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ON OSGOOD-YANG'S CONJECTURE AND MUES' CONJECTURE

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Abstract. In this paper, we deal with the relation between the characteristic functions of meromorphic functions that share three values CM. As applications of our main results, we shall affirmatively settle two conjectures proposed by Mues and Osgood-Yang.

§1. Introduction

In this paper, a meromorphic function always means a function that is meromorphic in the complex plane $\mathbb{C}$. We use the usual notations in the Nevanlinna theory of meromorphic functions as explained in [1]. Denote by E any set of finite Lebesgue measure on $(0, +\infty)$, which is not necessarily the same at each occurrence.

Let f and g be two nonconstant meromorphic functions. We say that f and g share a value $a \in \widehat{\mathbb{C}}$ provided that $f(z) = a$ if and only if $g(z) = a$. We say that they share the value a CM resp. IM, when we are counting the multiplicity, resp. ignoring the multiplicity (see [2]).

In 1976, C. F. Osgood and C. C. Yang [3] proved the following theorem:

THEOREM A. *Let f and g be two nonconstant entire functions of finite order. If f and g share $0, 1$ CM, then*

$$T(r, f) \sim T(r, g) \quad (r \to \infty).$$

In [3], C. F. Osgood and C. C. Yang proposed the following conjecture:

OSGOOD-YANG'S CONJECTURE. ([3, p. 409]) *Let f and g be two nonconstant entire functions sharing $0, 1$ CM. Then*

$$T(r, f) \sim T(r, g) \quad (r \to \infty, r \notin E).$$

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In 1989, G. Brosch [4] proved the following theorem:

Theorem B. *Let f and g be two nonconstant meromorphic functions sharing three values CM. Then*

$$\Big(\frac{3}{8}+o(1)\Big) \le \frac{T(r,f)}{T(r,g)} \le \Big(\frac{8}{3}+o(1)\Big) \quad (r\to\infty, r\notin E).$$

In 1990, W. Bergweiler [5] proved the following theorem:

Theorem C. *There exists a set $I \subset (0,\infty)$ of infinite Lebesgue measure and there exist meromorphic functions f and g sharing $0,1,\infty$ CM such that*

$$\liminf_{\substack{r\to\infty\\ r\in I}} \frac{T(r,f)}{T(r,g)} \ge 2.$$

Theorem C implies that the bound 8/3 in Theorem B cannot be replaced by any constant less than 2. In 1995, E. Mues [6] proposed the following conjecture:

Mues' Conjecture. ([6, p. 28]) *Let f and g be two nonconstant meromorphic functions sharing three values CM. Then*

$$\Big(\frac{1}{2}+o(1)\Big) \le \frac{T(r,f)}{T(r,g)} \le (2+o(1)) \quad (r\to\infty, r\notin E).$$

In 1998, P. Li and C. C. Yang [7] proved the following theorem:

Theorem D. *Let f and g be two nonconstant meromorphic functions sharing $0,1,\infty$ CM. Then, for any positive number ε,*

$$T(r,g) \le (2+\varepsilon)T(r,f)+S(r,f).$$

In this paper, we deal with the relation between the characteristic functions of meromorphic functions that share three values CM. As applications of our main results, we shall affirmatively settle two conjectures proposed by Mues and Osgood-Yang.

§2. Main results

Let f and g be two nonconstant meromorphic functions sharing $0, 1, \infty$ CM. In this paper, we denote by $N_0(r)$ the counting function of the zeros of $f - g$ that are not zeros of $f, f - 1$ and $1/f$.

THEOREM 2.1. *Let f and g be two nonconstant meromorphic functions sharing $0, 1, \infty$ CM. If*

$$\limsup_{\substack{r\to\infty \\ r\notin E}} \frac{N_0(r)}{T(r,f)} > \frac{1}{2}, \tag{2.1}$$

then

$$T(r,f) \sim T(r,g) \quad (r \to \infty). \tag{2.2}$$

If

$$0 < \limsup_{\substack{r\to\infty \\ r\notin E}} \frac{N_0(r)}{T(r,f)} \leq \frac{1}{2}, \tag{2.3}$$

then

$$T(r,f) \sim T(r,g) \quad (r \to \infty, r \notin E). \tag{2.4}$$

If

$$\limsup_{\substack{r\to\infty \\ r\notin E}} \frac{N_0(r)}{T(r,f)} = 0, \tag{2.5}$$

then

$$\left(\frac{1}{2} + o(1)\right) \leq \frac{T(r,f)}{T(r,g)} \leq (2 + o(1)) \quad (r \to \infty, r \notin E). \tag{2.6}$$

By Theorem 2.1, we immediately obtain the following corollary, which shows that Mues' conjecture is true.

