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L^p-CURVATURE AND THE CAUCHY-RIEMANN EQUATION NEAR AN ISOLATED SINGULAR POINT

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Abstract. Let X be a complex n-dimensional reduced analytic space with isolated singular point x_0, and with a strongly plurisubharmonic function $\rho : X \to [0, \infty)$ such that $\rho(x_0) = 0$. A smooth Kähler form on $X \setminus \{x_0\}$ is then defined by $\mathbf{i}\partial\bar{\partial}\rho$. The associated metric is assumed to have L^n_{loc}-curvature, to admit the Sobolev inequality and to have suitable volume growth near x_0. Let $E \to X \setminus \{x_0\}$ be a Hermitian-holomorphic vector bundle, and ξ a smooth $(0, 1)$-form with coefficients in E. The main result of this article states that if ξ and the curvature of E are both L^n_{loc}, then the equation $\bar{\partial}u = \xi$ has a smooth solution on a punctured neighbourhood of x_0. Applications of this theorem to problems of holomorphic extension, and in particular a result of Kohn-Rossi type for sections over a CR-hypersurface, are discussed in the final section.

§1. Introduction

During the 1960s the theory of L^2-cohomology for complex manifolds underwent a programme aimed at establishing the Hodge decomposition for manifolds with boundary. In the work of Kohn [11], Hörmander [10], Andreotti-Vesentini [1] and others, abstract methods from the study of unbounded linear operators on Hilbert space were again fundamental, and from this point of view it was necessary to work with a complete Riemannian or Kähler metric on the manifold. Reduced complex spaces with singularities were initially on the periphery of this development, although a theorem of Grauert [8] had shown that explicit construction of a complete Kähler metric on $V \setminus \{0\}$ is certainly possible when $V \subset \mathbb{C}^N$ is an analytic subvariety with isolated singularity at the origin. For the case of isolated singularities the approximation theorem of Artin had moreover demonstrated the existence of a projective algebraic variety for which the germ at a singular point is isomorphic to that of $(V, 0)$. It was from the perspective of intersection theory that the study of L^2-cohomology on punctured varieties consequently received a new impetus, when Goresky and MacPherson [7]

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conjectured that these groups and the intersection cohomology of the variety are canonically isomorphic with respect to some complete metric on the punctured variety. The general proof of this conjecture and the associated problem of defining Hodge structures on the intersection cohomology of Kähler varieties has subsequently been taken up in the work of Ohsawa [14], [15] and Saper [16].

The present article is concerned with the solvability of the equation $\bar{\partial} u = \xi$ when ξ is a $(0,1)$-form taking values in a Hermitian-holomorphic vector bundle E. Let X be a reduced complex n-dimensional analytic space with isolated singularity $x_0 \in X$, and let $\rho : X \to [0,\infty)$ be a strongly plurisubharmonic exhaustion function such that $\rho(x) = 0$ if and only if $x = x_0$. Here a Kähler metric g on $X \setminus \{x_0\}$ will be provided by the positive real form $\omega = \mathbf{i}\partial\bar{\partial}\rho$, and for $c > 0$, ω will be assumed to satisfy the following three conditions on $X_{0,c} = \{x \in X \mid 0 < \rho(x) < c\}$:

$$\text{(i)} \qquad \int_{X_{0,c}} |R_g|^n < \infty \ ,$$

where R_g denotes the canonical curvature form associated with g. It will further be assumed that the Sobolev inequality holds with respect to this metric, i.e.,

$$\text{(ii)} \qquad \left(\int_{X_{0,c}} |f|^{\frac{2n}{n-1}} \omega^n \right)^{\frac{n-1}{n}} \leq \mathrm{const}(n) \int_{X_{0,c}} |\nabla f|^2 \omega^n$$

for smooth compactly supported functions f. In addition,
(iii) let $\delta(x_0, x)$ denote the Riemannian metric distance function on $X_{0,c}$, and let $B_\delta(x_0, r)$ be the associated ball of radius r. For some sufficiently small $0 < c' < c$ it will be assumed that there exists a constant $\Omega > 0$ such that

$$\int_{B_\delta(x_0,r)} \omega^n \leq \Omega r^{2n}, \qquad \text{for all } \ 0 < r \leq c'.$$

