = \frac{1}{\tau(P)} \langle \varphi, M(w_0)^* \varphi_0\rangle_P = \frac{C_G d_P}{d_G \tau(G)} \langle \varphi, \varphi_0\rangle_P \,.$$

Here note that the constant $C_G d_P/(d_G\tau(G))$ is a positive real value. □

LEMMA 3. *Let $\tau \in W(M, M_Q)$, $\sigma = \tau^{-1} w_1 \in W_G$ and $\varphi \in \mathcal{A}_{0,P}^{DK^Q}$. If we fix a $\Lambda_1 \in \mathfrak{a}_Q$ with $(-\operatorname{Re}\Lambda_1, \alpha^\vee|_{Z_Q}) \gg 0$, then the function*

$$\Lambda_2 \longmapsto \langle (M(\tau, \tau^{-1}(\Lambda_1 + \Lambda_2))\varphi)|_{M(\mathbb{A})^1}, \varphi_0\rangle_M$$

is holomorphic at $\Lambda_2 = \rho_P^Q$. Moreover, one has

$$\begin{aligned}&\langle (M(\tau, \tau^{-1}(\Lambda_1 + \rho_P^Q))\varphi)|_{M(\mathbb{A})^1}, \varphi_0\rangle_M \\ &\quad = \frac{d_Q \tau(Q)}{C_Q d_P \tau(P)} \langle (M_1(\sigma^{-1}, \sigma(\Lambda_1 - \rho_P^Q))\varphi)|_{M(\mathbb{A})^1}, \varphi_0\rangle_M \,,\end{aligned}$$

where $M_1(\sigma^{-1}, \sigma(\Lambda_1 - \rho_P^Q))$ is defined by

$$\lim_{\substack{\Lambda_2 \in \mathfrak{c}_P^Q \\ \Lambda_2 \to \rho_P^Q}} \Bigg(\prod_{\beta \in \Delta_k - \{\alpha\}} (\Lambda_2 - \rho_P^Q, \beta^\vee) \Bigg) M(\sigma^{-1}, \sigma(\Lambda_1 - \Lambda_2)) \,.$$

Proof. By [MW, Lemma II.2.2], the function $M(\tau, \tau^{-1}(\Lambda_1 + \Lambda_2))\varphi$ in Λ_2 is holomorphic on the tube domain of the form $\{\Lambda_2 \in \mathfrak{a}_P^Q : (\operatorname{Re}\Lambda_2, \operatorname{Re}\Lambda_2) < c_0^2\}$, where c_0 is a positive real constant with $c_0^2 > (\rho_P, \rho_P)$. By the functional equations of $M(w, \Lambda)$,

$$\begin{aligned}&\langle (M(\tau, \tau^{-1}\Lambda)\varphi)|_{M(\mathbb{A})^1}, \varphi_0\rangle_M \\ &\quad = \langle (M(w_1, w_1^{-1}\Lambda) M(\sigma^{-1}, \sigma w_1^{-1}\Lambda)\varphi)|_{M(\mathbb{A})^1}, \varphi_0\rangle_M \\ &\quad = \langle (M(\sigma^{-1}, \sigma w_1^{-1}\Lambda)\varphi)|_{M(\mathbb{A})^1}, M(w_1, w_1^{-1}\Lambda)^* \varphi_0\rangle_M \\ &\quad = \langle (M(\sigma^{-1}, \sigma w_1^{-1}\Lambda)\varphi)|_{M(\mathbb{A})^1}, M(w_1^{-1}, -\overline{\Lambda})\varphi_0\rangle_M \,.\end{aligned}$$