COROLLARY 2.1. *Let f and g be two nonconstant meromorphic functions sharing three values CM. Then*

$$\left(\frac{1}{2} + o(1)\right) \leq \frac{T(r,f)}{T(r,g)} \leq (2 + o(1)) \quad (r \to \infty, r \notin E).$$

THEOREM 2.2. *Let f and g be two nonconstant entire functions sharing* $0, 1$ *CM. If*

$$\limsup_{\substack{r\to\infty \\ r\notin E}} \frac{N_0(r)}{T(r,f)} > \frac{1}{2}, \tag{2.7}$$

then

$$T(r,f) \sim T(r,g) \quad (r \to \infty). \tag{2.8}$$

If

$$\limsup_{\substack{r\to\infty \\ r\notin E}} \frac{N_0(r)}{T(r,f)} \leq \frac{1}{2}, \tag{2.9}$$

then

$$T(r,f) \sim T(r,g) \quad (r \to \infty, r \notin E). \tag{2.10}$$

By Theorem 2.2, we immediately obtain the following corollary, which shows that Osgood-Yang's conjecture is true.

COROLLARY 2.2. *Let f and g be two nonconstant entire functions sharing two finite values CM. Then*

$$T(r,f) \sim T(r,g) \quad (r \to \infty, r \notin E).$$

§3. Some lemmas

LEMMA 3.1. ([1, p. 8] or [8, Theorem 1.11]) *Let f and g be two nonconstant meromorphic functions. If f is a fractional linear transformation of g, then*

$$T(r,g) = T(r,f) + O(1). \tag{3.1}$$

LEMMA 3.2. ([8, Theorem 1.13] or [9]) *Let f be a nonconstant meromorphic function, and let*

$$R(f) = \sum_{k=0}^{n} a_k f^k \Big/ \sum_{j=0}^{m} b_j f^j$$

be an irreducible rational function in f with constant coefficients $\{a_k\}$ and $\{b_j\}$, where $a_n \neq 0$ and $b_m \neq 0$. Then

$$T(r, R(f)) = \max\{n, m\}\, T(r,f) + S(r,f). \tag{3.2}$$

LEMMA 3.3. ([10, Theorem 1]) *Let f and g be two nonconstant meromorphic functions sharing* $0, 1, \infty$ *CM. If*

$$\limsup_{\substack{r\to\infty \\ r\notin E}} \frac{N_0(r)}{T(r,f)} > \frac{1}{2}, \tag{3.3}$$

then f is a fractional linear transformation of g.

LEMMA 3.4. ([10, Theorem 2]) *Let f and g be two nonconstant meromorphic functions sharing* $0, 1, \infty$ *CM. If*

$$0 < \limsup_{\substack{r\to\infty \\ r\notin E}} \frac{N_0(r)}{T(r,f)} \leq \frac{1}{2}, \tag{3.4}$$

then f is not any fractional linear transformation of g and one of the following relations occurs:

$$(i) \quad f = \frac{e^{s\alpha}-1}{e^{(k+1)\alpha}-1}, \quad g = \frac{e^{-s\alpha}-1}{e^{-(k+1)\alpha}-1}, \tag{3.5}$$

$$(ii) \quad f = \frac{e^{(k+1)\alpha}-1}{e^{(k+1-s)\alpha}-1}, \quad g = \frac{e^{-(k+1)\alpha}-1}{e^{-(k+1-s)\alpha}-1}, \tag{3.6}$$

$$(iii) \quad f = \frac{e^{s\alpha}-1}{e^{-(k+1-s)\alpha}-1}, \quad g = \frac{e^{-s\alpha}-1}{e^{(k+1-s)\alpha}-1}, \tag{3.7}$$

where s and k (≥ 2) are positive integers such that and $1 \leq s \leq k$, s and $k+1$ are relatively prime, and α is a nonconstant entire function.

LEMMA 3.5. ([11, Lemma 1]) *Let h be a nonconstant entire function. Then*

$$T(r,h') = o(T(r,e^h)) \quad (r\to\infty, r\notin E). \tag{3.8}$$

LEMMA 3.6. ([12, Lemma 1]) *Let f and g be two nonconstant meromorphic functions sharing* $0, 1, \infty$ *CM. If $f \not\equiv g$, then*

$$f = \frac{e^q-1}{e^p-1}, \quad g = \frac{e^{-q}-1}{e^{-p}-1}, \tag{3.9}$$

where p and q are entire functions such that $e^p \not\equiv 1$, $e^q \not\equiv 1$, $e^{q-p} \not\equiv 1$, and

$$T(r,g) + T(r,e^p) + T(r,e^q) = O(T(r,f)) \quad (r\notin E). \tag{3.10}$$

LEMMA 3.7. *Let f and g be two nonconstant meromorphic functions sharing $0, 1, \infty$ CM. If f is not any fractional linear transformation of g, then*

