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CONVERGENCE OF THE ZETA FUNCTIONS OF PREHOMOGENEOUS VECTOR SPACES

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Abstract. Let (G, ρ, X) be a prehomogeneous vector space with singular set S over an algebraic number field F. The main result of this paper is a proof for the convergence of the zeta fucntions $Z(\Phi, s)$ associated with (G, ρ, X) for large $\mathrm{Re}\, s$ under the assumption that S is a hypersurface. This condition is satisfied if G is reductive and (G, ρ, X) is regular. When the connected component G_x^0 of the stabilizer of a generic point x is semisimple and the group Π_x of connected components of G_x is abelian, a clear estimate of the domain of convergence is given.

Moreover when S is a hypersurface and the Hasse principle holds for G, it is shown that the zeta fucntions are sums of (usually infinite) Euler products, the local components of which are orbital local zeta functions. This result has been proved in a previous paper by the author under the more restrictive condition that (G, ρ, X) is irreducible, regular, and reduced, and the zeta function is absolutely convergent.

§1. Introduction

Let (G, ρ, X) be a prehomogeneous vector space for a connected algebraic group G, a vector space X and a rational representation ρ of G on X, all defined over an algebraic number field F. Then for a closed subset S of X, $X^0 = X - S$ is an orbit of G. For $x \in X^0$, let G_x be the stabilizer of x and G_x^0 its connected component. Set

$$X^*(F) = \{\, x \in X^0(F) \mid \mathbb{X}(G_x^0)_F = \{1\} \,\},$$

where $\mathbb{X}(G_x^0)$ is the group of characters of G_x^0 and $\mathbb{X}(G_x^0)_F$ is its subset consisting of elements defined over F. For $x \in X^0$, we set $\Pi_x = G_x/G_x^0$.

Let $S^1 = \bigcup_{i=1}^n S_i$ be the disjoint union of all F-irreducible hypersurfaces S_i in S and let P_i be an F-irreducible homogeneous polynomial on X which defines S_i. Then P_i is a relative invariant of (G, ρ, X) and the group of F-rational relative invariants is generated by P_i up to $F^\times$. We denote by

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$\chi_1,\ \chi_2,\dots,\ \chi_n$ the characters of G associated to them. Throughout this paper, we assume

$$S = S^1, \quad X^*(F) \neq \emptyset. \tag{1.1}$$

Then there exist integers $m,\ d_1,\ d_2,\dots,\ d_n$ satisfying

$$\det \rho^m = \chi_1^{d_1}\chi_2^{d_2}\cdots\chi_n^{d_n}$$

(cf. (3.15)), and we set $\kappa_i = d_i/m$.

For a Schwartz-Bruhat function $\Phi \in \mathcal{S}(X(\mathbb{A}))$, a zeta function $Z(\Phi, s)$ in $s = (s_1, s_2, \dots, s_n) \in \mathbb{C}^n$ is defined by

$$Z(\Phi, s) = \int_{G(\mathbb{A})/G(F)} \prod_{i=1}^{n} |\chi_i(g)|_{\mathbb{A}}^{s_i} \sum_{x \in X^*(F)} \Phi(\rho(g)x)dg. \tag{1.2}$$

Here $\mathbb{A}$ is the adele ring of F. The purpose of this paper is to prove the convergence of $Z(\Phi, s)$ and an explicit expression of $Z(\Phi, s)$ by local orbital integrals in [Sa1] under a general condition.

On the convergence, we prove

Theorem 1.1. *Assume that S is a hypersurface and $X^*(F) \neq \emptyset$. Then $Z(\Phi, s)$ converges absolutely if $\operatorname{Re} s_i$ is sufficiently large for $i = 1, 2, \dots, n$.*

If we assume more conditions, we can give a clear estimate for the domain of convergence as follows.

Theorem 1.2. *Assume that S is a hypersurface, $X^*(F) \neq \emptyset$, and that G_x^0 is semisimple and Π_x is abelian. Then $Z(\Phi, s)$ converges absolutely if $\operatorname{Re} s_i > \kappa_i$ for $i = 1, 2, \dots, n$.*

These two theorems are proved in a unified way. The difference is caused by the fact that in the proof of Theorem 1.1, special values of L-functions and cardinalities of some algebraic extensions have to be estimated.

As shown in p.600-601 of [Sa1], for example, $Z(\Phi, s)$ can be written as a finite sum of products of Dirichlet series and local zeta functions at infinite places. These Dirichlet series are also called zeta functions of prehomogeneous vector spaces. Our theorems imply also the convergence of these Dirichlet series.

The convergence of zeta functions of prehomogeneous vector space was treated by Sato-Shintani [S-S], Sato [S2], Yukie [Yu1], [Yu2] and Yin [Y] under restrictive conditions (cf. [S3]). Yukie's results prove estimates on theta functions on Siegel sets and give stronger results.

The method of our proof is a modification of that in [S2] based on the description of orbits in prehomogeneous vector spaces given in [Sa1]. For $\Phi \in \mathcal{S}(X(\mathbb{A}))$, let

$$Z_m(\Phi, s) = \int_{X^0(\mathbb{A})} \prod_{i=1}^{n} |P_i(x)|_{\mathbb{A}}^{s_i} \Phi(x) dX \tag{1.3}$$

be the multiplicative zeta function associated to Φ. Then it is not difficult to prove the convergence of $Z_m(\Phi, s)$(cf. [O], [S2]). We show that the convergence of $Z(\Phi, s)$ can be reduced to that of $Z_m(\Phi, s)$. Sato proved the convergence under the asumption that G_x^0 is semi-simple and $H_x = G_x \cap H$ is connected for $H = G_{der} R_u(G) G_x^0$. The second condition implies that Π_x is abelian and our Theorem 1.2 contains his result.

In Section 5, we assume that the Hasse principle holds for $H^1(F, G)$, and for $\Phi = \prod_v \Phi_v$ we give an expression of $Z(\Phi, s)$ as a sum of products of local integrals of the form

$$\int_{\mathcal{O}_v} \tilde{\varepsilon}_v(x_v) \Phi_v(x_v) \prod_{i=1}^{n} |P_i(x_v)|_v^{s_i - \kappa_i} dX_v$$

as in [Sa1]. Here $\mathcal{O}_v$ is a union of $G(F_v)$-orbits in $X^0(F_v)$ for a place v of F, and $\tilde{\varepsilon}_v$ is a function on $\mathcal{O}_v$.

The idea to apply the method in [Sa1] to the proof of the convergence of the zeta functions of prehomogeneous vector spaces was suggested by Professor Fumihiro Sato. I wish to express my sincere thanks to him.

§2. Orbits in prehomegeneous vector spaces

Let F be a field of characteristic 0 and let $\Gamma = \mathrm{Gal}(\bar{F}/F)$ with the algebraic closure $\bar{F}$ of F. Let (G, ρ, X) be a prehomogeneous vector space defined over F. We write $\rho(g)x = gx$ for short. Let S, X^0, G_x, G_x^0 and Π_x be as in Introduction.

First we recall the results on orbits in prehomogeneous vector space in [Sa1] with some comments. There we assumed that (G, ρ, X) is regular, irreducible, and reduced and that $\operatorname{Ker} \rho = \{1\}$. But the results (2.4) and (2.5) stated below are valid without these assumptions.

Let $\iota_x\colon G_x \to G$ and $\iota_x^0\colon G_x^0 \to G$ be the inclusions. Then for $x \in X^0(F)$ they induce the canonical maps

$$\iota_x\colon H^1(F, G_x) \to H^1(F, G), \quad \iota_x^0\colon H^1(F, G_x^0) \to H^1(F, G).$$

Let $Y_x = G/G_x^0$. Then by [Se, Corollary 1 of Proposition 35], we have two bijections

$$\delta_x\colon G(F)\backslash X^0(F) \to \operatorname{Ker}\iota_x, \quad \delta_x^0\colon G(F)\backslash Y_x(F) \to \operatorname{Ker}\iota_x^0.$$

From the exact sequence

$$1 \longrightarrow G_x^0 \longrightarrow G_x \longrightarrow \Pi_x \longrightarrow 1,$$

we obtain an exact sequence

$$H^1(F, G_x^0) \longrightarrow H^1(F, G_x) \longrightarrow H^1(F, \Pi_x).$$

Connecting the maps

$$X^0(F) \to G(F)\backslash X^0(F) \simeq \operatorname{Ker}\iota_x \hookrightarrow H^1(F, G_x) \to H^1(F, \Pi_x),$$

we define a map φ_x of $X^0(F)$ to $H^1(F, \Pi_x)$. Define an equivalence relation $\sim$ in $X^0(F)$ by

$$a \sim b \Longleftrightarrow \varphi_x(a) = \varphi_x(b)$$

for $a,\ b \in X^0(F)$. Then this equivalence relation is independent of the choice of x. For a class $\tilde{\alpha} \in H^1(F, \Pi_x)$, set

$$X^0(F, \tilde{\alpha}) = \{\ a \in X^0(F) \mid \varphi_x(a) = \tilde{\alpha}\ \}.$$

Then we have a disjoint union

$$X^0(F) = \bigcup_{\tilde{\alpha} \in H^1(F, \Pi_x)} X^0(F, \tilde{\alpha}). \tag{2.4}$$

The set $X^0(F, \tilde{\alpha})$ may be empty for some $\tilde{\alpha}$. We note that for $a,\ b \in X^0(F, \tilde{\alpha})$, G_a^0 is an inner form of G_b^0(cf. [Sa1, Lemma 1.1]). Hence $\mathbb{X}(G_a^0) \simeq \mathbb{X}(G_b^0)$ as Γ-groups and $X^*(F) \cap X^0(F, \tilde{\alpha})$ is empty or is equal to $X^0(F, \tilde{\alpha})$.

Let $a \in X^0(F, \tilde{\alpha})$. Then $\Pi_a(F)$ acts $Y_a(F)$ on the right, and the natural morphism $\mu_a\colon gG_a^0 \longmapsto ga$ of Y_a to X^0 induces a bijection

$$Y_a(F)/\Pi_a(F) \simeq X^0(F, \tilde{\alpha}) \tag{2.5}$$

(cf. [Sa1, Lemma 1.1]). We note that $\mu_a\colon Y_a \to X^0$ is the normalization of X^0 in $F(Y_a)$.

For $a \in X^0(F)$, let $a = hx$ with $h \in G(\bar{F})$. Then $\alpha = (h^{-1\sigma}h)$ defines a 1-cocycle with values in G_x. We note that the inner automorphism Int_h of G induces an isomorphism of Y_x to Y_a which satisfies

$$\begin{array}{ccc} Y_x & \xrightarrow{\mathrm{Int}_h} & Y_a \\ {\scriptstyle\mu_x}\downarrow & & \downarrow{\scriptstyle\mu_a} \\ X^0 & \xrightarrow{h} & X^0. \end{array}$$

We can make G_x act on Y_x by the inner action

$$\mathrm{Int}_k(gG_x^0) = kgk^{-1}G_x^0$$

for $g \in G$ and $k \in G_x$. By this action we can consider the twist ${}_{\alpha}(Y_x)$ of Y_x by α, and we see that ${}_{\alpha}(Y_x) \simeq Y_a$.

Now let F be an algebraic number field, and let $\mathbb{A}$ be the adele ring of F. Let Σ be the set of all places of F and let Σ_f and Σ_∞ be the subsets of consisting of all finite and infinite places respcetively. For $v \in \Sigma$, let F_v be the completion of F at v, and $\bar{F}_v$ the algebraic closure of F_v. For $v \in \Sigma_f$, let O_v the ring of integers in F_v, $\mathfrak{p}_v$ its maximal ideal and $q_v = |O_v/\mathfrak{p}_v|$.