Here we identify $\mathcal{A}_{0,P}^{K}$ with $\mathcal{A}_0(A_P^Q U(\mathbb{A})M(k)\backslash Q(\mathbb{A})^1)^{K^{M_Q}}$ and regard $M(w_1, w_1^{-1}\Lambda)$ as an operator on it. Therefore,

$$\langle (M(\tau, \tau^{-1}(\Lambda_1 + \rho_P^Q))\varphi)|_{M(\mathbb{A})^1}, \varphi_0\rangle_M$$

equals

$$\Big\langle (M_1(\sigma^{-1}, \sigma(\Lambda_1 - \rho_P^Q))\varphi)|_{M(\mathbb{A})^1},\\ \lim_{\substack{\Lambda_2 \in \mathfrak{c}_P^Q \\ \Lambda_2 \to \rho_P^Q}} \overline{\Big(\prod_{\beta \in \Delta_k - \{\alpha\}} (\Lambda_2 - \rho_P^Q, \beta^\vee)\Big)}^{-1} M(w_1^{-1}, -\overline{\Lambda}_2)\varphi_0 \Big\rangle_M.$$

If we regard $\overline{M(w_1^{-1}, -\overline{\Lambda}_2)}$ acting on $\mathbb{C}\varphi_0$ as a scalar valued function, then

$$\begin{aligned} &\lim_{\substack{\Lambda_2 \in \mathfrak{c}_P^Q \\ \Lambda_2 \to \rho_P^Q}} \Big(\prod_{\beta \in \Delta_k - \{\alpha\}} (\Lambda_2 - \rho_P^Q, \beta^\vee)\Big)^{-1} \overline{M(w_1^{-1}, -\overline{\Lambda}_2)} \\ &\quad = \lim_{\substack{\Lambda_2 \in \mathfrak{c}_P^Q \\ \Lambda_2 \to \rho_P^Q}} \Big(\prod_{\beta \in \Delta_k - \{\alpha\}} (\Lambda_2 - \rho_P^Q, \beta^\vee)\Big)^{-1} \overline{M(w_1, -w_1^{-1}\overline{\Lambda}_2)}^{-1} \\ &\quad = \overline{M(w_1)}^{-1}. \end{aligned}$$

This implies the assertion. ∎

Lemma 4. *Being the notation as above, one has*

$$\lim_{\substack{\Lambda_1 \in -\mathfrak{c}_Q \\ \Lambda_1 \to -\rho_Q}} (\Lambda_1 + \rho_Q, \alpha^\vee) M_1(\sigma^{-1}, \sigma(\Lambda_1 - \rho_P^Q))\varphi = \begin{cases} M(w_0)\varphi & (\sigma = w_0) \\ 0 & (\sigma \neq w_0) \end{cases}$$

If $0 < \varepsilon$ *is sufficiently small, then the function*

$$\Lambda_1 \longmapsto \langle (M_1(\sigma^{-1}, \sigma(\Lambda_1 - \rho_P^Q))\varphi)|_{M(\mathbb{A})^1}, \varphi_0\rangle_M$$

is holomorphic on $\{\Lambda_1 \in \mathfrak{a}_Q : 1 - \epsilon < (\operatorname{Re}\Lambda_1, \rho_Q)/(\rho_Q, \rho_Q) < 1\}$ *with polynomial growth as* $|\Im\Lambda_1| \to \infty$.

Proof. For any $\psi \in \mathcal{A}_{0,P}^{DK^Q}$,

$$\begin{aligned}
&\Big\langle \lim_{\substack{\Lambda_1 \in -\mathfrak{c}_Q \\ \Lambda_1 \to -\rho_Q}} (\Lambda_1 + \rho_Q, \alpha^\vee) M_1(\sigma^{-1}, \sigma(\Lambda_1 - \rho_P^Q))\varphi, \psi \Big\rangle_P \\
&\quad = \Big\langle \varphi, \lim_{\substack{\Lambda_1 \in -\mathfrak{c}_Q \\ \Lambda_1 \to -\rho_Q}} \overline{(\Lambda_1 + \rho_Q, \alpha^\vee)} M_1(\sigma^{-1}, \sigma(\Lambda_1 - \rho_P^Q))^* \psi \Big\rangle_P \\
&\quad = \Big\langle \varphi, \lim_{\substack{\Lambda_1 \in -\mathfrak{c}_Q \\ \Lambda_1 \to -\rho_Q}} \overline{(\Lambda_1 + \rho_Q, \alpha^\vee)} M_1(\sigma, -\overline{\Lambda}_1 + \rho_P^Q))\psi \Big\rangle_P \\
&\quad = \Big\langle \varphi, \lim_{\substack{\Lambda \in \mathfrak{c}_P \\ \Lambda \to \rho_P}} \overline{\Big(\prod_{\beta \in \Delta_k} (\Lambda - \rho_P, \beta^\vee) \Big)} M(\sigma, \overline{\Lambda})\psi \Big\rangle_P .
\end{aligned}$$

