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ON PURELY PERIODIC BETA-EXPANSIONS OF PISOT NUMBERS

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Abstract. We characterize numbers having purely periodic β-expansions where β is a Pisot number satisfying a certain irreducible polynomial. The main tool of the proof is to construct a natural extension on a d-dimensional domain with a fractal boundary.

§1. Introduction

Let $\beta > 1$ be a real number and let T_β be the β-transformation on the unit interval $[0, 1)$ given by

$$T_\beta x = \beta x - [\beta x],$$

where $[x]$ denotes the integer part of x. Then every $x \in [0, 1)$ can be written as

$$x = \sum_{k=1}^{\infty} b_k \beta^{-k}, \quad b_k = [\beta T_\beta^{k-1} x].$$

We call this representation in base β the β-expansion, which was introduced by Rényi [16]. It is denoted by

$$x = .b_1 b_2 \dots .$$

A real number $x \in [0, 1)$ is said to have an eventually periodic β-expansion with period p if there exist integers $m \geq 0$ and $p \geq 1$ such that

$$x = .b_1 b_2 \dots b_m (b_{m+1} b_{m+2} \dots b_{m+p})^\infty,$$

where w^∞ will denote the sequence $www \dots$. In particular, if we can choose $m = 0$, we say that x has a purely periodic β-expansion with period p, that is,

$$x = .(b_1 b_2 \dots b_p)^\infty.$$

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We know that x has a purely periodic β-expansion with period p if and only if $T_\beta^p x = x$.

For $x = 1$, we can define the β-expansion of 1 in the same way:

$$d(1,\beta) = .t_1 t_2 \ldots, \quad t_k = [\beta T_\beta^{k-1} 1].$$

Let D_β be the set of β-expansions of numbers in $[0,1)$. Parry characterized the set D_β in [13]. By $<_{lex}$ will be denoted the lexicographical order, that is, $(v_i)_{i=1}^\infty <_{lex} (w_i)_{i=1}^\infty$ means that there exists $k \geq 1$ such that $v_j = w_j$ for any $1 \leq j < k$ and $v_k \lneq w_k$. The (one-sided) shift σ_s maps a point $(v_i)_{i=1}^\infty$ to the point $(v'_i)_{i=1}^\infty = \sigma_s\left((v_i)_{i=1}^\infty\right)$ whose ith coordinate is given by $v'_i = v_{i+1}$.

Theorem (Parry). *Let $\beta > 1$ be a real number, and let $d(1,\beta) = .t_1 t_2 \ldots$. Let w be an infinite sequence of positive integers.*

(1) *If $d(1,\beta)$ is infinite,*

$$w \in D_\beta \Longleftrightarrow \forall u \geq 0,\ \sigma_s^u(w) <_{lex} d(1,\beta).$$

(2) *If $d(1,\beta)$ is finite, $d(1,\beta) = .t_1 \ldots t_{n-1} t_n$, say, then*

$$w \in D_\beta \Longleftrightarrow \forall u \geq 0,\ \sigma_s^u(w) <_{lex} d^*(1,\beta) = \left(t_1 \ldots t_{n-1}(t_n - 1)\right)^\infty.$$

Bertrand [3] and K. Schmidt [18] investigated eventually periodic β-expansions. A Pisot number is an algebraic integer (> 1) whose conjugates other than itself have modulus less than one. Let $\mathbb{Q}(\beta)$ be the smallest extension field of rational numbers $\mathbb{Q}$ containing β.

Theorem (Bertrand, K. Schmidt). *Let β be a Pisot number and let x be a real number in $[0,1)$. Then x has an eventually periodic β-expansion if and only if $x \in \mathbb{Q}(\beta)$.*

In [1], Akiyama gives a sufficient condition for pure periodicity where β belongs to a certain class of Pisot numbers. Hara and Ito characterized purely periodic modified β-expansions for a quadratic irrational number β in [8]. The present author studied necessary and sufficient condition for pure periodicity in [11] where β is a cubic Pisot number whose minimal polynomial is given by

$$Irr(\beta) = x^3 - k_1 x^2 - k_2 x - 1, \quad k_1 (\neq 0),\ k_2 \in \mathbb{N} \cup \{0\}, \text{ and } k_1 \geq k_2.$$

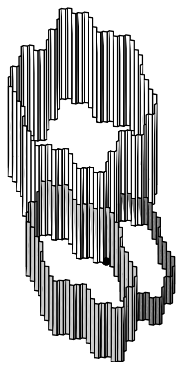

Figure 1: Figure of $\widehat{Y}$ in case $d = 3$.

In this paper, we will generalize the results of [11]. Hereafter, β is a positive root of the polynomial:

$$Irr(\beta) = x^d - k_1x^{d-1} - k_2x^{d-2} - \cdots - k_{d-1}x - 1,$$
$$k_i \in \mathbb{Z}, \text{ and } k_1 \geq k_2 \geq \cdots \geq k_{d-1} \geq 1.$$

Then β is a Pisot number. We have the following result:

MAIN THEOREM. *Let x be a real number in $\mathbb{Q}(\beta) \cap [0, 1)$. Then x has a purely periodic β-expansion if and only if x is reduced.*

We define reduced numbers in Section 5. For our purpose, we introduce a d-dimensional domain $\widehat{Y}$ with a fractal boundary (see Figure 1 and the definition in Section 4) and a natural extension of T_β on $\widehat{Y}$, which were originally discussed in [14] and [19]. In [8] and [9], you can find the basic idea of the proof.

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§2. Admissible sequences of β-expansions

Recall that β is a positive root of the irreducible polynomial

$$Irr(\beta) = x^d - k_1x^{d-1} - k_2x^{d-2} - \cdots - k_{d-1}x - 1, \quad (2.1)$$
$$k_i \in \mathbb{Z}, \text{ and } k_1 \geq k_2 \geq \cdots \geq k_{d-1} \geq 1.$$

From [4], we know that β is a Pisot number. From Theorem (Parry) it follows that

$$d(1,\beta) = .k_1k_2\dots k_{d-1}1.$$

Let

$$\beta = \beta^{(1)}, \beta^{(2)}, \dots, \beta^{(r_1)}$$

be the real Galois conjugates and

$$\beta^{(r_1+1)}, \overline{\beta^{(r_1+1)}}, \beta^{(r_1+2)}, \overline{\beta^{(r_1+2)}}, \dots, \beta^{(r_1+r_2)}, \overline{\beta^{(r_1+r_2)}}$$

be the complex Galois conjugates of β, where $r_1 + 2r_2 = d$ and $\bar{v}$ is the complex conjugate of a complex number v. The corresponding conjugates of $x \in \mathbb{Q}(\beta)$ are also denoted by

$$x = x^{(1)}, \dots, x^{(r_1)}, x^{(r_1+1)}, \overline{x^{(r_1+1)}}, \dots, x^{(r_1+r_2)}, \overline{x^{(r_1+r_2)}}.$$

Let M be the companion matrix of the polynomial (2.1), that is,

$$M = \begin{bmatrix} k_1 & k_2 & \dots & k_{d-1} & 1 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix}.$$