For a connected algebraic group H defined over F, let $\tau(H)$ be the Tamagawa number of H and let $\ker^1(H)$ be the cardinality of the kernel of the Hasse map

$$H^1(F,H) \longrightarrow \prod_v H^1(F_v,H).$$

We recall a group $A(H)$ associated to H, which was introduced by Kottwitz [K1], [K2], following Borovoi [B]. For a while, F is a field of characteristic 0. Assume H is reductive. Let $\pi_1(\bar{H})$ be the algebraic fundamental group of H(cf. [B, 1.4]). We set

$$A(H) = (\pi_1(\bar{H})_\Gamma)_{\mathrm{tor}}.$$

Here $\pi_1(\bar{H})_\Gamma$ is the group of coinvariants of $\pi_1(\bar{H})$ and $A(H)$ is its subgroup of torsion elements. When F is an algebraic number field, it is known that

$$|A(H)| = \tau(H)\ker^1(H) \qquad (2.6)$$

by Kottwitz [K1, (5.5.1)]. When H is not reductive, we set $A(H) = A(H/R_u(H)))$. We note that (2.6) is valid also in this case if F is an algebraic number field.

Let F be an algebraic number field again. For $a \in X^0(F)$, we have a surjective map

$$\eta_a \colon G(F)\backslash Y_a(F) \longrightarrow G(\mathbb{A})\backslash G(\mathbb{A})Y_a(F),$$

and for $zG_a^0 \in Y_a(F)$, we have (cf. [Sa1, Proposition 1.4])

$$|\eta_a^{-1}(\eta_a(zG_a^0))| \leq \ker^1(G_a^0). \tag{2.7}$$

The equality holds if G satisfies the Hasse principle. We note $G(\mathbb{A})Y_a(F)$ is an open subset of $Y_a(\mathbb{A})$ since G_a^0 is connected.

Next for $v \in \Sigma_f$, we consider O_v-structure of the relavant varieties, and describe the orbits $G(O_v)\backslash X^0(O_v)$ via Galois cohomology. Let F_v^{ur} be the maximal unramified extension of F_v, O_v^{ur} its ring of integers, k_v the residue field of v and $\bar{k}_v$ its algebraic closure. Let $\mathit{\Gamma}_v^{ur} = \mathrm{Gal}(F_v^{ur}/F_v)$. Then $\mathit{\Gamma}_v^{ur} \simeq \mathrm{Gal}(\bar{k}_v/k_v)$ and we identify these two groups.

For $x \in X^0(F_v)$, let $\mu_x \colon Y_x \to X^0$ be as above and consider two other morphisms

$$\lambda_x \colon G \to Y_x,\ \nu_x \colon G \to X^0$$

defined by

$$g \mapsto gG_x^0,\ g \mapsto gx.$$

Then $\nu_x = \mu_x\lambda_x$.

We fix a system of coordinates $x_1, x_2, \ldots, x_d$ of X, $d = \dim X$, and assume that $P_i \in O_v[X] = O_v[x_1, x_2, \ldots, x_d]$ and $P_i \notin \mathfrak{p}_v[X]$ for all i. We set $P = \prod_i P_i$. Let $\tilde{X}^0 = \operatorname{Spec} O_v[X]_P$ and $\tilde{S} = \operatorname{Spec} O_v[X]/(P)$. Then $\tilde{X}^0 \otimes_{O_v} F_v = X^0 \otimes_F F_v$ and $\tilde{S} \otimes_{O_v} F_v = S \otimes_F F_v$. Assume $x \in \tilde{X}^0(O_v)$. Here we see $\tilde{X}^0(O_v)$ as a subset of $X^0(F_v)$. Assume that there exist an affine group scheme $\tilde{G}$, integral noetherian smooth over O_v such that $\tilde{G} \otimes_{O_v} F_v \simeq G \otimes_F F_v$, and also assume that the actions of G on X^0 is extended to that of $\tilde{G}$ on $\tilde{X}^0$ and the morphism ν_x is extended to a smooth morphism $\tilde{\nu}_x$ of $\tilde{G}$ to X^0. Let e the unit point of $\tilde{G}(O_v)$. Then $\tilde{\nu}_x(e) = x$. Let be $\tilde{G}_x$ be the stabilizer of x.

Let $\tilde{Y}_x$ is the normalization of $\tilde{X}^0$ in the function field $F(Y_x)$ of Y_x. Then there exist morphisms $\tilde{\lambda}_x \colon \tilde{G} \to \tilde{Y}_x$ and $\tilde{\mu} \colon \tilde{Y}_x \to \tilde{X}^0$ extending λ_x, μ_x, which satisfy $\tilde{\nu}_x = \tilde{\mu}_x\tilde{\lambda}_x$. The morphism $\tilde{\mu}_x$ is finite. Let $y = \tilde{\lambda}_x(e)$. Also the action of G on Y_x and the inner action of G_x are extended to that of $\tilde{G}$ on $\tilde{Y}_x$ and the of $\tilde{G}_x$ on $\tilde{Y}_x$. We denote by $\tilde{G}_y$ the stabilizer of y. For the object A over O_v, we denote by $\bar{A}$ the object over the spcial point of $\mathrm{Spec}(O_v)$. For the unit element $\bar{e}$, we set $\bar{x} = \bar{\nu}_x(\bar{e})$ and $\bar{y} = \bar{\lambda}_x(\bar{e})$. Then $\bar{\mu}_x(\bar{y}) = \bar{x}$.

Proposition 2.1. *Let the notation be as above. Assume the following conditions:*

(1) $P_i \in O_v[X], \notin \mathfrak{p}_v[X]$ *for all* i;

(2) $\bar{G}$ *and* $\bar{G}_y$ *are connected*;

(3) $\tilde{\lambda}_x$ *and* $\tilde{\nu}_x$ *are smooth, and* $\tilde{\mu}_x$ *is étale*;

(4) $\bar{\lambda}_x$ *is surjective.*

Then one has

$$\tilde{G}(O_v^{ur})y = \tilde{Y}_x(O_v^{ur}), \quad \tilde{G}(O_v^{ur})x = \tilde{X}^0(O_v^{ur}). \tag{2.8}$$

Proof. First we show $\tilde{G}(O_v^{ur})y = \tilde{Y}_x(O_v^{ur})$. For $z \in \tilde{Y}_x(O_v^{ur})$, let $f_z\colon \mathrm{Spec}(O_v^{ur}) \to \tilde{Y}_x$ the morphism corresponding to z. Then $\tilde{G} \times_{\tilde{Y}_x} \mathrm{Spec}(O_v^{ur})$ is smooth over $\mathrm{Spec}(O_v^{ur})$. It is enough to prove that the O_v^{ur}-valued point of this scheme is non-empty. The set of $\bar{k}_v$-valued points of the special fibre of this scheme is not empty, since $\bar{\lambda}_x$ is surjective. The assertion follows from this since O_v^{ur} is Henselian.

By the assumption (3), $\bar{\mu}_x$ is surjective, hence $\bar{\nu}_x$ is also surjective. Since $\tilde{\nu}_x$ is smooth, we obtain the assertion for $\tilde{X}^0$ in the same way as above.

We set $G(O_v^{ur}) = \tilde{G}(O_v^{ur})$, $G(O_v) = \tilde{G}(O_v)$, $G_x(O_v^{ur}) = \tilde{G}_x(O_v^{ur})$, etc. for short.

Corollary 2.2. *Under the assumptions in Proposition* 2.1*,* $Y_x(O_v)$ *consists of a single* $G(O_v)$*-orbit, and there exists a bijection*

$$G(O_v)\backslash X^0(O_v) \simeq H^1(\Gamma_v^{ur}, G_x(O_v^{ur})). \tag{2.9}$$

Proof. From $G(O_v^{ur})/G_x(O_v^{ur}) = X^0(O_v^{ur})$, we obtain an exact sequence

$$G(O_v)\backslash X^0(O_v) \longrightarrow H^1(\Gamma_v^{ur}, G_x(O_v^{ur})) \longrightarrow H^1(\Gamma_v^{ur}, G(O_v^{ur})).$$

We note the first map is injective. We see that this is bijective since $H^1(\Gamma_v^{ur}, G(O_v^{ur})) = \{1\}$(cf. [P-R, Theorem 6.8′]).

The assertion for Y_x follows form the fact that $H^1(\Gamma^{ur}, G_y(O_v^{ur})) = \{1\}$ in the same way as above.

By the reduction modulo $\mathfrak{p}_v$, we can define surjective maps

$$G(O_v^{ur}) \longrightarrow \bar{G}(\bar{k}_v),\ G(O_v) \longrightarrow \bar{G}(k_v),\ X^0(O_v) \to \bar{X}^0(k_v),$$

etc. The inclusion and the reduction map give rise to a commutative diagram

$$\tag{2.10} \begin{array}{ccccc} \bar{G}(k_v)\backslash\bar{X}^0(k_v) & \longleftarrow & G(O_v)\backslash X^0(O_v) & \longrightarrow & G(F_v)\backslash X^0(F_v) \\ \downarrow\simeq & & \downarrow\simeq & & \downarrow \\ H^1(k_v, \bar{G}_{\bar{x}}) & \longleftarrow & H^1(\Gamma_v^{ur}, G_x(O_v^{ur})) & \longrightarrow & H^1(F_v, G_x). \end{array}$$

The first vertical map is bijective since $\bar{G}(\bar{k}_v)$ acts transitively on $\bar{X}^0(\bar{k}_v)$ and $H^1(k_v, \bar{G}) = \{1\}$. The second vertical map is also bijective as we have seen in Corollary 2.2. The upper left map is surjective since the reduction map of $X^0(O_v)$ to $\bar{X}^0(k_v)$ is surjective.

Lemma 2.3. *Let $\tilde{\Pi}_x = G_x(O_v^{ur})/G_y(O_v^{ur})$ and $\bar{\Pi}_x = \bar{G}_x(\bar{k}_v)/\bar{G}_y(\bar{k}_v)$ and assume the conditions in Proposition* 2.1. *Then*

$$\Pi_x \simeq \tilde{\Pi}_x \simeq \bar{\Pi}_x$$

as abstract groups. Hence $\mathrm{Gal}(\bar{F}_v/F_v^{ur})$ *acts trivially on* Π_x, *and* $\tilde{\Pi}_x \simeq \bar{\Pi}_x$ *as* Γ_v^{ur}*-groups.*

Proof. The inclusion of $G_x(O_v^{ur})$ into $G_x(\bar{F}_v)$ induces an injective homomorphism of $\tilde{\Pi}_x$ to $\Pi_x(= G_x(\bar{F}_v)/G_x^0(\bar{F}_v))$. On the other hand the reduction map induces a surjective homomorphism of $\tilde{\Pi}_x$ to $\bar{\Pi}_x$, since $\tilde{G}_x$ is smooth over $\mathrm{Spec}(O_v)$ by the condition (3) of Proposition 2.1, and we have an inequality

$$|\bar{\Pi}_x| \leq |\tilde{\Pi}_x| \leq |\Pi_x|.$$

We note that $|\Pi_x|$ and $|\bar{\Pi}_x|$ are degrees of μ_x and $\bar{\mu}_x$ respectively. But $\tilde{\mu}_x$ is finite étale. Hence we have $|\Pi_x| = |\bar{\Pi}_x|$. The assertion follows form this.