It is known that

$$\lim_{\substack{\Lambda \in \mathfrak{c}_P \\ \Lambda \to \rho_P}} \Big(\prod_{\beta \in \Delta_k} (\Lambda - \rho_P, \beta^\vee) \Big) M(\sigma, \Lambda) = \begin{cases} M(w_0) & (\sigma = w_0) \\ 0 & (\sigma \neq w_0) \end{cases}$$

(cf. [FMT, Lemma 7]). By this and Lemma 2, the equalities

$$\begin{aligned}
\langle M(w_0)\varphi, \psi \rangle_P &= \langle \varphi, M(w_0)\psi \rangle_P \\
&= \Big\langle \lim_{\substack{\Lambda_1 \in -\mathfrak{c}_Q \\ \Lambda_1 \to -\rho_Q}} (\Lambda_1 + \rho_Q, \alpha^\vee) M_1(\sigma^{-1}, \sigma(\Lambda_1 - \rho_P^Q))\varphi, \psi \Big\rangle_P
\end{aligned}$$

hold for all $\psi \in \mathcal{A}_{0,P}^{DK^Q}$. The remains of the assertion follows from [H, Lemma 118]. ∎

Proposition 3. *Let $\varphi \in \mathcal{A}_{0,P}$ and $\eta \in C_0^\infty(A_P^G)$. Then one has*

$$\lim_{T \to \infty} \langle \theta_{\varphi,\eta}, F_T \rangle = \frac{\tau(Q)}{\tau(P)} \langle \theta_{\varphi,\eta}, 1 \rangle .$$

Proof. It is sufficient to prove the assertion for right DK^Q-invariant $\varphi \in \mathcal{A}_{0,P}$. The calculations of $\langle \theta_{\varphi,\eta}, 1 \rangle$ and $\Pi_Q(\theta_{\varphi,\eta})$ are the same as in the proof of Proposition 2. We have

$$\langle \theta_{\varphi,\eta}, 1 \rangle = \frac{C_G d_P}{C_P d_G} \widehat{\eta}(\rho_P) \langle \varphi, 1 \rangle_P .$$

We need a further calculation of $\Pi_Q(\theta_{\varphi,\eta})$. Since φ is right DK^Q-invariant, $\Pi_Q(\theta_{\varphi,\eta})(z)$ equals

$$\frac{C_Q d_P}{C_P d_Q} \sum_{\tau\in W(M,M_Q)} \int_{\Lambda_1\in\Lambda_{1,0}+\sqrt{-1}\,\mathrm{Re}\,\mathfrak{a}_Q} z^{\Lambda_1+\rho_Q} \widehat{f}_\tau(\Lambda_1)\, d\Lambda_1\,, \tag{5.3}$$

where

$$\begin{aligned}\widehat{f}_\tau(\Lambda_1) = \int_{A_P^Q}\int_{\Lambda_2\in\Lambda_{\tau,0}+\sqrt{-1}\,\mathrm{Re}\,\mathfrak{a}_P^Q} &\widehat{\eta}(\tau^{-1}(\Lambda_1+\Lambda_2)) \\ &\times \langle (M(\tau,\tau^{-1}(\Lambda_1+\Lambda_2))\varphi)|_{M(\mathbb{A})^1}, \varphi_0\rangle_M z_2^{\Lambda_2-\rho_P^Q} \\ &\times d\Lambda_2 d(\mu_{A_Q}\backslash\mu_{A_P})(z_2)\,.\end{aligned}$$