We know that M is a $d \times d$ integer matrix with determinant $(-1)^{d-1}$. It is easily checked that the matrix M is irreducible. Here a nonnegative matrix A is irreducible if for each ordered pair of indices I, J, there exists some $n \geq 0$ such that $A^n_{IJ} > 0$, where A_{IJ} means the (I,J)-element of the matrix A. An eigenvector $\boldsymbol{\alpha}$ corresponding to the eigenvalue β of M and an eigenvector $\boldsymbol{\gamma}$ corresponding to β of the transpose of M are vectors $\boldsymbol{\alpha} = {}^t[\alpha_1, \alpha_2, \dots, \alpha_d]$ and $\boldsymbol{\gamma} = {}^t[\gamma_1, \gamma_2, \dots, \gamma_d]$, satisfying

$$M\boldsymbol{\alpha} = \beta\boldsymbol{\alpha} \quad \text{and} \quad {}^tM\boldsymbol{\gamma} = \beta\boldsymbol{\gamma}, \quad \text{respectively,} \tag{2.2}$$

where t indicates the transpose. From the Perron-Frobenius theory, irreducibility implies both eigenvectors are positive. We normalize $\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$ by putting $\gamma_1 = 1$ and choosing α_i $(1 \leq i \leq d)$ to satisfy $\langle \boldsymbol{\alpha}, \boldsymbol{\gamma} \rangle = 1$, where $\langle\ ,\ \rangle$ denotes the standard inner product. By using (2.2), we can see that α_i and γ_i $(1 \leq i \leq d)$ are given by

$$\alpha_i = \beta^{1-i} \Big/ \sum_{n=0}^{d-1} \beta^{-n} T^n_\beta 1, \tag{2.3}$$

$$
\begin{aligned}
\gamma_1 &= 1 = .k_1 k_2 \dots k_{d-1} 1, \\
\gamma_2 &= T_\beta 1 = .k_2 \dots k_{d-1} 1, \\
&\vdots \\
\gamma_{d-1} &= T_\beta^{d-2} 1 = .k_{d-1} 1, \\
\gamma_d &= T_\beta^{d-1} 1 = .1 = \frac{1}{\beta}.
\end{aligned}
\tag{2.4}
$$

By $\mathbb{Z}[\beta]$ will be denoted the set of polynomials in β with integral coefficients. Then both $\{\alpha_1, \dots, \alpha_d\}$ and $\{\gamma_1, \dots, \gamma_d\}$ generate $\mathbb{Z}[\beta]$ and both are bases of $\mathbb{Q}(\beta)$.

It follows from either (2.4) or Theorem (Parry) in Section 1 that a sequence $(b_i)_{i=1}^\infty \in D_\beta$ if and only if for all i

$$0 \le b_i \le k_1, \tag{2.5}$$

(2.6)

$$
\begin{cases}
b_i = k_1 & \Longrightarrow b_{i+1} \le k_2, \\
b_i = k_1, b_{i+1} = k_2 & \Longrightarrow b_{i+2} \le k_3, \\
\quad\vdots & \quad\vdots \\
b_i = k_1, b_{i+1} = k_2, \dots, b_{i+d-3} = k_{d-2} & \Longrightarrow b_{i+d-2} \le k_{d-1}, \\
b_i = k_1, b_{i+1} = k_2, \dots, b_{i+d-3} = k_{d-2}, b_{i+d-2} = k_{d-1} & \Longrightarrow b_{i+d-1} = 0.
\end{cases}
$$

Thus D_β is represented by the labeled graph $\mathcal{G}$ in Figure 2. In other words, the admissible sequence $(b_i)_{i=1}^\infty$ of β-expansions is an infinite label of the walk in the sofic shift $\mathsf{X}_\mathcal{G}$. See [12] concerning a labeled graph and a sofic shift.

§3. Substitutions

Let σ be the substitution of the alphabet $\mathcal{A} = \{1, 2, \dots, d\}$ given by:

$$
\begin{aligned}
\sigma: \quad 1 &\longrightarrow \underbrace{1 \dots 1}_{k_1} 2 \\
2 &\longrightarrow \underbrace{1 \dots 1}_{k_2} 3 \\
&\dots \\
d-1 &\longrightarrow \underbrace{1 \dots 1}_{k_{d-1}} d
\end{aligned}
$$

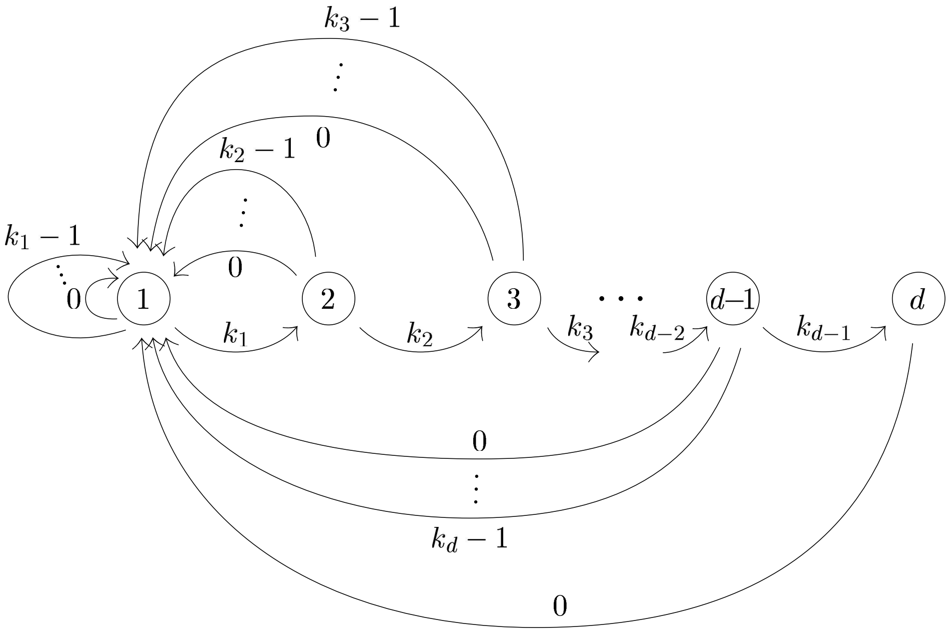

Figure 2: Labeled graph $\mathcal{G}$.

$$d \quad \longrightarrow 1.$$

The free monoid on $\mathcal{A}$, that is to say, the set of finite words on $\mathcal{A}$, is denoted by $\mathcal{A}^* = \bigcup_{n=0}^{\infty} \mathcal{A}^n$.

There is a natural homomorphism (abelianization) $f : \mathcal{A}^* \to \mathbb{Z}^d$ given by $f(i) = \mathbf{e}_i$ for any $i \in \mathcal{A}$ where $\{\mathbf{e}_1, \ldots, \mathbf{e}_d\}$ is the canonical basis of $\mathbb{R}^d$. Then there exists a unique linear transformation ${}^0\sigma$ satisfying the following commutative diagram:

$$\begin{array}{ccc} \mathcal{A}^* & \xrightarrow{\sigma} & \mathcal{A}^* \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ \mathbb{Z}^d & \xrightarrow[{}^0\sigma]{} & \mathbb{Z}^d. \end{array}$$

We know that ${}^0\sigma$ is given by the matrix M in Section 2 in our case.

Let $\mathcal{P}$ be the contractive invariant plane of M, that is,

$$\mathcal{P} = \left\{ \mathbf{x} \in \mathbb{R}^d \;\middle|\; \langle \mathbf{x}, \boldsymbol{\gamma} \rangle = 0 \right\}.$$

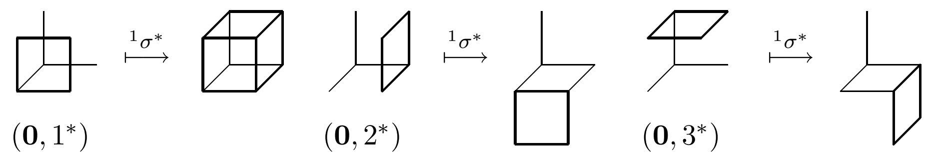

Figure 3: The figure for ${}^1\sigma^*$ for the Rauzy fractal ($k_1 = k_2 = 1$).