PROPOSITION 2.4. *Let the notation and the assumptions be as in Proposition* 2.1. *Then the reduction map induces a bijection*

$$G(O_v)\backslash X^0(O_v) \longrightarrow \bar{G}(k_v)\backslash \bar{X}^0(k_v).$$

Moreover the inclusion induces an injection

$$G(O_v)\backslash X^0(O_v) \longrightarrow G(F_v)\backslash X^0(F_v),$$

and φ_x *induces an injection*

$$G(O_v)\backslash X^0(O_v) \longrightarrow H^1(F_v, \Pi_x).$$

Proof. By the definition of Π_x, $\tilde{\Pi}_x$ and $\bar{\Pi}_x$, we have the following commutative diagram

$$\begin{array}{ccccc} H^1(k_v, \bar{G}_y) & \longleftarrow & H^1(\mathit{\Gamma}_v^{ur}, G_y(O_v^{ur})) & \longrightarrow & H^1(F_v, G_x^0) \\ \downarrow & & \downarrow & & \downarrow \\ H^1(k_v, \bar{G}_x) & \longleftarrow & H^1(\mathit{\Gamma}_v^{ur}, G_x(O_v^{ur})) & \longrightarrow & H^1(F_v, G_x) \\ \downarrow & & \downarrow & & \downarrow \\ H^1(k_v, \bar{\Pi}_x) & \xleftarrow{\sim} & H^1(\mathit{\Gamma}_v^{ur}, \tilde{\Pi}_x) & \longrightarrow & H^1(F_v, \Pi_x). \end{array} \tag{2.11}$$

Here the right horizontal arrows are inflation maps. Since $\bar{G}_y$ is connected, $H^1(\mathit{\Gamma}_v^{ur}, G_y(O_v^{ur})) = H^1(k_v, \bar{G}_y) = \{1\}$ and by the argument of twist, we see the map of $H^1(k_v, \bar{G}_x)$ to $H^1(k_v, \bar{\Pi}_x)$ and that of $H^1(\mathit{\Gamma}_v^{ur}, G_x(O_v^{ur}))$ to $H^1(\mathit{\Gamma}_v^{ur}, \tilde{\Pi}_x)$ are injective. From this it follows that the map of $H^1(\mathit{\Gamma}_v^{ur}, G_x(O_v^{ur}))$ to $H^1(k_v, \bar{G}_x)$ is injective. By (2.10), this map is surjective. From this we conclude that this map is bijective and that the reduction map of $G(O_v)\backslash X^0(O_v)$ to $\bar{G}(k_v)\backslash \bar{X}^0(k_v)$ is also bijective. This assertion can be proved also by (2.10) and [B-T, Lemma 2].

By Lemma 2.3, we see that the map

$$H^1(\mathit{\Gamma}_v^{ur}, \tilde{\Pi}_x) \longrightarrow H^1(F_v, \Pi_x)$$

is injective. From this it follows that the map of $H^1(\mathit{\Gamma}_v^{ur}, G_x(O_v^{ur}))$ to $H^1(F_v, G_x)$ is injective and that the inclusion map of $G(O_v)\backslash X^0(O_v)$ to $G(F_v)\backslash X^0(F_v)$ is injective. The last assertion follows also from this. This completes the proof.

Under the condition of Proposition 2.1, we may and will consider $H^1(\mathit{\Gamma}_v^{ur}, G_x(O_v^{ur}))$ as a subset of $H^1(F_v, G_x)$, also $H^1(\mathit{\Gamma}_v^{ur}, \tilde{\Pi}_x)$ as that of $H^1(F_v, \Pi_x)$ by inflations.

Proposition 2.5. *Assume the conditions in Propositiion 2.1. For $a \in X^0(O_v)$, let $a = gx$ for $g \in G(O_v^{ur})$ and let $\tilde{\alpha}$ be the image in $H^1(\mathit{\Gamma}_v^{ur}, \tilde{\Pi}_x)$ of the class in $H^1(\mathit{\Gamma}_v^{ur}, G_x(O_v^{ur}))$ of the 1-cocycle $\alpha = (g^{-1\sigma}g)$ of $\mathit{\Gamma}_v^{ur}$. Let $\tilde{\mu}_a \colon \tilde{Y}_a \to \tilde{X}^0$ be the normalization of $\tilde{X}^0$ in the function field $F_v(Y_a)$ of Y_a, which is seen as an extension of $F_v(X^0)$ via μ_a, and set*

$$X^0(O_v, \tilde{\alpha}) = X^0(O_v) \bigcap X^0(F_v, \tilde{\alpha}).$$

Then $\tilde{\mu}_a$ induces a covering

$$Y_a(O_v) \longrightarrow X^0(O_v, \tilde{\alpha})$$

of degree $|\Pi_a(F_v)|$, and $\bar{\mu}_a$ induces a covering

$$\bar{Y}_a(k_v) \longrightarrow X^0(O_v, \tilde{\alpha}) \bmod \mathfrak{p}_v$$

of degree $|\Pi_a(F_v)|$.

Proof. We note that $X^0(O_v, \tilde{\alpha})$ is a single $G(O_v)$-orbit by Proposition 2.4. First we prove the assertion for $\tilde{\mu}_a$.

Let $\tilde{1}$ be the pointed class in $H^1(\mathit{\Gamma}_v^{ur}, \tilde{\Pi}_x)$. Then $\tilde{\mu}_x$ induces a map

$$\tilde{\mu}_x \colon Y_x(O_v) \longrightarrow X^0(O_v, \tilde{1})$$

For $z \in X^0(O_v, \tilde{1})$, we can choose $g \in G(O_v^{ur})$ so that $gx = z$ and $g^{-1\sigma}g \in G_y(O_v^{ur})$ for $\sigma \in \mathit{\Gamma}_v^{ur}$. We see $\tilde{\mu}_x^{-1}(z) = gG_x(O_v^{ur})y$ and for $h \in G_x(O_v^{ur})$, $ghy \in Y_x(O_v)$ if and only if $h^{-1\sigma}h \in G_y(O_v^{ur})$ for $\sigma \in \mathit{\Gamma}_v^{ur}$. This shows the degree of $\tilde{\mu}_x$ is $|\tilde{\Pi}_x^{\mathit{\Gamma}_v^{ur}}| = |\Pi_x(F_v)|$.

For $a \in X^0(O_v, \tilde{\alpha})$, let $a = gx$ for $g \in G(O_v^{ur})$, and let α be the 1-cocycle $(g^{-1\sigma}g)$ of $\mathit{\Gamma}_v^{ur}$ in $G_x(O_v^{ur})$. We compare the schemes $g \circ \tilde{\mu}_x \colon \tilde{Y}_x \to \tilde{X}^0$ and $\tilde{\mu}_a \colon \tilde{Y}_a \to \tilde{X}^0$ over $\tilde{X}^0$. If we identify the function fields $F_v^{ur}(Y_x)$ and $F_v^{ur}(Y_a)$ via Int_g, they give the normalization of $\tilde{X}^0$ in the field $F_v^{ur}(Y_a)$. Hence there exists an isomorphism I_g over O_v^{ur} which allows the commutative diagram

$$\begin{array}{ccc} \tilde{Y}_x & \xrightarrow{I_g} & \tilde{Y}_a \\ {\scriptstyle \tilde{\mu}_x}\downarrow & & \downarrow{\scriptstyle \tilde{\mu}_a} \\ \tilde{X}^0 & \xrightarrow{g} & \tilde{X}^0 \end{array}$$

such that the restriction of I_g to the generic fiber is Int_g. Let ${}_\alpha(\tilde{Y}_x)$ and ${}_\alpha(\tilde{X}^0)$ be the twists of $\tilde{Y}_x$ and $\tilde{X}^0$ by the 1-cocycle α. Then we have

$$\begin{array}{ccc} {}_\alpha(\tilde{Y}_x) & \xrightarrow{{}_\alpha I_g} & \tilde{Y}_a \\ {\scriptstyle {}_\alpha(\tilde{\mu}_x)}\downarrow & & \downarrow{\scriptstyle \tilde{\mu}_a} \\ {}_\alpha(\tilde{X}^0) & \xrightarrow{{}_\alpha g} & \tilde{X}^0, \end{array}$$

where ${}_\alpha I_g$ and ${}_\alpha g$ are isomorphisms over O_v. The point $y \in Y_x(O_v)$ is invariant under the inner action of $G_x(O_v^{ur})$ on $Y_x(O_v^{ur})$. Hence y belongs to ${}_\alpha(\tilde{Y}_x)(O_v)$ and $\tilde{\mu}_a \circ {}_\alpha I_g(y) = a$. Since the condition in Proposition 2.1 is satisfied for $a = x$, $\tilde{Y}_a(O_v)$ consists of a single $G(O_v)$-orbit.

For $z \in X^0(O_v, \tilde{\alpha})$, let $z = hx$ with $h \in G(O_v^{un})$. Noting $\Pi_a \simeq {}_\alpha\Pi_x$, we see that under the bijections

$$H^1(F, G_x) \simeq H^1(F, {}_\alpha G_x) \simeq H^1(F, G_a)$$

the 1-cocycle $(h^{-1\sigma}h)$ corresponds to

$$g(h^{-1\sigma}h(g^{-1\sigma}g)^{-1})g^{-1} = (hg^{-1})^{-1\sigma}(hg^{-1}),$$

and we see that $\varphi_a(z) = \tilde{1}$. The assertion follows from this in the same way as above.

The assertion for $\bar{\mu}_a$ can be deduced easily from the above argument and Lemma 2.3.

We note that for $z \in X^0(O_v)$, $z \in X^0(O_v, \tilde{\alpha})$ if and only if $z \bmod \mathfrak{p}_v \in X^0(O_v, \tilde{\alpha}) \bmod \mathfrak{p}_v$. This can be seen easily by the proof of Proposition 2.4.

For a set of finite places Σ_0 such that $\Sigma \setminus \Sigma_0$ is finite, let O_{Σ_0} be the ring of $\Sigma_f \setminus \Sigma_0$-integers in F, that is, the ring consisting of all elements z of F satisfying $z \in O_v$ for all $v \in \Sigma_0$.

In the rest of the paper, we fix $x \in X^*(F)$, and assume the conditions in Proposition 2.1 are satisfied for each $v \in \Sigma_0$. Namely, we assume that there exists an affine group scheme $\tilde{G}$, integral noetherian smooth over $\operatorname{Spec} O_{\Sigma_0}$, whose fibre at each $v \in \Sigma_0$ is connected. We assume $P_i \in O_{\Sigma_0}[X]$, $\notin \mathfrak{p}_v[X]$ for each $v \in \Sigma_0$. We set $\tilde{X}^0 = \operatorname{Spec} O_{\Sigma_0}[X]_P$, $\tilde{S} = \operatorname{Spec} O_{\Sigma_0}[X]/(P)$ for $P = \prod_i P_i$, and assume $x \in \tilde{X}^0(O_{\Sigma_0})$. The action on G on X^0 is extended to that of $\tilde{G}$ on $\tilde{X}^0$, and the morphism ν_x is extended to a smooth morphism $\tilde{\nu}_x$ of $\tilde{G}$ to $\tilde{X}^0$.