If $\Lambda_1\in\Lambda_{1,0}+\sqrt{-1}\,\mathrm{Re}\,\mathfrak{a}_Q$ is fixed, the function

$$\Lambda_2 \longmapsto \widehat{\eta}(\tau^{-1}(\Lambda_1+\Lambda_2))\langle (M(\tau,\tau^{-1}(\Lambda_1+\Lambda_2))\varphi)|_{M(\mathbb{A})^1}, \varphi_0\rangle_M$$

is holomorphic on the tube domain $\{\Lambda_2\in\mathfrak{a}_P^Q : (\mathrm{Re}\,\Lambda_2, \mathrm{Re}\,\Lambda_2) < c_0^2\}$ as mentioned in the proof of Lemma 3. We can take $\Lambda_{\tau,0}$ in this domain. Then, from the inversion formula, it follows

$$\widehat{f}_\tau(\Lambda_1) = \widehat{\eta}(\tau^{-1}(\Lambda_1+\rho_P^Q))\langle (M(\tau,\tau^{-1}(\Lambda_1+\rho_P^Q))\varphi)_{M(\mathbb{A})^1}, \varphi_0\rangle_M\,.$$

We shift the integral domain in (5.3) from $\Lambda_{1,0}+\sqrt{-1}\,\mathrm{Re}\,\mathfrak{a}_Q$ to $(\epsilon-1)\rho_Q+\sqrt{-1}\,\mathrm{Re}\,\mathfrak{a}_Q$, where ϵ is a sufficiently small positive number so that all $\widehat{f}_\tau$ are holomorphic on the domain $B_\epsilon=\{\Lambda_1\in\mathfrak{a}_Q : 1-2\epsilon < (-\mathrm{Re}\,\Lambda_1,\rho_Q)/(\rho_Q,\rho_Q) < 1\}$. Taking account the residue at $-\rho_Q$, we obtain

$$\begin{aligned}&\int_{\Lambda_1\in\Lambda_{1,0}+\sqrt{-1}\,\mathrm{Re}\,\mathfrak{a}_Q} z^{\Lambda_1+\rho_Q}\widehat{f}_\tau(\Lambda_1)\,d\Lambda_1 \\ &\quad = \int_{\Lambda_1\in(\epsilon-1)\rho_Q+\sqrt{-1}\,\mathrm{Re}\,\mathfrak{a}_Q} z^{\Lambda_1+\rho_Q}\widehat{f}_\tau(\Lambda_1)\,d\Lambda_1 + \mathrm{Res}_{\Lambda_1=-\rho_Q}\,\widehat{f}_\tau(\Lambda_1)\,.\end{aligned}$$

We write $f_\tau(z)$ for the first term. By Lemmas 2, 3 and 4, $\Pi_Q(\theta_{\varphi,\eta})(z)$ equals

$$\begin{aligned}&\frac{C_Q d_P}{C_P d_Q}\sum_{\tau\in W(M,M_Q)} f_\tau(z) + \frac{C_Q d_P}{C_P d_Q}\cdot\frac{d_Q\tau(Q)}{C_Q d_P\tau(P)}\widehat{\eta}(\rho_P)\langle M(w_0)\varphi|_{M(\mathbb{A})^1},\phi_0\rangle_M \\ &\quad = \frac{C_Q d_P}{C_P d_Q}\sum_{\tau\in W(M,M_Q)} f_\tau(z) + \frac{C_G d_P\tau(Q)}{C_P d_G\tau(G)}\widehat{\eta}(\rho_P)\langle\varphi,1\rangle_P\,.\end{aligned}$$

Here note that $\langle\varphi_0,\varphi_0\rangle_M=\tau(M)=\tau(P)$. Since $\widehat{\eta}$ is a function of Paley – Wiener type and $\widehat{f}_\tau(\Lambda_1)/\widehat{\eta}(\tau^{-1}(\Lambda_1+\rho_P^Q))$ is of polynomial growth on B_ϵ as $|\Im\Lambda_1|\to\infty$ by Lemma 4, we have an estimate of the formula

$$|f_\tau(z)|\le z^{\epsilon\rho_Q}\int_{\sqrt{-1}\,\mathrm{Re}\,\mathfrak{a}_Q}|z^\Lambda||\widehat{f}_\tau((\epsilon-1)\rho_Q+\Lambda)|\,d\Lambda\le c_1z^{\epsilon\rho_Q}\,, \tag{5.4}$$