Let $\pi : \mathbb{R}^d \to \mathcal{P}$ be the projection along the eigenvector $\boldsymbol{\alpha}$. In [15], Rauzy constructed a curious compact domain with a fractal boundary, called the Rauzy fractal, by using the Pisot number β for which $Irr(\beta) = x^3 - x^2 - x - 1$ ($k_1 = k_2 = 1$). Arnoux and Ito in [2] showed that for any Pisot substitution σ a compact domain X with a fractal boundary can be similarly constructed, using the following mapping ${}^1\sigma^*$:

$$ {}^1\sigma^*(\mathbf{x}, i^*) = \sum_{j=1}^{d} \sum_{W_n^{(j)}=i} \big(M^{-1}\big(\mathbf{x} - f(P_n^{(j)})\big), j^*\big), \tag{3.1} $$

where $\sigma(j) = W_1^{(j)} \cdots W_{l_j}^{(j)}$, $W_n^{(j)} \in \{1, \ldots, d\}$, $P_n^{(j)}$ is the prefix of the letter $W_n^{(j)}$, and $(\mathbf{x}, i^*)$ is the set $\big\{\mathbf{x} + \mathbf{e}_i + \sum_{j\neq i} \lambda_j \mathbf{e}_j \;\big|\; \lambda_j \in [0,1]\big\}$. (See Figure 3.) We remark that we use the notation ${}^1\sigma^*$ in stead of $E_1^*(\sigma)$ which was used in [2].

In [17], the authors define higher dimensional extensions ${}^k\sigma$ ($1 \le k \le d$) of σ, acting on formal sums of weighted k-dimensional faces of unit cubes with vertices in $\mathbb{Z}^d$, and their dual maps ${}^k\sigma^*$. Moreover, they proved that these maps commute with the natural boundary morphisms and establish some basic properties.

THEOREM. *The following limit sets exist in the sense of Hausdorff metric*:

$$ \begin{aligned} X_i &:= \lim_{n\to\infty} M^n\big(\pi\big({}^1\sigma^{*n}(\mathbf{0}, i^*)\big)\big) \\ &= \lim_{n\to\infty} M^n\big(\pi\big({}^1\sigma^{*n}(-\mathbf{e}_i, i^*)\big)\big), \quad (1 \le i \le d) \\ X &= \bigcup_{i=1}^{d} X_i. \end{aligned} $$

X_i *are bounded, closed, and disjoint, up to a set of measure* 0.

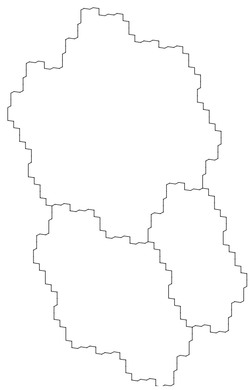

Figure 4: The Rauzy fractal ($k_1 = k_2 = 1$).

Note that the origin of $\mathbb{R}^{d-1}$ belongs to X. (See the details in [2].) See Figure 4 in case $k_1 = k_2 = 1$.

From the equation (3.1) and $M^{-1}\mathbf{e}_1 = \mathbf{e}_d$, we see that the mapping ${}^1\sigma^*(\mathbf{0}, i^*)$ $(1 \le i \le d)$ in our case are given by

$$\begin{aligned}
{}^1\sigma^* : (\mathbf{0}, 1^*) &\longmapsto \sum_{i_1=0}^{k_1-1}(-i_1\mathbf{e}_d, 1^*) + \sum_{i_2=0}^{k_2-1}(-i_2\mathbf{e}_d, 2^*) \\
&\qquad\qquad + \cdots + \sum_{i_{d-1}=0}^{k_{d-1}-1}(-i_{d-1}\mathbf{e}_d, d-1^*) + (\mathbf{0}, d^*), \\
(\mathbf{0}, 2^*) &\longmapsto (-k_1\mathbf{e}_d, 1^*), \\
(\mathbf{0}, 3^*) &\longmapsto (-k_2\mathbf{e}_d, 2^*), \\
&\ \ \vdots \\
(\mathbf{0}, d^*) &\longmapsto (-k_{d-1}\mathbf{e}_d, d-1^*).
\end{aligned}$$

Hence, $M^{-1}X_i$ $(1 \le i \le d)$ are given by

$$\begin{aligned}
M^{-1}X_1 &= \lim_{n\to\infty} M^{n-1}\pi\Big({}^1\sigma^{*\,n-1}\big({}^1\sigma^*(\mathbf{0}, 1^*)\big)\Big) \\
&= \lim_{n\to\infty} M^{n-1}\pi\Bigg({}^1\sigma^{*\,n-1}\Bigg(\sum_{i_1=0}^{k_1-1}(-i_1\mathbf{e}_d, 1^*) + \cdots \\
&\qquad\qquad + \sum_{i_{d-1}=0}^{k_{d-1}-1}(-i_{d-1}\mathbf{e}_d, d-1^*) + (\mathbf{0}, d^*)\Bigg)\Bigg)
\end{aligned}$$

$$= \bigcup_{i_1=0}^{k_1-1} (X_1 - i_1\pi\mathbf{e}_d) \cup \cdots \cup \bigcup_{i_{d-1}=0}^{k_{d-1}-1} (X_{d-1} - i_{d-1}\pi\mathbf{e}_d) \cup X_d,$$

$$\begin{aligned}
M^{-1}X_2 &= \lim_{n\to\infty} M^{n-1}\pi\left({}^1\sigma^{*n-1}\left({}^1\sigma^*(\mathbf{0},2^*)\right)\right) \\
&= \lim_{n\to\infty} M^{n-1}\pi\left({}^1\sigma^{*n-1}(-k_1\mathbf{e}_d,1^*)\right) \\
&= X_1 - k_1\pi\mathbf{e}_d, \\
&\ \vdots \\
M^{-1}X_d &= \lim_{n\to\infty} M^{n-1}\pi\left({}^1\sigma^{*n-1}\left({}^1\sigma^*(\mathbf{0},d^*)\right)\right) \\
&= \lim_{n\to\infty} M^{n-1}\pi\left({}^1\sigma^{*n-1}(-k_{d-1}\mathbf{e}_d,d-1^*)\right) \\
&= X_{d-1} - k_{d-1}\pi\mathbf{e}_d.
\end{aligned}$$

Then applying M, from the property $M\pi\mathbf{e}_d = \pi M\mathbf{e}_d = \pi\mathbf{e}_1$, we have

(3.2)

$$\begin{cases}
X_1 = \bigcup_{i_1=0}^{k_1-1} (MX_1 - i_1\pi\mathbf{e}_1) \cdots \bigcup_{i_{d-1}=0}^{k_{d-1}-1} (MX_{d-1} - i_{d-1}\pi\mathbf{e}_1) \cup MX_d, \\
X_2 = MX_1 - k_1\pi\mathbf{e}_1, \\
\quad\vdots \\
X_d = MX_{d-1} - k_{d-1}\pi\mathbf{e}_1,
\end{cases}$$

(3.3)
$$\begin{aligned}
X &= \bigcup_{i=1}^{d} X_i \\
&= \bigcup_{i_1=0}^{k_1} (MX_1 - i_1\pi\mathbf{e}_1) \cdots \bigcup_{i_{d-1}=0}^{k_{d-1}} (MX_{d-1} - i_{d-1}\pi\mathbf{e}_1) \cup MX_d.
\end{aligned}$$

Since X_i are disjont up to a set of measure 0, the partition of X is constructed. By using the partition (3.3), the transformation T^*_β on X without boundaries can be defined as follows:

(3.4) $$T^*_\beta\mathbf{x} = M^{-1}\mathbf{x} + b^*\pi\mathbf{e}_d \text{ if } \mathbf{x} \in MX_j - b^*\pi\mathbf{e}_1 \text{ for some } j \text{ and } b^*.$$