Let $\tilde{Y}_x$ be the normalization of $\tilde{X}_0$ in $F(Y_x)$. The morphism λ_x is extended to a smooth morphism of $\tilde{G}$ to $\tilde{Y}_x$ and μ_x is extended to an étale morphism $\tilde{\mu}_x$ of $\tilde{Y}_x$ to $\tilde{X}^0$. Let $y = \tilde{\lambda}_x(\tilde{e})$ for the unit element of $\tilde{G}(O_{\Sigma_0})$ and $\tilde{G}_y$ the stabilizer of y. We assume that the fibres of $\tilde{G}$ and $\tilde{G}_y$ are connected and $\tilde{\lambda}_x$ is surjective at each $v \in \Sigma_0$.

We also assume the following:

(1) The ℓ-adic Betti numbers of each fibres of $\tilde{S}$ at $v \in \Sigma_0$ are independent of v;

(2) For $v \in \Sigma_0$, v is unramified in the representations of $\mathrm{Gal}(\bar{F}/F)$ in Π_x, $\mathbb{X}(G)$ and $\mathbb{X}(G_x^0)$;

(3) For $v \in \Sigma_0$, $N\mathfrak{p}_v$ is prime to $|\Pi_x|$ and to the orders of finite groups in $\mathrm{GL}_l(\mathbb{Z})$ for $l = \mathrm{rank}(\mathbb{X}(G_x^0))$;

(4) For $v \in \Sigma_0$, $\mathbb{X}(G) \simeq \mathbb{X}(\bar{G})$ as $\mathit{\Gamma}_v^{ur}$-modules, $\dim R_u(\bar{G})$ is independent of v, and the exponents of H_{der} for $H = \bar{G}/R_u(\bar{G})$ are given by $a(1)-1, a(2)-1, \dots, a(r)-1$, $a(i) \geq 2$.;

(5) For $v \in \Sigma_0$, $\mathbb{X}(G_x^0) \simeq \mathbb{X}(\bar{G}_y)$ as $\mathit{\Gamma}_v^{ur}$-modules, $\dim R_u(\bar{G}_y)$ is independent of v, and the exponents of H_{der} for $H = \bar{G}_y/R_u(\bar{G}_y)$ are given by $b(1)-1, b(2)-1, \dots, b(r')-1$, $b(i) \geq 2$.

We can easily verify there exists Σ_0 satisfying these conditions excluding bad places. For the condition (4) of Proposition 2.1, we refer to [P-R, Proposition 3.22].

By the conditions (4), (5), we have estimates

(2.12)
$$\prod_{i=1}^{r}(1 - q_v^{-a(i)}) \leq q_v^{-\dim G} L_v(1, \chi_{\mathbb{X}(G)}) |\bar{G}(k_v)| \leq \prod_{i=1}^{r}(1 + q_v^{-a(i)}),$$

(2.13)
$$\prod_{i=1}^{r'}(1 - q_v^{-b(i)}) \leq q_v^{-\dim G_x} L_v(1, \chi_{\mathbb{X}(G_x^0)}) |(\bar{G}_x)^0(k_v)| \leq \prod_{i=1}^{r'}(1 + q_v^{-b(i)}),$$

where $L_v(s, \chi_{\mathbb{X}(G)})$ and $L_v(s, \chi_{\mathbb{X}(G_x^0)})$ are the v-components of the Artin L-functions for representations of Γ on $\mathbb{X}(G)$ and $\mathbb{X}(G_x^0)$ respectively.

For $a \in X^*(F)$, let Σ_a be the maximal subset of Σ_0 such that $a \in X^0(O_{\Sigma_a})$. Define $\tilde{Y}_a$ to be the normalization of $\tilde{X}^0 \otimes_{O_{\Sigma_0}} O_{\Sigma_a}$ in the function field $F(Y_a)$ of Y_a. Then the action of G on Y_a extends to that of $\tilde{G}\times_{O_{\Sigma_0}} O_{\Sigma_a}$ on $\tilde{Y}_a$. If $v\in\Sigma_a$, $(\bar{G}_a)^0$ is a twist of $(\bar{G}_x)^0$ and the estimate (2.13) holds also for $(\bar{G}_a)^0$. We also note that for $v\in\Sigma_a$, $\Pi_a\simeq\bar{\Pi}_a$ as Γ_v^{ur}-groups for $\bar{\Pi}_a=\bar{G}_a/(\bar{G}_a)^0$.

§3. Multiplicative zeta functions

In this section, we prove the convergence of the multiplicative zeta function (1.3) and an integral on $Y_a(\mathbb{A})$.

Let $G_1=G_{der}G_a$ for $a\in X^0$. Then the group G_1 is defined over F and independent of the choice of $a\in X^0$. Set

$$\mathbb{X}_1(G)=\{\ \chi\in\mathbb{X}(G)\ :\ \chi|_{G_1}=1\ \}.$$

Then it is known that for $\chi\in\mathbb{X}(G)$, there exists a relative invariant with the associated character χ if and only if $\chi\in\mathbb{X}_1(G)$(cf. [S-K, Proposition 19]). Let S_i, P_i and χ_i be as in Introduction. We know that the characters χ_i's make a basis of $\mathbb{X}_1(G)_F$(cf. [S1, Lemma 1.3]).

Let $G_0=G_{der}G_a^0$ for $a\in X^0$. Then this group is also defined over F and is independent of the choice of a. By the assumption (1.1), $\mathbb{X}(G_0)_F=\{1\}$. Let

$$\mathbb{X}_0(G)=\{\ \chi\in\mathbb{X}(G)\ :\ \chi|_{G_0}=1\ \}.$$

Then we have an exact sequence

$$1\longrightarrow\mathbb{X}_0(G)_F\longrightarrow\mathbb{X}(G)_F\longrightarrow\mathbb{X}(G_0)_F.$$

The second map is the inclusion and the third one is the restriction. From $\mathbb{X}(G_0)_F=\{1\}$, we see

$$\mathbb{X}_0(G)_F=\mathbb{X}(G)_F. \tag{3.14}$$

Now we define measures on $G(\mathbb{A})$, $G_a^0(\mathbb{A})$, $Y_a(\mathbb{A})$ for $a\in X^*(F)$ and $X^0(\mathbb{A})$. Let ω be a gauge form on G defined over F. Let ω_v be the measure on $G(F_v)$ associated to ω. On $G(\mathbb{A})$, we take the Tamagawa measure

$$dg=\gamma_G^{-1}|\Delta_F|^{-\dim G/2}\prod_{v\in\Sigma}c_v\omega_v.$$

Here Δ_F is the discriminant of F, $L(s,\chi_{\mathbb{X}(G)})$ is the Artin L-function of the representation of Γ in $\mathbb{X}(G)$,

$$\gamma_G=\lim_{s\to1}(s-1)^tL(s,\chi_{\mathbb{X}(G)})$$

for the order t of the pole at $s=1$ of $L(s,\chi_{\mathbb{X}(G)})$ and c_v is the convergence factor given by

$$c_v = \begin{cases} L_v(1,\chi_{\mathbb{X}(G)}) & \text{if } v\in\Sigma_f, \\ 1 & \text{if } v\in\Sigma_\infty. \end{cases}$$

To define a measure on $Y_a(\mathbb{A})$, we need the following.

LEMMA 3.1. *Let dX be the differential form $dx_1 dx_2\cdots dx_d (d{=}\dim X)$ on X for a system of coordinates $x_1,x_2,\ldots,x_d$ of $\tilde{X}$ over O_{Σ_0}. Let $dY=\mu_a^* dX$. Then there exists a function $f(y)$ on Y_a such that*

$$\eta = \frac{1}{f(y)}dY$$

defines a G-invariant gauge form on Y_a. Moreover there exists an integer m such that

$$f^m = c_a\mu_a^* R$$

for a relative invariant R on X^0 defined over F and a constant $c_a\in F^\times$.

Proof. By the definition of dY, we have

$$g^* dY = \det\rho(g)dY$$

for $g\in G$ and $\det\rho\in\mathbb{X}(G)_F$. By (3.14), $\det\rho\in\mathbb{X}_0(G)_F$. Hence $\det\rho$ is trivial on $G_0=G_{der}G_a^0$. Define a function f on Y_a by

$$f(gG_a^0)=\det\rho(g).$$

Then f is a function rational over F and satisfies

$$f(gy)=\det\rho(g)f(y).$$

From this, we see η satisfies the required condition.

Since $[\mathbb{X}_0(G)_F:\mathbb{X}_1(G)_F]<\infty$, there exists a positive integer m such that $\det\rho^m\in\mathbb{X}_1(G)_F$. Hence there exist integers $d_1,\ d_2,\ldots,\ d_n$ satisfying

$$\det\rho^m=\chi_1^{d_1}\chi_2^{d_2}\cdots\chi_n^{d_n} \tag{3.15}$$

Set $R=P_1^{d_1}P_2^{d_2}\cdots P_n^{d_n}$. Then we see that $\mu_a^* R/f^m$ is a constant. This proves the second assertion.

Let $\xi = \omega/\lambda_a^*\eta$ be the gauge form on G_a^0 determined by ω and $\lambda_a^*\eta$. We note that ξ is bi-invariant and $L(1, \chi_{\mathbb{X}(G_a^0)})$ is finite if $a \in X^*(F)$. On $G_a^0(\mathbb{A})$, we employ the Tamagawa measure

$$dh = L(1, \chi_{\mathbb{X}(G_a^0)})^{-1}|\Delta_F|^{-\dim G_a^0/2} \prod_{v\in\Sigma} d_v\xi_v,$$

where ξ_v is the measure on $G_x^0(F_v)$ associated to ξ and d_v is the convergence factor given by

$$d_v = \begin{cases} L_v(1, \chi_{\mathbb{X}(G_a^0)}) & \text{if } v \in \Sigma_f, \\ 1 & \text{if } v \in \Sigma_\infty. \end{cases}$$

On $Y_a(\mathbb{A})$, we take the measure

$$\begin{aligned} dy &= \gamma_G^{-1} L(1, \chi_{\mathbb{X}(G_a^0)})|\Delta_F|^{-\dim X/2} \prod_{v\in\Sigma} c_v d_v^{-1}\eta_v \\ &= \gamma_G^{-1} L(1, \chi_{\mathbb{X}(G_a^0)})|\Delta_F|^{-\dim X/2} \prod_{v\in\Sigma} |c_a|_v^{1/m} c_v d_v^{-1}\eta_v. \end{aligned}$$

Here η_v is the measure on $Y_a(F_v)$ associated to η and c_a is the constant in Lemma 3.1. Then we see the measures dg, dh and dy are compatible. We note that $|c_a|^{1/m}\eta_v$ is independent of the choice of c_a and f for a fixed R.

Let dX be as in Lemma 3.1 and let dX_v be the measure on $X(F_v)$ associated to dX. We define a convergence factor e_v by

$$e_v = \begin{cases} L_v(1, \chi_{\mathbb{X}_1(G)}) & \text{if } v \in \Sigma_f, \\ 1 & \text{if } v \in \Sigma_\infty, \end{cases}$$

and define a measure on $X^0(\mathbb{A})$ by $dX = \gamma_G^{-1}\prod_v e_v dX_v$. Let d_i and m be as in the proof of Lemma 3.1 and set $\kappa_i = d_i/m$. Then

$$\prod_{i=1}^n |P_i(x_v)|_v^{-\kappa_i} dX_v \tag{3.16}$$

defines a $G(F_v)$-invariant measure on $X^0(F_v)$. For a function Ψ on $X^0(F_v)$, we have

$$\begin{aligned} &\int_{Y_a(F_v)} \Psi(\mu_a(y_v))|c_a|_v^{1/m}\eta_v \\ &\qquad = |\Pi_a(F_v)| \int_{X^0(F_v,\tilde{\alpha}_v)} \Psi(x_v) \prod_{i=1}^n |P_i(x_v)|_v^{-\kappa_i} dX_v \end{aligned} \tag{3.17}$$

with the notation in Proposition 2.5. Hence the integral on the left hand side depends only on the class $\tilde{\alpha}_v$ and is independent of the choice of a.