(3.11)
$$T(r,f)+T(r,g)=N\Big(r,\frac{1}{f}\Big)+N\Big(r,\frac{1}{f-1}\Big)+N(r,f)+N_0(r)+S(r,f).$$

Proof. Suppose that $f \not\equiv g$. Since f and g share $0, 1, \infty$ CM, by Lemma 3.6, we obtain (3.9) and (3.10). From (3.9) we get

$$\frac{f}{g}=e^{q-p}, \qquad \frac{f-1}{g-1}=e^{q}, \qquad \frac{(f-1)g}{f(g-1)}=e^{p}. \tag{3.12}$$

If e^p is a constant, from (3.12) we obtain that f is a fractional linear transformation of g, which is a contradiction. Thus e^p is not constant. In the same manner as above, we have e^q and e^{q-p} are not constants. In this case, we have the following ([12, p. 309, (18)]):

(3.13)
$$T(r,f)+T(r,g)=N\Big(r,\frac{1}{g}\Big)+N\Big(r,\frac{1}{g-1}\Big)+N(r,g)+N_0^{*}(r)+S(r,f),$$

where $N_0^*(r)$ denotes the counting function of the zero of $f-g$ that are not zeros of f and $f-1$ (in [12], we use $N_0(r)$ for $N_0^*(r)$ in this paper).

Let

$$N_0^{**}(r):=N_0^{*}(r)-N_0(r). \tag{3.14}$$

Next we proceed to estimate $N_0^{**}(r)$. It is obvious that $N_0^{**}(r)$ denotes the counting function of the zero of $f-g$ that are zeros of $1/f$. Suppose that z_0 is a zero of $1/f$ that is a zero of $f-g$. Since z_0 is a zero of $1/f$, by (3.9), we have

$$e^{p(z_0)}=1. \tag{3.15}$$

By (3.9), we also have

$$f-g=\frac{e^q-e^p+e^{p-q}-1}{e^p-1}. \tag{3.16}$$

Note that z_0 is a zero of $f-g$. By (3.16), we have

$$e^{q(z_0)}-e^{p(z_0)}+e^{p(z_0)-q(z_0)}=1. \tag{3.17}$$

By (3.15) and (3.17), we obtain

$$e^{q(z_0)} = 1. \tag{3.18}$$

Since z_0 is a zero of $1/f$, by (3.9) and (3.18), we get that z_0 is a zero of $e^p - 1$ with multiplicity ≥ 2, and hence z_0 is a zero of $(e^p - 1)' = p'e^p$. Thus

$$N_0^{**}(r) \leq N\Big(r, \frac{1}{p'}\Big) \leq T(r, p') + O(1). \tag{3.19}$$

By Lemma 3.5, (3.10) and (3.19), we have

$$N_0^{**}(r) = S(r, f). \tag{3.20}$$

By (3.13), (3.14) and (3.20), we get (3.11), which proves Lemma 3.7. □

§4. Proof of Theorem 2.1 and Theorem 2.2

4.1. Proof of Theorem 2.1

Suppose that $f \not\equiv g$. We consider the following three cases.

Case 1. Suppose that $N_0(r)$ satisfies (2.1). By Lemma 3.3, we have that f is a fractional linear transformation of g. Hence, by Lemma 3.1, we obtain (2.2).

Case 2. Suppose that $N_0(r)$ satisfies (2.3). By Lemma 3.4, we have that f is not any fractional linear transformation of g and one of (3.5), (3.6) and (3.7) occurs. We consider the following three subcases.

Subcase 2.1. Assume that f and g satisfy (3.5). Let

$$R(w) := \frac{w^s - 1}{w^{k+1} - 1}. \tag{4.1}$$

By (3.5), we have

$$f = R(e^{\alpha}), \quad g = R(e^{-\alpha}). \tag{4.2}$$

Since $1 \leq s \leq k$, and s and $k+1$ are relatively prime, by Lemma 3.2, (4.1) and (4.2), we get

$$T(r, f) = k\,T(r, e^{\alpha}) + S(r, e^{\alpha}), \tag{4.3}$$

and

$$T(r, g) = k\,T(r, e^{-\alpha}) + S(r, e^{-\alpha}) = k\,T(r, e^{\alpha}) + S(r, e^{\alpha}). \tag{4.4}$$

By (4.3) and (4.4), we obtain (2.4).

Subcase 2.2. Assume that f and g satisfy (3.6). Let

$$R(w) := \frac{w^{k+1} - 1}{w^{k+1-s} - 1}. \tag{4.5}$$

By (3.6), we have

$$f = R(e^{\alpha}), \quad g = R(e^{-\alpha}). \tag{4.6}$$

By Lemma 3.2, (4.5) and (4.6), we get

$$T(r, f) = k\,T(r, e^{\alpha}) + S(r, e^{\alpha}), \tag{4.7}$$

and

$$T(r, g) = k\,T(r, e^{-\alpha}) + S(r, e^{-\alpha}) = k\,T(r, e^{\alpha}) + S(r, e^{\alpha}). \tag{4.8}$$

By (4.7) and (4.8) we obtain (2.4).