Then for $\mathbf{x} \in X$ satisfying the condition that ${T_\beta^*}^k x$ are not on the boundaries of X^i for any k, there exists an infinite sequence $(b_k^*)_{k=1}^\infty$ such that

$$ {T_\beta^*}^{k-1}\mathbf{x} \in MX_{j(k)} - b_k^* \pi \mathbf{e}_1, \tag{3.5} $$

and $\mathbf{x}$ is represented by

$$ \mathbf{x} = -\sum_{k=1}^{\infty} b_k^* M^{k-1} \pi \mathbf{e}_1. $$

Note that $(j(k))_{k=1}^\infty$ is the orbit of the point $\mathbf{x}$, that is,

$$ {T_\beta^*}^k \mathbf{x} \in X_{j(k)}. $$

From the set equations (3.2) and (3.5) we can see that

$$ \begin{aligned} T_\beta^*(X_1^\circ) &= X_1^\circ \cup X_2^\circ \cup \cdots \cup X_d^\circ, \\ T_\beta^*(X_2^\circ) &= X_1^\circ, \\ T_\beta^*(X_3^\circ) &= X_2^\circ, \\ &\vdots \\ T_\beta^*(X_d^\circ) &= X_{d-1}^\circ, \end{aligned} $$

where for each i X_i° is given by

$$ X_i^\circ = \left\{ \mathbf{x} \in X_i \;\middle|\; \begin{array}{l} {T_\beta^*}^k \mathbf{x} \text{ are not on the boundaries of } X_j \text{ for any } k \\ \text{and any } j \end{array} \right\}. $$

Hence, an infinite walk $(b_k^*)_{k=1}^\infty$ is obtained from the labeled graph $\mathcal{G}^*$, which is the dual graph of $\mathcal{G}$. Here, the dual graph G^* is the graph with the same vertices as G, but with each edge in G reversed in direction. We can deal with all points of X_i successfully. As a consequence, we know that the domains X_is ($1 \le i \le d$) are given by

$$ X_i = \left\{ -\sum_{k=1}^{\infty} b_k^* M^{k-1} \pi \mathbf{e}_1 \;\middle|\; \begin{array}{l} (b_k^*)_{k=1}^\infty \text{ is an admissible walk starting} \\ \text{at } i \text{ in } \mathcal{G}^* \end{array} \right\}. \tag{3.6} $$

See details in [2] and [6].

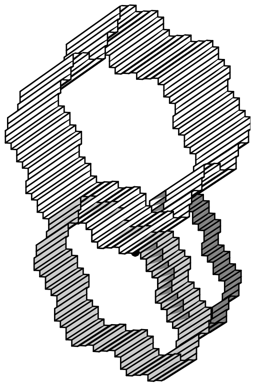

Figure 5: The figure of $\widehat{X} = \bigcup_{i=1}^{d} \widehat{X_i}$ in case $k_1 = k_2 = 1$.

§4. The natural extension of the β-transformation T_β

Let for each i $(1 \le i \le d)$ $\widehat{X_i} \subset \mathbb{R}^d$ be the following domain:

$$\widehat{X_i} = \left\{ t\boldsymbol{\alpha} + \mathbf{x} \mid 0 \le t < \gamma_i \text{ and } \mathbf{x} \in X_i \right\}. \tag{4.1}$$

And we define $\widehat{X}$ by

$$\widehat{X} = \bigcup_{i=1}^{d} \widehat{X_i}.$$

See Figure 5.

Let $\widehat{T_\beta}$ be the transformation on $\widehat{X}$ given by

$$\widehat{T_\beta}\left(\underline{\underline{t}}\boldsymbol{\alpha} + \mathbf{x}\right) = \underbrace{\underline{\underline{(\beta t - [\beta t])}}}_{T_\beta t}\boldsymbol{\alpha} + M\mathbf{x} - [\beta t]\pi\mathbf{e}_1. \tag{4.2}$$

Note that $\widehat{T_\beta}$ is just the β-transformation T_β on the direction $\boldsymbol{\alpha}$. We know that for any $\mathbf{z} = t\boldsymbol{\alpha} + \mathbf{x} \in \widehat{X}$,

$$\widehat{T_\beta}(\mathbf{z}) = M\mathbf{z} - [\beta t]\mathbf{e}_1.$$

$\widehat{T_\beta}$ will be a toral automorphism associated with M on the fundamental domain $\widehat{X}$.

From the partition (3.3) and the property (2.4) we have the following result.

Proposition 4.1. *$\widehat{T_\beta}$ is surjective and injective on $\widehat{X}$ except on the boundary.*

Proof. For any $\mathbf{y} \in \widehat{X}$ there exist $\mathbf{y}' \in X_i$ $(1 \le i \le d)$ and t' $(0 \le t' < \gamma_i \le 1)$ such that

$$\mathbf{y} = t'\boldsymbol{\alpha} + \mathbf{y}'.$$

From the partition (3.2), we have

$$\mathbf{y} = t'\boldsymbol{\alpha} + M\mathbf{x}' - k\pi\mathbf{e}_1 \quad \text{for some } \mathbf{x}' \in X_j \text{ and } 0 \le k \le k_1.$$

Let

$$\mathbf{x} = \left(\frac{k}{\beta} + \frac{t'}{\beta}\right)\boldsymbol{\alpha} + \mathbf{x}'.$$

Then

$$\widehat{T_\beta}\mathbf{x} = t'\boldsymbol{\alpha} + M\mathbf{x}' - k\pi\mathbf{e}_1 = \mathbf{y}.$$

If $i = 1$, $0 \le t' < 1$ and $k = 0, 1, \dots, k_j - 1$. Here we set $k_d = 1$. Then

$$0 \le \frac{k}{\beta} + \frac{t'}{\beta} < \frac{k_j - 1}{\beta} + \frac{1}{\beta} = \frac{k_j}{\beta} = .k_j \le \gamma_j.$$

If $i = 2, \dots, d$, we know that $0 \le t' < \gamma_i = .k_i \dots k_{d-1}1$, $j = i - 1$, and $k = k_{i-1}$. Hence

$$0 \le \frac{k}{\beta} + \frac{t'}{\beta} < \frac{k_{i-1}}{\beta} + \frac{\gamma_i}{\beta} = .k_{i-1} \dots k_{d-1}1 = \gamma_{i-1} = \gamma_j.$$

Therefore for any i, we see that $\mathbf{x} \in \widehat{X_j} \subset \widehat{X}$. Hence $\widehat{T_\beta}$ is surjective. And except for the boundary, i, t', j, and k are uniquely determined by $\mathbf{y}$. Therefore $\widehat{T_\beta}$ is almost everywhere injective. $\square$

Therefore $\widehat{T_\beta}$ is the natural extension of the transformation T_β.

Recall that the domain X is on the plane $\mathcal{P}$, which is orthogonal to $\boldsymbol{\gamma}$. We put

$$Q := \begin{bmatrix} \alpha_1^{(1)} & \cdots & \alpha_1^{(r_1)} & \Re\alpha_1^{(r_1+1)} & -\Im\alpha_1^{(r_1+1)} & \cdots & \Re\alpha_1^{(r_1+r_2)} & -\Im\alpha_1^{(r_1+r_2)} \\ \alpha_2^{(1)} & \cdots & \alpha_2^{(r_1)} & \Re\alpha_2^{(r_1+1)} & -\Im\alpha_2^{(r_1+1)} & \cdots & \Re\alpha_2^{(r_1+r_2)} & -\Im\alpha_2^{(r_1+r_2)} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \alpha_d^{(1)} & \cdots & \alpha_d^{(r_1)} & \Re\alpha_d^{(r_1+1)} & -\Im\alpha_d^{(r_1+1)} & \cdots & \Re\alpha_d^{(r_1+r_2)} & -\Im\alpha_d^{(r_1+r_2)} \end{bmatrix}$$
$$=: [\boldsymbol{\alpha}, \boldsymbol{\alpha}_2, \dots, \boldsymbol{\alpha}_d],$$

where $\Re$ indicates the real part and $\Im$ indicates the imaginary part. The plane $\mathcal{P}$ is spanned by $\boldsymbol{\alpha}_2$, $\boldsymbol{\alpha}_3, \dots$, and $\boldsymbol{\alpha}_d$, because $\boldsymbol{\alpha}_i$ $(2 \le i \le d)$ and $\boldsymbol{\gamma}$ intersect orthogonally and $\boldsymbol{\alpha}_i$s are linearly independent.