We recall that the multiplicative zeta function $Z_m(\Phi, s)$ is defined by

$$Z_m(\Phi, s) = \int_{X^0(\mathbb{A})} \prod_{i=1}^{n} |P_i(x)|_{\mathbb{A}} \Phi(x) dX$$

for $\Phi \in \mathcal{S}(X(\mathbb{A}))$. To prove the convergence, we may assume $\Phi = \prod_v \Phi_v$. In the following, making Σ_0 smaller if necessary, we assume that the set of places Σ_0 satisfies the following conditions:

(1) For $v \in \Sigma_0$, Φ_v is the characteristic function of $X(O_v)$;

(2) $c_0 q_v^{-3/2} < 1$ for $v \in \Sigma_0$ for the constant c_0 in the following Lemma 3.2.

LEMMA 3.2. *Assume* $\mathrm{Re}\, s_i > 0$ *for* $i = 1, 2, \dots, n$. *There exists a constant* c_0 *independent of* $v \in \Sigma_0$ *such that for* $v \in \Sigma_0$

$$|\int_{X^0(O_v)} \prod_{i=1}^{n} |P_i(x_v)|_v^{s_i} \Phi_v(x_v) e_v dX_v - 1| \le c_0 q_v^{-3/2}. \tag{3.18}$$

Proof. Let $P = \prod_{i=1}^m Q_i$ be a decomposition into F_v-irreducible polynomials $Q_i \in O_v[X]$, and assume that each Q_i decomposes into t_i absolutely irreducible polynomials. Then we see

$$L_v(1, \chi_{\mathbb{X}_1(G)}) = \prod_{i=1}^{m} (1 - q_v^{-t_i})^{-1}.$$

On $X^0(O_v)$, we have $\prod_i |P_i(x_v)|_v^{s_i} \Phi_v(x_v) = 1$, and

$$\int_{X^0(O_v)} \prod_{i=1}^{n} |P_i(x_v)|_v^{s_i} \Phi_v(x_v) dX_v = q_v^{-\dim X} |\bar{X}^0(k_v)|.$$

To estimate $|\bar{X}^0(k_v)|$, it is enough to estimate $|\bar{S}(k_v)|$. This can be written as an alternating sum over the traces of the Frobenius endomorphim on $H_c^i(\bar{S}, \mathbb{Q}_\ell)$ for $i = 0,\ 1, \dots,\ 2(\dim X - 1)$. By the assumptions (1) and (4) in Section 2, $\bar{Q}_i$, the reduction modulo $\mathfrak{p}_v$ of Q_i, is irreducible over k_v and decomposes into t_i absolutely irreducible polynomials. Hence the trace on

$H_c^{2(\dim X-1)}(\bar{S}, \mathbb{Q}_\ell)$ is equal to $q_v^{\dim X-1}$ times the number l of t_i such that $t_i = 1$. From this, we see

$$||\bar{X}^0(k_v)| - q_v^{\dim X} + lq_v^{\dim X-1}| \leq Cq_v^{\dim X-3/2}$$

for a constant C independent of v and

$$\left|e_v q_v^{-\dim X}|\bar{X}^0(k_v)| - 1\right| \leq c_0 q_v^{-3/2}$$

for a constant c_0 independent of v. This completes the proof.

This lemma can be proved also by a coarser estimate by Lang and Weil [L-W].

PROPOSITION 3.3. *If* $\operatorname{Re} s_i > 0$ *for* $i = 1, 2, \ldots, n$, *then* $Z_m(\Phi, s)$ *converges absolutely.*

Proof. It is enough to prove that if $\operatorname{Re} s_i > 0$ for $i = 1,\ 2, \ldots,\ n$, then

$$\prod_{v \in \Sigma_0} \int_{X(O_v)} \prod_{i=1}^{n} |P_i(x_v)|_v^{s_i} e_v dX_v \tag{3.19}$$

converges absolutely. For $v \in \Sigma_0$, let $E_0 = X^0(O_v)$ and $E_1 = X(O_v) \setminus X^0(O_v)$. If $\operatorname{Re} s_i \geq \varepsilon > 0$ for $i = 1,\ 2, \ldots,\ n$, then for $z \in E_1$, we have $|\prod_i |P_i(z)|_v^{s_i}| \leq q_v^{-\varepsilon}$ and

$$\int_{E_1} \left|\prod_i |P_i(x_v)|_v^{s_i}\right| dX_v \leq C_1 q_v^{-1-\varepsilon},$$

since

$$\int_{E_1} dX_v \leq q_v^{-\dim X} |\{E_1 \bmod \mathfrak{p}_v\}|$$

and $|\{E_1 \bmod \mathfrak{p}_v\}| \leq C_1 q_v^{\dim X-1}$ with a constant C_1 independent of v. From this and Lemma 3.2, we have

$$\left|1 - \int_{X(O_v)} \prod_i |P_i(x_v)|_v^{s_i} e_v dX_v\right| \leq C_2 q_v^{-1-\varepsilon}.$$

Here C_2 is a constant independent of v. From this, our assertion follows easily.

For the application in Section 4, we modify this result as follows. Set

$$C_v = \frac{\prod_{i=1}^{r}(1+q_v^{-a(i)})}{\prod_{i=1}^{r'}(1-q_v^{-b(i)})(1-c_0 q_v^{-3/2})} \tag{3.20}$$

for the constant c_0 in Lemma 3.2, and choose M so that

$$|\Pi_x| c_v d_v^{-1} e_v^{-1} C_v^{-1} \leq M.$$

We can choose M independent of v. We note that $C_v > 0$ under the above assumption (2). Define a function Φ'_v on $X(F_v)$ by

$$\Phi'_v(z) = \begin{cases} 1 & \text{if } z \in X^0(O_v), \\ M & \text{if } z \in X(O_v) \setminus X^0(O_v), \\ 0 & \text{otherwise.} \end{cases} \tag{3.21}$$

Then we have

Corollary 3.4. *Let*

$$T_v(s) = \int_{X^0(F_v)} \prod_{i=1}^{n} |P_i(x_v)|_v^{s_i} \Phi'_v(x_v) e_v dX_v.$$

If $\operatorname{Re} s_i > 0$ *for* $i = 1, 2, \ldots, n$*, then the infinite product* $\prod_{v \in \Sigma_0} T_v(s)$ *converges.*

Proof. This follows from the proof of the proposition and the fact that the set of points z satisfying $\Phi'(z) \neq \Phi(z)$ is contained in $X(O_v) \setminus X^0(O_v)$.

Similarly we can prove

Proposition 3.5. *Let* κ_i *be as in* (3.16)*. If* $\operatorname{Re} s_i - \kappa_i > 0$ *for* $i = 1, 2, \ldots, n$*, the integral*

$$\int_{Y_a(\mathbb{A})} \prod_{i=1}^{n} |P_i(\mu_a(y))|_{\mathbb{A}}^{s_i} \Phi(\mu_a(y)) dy$$

converges absolutely.

Proof. First we estimate the integral

$$\begin{aligned} I_v &= \int_{Y_a(O_v)} \prod_i |P_i(\mu_a(y_v))|_v^{s_i} \Phi_v(\mu_a(y_v)) |c_a|_v^{1/m} c_v d_v^{-1} \eta_v \\ &= \int_{Y_a(O_v)} |c_a|_v^{1/m} c_v d_v^{-1} \eta_v \end{aligned}$$

for $v \in \Sigma_a$. By Proposition 2.5 and (3.17), we see

$$\begin{aligned} I_v &= c_v d_v^{-1} q_v^{-\dim X} |\bar{\Pi}_a(k_v)| |X^0(O_v, \tilde{\alpha}_v) \bmod \mathfrak{p}_v| \\ &= c_v d_v^{-1} q_v^{-\dim Y_a} |\bar{Y}_a(k_v)|. \end{aligned}$$

By the definition of Σ_a and (2.12), (2.13), we have $|\bar{Y}_a(k_v)| = |\bar{G}(k_v)|/|(\bar{G}_a)^0(k_v)|$, and

$$\frac{\prod_{i=1}^r (1 - q_v^{-a(i)})}{\prod_{i=1}^{r'} (1 + q_v^{-b(i)})} \le I_v \le \frac{\prod_{i=1}^r (1 + q_v^{-a(i)})}{\prod_{i=1}^{r'} (1 - q_v^{-b(i)})}.$$

Hence $|I_v - 1| \le C_1 q_v^{-2}$ with a constant C_1 independent of v.

Choose ε so that $\operatorname{Re} s_i - \kappa_i \ge \varepsilon > 0$. By Proposition 2.5 and (3.17), we see

$$\begin{aligned} &\int_{Y_a(F_v) \setminus Y_a(O_v)} \left| \prod_{i=1}^n |P_i(\mu_a(y_v))|_v^{s_i} \right| |\Phi_v(\mu_a(y_v))| |c_a|_v^{1/m} c_v d_v^{-1} \eta_v \\ &\qquad \le |\Pi_a(F_v)| c_v d_v^{-1} \int_{X^0(F_v) \setminus X^0(O_v)} \left| \prod_{i=1}^n |P_i(x_v)|_v^{s_i - \kappa_i} \right| |\Phi_v(x_v)| dX_v \\ &\qquad \le |\Pi_a(F_v)| c_v d_v^{-1} \int_{X(O_v) \setminus X^0(O_v)} \left| \prod_i |P_i(x_v)|_v^{s_i - \kappa_i} \right| dX_v \\ &\qquad \le C_2 q_v^{-1-\varepsilon} \end{aligned}$$

with a constant C_2 independent of v, since.

$$\left| \prod_{i=1}^n |P_i(\mu_a(y_v))|_v^{s_i - \kappa_i} \right| \le q_v^{-\varepsilon}.$$

on $X(O_v) \setminus X^0(O_v)$. The assertion follows in the same way as Proposition 3.3.