where c_1 is a constant depending on $\widehat{f}_\tau$. This implies

$$\begin{aligned}&\limsup_{T\to\infty}\frac{e_Q}{T^{e_Q[k:\mathbb{Q}]/e_\pi}}\int_0^{T^{[k:\mathbb{Q}]/e_\pi}}t^{e_Q}|f_\tau(\iota_Q(\overline{e},|\alpha_Q|_{\mathbb{A}}^{-1}(t)))|\frac{dt}{t}\\&\le\limsup_{T\to\infty}\frac{e_Q}{T^{e_Q[k:\mathbb{Q}]/e_\pi}}\int_0^{T^{[k:\mathbb{Q}]/e_\pi}}c_1t^{(1-\epsilon/2)e_Q}\frac{dt}{t}=0\,.\end{aligned}$$

As a consequence, we have

$$\lim_{T\to\infty}\langle\theta_{\varphi,\eta},F_T\rangle=\frac{C_Gd_P\tau(Q)}{C_Pd_G\tau(G)}\widehat{\eta}(\rho_P)\langle\varphi,1\rangle_P=\frac{\tau(Q)}{\tau(G)}\langle\theta_{\varphi,\eta},1\rangle\,.$$

This completes the proof of Proposition 3, and therefore we are led to Theorem 1. □

§6. Error terms

We give some estimates of error terms of (3.3).

LEMMA 5. *Let $a>0$ be a constant. If*

$$\lim_{T\to\infty}\Big\langle\psi,\frac{F_T-\tau(Q)/\tau(G)}{T^a}\Big\rangle=0$$

holds for any $\psi\in C_0(G(k)\backslash G(\mathbb{A})^1)$, then one has

$$\lim_{T\to\infty}\frac{F_T(g)-\tau(Q)/\tau(G)}{T^a}=0 \tag{6.1}$$

for every $g\in G(\mathbb{A})^1$.

Proof. Using the same notations as in the proof of Proposition 1, we have

$$\begin{aligned}&\beta_m^{-a-e_Q[k:\mathbb{Q}]/e_\pi}\frac{\langle\psi_m,F_{\beta_m^{-1}T}-\tau(Q)/\tau(G)\rangle}{(\beta_m^{-1}T)^a}+\frac{(\beta_m^{-e_Q[k:\mathbb{Q}]/e_\pi}-1)\tau(Q)/\tau(G)}{T^a}\\&\le\frac{F_T(g_0)-\tau(Q)/\tau(G)}{T^a}\\&\le\beta_m^{a+e_Q[k:\mathbb{Q}]/e_\pi}\frac{\langle\psi_m,F_{\beta_mT}-\tau(Q)/\tau(G)\rangle}{(\beta_mT)^a}+\frac{(\beta_m^{e_Q[k:\mathbb{Q}]/e_\pi}-1)\tau(Q)/\tau(G)}{T^a}\end{aligned}$$

The assertion follows immediately from this. ∎

By [DRS, Lemma 2.4] and Proposition 2, if

$$\lim_{T\to\infty}\left\langle \theta_{\varphi,\eta}, \frac{F_T - \tau(Q)/\tau(G)}{T^a}\right\rangle = 0$$

holds for all $\theta_{\varphi,\eta}$, $\varphi \in \mathcal{A}_{0,P}^{DK^Q}$, $\eta \in C_0^\infty(A_P^G)$, then we get (6.1). Let ϵ_0 be the superior of $\epsilon \in (0, 1/2)$ such that all $M(\tau, \tau^{-1}(\Lambda_1 + \delta_P^Q))$, $\tau \in W(M, M_Q)$ are holomorphic on B_ϵ, where B_ϵ is the same as in the proof of Proposition 3. Then, for any $0 < a < \epsilon_0$, we can shift the integral domain of (5.3) from $\Lambda_{1,0} + \sqrt{-1}\,\mathrm{Re}\,\mathfrak{a}_Q$ to $(2a-1)\rho_Q + \sqrt{-1}\,\mathrm{Re}\,\mathfrak{a}_Q$ and the estimate similar to (5.4) leads to

$$\lim_{T\to\infty}\frac{\langle F_T, f_\tau\rangle}{T^{(1-a)e_Q[k:\mathbb{Q}]/e_\pi}} = 0\,.$$

Thus we proved the following.