Subcase 2.3. Assume that f and g satisfy (3.7). Let

$$R(w) := \frac{w^{k+1-s}(w^{s} - 1)}{1 - w^{k+1-s}}. \tag{4.9}$$

By (3.7), we have

$$f = R(e^{\alpha}), \quad g = R(e^{-\alpha}). \tag{4.10}$$

By Lemma 3.2, (4.9) and (4.10), we get

$$T(r, f) = k\,T(r, e^{\alpha}) + S(r, e^{\alpha}), \tag{4.11}$$

and

$$T(r, g) = k\,T(r, e^{-\alpha}) + S(r, e^{-\alpha}) = k\,T(r, e^{\alpha}) + S(r, e^{\alpha}). \tag{4.12}$$

By (4.11) and (4.12) we obtain (2.4).

Case 3. Suppose that $N_0(r)$ satisfies (2.5). We consider the following two subcases.

Subcase 3.1. Assume that f is a fractional linear transformation of g. By Lemma 3.1, we obtain (2.2).

Subcase 3.2. Assume that f is not any fractional linear transformation of g. By Lemma 3.7, we obtain (3.11). By (2.5), we have

$$N_0(r) = S(r,f). \tag{4.13}$$

Combining (3.11) and (4.13), we get

$$T(r,f) + T(r,g) = N\Big(r,\frac{1}{f}\Big) + N\Big(r,\frac{1}{f-1}\Big) + N(r,f) + S(r,f). \tag{4.14}$$

It is clear that

$$N\Big(r,\frac{1}{f}\Big) + N\Big(r,\frac{1}{f-1}\Big) + N(r,f) \le 3\,T(r,f) + O(1). \tag{4.15}$$

Combining (4.14) and (4.15), we obtain

$$T(r,g) \le 2\,T(r,f) + S(r,f). \tag{4.16}$$

Similarly, we have

$$T(r,f) \le 2\,T(r,g) + S(r,f). \tag{4.17}$$

Combining (4.16) and (4.17), we get (2.6). This completes the proof of Theorem 2.1. ∎

4.2. Proof of Corollary 2.1

Let f and g share a_1, a_2, a_3 CM, where a_1, a_2, a_3 are three distinct elements in $\widehat{\mathbb{C}}$. Set

$$L(w) := \frac{(w-a_1)(a_2-a_3)}{(w-a_3)(a_2-a_1)}.$$

Let $F := L(f)$ and $G := L(g)$. Then F and G share $0, 1, \infty$ CM. By Theorem 2.1, we have

$$\Big(\frac{1}{2} + o(1)\Big) \le \frac{T(r,F)}{T(r,G)} \le (2 + o(1)) \quad (r \to \infty, r \notin E). \tag{4.18}$$

By Lemma 3.1, we have

$$T(r,f) = T(r,F) + O(1), \quad T(r,g) = T(r,G) + O(1). \tag{4.19}$$

Combining (4.18) and (4.19), we obtain

$$\Big(\frac{1}{2} + o(1)\Big) \le \frac{T(r,f)}{T(r,g)} \le (2 + o(1)) \quad (r \to \infty, r \notin E).$$

This completes the proof of Corollary 2.1. ∎

4.3. Proof of Theorem 2.2

Since f and g are two nonconstant entire functions, f and g share ∞ CM. By Theorem 2.1, if $N_0(r)$ satisfies either (2.7) or (2.3), then we have the desired conclusion. Next, we assume that $N_0(r)$ satisfies (2.5). Then we have

$$N_0(r) = S(r, f). \tag{4.20}$$

Since $N(r, f) = N(r, g) = 0$, we get (2.10) as in the proof of Theorem 2.1. This completes the proof of Theorem 2.2. □

4.4. Proof of Corollary 2.2

Let f and g share a_1, a_2 CM, where a_1, a_2 are two distinct points in $\mathbb{C}$. Set $L(w) = (w - a_1)/(a_2 - a_1)$. Let $F := L(f)$ and $G := L(g)$. Then F and G are two nonconstant entire functions sharing $0, 1$ CM. By Theorem 2.2, we get

$$T(r, F) \sim T(r, G) \quad (r \to \infty, r \notin E). \tag{4.21}$$

By Lemma 3.1, we see

$$T(r, f) = T(r, F) + O(1), \quad T(r, g) = T(r, G) + O(1). \tag{4.22}$$

Combining (4.21) and (4.22), we obtain

$$T(r, f) \sim T(r, g) \quad (r \to \infty, r \notin E).$$

This completes the proof of Corollary 2.2. □

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