Let us define the domains $\widehat{Y}$ and $\widehat{Y}_i$ $(1 \le i \le d)$ as follows:

$$\widehat{Y} := Q^{-1}(\widehat{X}) \quad \text{and} \quad \widehat{Y}_i := Q^{-1}(\widehat{X_i}). \tag{4.3}$$

We will make preparations for the explicit representation of $\widehat{Y}_i$.

Define a $d \times d$ matrix

$$\begin{aligned} P &:= \begin{bmatrix} \alpha_1^{(1)} & \cdots & \alpha_1^{(r_1)} & \alpha_1^{(r_1+1)} & \overline{\alpha_1^{(r_1+1)}} & \cdots & \alpha_1^{(r_1+r_2)} & \overline{\alpha_1^{(r_1+r_2)}} \\ \alpha_2^{(1)} & \cdots & \alpha_2^{(r_1)} & \alpha_2^{(r_1+1)} & \overline{\alpha_2^{(r_1+1)}} & \cdots & \alpha_2^{(r_1+r_2)} & \overline{\alpha_2^{(r_1+r_2)}} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \alpha_d^{(1)} & \cdots & \alpha_d^{(r_1)} & \alpha_d^{(r_1+1)} & \overline{\alpha_d^{(r_1+1)}} & \cdots & \alpha_d^{(r_1+r_2)} & \overline{\alpha_d^{(r_1+r_2)}} \end{bmatrix} \\ &=: [\boldsymbol{\alpha}, \mathbf{u}_2, \dots, \mathbf{u}_d]. \end{aligned}$$

Let

$$U := I_{r_1} \oplus \begin{bmatrix} 1 & 1 \\ -\sqrt{-1} & \sqrt{-1} \end{bmatrix} \oplus \begin{bmatrix} 1 & 1 \\ -\sqrt{-1} & \sqrt{-1} \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 1 & 1 \\ -\sqrt{-1} & \sqrt{-1} \end{bmatrix},$$

where $A \oplus B$ is a matrix of a form:

$$\begin{bmatrix} A & \mathbf{0} \\ \mathbf{0} & B \end{bmatrix}$$

and I_{r_1} is the identity matrix of size r_1. Then

$$QU = P.$$

From $I_d = P \cdot P^{-1}$, we have

$$\mathbf{e}_1 = (P^{-1})_{11}\boldsymbol{\alpha} + (P^{-1})_{21}\mathbf{u}_2 + \cdots + (P^{-1})_{d1}\mathbf{u}_d.$$

Each $\mathbf{u}_i$ $(2 \le i \le d)$ is also orthogonal to $\boldsymbol{\gamma}$. Therefore,

$$\langle \mathbf{e}_1, \boldsymbol{\gamma} \rangle = \langle (P^{-1})_{11}\boldsymbol{\alpha}, \boldsymbol{\gamma} \rangle = (P^{-1})_{11}\langle \boldsymbol{\alpha}, \boldsymbol{\gamma} \rangle = (P^{-1})_{11}.$$

It follows that

$$(P^{-1})_{11} = 1$$

and

$$P^{-1}\mathbf{e}_1 = {}^t[1, 1, \dots, 1].$$

You can see the detailed proof in [11]. According to the relation $QU = P$

$$Q^{-1}\mathbf{e}_1 = {}^t[\underbrace{1, 1, \dots, 1}_{r_1}, \underbrace{2, 0, \dots, 2, 0}_{2r_2}].$$

Moreover, from $I_d = Q \cdot Q^{-1}$

$$\mathbf{e}_1 = \boldsymbol{\alpha} + \boldsymbol{\alpha}_2 + \cdots + \boldsymbol{\alpha}_{r_1} + 2\boldsymbol{\alpha}_{r_1+1} + 2\boldsymbol{\alpha}_{r_1+3} + \cdots + 2\boldsymbol{\alpha}_{d-1}.$$

Since π is the projection along $\boldsymbol{\alpha}$,

$$\pi\mathbf{e}_1 = \boldsymbol{\alpha}_2 + \cdots + \boldsymbol{\alpha}_{r_1} + 2\boldsymbol{\alpha}_{r_1+1} + 2\boldsymbol{\alpha}_{r_1+3} + \cdots + 2\boldsymbol{\alpha}_{d-1}.$$

Hence

$$Q^{-1}\pi\mathbf{e}_1 = {}^t[\underbrace{0, 1, 1, \dots, 1}_{r_1}, \underbrace{2, 0, \dots, 2, 0}_{2r_2}]. \tag{4.4}$$

Lemma 4.2. *The following relation holds*:

$$MQ = Q \begin{bmatrix} \beta & & & \\ & \beta^{(2)} & & \\ & & \ddots & \\ & & & \beta^{(r_1)} \end{bmatrix} \oplus \begin{bmatrix} \Re\beta^{(r_1+1)} & -\Im\beta^{(r_1+1)} \\ \Im\beta^{(r_1+1)} & \Re\beta^{(r_1+1)} \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} \Re\beta^{(r_1+r_2)} & -\Im\beta^{(r_1+r_2)} \\ \Im\beta^{(r_1+r_2)} & \Re\beta^{(r_1+r_2)} \end{bmatrix}.$$

Proof. The relation (2.2) implies that

$$MP = PD,$$

where D is the diagonal matrix

$$D = \begin{bmatrix} \beta &&&&&&&& \\ & \beta^{(2)} &&&&&&& \\ && \ddots &&&&&& \\ &&& \beta^{(r_1)} &&&&& \\ &&&& \beta^{(r_1+1)} &&&& \\ &&&&& \overline{\beta^{(r_1+1)}} &&& \\ &&&&&& \ddots && \\ &&&&&&& \beta^{(r_1+r_2)} & \\ &&&&&&&& \overline{\beta^{(r_1+r_2)}} \end{bmatrix}.$$

Then, from the relation $P = QU$,

$$MQU = QUD.$$

Using

$$U^{-1} = I_{r_1} \oplus \frac{1}{2}\begin{bmatrix} 1 & \sqrt{-1} \\ 1 & -\sqrt{-1} \end{bmatrix} \oplus \cdots \oplus \frac{1}{2}\begin{bmatrix} 1 & \sqrt{-1} \\ 1 & -\sqrt{-1} \end{bmatrix},$$

we have

$$\begin{aligned} Q^{-1}MQ &= UDU^{-1} \\ &= \begin{bmatrix} \beta & & & \\ & \beta^{(2)} & & \\ & & \ddots & \\ & & & \beta^{(r_1)} \end{bmatrix} \\ &\qquad \oplus \begin{bmatrix} \Re\beta^{(r_1+1)} & -\Im\beta^{(r_1+1)} \\ \Im\beta^{(r_1+1)} & \Re\beta^{(r_1+1)} \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} \Re\beta^{(r_1+r_2)} & -\Im\beta^{(r_1+r_2)} \\ \Im\beta^{(r_1+r_2)} & \Re\beta^{(r_1+r_2)} \end{bmatrix}. \end{aligned}$$

□

Hereafter we represent $\widehat{Y}_i$s as the domains in $\mathbb{R} \times \mathbb{R}^{d-1}$.