Proposition 4. *For any* $0 < a < \epsilon_0$, *one has*

$$|E_\pi(D,T) \cap X_Q(k)g| = \frac{\tau(Q)}{\tau(G)}\omega_{Y_Q}(E_\pi(D,T)) + o(T^{(1-a)e_Q[k:\mathbb{Q}]/e_\pi})\,.$$

We note that, in some cases, the holomorphic domain of $M(\tau, \tau^{-1}(\Lambda_1 + \rho_O^Q))$ is extendable to the right side of the imaginary axis $\sqrt{-1}\,\mathrm{Re}\,\mathfrak{a}_Q$, however we do not know in general the asymptotic behavior of f_τ as $|\Im\Lambda_1| \to \infty$ in this region.

§7. Examples

Example 1. Let V be an n-dimensional vector space defined over k, G a group of linear automorphisms of V and $\pi : G \to G$ the natural representation. We fix a free $\mathfrak{O}$-lattice L in $V(k)$ and its $\mathfrak{O}$-basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$. Then $V(k)$ and G are identified with the column vector space k^n and the general linear group GL_n, respectively. Let P be the subgroup of upper triangular matrices and Q the stabilizer in G of the line spanned by $\mathbf{e}_1$. Then the map $g \mapsto \mathbf{e}_1 \cdot g = g^{-1}\mathbf{e}_1$ yields an isomorphism from $X_Q = Q\backslash G$ to the projective space $\mathbb{P}V = \mathbb{P}^{n-1}$. Let H_π be a height on $X_Q(k)$ defined as in Section 2. We take a maximal compact subgroup $K = \prod_{v\in\mathfrak{V}} K_v$ as follows:

$$K_v = \begin{cases} GL_n(\mathfrak{O}_v) & (v \in \mathfrak{V}_f) \\ O(n) & (v \text{ is a real place}) \\ U(n) & (v \text{ is an imaginary place}) \end{cases}$$

For each $v \in \mathfrak{V}_f$, $\mathfrak{p}_v$ and $\mathfrak{f}_v$ stand for the maximal ideal of $\mathfrak{O}_v$ and the residual field $\mathfrak{O}_v/\mathfrak{p}_v$, respectively. If we set

$$D_v = \left\{ g \in K_v : g \equiv \begin{pmatrix} * & * \\ 0 & \\ \vdots & * \\ 0 & \end{pmatrix} \mod \mathfrak{p}_v \right\},$$

then $D_v \backslash K_v$ is isomorphic to $\mathbb{P}^{n-1}(\mathfrak{f}_v)$ by the reduction homomorphism. For every $x \in \mathbb{P}^{n-1}(k_v)$, there is an $h_x \in K_v$ such that $x = k_v(\mathbf{e}_1 \cdot h_x)$. We denote by $[x]_v$ the reduction of x modulo $\mathfrak{p}_v$, i.e., $[x]_v = \mathfrak{f}_v(\mathbf{e}_1 \cdot h_x \mod \mathfrak{p}_v)$. Let $\mathfrak{S}$ be a finite subset of $\mathfrak{V}_f$. We fix a point $(a_v)_{v\in\mathfrak{S}}$ in $\prod_{v\in\mathfrak{S}} \mathbb{P}^{n-1}(k_v)$ and set

$$N(\mathbb{P}^{n-1}(k), T, (a_v)_{v\in\mathfrak{S}}) = |\{x \in \mathbb{P}^{n-1}(k) : H_\pi(x) \le T \text{ and } [x]_v = [a_v]_v \text{ for all } v \in \mathfrak{S}\}|\,.$$