§4. Convergence of zeta functions

In this section, we give a proof of Theorem 1.1 and Theorem 1.2, that is, the convergence of $Z(\Phi, s)$ in (1.2). As in Sections 2 and 3, we fix $x \in X^*(F)$ and assume that Φ and Σ_0 satisfy the condition in the previous sections

For $\tilde{\alpha} \in H^1(F, \Pi_x)$, set

$$X^*(F, \tilde{\alpha}) = X^0(F, \tilde{\alpha}) \bigcap X^*(F).$$

Then $X^*(F, \tilde{\alpha}) = X^0(F, \tilde{\alpha})$, or $\emptyset$. We define

$$Z(\Phi, s; \tilde{\alpha}) = \int_{G(\mathbb{A})/G(F)} \prod_{i=1}^{n} |\chi_i(g)|_{\mathbb{A}}^{s_i} \sum_{z \in X^*(F, \tilde{\alpha})} \Phi(gz) dg. \tag{4.22}$$

Then we have

$$Z(\Phi, s) = \sum_{\tilde{\alpha} \in H^1(F, \Pi_x)} Z(\Phi, s; \tilde{\alpha}).$$

First we show that $Z(\Phi, s; \tilde{\alpha})$ converges absolutely for $\operatorname{Re} s_i - \kappa_i > 0$. Assume $X^*(F, \tilde{\alpha}) \neq \emptyset$ and fix $a \in X^0(F, \tilde{\alpha})$. Assume $s_i \in \mathbb{R}$. Then by (2.5), (2.7), we have

$$\begin{aligned}
&\int_{G(\mathbb{A})/G(F)} \prod_i |\chi_i(g)|_{\mathbb{A}}^{s_i} \sum_{z \in X^0(F, \tilde{\alpha})} |\Phi(gz)| dg \\
&= \frac{1}{|\Pi_a(F)|} \int_{G(\mathbb{A})/G(F)} \prod_i |\chi_i(g)|_{\mathbb{A}}^{s_i} \sum_{w \in Y_a(F)} |\Phi(\mu_a(gw))| dg \\
&= \frac{1}{|\Pi_a(F)|} \sum_{w \in G(F)\backslash Y_a(F)} \int_{G(\mathbb{A})/G_w(F)} \prod_i |\chi_i(g)|_{\mathbb{A}}^{s_i} |\Phi(\mu_a(gw))| dg \\
&= \frac{\tau(G_a^0)}{|\Pi_a(F)|} \sum_{w \in G(F)\backslash Y_a(F)} \int_{G(\mathbb{A})w} \prod_i \Big| P_i(\mu_a(y)) \Big|_{\mathbb{A}}^{s_i} |\Phi(\mu_a(y))| dy \\
&\leq c(\tilde{\alpha}) \sum_{w \in G(\mathbb{A})\backslash G(\mathbb{A})Y_a(F)} \int_{G(\mathbb{A})w} \prod_i \Big| P_i(\mu_a(y)) \Big|_{\mathbb{A}}^{s_i} |\Phi(\mu_a(y))| dy \\
&= c(\tilde{\alpha}) \int_{G(\mathbb{A})Y_a(F)} \prod_i \Big| P_i(\mu_a(y)) \Big|_{\mathbb{A}}^{s_i} \Phi(\mu_a(y))| dy \\
&\leq c(\tilde{\alpha}) \int_{Y_a(\mathbb{A})} \prod_i \Big| P_i(\mu_a(y)) \Big|_{\mathbb{A}}^{s_i} |\Phi(\mu_a(y))| dy,
\end{aligned}$$

where

$$c(\tilde{\alpha}) = \frac{\tau(G_a^0)\ker^1(G_a^0)}{|\Pi_a(F)|} = \frac{|A(G_a^0)|}{|\Pi_a(F)|}.$$

We note that $c(\tilde{\alpha})$ depends only on $\tilde{\alpha}$. The last integral converges for $\operatorname{Re} s_i > \kappa_i$ by Proposition 3.5.

Now we turn to the proof of the convergence of $Z(\Phi, s)$. Assume $s_i \in \mathbb{R}$. Since $\prod_{v\in\Sigma\setminus\Sigma_0} H^1(F_v, \Pi_x)$ is a finite set, it is enough to show that the series

$$\sum_{\tilde{\alpha}\in H^1(F,\Pi_x),\ \tilde{\alpha}_{\Sigma\setminus\Sigma_0}=\tilde{\alpha}_0} Z(\Phi, s; \tilde{\alpha})$$

converges absolutely. Here $\tilde{\alpha}_0$ is a fixed element of $\prod_{v\in\Sigma\setminus\Sigma_0} H^1(F_v, \Pi_x)$, and $\tilde{\alpha}_{\Sigma\setminus\Sigma_0}$ is the image of $\tilde{\alpha}$ into $\prod_{v\in\Sigma\setminus\Sigma_0} H^1(F_v, \Pi_x)$. By the above inequality, it is enough to show that

$$\sum c(\tilde{\alpha})|L(1, \chi_{\mathbb{X}(G_a^0)})| \int_{Y_a(\mathbb{A})} \prod_i \Big|P_i(\mu_a(y))\Big|_{\mathbb{A}}^{s_i} |\Phi(\mu_a(y))| \prod_v |c_a|^{1/m} c_v d_v^{-1} \eta_v$$

converges absolutely, where the sum is extended over all $\tilde{\alpha} \in H^1(F, \Pi_x)$ satisfying $\tilde{\alpha}_{\Sigma\setminus\Sigma_0} = \tilde{\alpha}_0$. As noted after (3.17), the local integrals on $Y_a(F_v)$ in the above expression depends only on $\tilde{\alpha}_v$. Hence, we see that it is also enough to show the convergence of

$$\sum_{\tilde{\alpha}\in H^1(F,\Pi_x),\ \tilde{\alpha}_{\Sigma\setminus\Sigma_0}=\tilde{\alpha}_0} c(\tilde{\alpha})|L(1, \chi_{\mathbb{X}(G_a^0)})| I_{\tilde{\alpha}}, \tag{4.23}$$

where

$$\begin{aligned} I_{\tilde{\alpha}} &= \int_{Y_a(\mathbb{A}_{\Sigma_0})} \prod_i |P_i(\mu_a(y_{\Sigma_0}))|_{\Sigma_0}^{s_i} |\Phi_{\Sigma_0}(\mu_a(y_{\Sigma_0}))| dy_{\Sigma^0} \\ &= \prod_{v\in\Sigma_0} \int_{Y_a(F_v)} \prod_i |P_i(\mu_a(y_v))|_v^{s_i} |\Phi_v(\mu_a(y_v))|_v |c_a|_v^{1/m} c_v d_v^{-1} \eta_v. \end{aligned} \tag{4.24}$$

Here

$$\mathbb{A}_{\Sigma_0} = \prod_{v\in\Sigma_0}{}' F_v,\ y_{\Sigma_0} \in Y_a(\mathbb{A}_{\Sigma_0}),\ |P_i(\mu_a(y_{\Sigma_0}))|_{\Sigma_0} = \prod_{v\in\Sigma_0} |P_i(\mu_a(y_v))|_v,$$

$$\Phi_{\Sigma_0} = \prod_{v\in\Sigma_0} \Phi_v,\ dy_{\Sigma_0} = \prod_{v\in\Sigma_0} |c_a|^{1/m} c_v d_v^{-1} \eta_v.$$

To complete the proof, we need some estimates.

LEMMA 4.1. *There exists a constant C depending only on the rank of $\mathbb{X}(G_x^0)$ such that*

$$|A(G_a^0)| < C$$

for all $a \in X^*(F)$.

Proof. Let $H = G_a^0/R_u(G_a^0)$. Then we have an exact sequence

$$1 \longrightarrow \pi_1(\bar{H}_{der}) \longrightarrow \pi_1(\bar{H}) \longrightarrow \pi_1(\overline{H/H_{der}}) \longrightarrow 1$$

and from this we obtain another exact sequence

$$\pi_1(\bar{H}_{der}) \longrightarrow \pi_1(\bar{H})_\Gamma \longrightarrow \pi_1(\overline{H/H_{der}})_\Gamma \longrightarrow 1.$$

Since $|\pi_1(\bar{H}_{der})|$ is bounded, it is enough to show that $|\pi_1(\overline{H/H_{der}})_\Gamma|$ is bounded. For $a \in X^*(F)$, $\pi_1(\overline{H/H_{der}})_\Gamma$ is a finite group. Let $\pi_1(\overline{H/H_{der}}) \simeq \mathbb{Z}^l$ as $\mathbb{Z}$-modules. Then the order of $\pi_1(\overline{H/H_{der}})_\Gamma$ depends only on the $\mathrm{GL}_l(\mathbb{Z})$-conjugacy class of the image of $\mathrm{Gal}(\bar{F}/F)$ into $\mathrm{GL}_l(\mathbb{Z})$. The assertion follows from the fact that there exist only a finite number of finite groups in $\mathrm{GL}_l(\mathbb{Z})$ up to $\mathrm{GL}_l(\mathbb{Z})$-conjugacy(cf. [P-R, Theorem 4.3]).

Let $a \in X^*(F)$, and let K be the Galois extension K of F corresponding to the kernel of the representation of $\mathrm{Gal}(\bar{F}/F)$ on $\mathbb{X}(G_a^0)$. Then we can see $\mathbb{X}(G_a^0)$ as a $\mathrm{Gal}(K/F)$-module. By the proof of Lemma 4.1, we may assume that $[K : F]$ are less than a constant depending only on the rank of $\mathbb{X}(G_x^0)$.

LEMMA 4.2. *Let $a \in X^*(F)$ and let $\mathfrak{f}_a$ be the Artin conductor of the representation of* $\mathrm{Gal}(K/F)$ *on $\mathbb{X}(G_a^0)$. Then there exist positive constants C and e independent of $a \in X^*(F)$ such that*

$$|L(1, \chi_{\mathbb{X}(G_a^0)})| \leq CN(\mathfrak{f}_a)^e.$$

This can be deduced easily by expressing $L(s, \chi_{\mathbb{X}(G_a^0)})$ via L-functions for characters of degree 1 by means of Brauer's theorem on characters of finite groups and by using the lower and the upper bounds of their values or residues at $s = 1$ by the conductors of characters in [L, Hauptzatz, Satz5] and [Br, II, Lemma, §5]. In [L], it is assumed that the characters are complex-valued. In the case of real characters, we can use the results of [Br].

COROLLARY 4.3. *For $a \in X^*(F)$ let $\varphi_x(a) = \tilde{\alpha}$ and let $\tilde{\alpha}_v$ be the image of $\tilde{\alpha}$ into $H^1(F_v, \Pi_x)$. Then there exist positive constsnts β and C independent of $a \in X^*(F)$ such that*

$$|L(1, \chi_{\mathbb{X}(G_a^0)})| \prod_{v \in \Sigma_0} \prod_{i=1}^{n} |P_i(z_v)|_v^\beta \leq C$$

for $z = (z_v) \in \prod_{v \in \Sigma^0} X(O_v) \bigcap X^0(F_v, \tilde{\alpha}_v)$.

Proof. If $z_v \in X^0(O_v)$, by the assumption (2) in Section 2, we see that the representation of $\mathrm{Gal}(K/F)$ on $\mathbb{X}(G_a^0)$ is unramified at v since $z \in G(O_v^{ur})x$. Hence if the representation ramifies at $v \in \Sigma_0$, then $\prod_i |P_i(z_v)|_v \leq q_v^{-1}$. The conductor-discriminant theorem says that

$$\mathfrak{D}_{K/F} = \prod_{\chi \in Irr((\mathrm{Gal}(K/F))} \mathfrak{f}(\chi)^{\chi(1)},$$

where $Irr(\mathrm{Gal}(K/F))$ is the set of isomorphism calsses of irreducible representation of $\mathrm{Gal}(K/F)$, $\mathfrak{D}_{K/F}$ is the discriminant of K/F, and $\mathfrak{f}(\chi)$ is the Artin conductor of χ. Hence it implies that $\mathfrak{f}_a$ divides $\mathfrak{D}_{K/F}^l$, where $l = \mathrm{rank}\, \mathbb{X}(G_a^0)$. On the other hand, the places in Σ_0 ramify at most tamely by the assumption (3) in Section 2. Hence $\mathfrak{f}_a$ is divided by $\mathfrak{p}_v$ at most $l[K : F]$ times. We note that $\prod_{v \in \Sigma \setminus \Sigma_0} H^1(F_v, \Pi_x)$ is a finite set, and that the contribution from the places $v \notin \Sigma_0$ is bounded. The assertion follows from this.

From Lemma 4.1 and Corollary 4.3, we deduce the following.