Proposition 4.3. *The domains $\widehat{Y}_i$s are given by*

$$\widehat{Y}_i = \left\{ \left(t, -\sum_{k=1}^{\infty} b_k^* R^{k-1}\mathbf{v} \right) \,\middle|\, 0 \le t < \gamma_i \text{ and } (b_k^*)_{k=1}^{\infty} \text{ is an admissible walk starting at } i \text{ in } \mathcal{G}^* \right\},$$

where

$$\begin{aligned} R = &\begin{bmatrix} \beta^{(2)} & & \\ & \ddots & \\ & & \beta^{(r_1)} \end{bmatrix} \\ &\oplus \begin{bmatrix} \Re\beta^{(r_1+1)} & -\Im\beta^{(r_1+1)} \\ \Im\beta^{(r_1+1)} & \Re\beta^{(r_1+1)} \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} \Re\beta^{(r_1+r_2)} & -\Im\beta^{(r_1+r_2)} \\ \Im\beta^{(r_1+r_2)} & \Re\beta^{(r_1+r_2)} \end{bmatrix} \end{aligned}$$

and

$$\mathbf{v} = {}^t[\underbrace{1, \ldots, 1}_{r_1-1}, \underbrace{2, 0, \ldots, 2, 0}_{2r_2}].$$

Proof. The definitions of $\widehat{Y_i}$, $\widehat{X_i}$, and X_i, that is, (4.3), (4.1), and (3.6), show that

$$\begin{aligned}
\widehat{Y_i} &= Q^{-1}(\widehat{X_i}) \\
&= Q^{-1}\left\{ t\boldsymbol{\alpha} - \sum_{k=1}^{\infty} b_k^* M^{k-1}\pi\mathbf{e}_1 \;\middle|\; (*)\text{-condition} \right\} \\
&= \left\{ tQ^{-1}\boldsymbol{\alpha} - \sum_{k=1}^{\infty} b_k^* Q^{-1} M^{k-1}\pi\mathbf{e}_1 \;\middle|\; (*)\text{-condition} \right\}.
\end{aligned}$$

By Lemma 4.2

$$Q^{-1}M^{k-1} = \left(\beta^{k-1} \oplus R^{k-1}\right) Q^{-1}.$$

And using (4.4), we have

$$\begin{aligned}
\widehat{Y_i} &= \left\{ t\mathbf{e}_1 - \sum_{k=1}^{\infty} b_k^* \left(\beta^{k-1} \oplus R^{k-1}\right) Q^{-1}\pi\mathbf{e}_1 \;\middle|\; (*)\text{-condition} \right\} \\
&= \left\{ \left(t, -\sum_{k=1}^{\infty} b_k^* R^{k-1}\mathbf{v} \right) \;\middle|\; (*)\text{-condition} \right\}.
\end{aligned}$$

Here $(*)$-condition means that $0 \le t < \gamma_i$ and $(b_k^*)_{k=1}^{\infty}$ is an admissible walk starting at i in $\mathcal{G}^*$. Therefore we arrive at the conclusion of the assertion. $\square$

Naturally, we can define a transformation $\widehat{S_\beta}$ on $\widehat{Y}$ as follows:

$$\widehat{S_\beta} := Q^{-1} \circ \widehat{T_\beta} \circ Q. \tag{4.5}$$

Then $\widehat{S_\beta}$ is also a natural extension of T_β.

Proposition 4.4. *The transformation $\widehat{S_\beta}$ on $\widehat{Y}$ is given by*

$$\widehat{S_\beta}(t, \mathbf{x}) = (\beta t - [\beta t], R\mathbf{x} - [\beta t]\mathbf{v})$$

and surjective.

Proof. From (4.5) and (4.2), which are definitions of $\widehat{S_\beta}$ and $\widehat{T_\beta}$,

$$\begin{aligned}\widehat{S_\beta}(t,\mathbf{x}) &:= Q^{-1}\circ\widehat{T_\beta}\circ Q(t\mathbf{e}_1+0\oplus\mathbf{x})\\ &= Q^{-1}\circ\widehat{T_\beta}\,(t\boldsymbol{\alpha}+Q\,(0\oplus\mathbf{x}))\\ &= Q^{-1}\,((\beta t-[\beta t])\boldsymbol{\alpha}+MQ\,(0\oplus\mathbf{x})-[\beta t]\pi\mathbf{e}_1)\\ &= (\beta t-[\beta t],R\mathbf{x}-[\beta t]\mathbf{v})\,.\end{aligned}$$

Surjectivity of $\widehat{S_\beta}$ is obtained by Proposition 4.1. ∎

§5. The reduction theorem

In this section, we introduce reduced numbers and show our main theorem.

Let $\widetilde{Y}$ ($\subset\mathbb{R}\times\mathbb{R}^{d-1}$) be the following product space:

$$\widetilde{Y}:=[0,1)\times\mathbb{R}^{d-1}.$$

Let $\widetilde{S_\beta}$ be the transformation on $\widetilde{Y}$ defined by

$$\widetilde{S_\beta}\,(x,\mathbf{x}):=(\beta x-[\beta x],R\mathbf{x}-[\beta x]\mathbf{v})\,,\quad x\in[0,1).$$

Then the restriction of $\widetilde{S_\beta}$ on $\widehat{Y}$ ($\subset\widetilde{Y}$) is $\widehat{S_\beta}$.

Define a map $\rho:\mathbb{Q}(\beta)\to\mathbb{R}\times\mathbb{R}^{d-1}$ by

$$\rho(x)=\left(x,\begin{bmatrix}x^{(2)}\\ \vdots\\ x^{(r_1)}\\ 2\Re x^{(r_1+1)}\\ 2\Im x^{(r_1+1)}\\ \vdots\\ 2\Re x^{(r_1+r_2)}\\ 2\Im x^{(r_1+r_2)}\end{bmatrix}\right).$$

DEFINITION 5.1. A real number $x\in\mathbb{Q}(\beta)\cap[0,1)$ is reduced if $\rho(x)\in\widehat{Y}$.

In order to prove the main theorem, we will need some important lemmas.

LEMMA 5.1. *Let $x \in \mathbb{Q}(\beta) \cap [0,1)$. Then*

$$\widetilde{S_\beta}\left(\rho(x)\right) = \rho\left(T_\beta x\right).$$

Proof. From the definitions of $\widetilde{S_\beta}$, ρ, and T_β, we have

$$\widetilde{S_\beta}\left(\rho(x)\right) = \widetilde{S_\beta}\left(x, \begin{bmatrix} x^{(2)} \\ \vdots \\ x^{(r_1)} \\ 2\Re x^{(r_1+1)} \\ 2\Im x^{(r_1+1)} \\ \vdots \\ 2\Re x^{(r_1+r_2)} \\ 2\Im x^{(r_1+r_2)} \end{bmatrix} \right)$$

$$= \left(\beta x - [\beta x], \begin{bmatrix} \beta^{(2)} x^{(2)} \\ \vdots \\ \beta^{(r_1)} x^{(r_1)} \\ 2\left(\Re\beta^{(r_1+1)} \cdot \Re x^{(r_1+1)} - \Im\beta^{(r_1+1)} \cdot \Im x^{(r_1+1)}\right) \\ 2\left(\Im\beta^{(r_1+1)} \cdot \Re x^{(r_1+1)} + \Re\beta^{(r_1+1)} \cdot \Im x^{(r_1+1)}\right) \\ \vdots \\ 2\left(\Re\beta^{(r_1+r_2)} \cdot \Re x^{(r_1+r_2)} - \Im\beta^{(r_1+r_2)} \cdot \Im x^{(r_1+r_2)}\right) \\ 2\left(\Im\beta^{(r_1+r_2)} \cdot \Re x^{(r_1+r_2)} + \Re\beta^{(r_1+r_2)} \cdot \Im x^{(r_1+r_2)}\right) \end{bmatrix} \right.$$