It is obvious that

$$N(\mathbb{P}^{n-1}(k), T, (a_v)_{v\in\mathfrak{S}}) = |E_\pi(D,T)\cdot h \cap X(k)|\,,$$

where $D = K_\infty \times \prod_{v\in\mathfrak{S}} D_v \times \prod_{v\in\mathfrak{V}_f-\mathfrak{S}} K_v$ and $h = (h_{a_v})_{v\in\mathfrak{S}} \times (e)_{v\in\mathfrak{V}-\mathfrak{S}} \in K$. By Theorem 1 and the calculation of [W, Example 2], we have

$$N(\mathbb{P}^{n-1}(k), T, (a_v)_{v\in\mathfrak{S}}) \sim \prod_{v\in\mathfrak{S}} \frac{|\mathfrak{f}_v| - 1}{|\mathfrak{f}_v|^n - 1} \cdot \frac{\mathrm{Res}_{s=1}\,\zeta_k(s)}{|D_k|^{(n-1)/2} n Z_k(n)} \cdot T^{n[k:\mathbb{Q}]} \quad \text{as } T \to \infty\,.$$

Here $\zeta_k(s)$ is the Dedekind zeta function of k,

$$Z_k(s) = (\pi^{-s/2}\Gamma(s/2))^{r_1}((2\pi)^{1-s}\Gamma(s))^{r_2}\zeta_k(s)$$

and r_1 (resp. r_2) denotes a number of real (resp. imaginary) places of k. If $k = \mathbb{Q}$, this formula was proved in [S].

Example 2. Let V, L and $\mathbf{e}_1, \dots, \mathbf{e}_n$ be the same as in Example 1. Let Φ be a non-degenerate isotropic quadratic form on $V(k)$, $G = SO_\Phi$ the special orthogonal group of Φ and $\pi : G \to GL(V)$ the natural representation. The height H_π is the same as Example 1. We assume $n \ge 4$ and Φ has the following matrix form with respect to the basis $\mathbf{e}_1, \dots, \mathbf{e}_n$:

$$\Phi = \begin{pmatrix} & & 1 \\ & \Phi_0 & \\ 1 & & \end{pmatrix},$$

where Φ_0 is a non-degenerate $(n-2)\times(n-2)$ symmetric matrix. Thus $\mathbf{e}_1$ is an isotropic vector of Φ. Let Q be the stabilizer in G of the isotropic line spanned by $\mathbf{e}_1$. The map $g\mapsto \mathbf{e}_1\cdot g = g^{-1}\mathbf{e}_1$ gives rise to a k-rational embedding from $X_\Phi = Q\backslash G$ into $\mathbb{P}^{n-1}$. The image of $X_\Phi(k)$ is the set of all Φ-isotropic lines $x\in\mathbb{P}^{n-1}(k)$. We put

$$N(X_\Phi(k),T) = \left|\{x\in X_\Phi(k) : H_\pi(x)\le T\}\right| .$$

Since the Levi-subgroup M_Q is isomorphic to $GL_1\times SO_{\Phi_0}$, we have $\tau(G) = \tau(Q) = 2$ and $d_G = d_Q = 1$, and furthermore, $e_Q = \dim U_Q = n-2$ and $e_\pi = 1$. Therefore, Theorem 1 implies

$$N(X_\Phi(k),T) \sim \frac{C_G}{(n-2)C_Q} T^{(n-2)[k:\mathbb{Q}]} \quad \text{as } T\to\infty .$$

Here we supposed that H_π is invariant by a good maximal compact subgroup K of $G(\mathbb{A})$. The formula due to Ikeda [I, Theorems 9.6 and 9.7] deduces an explicit value of C_G/C_Q for some choice of K. In the following, we state this formula. Let $\mathfrak{V}'_\infty$ be the set of all real places of k. For every $v\in\mathfrak{V}$, $\mathbb{H}(k_v)$ denotes the hyperbolic plane k_v^2 endowed with the quadratic form $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$. Then $V(k_v)$ is decomposed into the following form on k_v:

$$V(k_v) = \mathbb{H}(k_v)^{m_v}\oplus V_v^0 ,$$

where V_v^0 is a Φ-anisotropic subspace. We put $\ell_v = \dim V_v^0$. In other words, $(n-\ell_v)/2$ is the Witt index of Φ on $V(k_v)$. If $v\in\mathfrak{V}_f$, then ℓ_v is at most 4. If $v\in\mathfrak{V}_f$ and $\ell_v = 3$, then V_v^0 is identified with the space of pure quaternions of the division quaternion algebra $\mathbb{D}_v$ over k_v.