COROLLARY 4.4. *Let $I_{\tilde{\alpha}}$ be as in (4.24). Then there exists a constant C such that*

$$\begin{aligned} &c(\tilde{\alpha})|L(1, \chi_{\mathbb{X}(G_a^0)})| I_{\tilde{\alpha}} \\ &\qquad \leq C \prod_{v \in \Sigma_0} \int_{Y_a(F_v)} \prod_{i=1}^{n} |P_i(\mu_a(y_v))|_v^{s_i - \beta} |\Phi_v(\mu_a(y_v))| |c_a|_v^{1/m} c_v d_v^{-1} \eta_v \end{aligned}$$

for all $a \in X^*(F)$.

Let

$$X_v^{ur} = \bigcup_{\tilde{\alpha}_v \in H^1(\Gamma_v^{ur}, \tilde{\Pi}_x)} X^0(F_v, \tilde{\alpha}_v).$$

Then $X_v^{ur} \supset X^0(O_v)$.

Lemma 4.5. *For $v \in \Sigma_0$, let Φ'_v and C_v be as in* (3.20) *and* (3.21)*. If $\tilde{\alpha}_v \in H^1(\Gamma_v^{ur}, \tilde{\Pi}_x)$, then*

$$\int_{Y_a(F_v)} \prod_{i=1}^{n} |P_i(\mu_a(y_v))|_v^{s_i-\beta} \Phi_v(\mu_a(y_v)) |c_a|_v^{1/m} c_v d_v^{-1} \eta_v \leq C_v \int_{X_v^{ur}} \prod_i |P_i(x_v)|_v^{s_i-\kappa_i-\beta} \Phi'_v(x_v) e_v dX_v,$$

and if $\tilde{\alpha}_v \notin H^1(\Gamma_v^{ur}, \tilde{\Pi}_x)$, then

$$\int_{Y_a(F_v)} \prod_i |P_i(\mu_a(y_v))|_v^{s_i-\beta} \Phi_v(\mu_a(y_v)) |c_a|_v^{1/m} c_v d_v^{-1} \eta_v \leq C_v \int_{X^0(F_v)} \prod_i |P_i(x_v)|_v^{s_i-\kappa_i-\beta} \Phi'_v(x_v) e_v dX_v.$$

Proof. First assume $\tilde{\alpha}_v \in H^1(\Gamma_v^{ur}, \tilde{\Pi}_x)$. The integral on the left hand side is equal to

$$|\Pi_a(F_v)| \int_{X^0(F_v, \tilde{\alpha}_v)} \prod_{i=1}^{n} |P_i(x_v)|_v^{s_i-\kappa_i-\beta} \Phi_v(x_v) c_v d_v^{-1} dX_v$$

by (3.17). By the assumption, there exists $b \in X^0(O_v, \tilde{\alpha}_v)$, and the above integral is equal to that on the left hand side of the lemma for $a = b$. Hence we may assume that $a \in X^0(O_v, \tilde{\alpha}_v)$. Let $Z = X^0(F_v, \tilde{\alpha}_v) \bigcap (X(O_v) \setminus X^0(O_v))$. Then we get

$$X_v^{ur} \supset X^0(F_v, \tilde{\alpha}_v) \bigcap X(O_v) = X^0(O_v, \tilde{\alpha}_v) \bigcup Z,$$

where the union of the last member is disjoint. On $Y_a(O_v) = \mu_a^{-1}(X^0(O_v, \tilde{\alpha}_v))$, by Lemma 3.2 and the proof of Proposition 3.5, we have

$$\begin{aligned} \int_{Y_a(O_v)} \prod_i |P(\mu_a(y_v))|_v^{s_i-\beta} \Phi_v(\mu_a(y_v)) |c_a|_v^{1/m} c_v d_v^{-1} \eta_v &= \int_{Y_a(O_v)} |c_a|_v^{1/m} c_v d_v^{-1} \eta_v \\ &\leq C_v \int_{X^0(O_v)} \prod_i |P(x_v)|^{s_i-\kappa_i-\beta} \Phi'_v(x_v) e_v dX_v. \end{aligned}$$

By the definition of Φ' and C_v, and (3.17), we see

$$\begin{aligned}\int_{\mu_a^{-1}(Z)} \prod_i |P_i(\mu_a(y_v))|_v^{s_i-\beta}\Phi_v(\mu_a(y_v))|c_a|_v^{1/m}c_v d_v^{-1}\eta_v \\ \leq |\Pi_a(F_v)| \int_Z \prod_i |P_i(x_v)|_v^{s_i-\kappa_i-\beta}\Phi_v(x_v)c_v d_v^{-1}dX_v \\ \leq C_v \int_Z \prod_i |P_i(x_v)|_v^{s-\kappa_i-\beta}\Phi'_v(x_v)e_v dX_v.\end{aligned}$$

The assertion follows from this.

When $\tilde{\alpha}_v \notin H^1(\varGamma_v^{ur}, \tilde{\Pi}_x)$, $X^0(F_v, \tilde{\alpha}_v) \cap X^0(O_v) = \emptyset$. The second assertion follows from this in the same way as above.

Let J_v be the set obtained from $H^1(F_v, \Pi_x)$ by contracting $H^1(\varGamma_v^{ur}, \tilde{\Pi}_x)$ to one element, and consider $\Pi'_{v\in\Sigma_0} J_v$. Here the prime indicates that all components are the pointed element $H^1(\varGamma_v^{ur}, \tilde{\Pi}_x)$ except for a finite number of places. For $j = (j_v) \in \Pi'_{v\in\Sigma_0} J_v$, we define a subset X_j of $X^0(\mathbb{A}_{\Sigma_0})$ by

$$X_j = \prod_{j_v = H^1(\varGamma_v^{ur}, \tilde{\Pi}_x)} X_v^{ur} \times \prod_{j_v \neq H^1(\varGamma_v^{ur}, \tilde{\Pi}_x)} X^0(F_v, j_v).$$

It is easy to see that the union $\cup_j X_j$ is disjoint. We note that X_j is an open subset of $X^0(\mathbb{A}_{\Sigma_0})$. We define a map of $H^1(F, \Pi_x)$ to $\prod'_{v\in\Sigma_0} J_v$ by

$$\psi\colon H^1(F, \Pi_x) \longrightarrow \prod_{v\in\Sigma_0}{}' H^1(F_v, \Pi_v) \longrightarrow \prod_{v\in\Sigma_0}{}' J_v.$$

Lemma 4.6. *There exist positive constants γ and C independent of $\tilde{\alpha} \in H^1(F, \Pi_x)$ such that*

$$|\psi^{-1}(\psi(\tilde{\alpha}))| \prod_{i=1}^n |P_i(a_{\Sigma_0})|_{\Sigma_0}^{\gamma} \leq C$$

for all $a_{\Sigma_0} = (a_v)_{v\in\Sigma_0} \in X_j \cap \prod_{v\in\Sigma_0} X(O_v)$ with $j = \psi(\tilde{\alpha})$.

Proof. Let K be the Galois extension of F which corresponds to the kernel of the homomorphism of $\mathrm{Gal}(\bar{F}/F)$ to the automrophism group of Π_x. We note that $v \in \Sigma_0$ is unramified in K by the assumption (2) in Section 2. Let Σ^K be the set of places of K and Σ_0^K the set of places of

K lying above Σ_0. Define $J_w,\ w \in \Sigma_0^K$, and ψ_K in the same way as above taking K instead of F. Then we have a commutative diagram

$$\begin{array}{ccc} H^1(F,\Pi_x) & \xrightarrow{\psi} & \prod_{v\in\Sigma_0} J_v \\ \iota\big\downarrow & & \big\downarrow \\ H^1(K,\Pi_x) & \xrightarrow[\psi_K]{} & \prod_{w\in\Sigma_0^K} J_w. \end{array} \tag{4.25}$$

The vertical maps are induced by the restriction of $\mathrm{Gal}(\bar{F}/F)$ to $\mathrm{Gal}(\bar{K}/K)$. Let ι be the first vertical map. The cardinality of each fibre of ι does not exceed $[K:F]^{|\Pi_x|}$. Let $\tilde{\beta} = \iota(\tilde{\alpha})$ and set

$$N(\tilde{\beta}) = \{\ \tilde{\beta}' \in \iota(H^1(F,\Pi_x)) \mid \tilde{\beta}'_{\Sigma^K\setminus\Sigma_0^K} = \tilde{\beta}_0,\ \psi_K(\tilde{\beta}) = \psi_K(\tilde{\beta}')\ \}.$$

Here $\tilde{\beta}_0$ is the image of $\tilde{\alpha}_0$ into $\prod_{w\in\Sigma^K\setminus\Sigma_0^K} H^1(K_w,\Pi_x)$. It is enough to show that there exist constants ε and C such that

$$|N(\tilde{\beta})| \prod_i |P_i(a_{\Sigma_0})|_{\Sigma_0}^{\varepsilon} \le C$$

for all $a_{\Sigma_0} \in X_j \cap \prod_{v\in\Sigma_0} X(O_v)$.

Since $\mathrm{Gal}(\bar{K}/K)$ acts trivially on Π_x, a 1-cocycle of $\mathrm{Gal}(\bar{K}/K)$ in Π_x is a homomorphism of $\mathrm{Gal}(\bar{K}/K)$ to Π_x and two homorphisms $\tau_1,\ \tau_2$ are equivalent if and only if there exists $h \in \Pi_x$ such that $\tau_1 = h\tau_2 h^{-1}$. Let $K_{\tilde{\beta}'}$ be the field corresponding to the kernel of a homomorphism in the class of $\tilde{\beta}'$. Let $Inj(\mathrm{Gal}(K_{\tilde{\beta}'}/K),\Pi_x)$ be the set of injective homomorphisms of $\mathrm{Gal}(K_{\tilde{\beta}'}/K)$ into Π_x and let $Inj(\mathrm{Gal}(K_{\tilde{\beta}'}/K),\Pi_x)/\sim$ be the set of conjugacy classes with respect to Π_x. Then we can find a constant C_1 such that

$$|Inj(\mathrm{Gal}(K_{\tilde{\beta}'}/K),\Pi_x)/\sim| \le C_1$$

for all $\tilde{\beta}' \in H^1(F,\Pi_x)$, and from this we obtain

$$|N(\tilde{\beta})| \le C_1 |\{\ K_{\tilde{\beta}'} \mid \tilde{\beta}' \in N(\tilde{\beta})\}|.$$

Since the number of subgroups of Π_x is finite, it is enough to count $K_{\tilde{\beta}'}$ which satisfy $\mathrm{Gal}(K_{\tilde{\beta}'}/K) \simeq H$ for a fixed subgroup H of Π_x. For $\tilde{\beta}' \in N(\tilde{\beta})$ satisfying this condition, $[K_{\tilde{\beta}'}:\mathbb{Q}]$ and $|\Delta_{K_{\tilde{\beta}'}}|$ are identical. We know that the cardinality of fields having the same degree and the same discriminant as $K_{\tilde{\beta}'}$ is less than $C_2|\Delta_{K_{\tilde{\beta}'}}|^{\varepsilon_1}$ for positive constants ε_1 and C_2 depending only on Π_x by the classical theorem of Hermite-Minkowski.