$$\left. - [\beta x] \begin{bmatrix} 1 \\ \vdots \\ 1 \\ 2 \\ 0 \\ \vdots \\ 2 \\ 0 \end{bmatrix} \right)$$

using the relations $\Re(xy) = \Re x \Re y - \Im x \Im y$ and $\Im(xy) = \Re x \Im y + \Im x \Re y$,

$$
\begin{aligned}
&= \left(\beta x - [\beta x], \begin{bmatrix} (\beta x)^{(2)} - [\beta x] \\ \vdots \\ (\beta x)^{(r_1)} - [\beta x] \\ 2\Re\left((\beta x)^{(r_1+1)} - [\beta x]\right) \\ 2\Im\left((\beta x)^{(r_1+1)} - [\beta x]\right) \\ \vdots \\ 2\Re\left((\beta x)^{(r_1+r_2)} - [\beta x]\right) \\ 2\Im\left((\beta x)^{(r_1+r_2)} - [\beta x]\right) \end{bmatrix} \right) \\
&= \rho(\beta x - [\beta x]) \\
&= \rho\left(T_\beta x\right).
\end{aligned}
$$

Therefore we arrive at the conclusion. ∎

LEMMA 5.2. *Let $x \in \mathbb{Q}(\beta) \cap [0,1)$ be reduced. Then*

(1) *$T_\beta x$ is reduced,*
(2) *there exists x^* such that x^* is reduced and $T_\beta x^* = x$.*

Proof. Since $x \in \mathbb{Q}(\beta) \cap [0,1)$ is reduced, $\rho(x) \in \widehat{Y}$.
(1) From Lemma 5.1,

$$\widehat{S_\beta}\left(\rho(x)\right) = \rho\left(T_\beta x\right) \in \widehat{Y}.$$

Hence $T_\beta x$ is reduced.
(2) From Proposition 4.4, $\widehat{S_\beta}$ is surjective on $\widehat{Y}$. Thus there exist $(x^*, \mathbf{x}) \in \widehat{Y}$ such that

$$\widehat{S_\beta}\left(x^*, \mathbf{x}\right) = \rho(x). \tag{5.1}$$

Comparing first coordinates in both sides, we see that

$$T_\beta x^* = x.$$

To verify x^* is reduced, we will only show

$$(x^*, \mathbf{x}) = \rho\left(x^*\right).$$

Then $\rho\left(x^*\right) \in \widehat{Y}$ implies that x^* is reduced.
We put

$$\mathbf{x} = {}^t\left[x_2, \dots, x_{r_1}, x_{r_1+1}, \widetilde{x_{r_1+1}}, \dots, x_{r_1+r_2}, \widetilde{x_{r_1+r_2}}\right].$$

Then (5.1) shows that

$$\beta x^* - [\beta x^*] = x$$

and

$$R\mathbf{x} - [\beta x^*]\mathbf{v} = \begin{bmatrix} x^{(2)} \\ \vdots \\ x^{(r_1)} \\ 2\Re x^{(r_1+1)} \\ 2\Im x^{(r_1+1)} \\ \vdots \\ 2\Re x^{(r_1+r_2)} \\ 2\Im x^{(r_1+r_2)} \end{bmatrix}.$$

So that,

$$\begin{bmatrix} \beta^{(2)} x_2 - [\beta x^*] \\ \vdots \\ \beta^{(r_1)} x_{r_1} - [\beta x^*] \\ \Re\beta^{(r_1+1)} \cdot x_{r_1+1} - \Im\beta^{(r_1+1)} \cdot \widetilde{x_{r_1+1}} - 2[\beta x^*] \\ \Im\beta^{(r_1+1)} \cdot x_{r_1+1} + \Re\beta^{(r_1+1)} \cdot \widetilde{x_{r_1+1}} \\ \vdots \\ \Re\beta^{(r_1+r_2)} \cdot x_{r_1+r_2} - \Im\beta^{(r_1+r_2)} \cdot \widetilde{x_{r_1+r_2}} - 2[\beta x^*] \\ \Im\beta^{(r_1+r_2)} \cdot x_{r_1+r_2} + \Re\beta^{(r_1+r_2)} \cdot \widetilde{x_{r_1+r_2}} \end{bmatrix}$$

$$= \begin{bmatrix} \beta^{(2)} x^{*(2)} - [\beta x^*] \\ \vdots \\ \beta^{(r_1)} x^{*(r_1)} - [\beta x^*] \\ 2\Re\left(\beta^{(r_1+1)} x^{*(r_1+1)} - [\beta x^*]\right) \\ 2\Im\left(\beta^{(r_1+1)} x^{*(r_1+1)} - [\beta x^*]\right) \\ \vdots \\ 2\Re\left(\beta^{(r_1+r_2)} x^{*(r_1+r_2)} - [\beta x^*]\right) \\ 2\Im\left(\beta^{(r_1+r_2)} x^{*(r_1+r_2)} - [\beta x^*]\right) \end{bmatrix}.$$

Thus

$$x_2 = x^{*(2)}, \ldots, x_{r_1} = x^{*(r_1)},$$

and for $1 \le j \le r_2$

$$\begin{aligned}
&\Re\beta^{(r_1+j)} \cdot x_{r_1+j} - \Im\beta^{(r_1+j)} \cdot \widetilde{x_{r_1+j}} \\
&\quad = 2\Re\Big(\beta^{(r_1+j)} x^{*(r_1+j)}\Big) \\
&\quad = 2\Re\beta^{(r_1+j)}\Re x^{*(r_1+j)} - 2\Im\beta^{(r_1+j)}\Im x^{*(r_1+j)}, \\
&\Im\beta^{(r_1+j)} \cdot x_{r_1+j} + \Re\beta^{(r_1+j)} \cdot \widetilde{x_{r_1+j}} \\
&\quad = 2\Im\Big(\beta^{(r_1+j)} x^{*(r_1+j)}\Big) \\
&\quad = 2\Re\beta^{(r_1+j)}\Im x^{*(r_1+j)} + 2\Im\beta^{(r_1+j)}\Re x^{*(r_1+j)}.
\end{aligned}$$

Then for $1 \le j \le r_2$, we have

$$\begin{aligned}
&\Big(x_{r_1+j} - 2\Re x^{*(r_1+j)}\Big)\Re\beta^{(r_1+j)} - \Big(\widetilde{x_{r_1+j}} - 2\Im x^{*(r_1+j)}\Big)\Im\beta^{(r_1+j)} = 0, \\
&\Big(x_{r_1+j} - 2\Re x^{*(r_1+j)}\Big)\Im\beta^{(r_1+j)} + \Big(\widetilde{x_{r_1+j}} - 2\Im x^{*(r_1+j)}\Big)\Re\beta^{(r_1+j)} = 0.
\end{aligned}$$

Thus

$$x_{r_1+j} = 2\Re x^{*(r_1+j)} \quad \text{and} \quad \widetilde{x_{r_1+j}} = 2\Im x^{*(r_1+j)}.$$

Therefore

$$(x^*, \mathbf{x}) = \rho(x^*).$$

Thus we obtain the assertion (2). ∎

By the lemmas above, we can get a sufficient condition for pure periodicity of β-expansions.