First, let n be odd. We may assume without loss of generality that $\det\Phi_0\equiv 2(-1)^{(n-3)/2}$ module $(k^\times)^2$ ([I, p. 207]). For every $v\in\mathfrak{V}_f$ with $\ell_v = 3$, we take a maximal compact subgroup K_v as the stabilizer in $G(k_v)$ of the lattice $\mathbb{H}(\mathfrak{O}_v)^{(n-3)/2}\oplus(\mathfrak{O}_{\mathbb{D}_v}\cap V_v^0)$. Here $\mathfrak{O}_{\mathbb{D}_v}$ denotes the maximal order of $\mathbb{D}_v$. In other places v, we take K_v as in [I, pp. 209–210]. Then

$$\frac{C_G}{C_Q} = \frac{\operatorname{Res}_{s=1}\zeta_k(s)}{|D_k|^{(n-2)/2}Z_k(n-1)} \prod_{\substack{v\in\mathfrak{V}_f\\ \ell_v=3}} \frac{1-|\mathfrak{f}_v|^{-n+3}}{|\mathfrak{f}_v|(1-|\mathfrak{f}_v|^{-n+1})}$$

$$\times \prod_{v\in\mathfrak{V}'_\infty}\prod_{i=1}^{[(\ell_v-1)/4]} \frac{n-\ell_v+4i-2}{n+\ell_v-4i-2} .$$

Next, let n be even. We take a maximal compact subgroup K_v as in [I, pp. 209–210] for every $v \in \mathfrak{V}$. Let $k' = k(\sqrt{(-1)^{n/2}\det\Phi})$ be an extension of degree at most 2 over k and let $\mathfrak{V}'_f$ (resp. $\mathfrak{V}''_f$) be the set of $v \in \mathfrak{V}_f$ such that $\ell_v = 2$ (resp. $\ell_v = 4$), v is unramified (resp. split) over k'/k and $\Phi|_{V_v^0}$ is equivalent to the form $2\varpi_v \cdot \mathrm{Norm}_{k'_v/k_v}$, where ϖ_v is a prime element of k_v and $\mathrm{Norm}_{k'_v/k_v}$ the norm form of the unramified quadratic extension k'_v/k_v. Then

$$\frac{C_G}{C_Q} = \frac{1}{|\mathfrak{f}_{\chi_\Phi}|^{1/2}|D_k|^{(n-2)/2}} \frac{\mathrm{Res}_{s=1}\zeta_k(s)}{Z_k(n-2)} \frac{L(-1+n/2,\chi_\Phi)}{L(n/2,\chi_\Phi)} \times \prod_{v\in\mathfrak{V}'_f} |\mathfrak{f}_v|^{1-n/2} \prod_{v\in\mathfrak{V}''_f} \frac{1-|\mathfrak{f}_v|^{2-n/2}}{|\mathfrak{f}_v|(1-|\mathfrak{f}_v|^{-n/2})} \times \prod_{\substack{v\in\mathfrak{V}'_\infty \\ \ell_v\equiv 0\ (4)}} \prod_{i=1}^{\ell_v/4} \frac{n-4i}{n+4i-4} \prod_{\substack{v\in\mathfrak{V}'_\infty \\ \ell_v\equiv 2\ (4)}} \prod_{i=1}^{(\ell_v-2)/4} \frac{n-4i-2}{n+4i-2}.$$

Here χ_Φ is the quadratic character of $\mathbb{A}^\times$ associated with Φ, i.e.,

$$\chi_\Phi(a) = \langle (-1)^{n/2}\det\Phi, a\rangle$$

for $a \in \mathbb{A}^\times$, where $\langle\,\cdot\,,\,\cdot\,\rangle$ is the Hilbert symbol, and $\mathfrak{f}_{\chi_\Phi}$ denotes the conductor of χ_Φ and $L(s,\chi_\Phi)$ the Hecke L-function of χ_Φ.

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Department of Mathematics
Graduate School of Science
Osaka University
Toyonaka
Osaka, 560-0043
Japan
watanabe@math.wani.osaka-u.ac.jp