Let w be a place of K lying above v and let $\Gamma_w^{ur} = \mathrm{Gal}(K_w^{ur}/K_w)$. The inflations give rise to the commutative diagram

$$\begin{array}{ccc} H^1(\Gamma_v^{ur}, \tilde{\Pi}_x) & \longrightarrow & H^1(F_v, \Pi_x) \\ \big\downarrow & & \big\downarrow \\ H^1(\Gamma_w^{ur}, \tilde{\Pi}_x) & \longrightarrow & H^1(K_w, \Pi_x). \end{array}$$

Let $\tilde{\beta}' = \iota(\tilde{\alpha}')$. From this diagram, we see that if $X^0(F_v, \tilde{\alpha}'_v) \cap X^0(O_v) \neq \emptyset$, that is, if $\tilde{\alpha}'_v \in H^1(\Gamma_v^{ur}, \tilde{\Pi}_x)$, then w is unramified in $K_{\tilde{\beta}'}$. Hence if $w \in \Sigma_0^K$ ramifies in $K_{\tilde{\beta}'}$, then $\tilde{\alpha}'_v \notin H^1(\Gamma_v^{ur}, \tilde{\Pi}_x)$. Therefore $a_v \in X^0(F, \tilde{\alpha}'_v) \cap X(O_v)$ does not belong to X_v^{ur}, hence $\prod_i P_i(a_v) \in \mathfrak{p}_v$. Since $w \in \Sigma_0^K$ is unramified or ramifies at most tamely in $K_{\tilde{\beta}'}$ by the assumption (3) in Section 2, there exist constants ε_2 depending only on Π_x such that

$$|N_{K_w/F_v}(\mathfrak{d}_{K_{\tilde{\beta}',\tilde{w}}/K_w})|_v^{-1} \prod_{i=1}^{n} |P_i(a_v)|_v^{\varepsilon_2} \leq 1$$

for the relative different $\mathfrak{d}_{K_{\tilde{\beta}',\tilde{w}}/K_w}$ of $K_{\tilde{\beta}',\tilde{w}}/K_w$, where $\tilde{w}$ is a place of $K_{\tilde{\beta}'}$ lying above w. For $w \in \Sigma^K \setminus \Sigma_0^K$, the completions of $K_{\tilde{\beta}'}$ at places lying above w are contained in a finte set of extensions of K_w, since their degrees do not exceed $|\Pi_x|$. Hence we have

$$|\Delta_{K_{\tilde{\beta}'}}|^{\varepsilon_1} \prod_{v \in \Sigma_0} \prod_{i=1}^{n} |P_i(a_v)|_v^{\varepsilon_3} \leq C_3$$

for positive constants ε_3 and C_3 independent of $\tilde{\alpha}$. This proves the assertion.

When Π_x is abelian, by class field theory, $|\{K_{\beta'} | \tilde{\beta}' \in N(\tilde{\beta})\}| \leq C_5$ with a constant C_5 and by the proof of the above lemma, we obtain

LEMMA 4.7. *Assume Π_x is abelian. Then there exists a constant C which satisfies*

$$|\psi^{-1}(\psi(\tilde{\alpha}))| \leq C$$

for all $\tilde{\alpha} \in H^1(F, \Pi_x)$.

We are ready to prove the convergence. First we prove Theorem 1.1. Assume $s_i \in \mathbb{R}$. We note $\prod_{v \in \Sigma_0} C_v$ converges. Hence by Corollary 4.4 and

Lemma 4.5, we see

$$c(\tilde{\alpha})|L(1,\chi_{\mathbb{X}(G_a^0)})|I_{\tilde{\alpha}} \\ \leq C' \int_{X_j} \prod_i |P_i(x_{\Sigma_0})|_{\Sigma_0}^{s_i-\kappa_i-\beta} \Phi'_{\Sigma_0}(x_{\Sigma_0}) dX_{\Sigma_0}.$$

for a constant C'. Here $j=\psi(\tilde{\alpha})$ and

$$dX_{\Sigma_0} = \prod_{v\in\Sigma_0} e_v dX_v, \quad \Phi'_{\Sigma_0} = \prod_{v\in\Sigma_0} \Phi'_v.$$

To estimate (4.23), it is enough to count the above integral $|\psi^{-1}(\psi(\tilde{\alpha}))|$ times. Hence by Lemma 4.6, (4.23) has

$$C'' \sum_j \int_{X_j} \prod_i |P_i(x_{\Sigma_0})|_{\Sigma_0}^{s_i-\kappa_i-\beta-\gamma} \Phi'_{\Sigma_0}(x_{\Sigma_0}) dX_{\Sigma_0}$$

as its upper bound for a constant C''. This converges if $s_i-\kappa_i-\beta-\gamma>0$ by Corollary 3.4. This completes the proof of Theorem 1.1.

For the proof of Theorem 1.2, it is enough to notice that $\beta=0$, since G_x^0 is semisimple, and also that we may take $\gamma=0$, since we can apply Lemma 4.7.

§5. Explicit form of zeta functions

For $\tilde{\alpha}\in H^1(F,\Pi_x)$, let $Z(\Phi,s;\tilde{\alpha})$ be as in (4.22). We recall

$$Z(\Phi,s) = \sum_{\tilde{\alpha}\in H^1(F,\Pi_x)} Z(\Phi,s;\tilde{\alpha}).$$

In this section, we show that for $\Phi=\prod_v \Phi_v \in \mathcal{S}(X(\mathbb{A}))$, $Z(\Phi,s;\tilde{\alpha})$ is a finite sum of Euler products under the assumption that the Hasse principle holds for G.

For $a\in X^*(F)$, let $\iota^0_{a,A}\colon A(G_a^0)\to A(G)$ be the canonical map. When G or G_a^0 is not reductive, we note that there exists a homorphism ι' of $G_a^0/R_u(G_a^0)$ to $G/R_u(G)$ such that the diagram

$$\begin{array}{ccc} G_a^0 & \xrightarrow{\iota} & G \\ \downarrow & & \downarrow \\ G_a^0/R_u(G_a^0) & \xrightarrow{\iota'} & G/R_u(G) \end{array}$$

is commutative. The map ι' induces a map $\iota^0_{a,A}$ of $A(G^0_a)$ to $A(G)$. This depends on the choice of ι', but $\operatorname{Ker}\iota^0_{a,A}$ is indepedent of the choice of ι'.

Let $\mathbb{X}(\operatorname{Ker}\iota^0_{a,A})$ be the group of characters of $\operatorname{Ker}\iota^0_{a,A}$. For each $\varepsilon \in \mathbb{X}(\operatorname{Ker}\iota^0_{a,A})$, we define a function ε_v on $Y_a(F_v)$ by connecting the following maps

$$Y_a(F_v) \longrightarrow H^1(F_v, G^0_a) \longrightarrow A(G^0_{a,v}) \longrightarrow A(G^0_a)$$

and ε. Here $G^0_{a,v} = G^0_a \otimes_F F_v$. We note that the image of $Y_a(F_v)$ is contained in $\operatorname{Ker}\iota^0_{a,A}$ and for $y = (y_v) \in Y_a(\mathbb{A})$, $\prod_v \varepsilon_v(y_v)$ is well-defined. Then by the assumption on the Hasse principle we have(cf. [Sa1, Proposition 1.7])

$$\sum_{\varepsilon\in\mathbb{X}(\operatorname{Ker}\iota^0_{a,A})} \prod_v \varepsilon_v(y_v) = \begin{cases} |\operatorname{Ker}\iota^0_{a,A}| & y \in G(\mathbb{A})Y_a(F), \\ 0 & \text{otherwise.} \end{cases}$$

We define a fucntion $\tilde{\varepsilon}_v$ on $X^*(F_v, \tilde{\alpha}_v)$ by

$$\tilde{\varepsilon}_v(x_v) = \frac{1}{|\Pi_a(F_v)|} \sum_{y_v\in\mu_a^{-1}(x_v)} \varepsilon_v(y_v).$$

Then we can prove the following theorem in the same way as [Sa1, Theorem 2.1].

THEOREM 5.1. *Assume that S is a hypersurface and the Hasse principle holds for G. Then for $\tilde{\alpha} \in H^1(F, \Pi_x)$ with $X^*(F, \tilde{\alpha}) \neq \emptyset$ and for $\Phi = \prod_v \Phi_v \in \mathcal{S}(X(\mathbb{A}))$, one has*

$$\begin{aligned} Z(\Phi, s; \tilde{\alpha}) = \gamma_G^{-1} |\Delta_F|^{-\dim X/2} &\frac{|A(G^0_a)|}{|\Pi_a(F)|} \frac{L(1, \chi_{\mathbb{X}(G^0_a)})}{|\operatorname{Ker}\iota^0_{a,A}|} \\ &\times \sum_{\varepsilon\in\mathbb{X}(\operatorname{Ker}\iota^0_{a,A})} \prod_v Z_v(\Phi_v, s; \tilde{\alpha}_v, \tilde{\varepsilon}_v), \end{aligned}$$

where $a \in X^*(F, \tilde{\alpha})$ and

$$\begin{aligned} &Z_v(\Phi_v, s; \tilde{\alpha}_v, \tilde{\varepsilon}_v) \\ &\quad = |\Pi_a(F_v)| \int_{X^0(F_v,\tilde{\alpha}_v)} \tilde{\varepsilon}_v(x_v)\Phi_v(x_v) \prod_{i=1}^n |P_i(x_v)|_v^{s_i-\kappa_i} c_v d_v^{-1} dX_v. \end{aligned}$$

Remark 5.2. In [Sa1], we assumed that $\operatorname{Ker}\rho = \{1\}$. This assumption is unnecesary. But the zeta function $Z(\Phi, s)$ and the expression via Euler products depend on the choice of G. For example, let $G = \mathrm{GL}_n,\ X = S_n$ the space of symmetric matrices of degree n, and define

$$\rho(g)x = gx\,{}^t g, \quad g \in G,\ x \in X.$$

Then $\operatorname{Ker}\rho = \{\pm 1\}$. In [Sa1], we gave an explicit expression of $Z(\Phi, s)$ of $(G/\operatorname{Ker}\rho, \bar{\rho}, X)$ for a good Φ. Here $\bar{\rho}$ is the representation of $G/\operatorname{Ker}\rho$ on X induced by ρ. For example, if n is odd, then $\Pi_x = \{1\},\ \operatorname{Ker}\iota^0_{a,A} = \{\pm 1\}$, and $Z(\Phi, s)$ is a sum of two Euler products.

Let us consider $Z(\Phi, s)$ of (G, ρ, X) for n odd. For $a \in X^0(F)$, $G_a = O_a (= SO_a\{\pm 1\})$ and $G^0_a = SO_a$, where

$$O_a = \{\ g \in \mathrm{GL}_n \ \mid ga\,{}^t g = a\ \},$$
$$SO_a = \{\ g \in \mathrm{SL}_n \ \mid ga\,{}^t g = a\ \}.$$

Hence $\Pi_x = \{\pm 1\}$, $H^1(F, \Pi_x) \simeq F^\times/F^{\times 2}$ and $Z(\Phi, s)$ is a sum of infinitely many Euler products. We see that $A(G^0_a) = \{\pm 1\}$, $A(G) = \{1\}$ and $\operatorname{Ker}\iota^0_{a,A} = A(G^0_a) = \{\pm 1\}$, and that $\tilde{\varepsilon}_v$ for the non-trivial element in $\mathbb{X}(\operatorname{Ker}\iota^0_{a,A})$ is given by

$$\tilde{\varepsilon}_v(z) = \frac{s_v(z)}{s_v(a)}$$

on $X^0(F_v, \tilde{\alpha}_v)$. Here $\tilde{\alpha} = \varphi_x(a)$ and s_v is the Hasse symbol. We can easily compute $Z(\Phi_v, s; \tilde{\alpha}_v, \varepsilon_v)$ for $\varepsilon \in \mathbb{X}(\operatorname{Ker}\iota^0_{a,A})$ at good v using [Sa2, Theorem 2.2].

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