PROPOSITION 5.3. *Let $x \in \mathbb{Q}(\beta) \cap [0,1)$ be reduced. Then x has a purely periodic β-expansion.*

Proof. Lemma 5.2 (2) shows that there exist x_i^*s such that x_i^*s are reduced and $T_\beta x_i^* = x_{i-1}^*$, where we set $x_0^* = x$. Here, we put

$$x = \frac{p_0}{q} \quad \text{for some} \quad q \in \mathbb{Z},\ p_0 \in \mathbb{Z}[\beta].$$

Then $T_\beta x_1^* = x$ implies that

$$\beta x_1^* - [\beta x_1^*] = x.$$

So that

$$x_1^* = \frac{[\beta x_1^*]}{\beta} + \frac{x}{\beta} = \frac{p_1}{q} \quad \text{for some} \quad p_1 \in \mathbb{Z}[\beta].$$

Inductively we can see for every k

$$x_k^* = \frac{p_k}{q} \quad \text{for some} \quad p_k \in \mathbb{Z}[\beta].$$

Let b_j be positive real numbers. Only in this proof, we denote by $x^{(j)}$ $(1 \le j \le d)$ algebraic conjugates of x and $x^{(1)} = x$. Let

$$C = \left\{ x \in \mathbb{Z}[\beta] \;\middle|\; |x^{(j)}| \le b_j \right\}.$$

Obviously, C is a finite set. As $\widehat{Y}$ is bounded, we can see the set $\{x_i^*\}_{i=0}^{\infty}$ is a finite set. Hence there exist j and k $(j > k)$ such that

$$x_j^* = x_{j-k}^*.$$

Applying T_β^{j-k} we get

$$x_k^* = x.$$

Hence

$$T_\beta^k x = x.$$

Therefore x has a purely periodic β-expansion. □

Lemma 5.2 and Proposition 5.3 show that the transformation T_β restricted to $\mathbb{Q}(\beta) \cap [0,1)$ is bijective.

To complete the proof of our main theorem, the following proposition is positively necessary.

Proposition 5.4. *Let $x \in \mathbb{Q}(\beta) \cap [0,1)$. Then there exists $N_1 > 0$ such that $T_\beta^N x$ are reduced for any $N \ge N_1$.*

Proof. Consider the Euclidean distance d between $\widetilde{S_\beta}^k\big(\rho(x)\big)$ and $\widetilde{S_\beta}^k(x, \mathbf{0})$ for $k \ge 0$. Since the first coordinates of these points are equal, this coincides with the distance between the origin of $\mathbb{R}^{d-1}$ and

$$R^k \begin{bmatrix} x^{(2)} \\ \vdots \\ x^{(r_1)} \\ 2\Re x^{(r_1+1)} \\ 2\Im x^{(r_1+1)} \\ \vdots \\ 2\Re x^{(r_1+r_2)} \\ 2\Im x^{(r_1+r_2)} \end{bmatrix}.$$

By $s(x)$ we denote this distance. As R^ks are given by

$$\begin{aligned} R^k = & \begin{bmatrix} (\beta^{(2)})^k & & \\ & \ddots & \\ & & (\beta^{(r_1)})^k \end{bmatrix} \\ & \oplus \left|\beta^{(r_1+1)}\right|^{2k} \begin{bmatrix} \Re\beta^{(r_1+1)}/\left|\beta^{(r_1+1)}\right|^2 & -\Im\beta^{(r_1+1)}/\left|\beta^{(r_1+1)}\right|^2 \\ \Im\beta^{(r_1+1)}/\left|\beta^{(r_1+1)}\right|^2 & \Re\beta^{(r_1+1)}/\left|\beta^{(r_1+1)}\right|^2 \end{bmatrix}^k \\ & \oplus \cdots \oplus \left|\beta^{(r_1+r_2)}\right|^{2k} \begin{bmatrix} \Re\beta^{(r_1+r_2)}/\left|\beta^{(r_1+r_2)}\right|^2 & -\Im\beta^{(r_1+r_2)}/\left|\beta^{(r_1+r_2)}\right|^2 \\ \Im\beta^{(r_1+r_2)}/\left|\beta^{(r_1+r_2)}\right|^2 & \Re\beta^{(r_1+r_2)}/\left|\beta^{(r_1+r_2)}\right|^2 \end{bmatrix}^k, \end{aligned}$$

we have

$$\begin{aligned} s(x)^2 = & \left(\beta^{(2)}\right)^{2k}\left(x^{(2)}\right)^2 + \cdots + \left(\beta^{(r_1)}\right)^{2k}\left(x^{(r_1)}\right)^2 \\ & + \left|\beta^{(r_1+1)}\right|^{2k}\left\{\left(2\Re x^{(r_1+1)}\right)^2 + \left(2\Im x^{(r_1+1)}\right)^2\right\} \\ & + \cdots + \left|\beta^{(r_1+r_2)}\right|^{2k}\left\{\left(2\Re x^{(r_1+r_2)}\right)^2 + \left(2\Im x^{(r_1+r_2)}\right)^2\right\}. \end{aligned}$$

If we put

$$u = \max\left\{\left|\beta^{(2)}\right|, \dots, \left|\beta^{(r_1)}\right|, \left|\beta^{(r_1+1)}\right|, \dots, \left|\beta^{(r_1+r_2)}\right|\right\},$$

then $0 < u < 1$ and

$$\begin{aligned} s(x) \le u^k \cdot \Big\{ & \left(x^{(2)}\right)^2 + \cdots + \left(x^{(r_1)}\right)^2 + \left(2\Re x^{(r_1+1)}\right)^2 + \left(2\Im x^{(r_1+1)}\right)^2 \\ & + \cdots + \left(2\Re x^{(r_1+r_2)}\right)^2 + \left(2\Im x^{(r_1+r_2)}\right)^2\Big\}^{1/2}. \end{aligned}$$

Thus

$$d\Big(\widetilde{S_\beta}^k\big(\rho(x)\big), \widetilde{S_\beta}^k(x, \mathbf{0})\Big) \leq u^k \cdot d\big(\rho(x), (x, \mathbf{0})\big).$$

From the fact $(x, \mathbf{0}) \in \widehat{Y}$ and $\widetilde{S_\beta}|\widehat{Y} = \widehat{S_\beta}$, we know that

$$\widetilde{S_\beta}^k(x, \mathbf{0}) \in \widehat{Y}.$$

It follows that $\widetilde{S_\beta}^k\big(\rho(x)\big)$ comes exponentially close to $\widehat{Y}$ as $k \to \infty$. By the same reason that we used in the proof of Proposition 5.3, we can conclude that there exists a finite number of $\rho\big(T_\beta^k\big)$ in a certain bounded domain. Hence

$$\widetilde{S_\beta}^{N_1}\big(\rho(x)\big) = \rho\big(T_\beta^{N_1}x\big) \in \widehat{Y}$$

for a sufficiently large N_1. Then $T_\beta^{N_1}x$ is reduced. From Lemma 5.2 (1), we see that $T_\beta^N x$ are reduced for any $N \geq N_1$.

At last we attain our goal.

Theorem 5.5. *Let $x \in [0, 1)$. Then*

(1) *$x \in \mathbb{Q}(\beta)$ if and only if x has an eventually periodic β-expansion,*
(2) *$x \in \mathbb{Q}(\beta)$ is reduced if and only if x has a purely periodic β-expansion.*

Proof. (1) Assume that $x \in \mathbb{Q}(\beta)$. By Proposition 5.4, there exists $N > 0$ such that $T_\beta^N x$ is reduced. Proposition 5.3 says that $T_\beta^N x$ has a purely periodic β-expansion. Hence x has an eventually periodic β-expansion. The opposite implication is trivial.
(2) Necessity is obtained by Proposition 5.3. Conversely, assume that x has a purely periodic β-expansion. According to Proposition 5.4, there exists $N > 0$ such that $T_\beta^N x$ is reduced. The pure periodicity of x implies that there exists $j > 0$ such that $T_\beta^{N+j} x = x$. Lemma 5.2 (1) says that x is reduced. □

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