Throughout our discussion $X_{0,c}$ will be understood to be connected, i.e., the local affine embedding of X is irreducible. Analogous assumptions were introduced by Bando, Kasue and Nakajima [3] in their proof of a removable singularities theorem for Einstein orbifold metrics. Part of the analysis used in their argument (cf. [3], lemmata 5.8, 5.9) was further applied by Bando in his proof of removable singularities for Hermitian-holomorphic vector bundles E, initially defined over the punctured ball in $\mathbb{C}^2$ [2]. In this case the metric on the base manifold was simply the Euclidean metric on $\mathbb{C}^2$, for which the conditions (i)–(iii) are satisfied automatically such that the punctured ball $B(0, \sqrt{c})^* = X_{0,c}$. It remained then to make the single condition that the Hermitian metric on E has L^2-curvature in order to obtain a locally free extension of E across the origin. Embedded in the proof is Bando's solution of the Cauchy-Riemann equation for $(0,1)$-forms on the punctured ball in $\mathbb{C}^2$. His method requires a combination of the $\bar{\partial}$-Neumann and Dirichlet conditions for solution of the Laplace-Beltrami equation on an annular region, before taking the uniform limit of this solution as the radius of the inner (Dirichlet) boundary goes to zero. Standard methods for extracting curvature terms from the complex Hodge Laplacian of E are fundamental in obtaining the "basic estimate" necessary for existence and regularity of solutions. As a means of solving the Cauchy-Riemann equation via Laplace's equation, Bando's method is analogous to the theory of Kohn and Hörmander for manifolds with strictly pseudoconvex boundary [11]. On a punctured domain, however, additional techniques from [3], as mentioned above, are required. Moreover, by contrast with the methods of Andreotti-Vesentini, or the main results of Ohsawa and Saper, the property of metric completeness on $X_{0,c}$ is not used.

In obtaining a previous removable singularities theorem for Hermitian-holomorphic vector bundles, defined initially on the complement of an analytic subset A of an n-dimensional complex manifold [9], the authors of the present article verified a straightforward extension of Bando's method to the solution of the Cauchy-Riemann equation on a punctured ball in $\mathbb{C}^d$, where $d \geq 2$ corresponds to the complex codimension of A. The main result of the following sections is a similarly straightforward extension of this method to the punctured neighbourhood of an isolated singular point. It should be remarked, however, that since the base metric is not smoothly defined at a singularity, we pass to a situation in some respects resembling that of [3]. For affine-analytic or projective varieties V the most natural examples of a strongly plurisubharmonic function ρ are provided by the restriction of $|z|^2$ or $\log(1+|z|^2)$ from the ambient $\mathbb{C}^N$. In the first instance ω then corresponds simply to the standard Kähler form on Euclidean space restricted to V, while in the second ω corresponds to a local restriction of the Kähler form associated with the Fubini-Study metric on $\mathbb{CP}_{N+1}$. For any $V \subset \mathbb{C}^N$ the restriction of the Euclidean metric is area-minimizing (cf. [6]), and the intrinsic Riemannian distance $\delta(0,x) \geq |x|$ implies that

$B_\delta(0,r)$ is contained in $\{x \in V \mid |x| < r\}$, hence conditions (ii) and (iii) above are satisfied at once. The curvature R_g of the restricted metric corresponds to $\beta \wedge \beta^*$, where β denotes the second fundamental form of the embedded variety. It is an easy computation to show that $\int_{V\cap B(0,r)} |\beta|^4 < \infty$ when, for example, $V : z^k = f(x,y)$ is an analytic surface in $\mathbb{C}^3$ such that $2 \le k < \mathrm{Ord}_f(0)$. Hence in this case $|R_g|$ belongs to $L^2(V \cap B(0,r))$. On the other hand surfaces defined by an equation of the form $z^k = xy$, which constitute a special class of orbifold singularities (cf., e.g., [5]), do not satisfy condition (i) with respect to the restricted ambient metric. For any singular space X of this type, corresponding to the quotient of $\mathbb{C}^n$ by a finite subgroup of $\mathbb{SU}(n)$, the most natural choice of ρ is that induced by $|z|^2$ on the Euclidean covering space, since the associated orbifold metric is flat.

In Sections two and three of the present article, we make the necessary modifications of Bando's method in order to treat the case of a reduced singular space X. For ρ satisfying conditions (i) – (iii) with both ξ and the curvature of the Hermitian metric of E belonging to $L^n_{\mathrm{loc}}(X)$, it is shown that the equation $\bar\partial u = \xi$ admits a smooth solution on $X_{0,c}$. Applications of this result to problems of holomorphic extension are discussed in section four. In particular, an extension theorem of Kohn-Rossi type [11], [12] for $\bar\partial_b$-closed sections of E defined initially on the CR-hypersurface corresponding to $\{\rho = c\}$ is obtained.

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§2. Weitzenböck formulae and the Neumann/Dirichlet condition

Let X be an n-dimensional reduced complex analytic space with isolated singularity $x_0 \in X$, equipped with a strongly plurisubharmonic exhaustion function $\rho : X \to [0,\infty)$ such that $\rho(x) = 0$ if and only if $x = x_0$. A smooth Kähler metric g on $X \setminus \{x_0\}$ will then be assumed to correspond to the positive, real closed form $\omega = \mathbf{i}\partial\bar\partial\rho$. Consider a holomorphic vector bundle $E \to X \setminus \{x_0\}$ equipped with a Hermitian metric h, and for $\varepsilon \geq 0$ define $X_{\varepsilon,c} = \{x \in X \mid \varepsilon < \rho(x) < c\}$. Given $\xi \in C^\infty(X_{0,r}, \Omega_X^{0,1}(E))$ such that $\bar\partial\xi = 0$, our aim will be to solve the Cauchy-Riemann equation for ξ by way of the Laplace equation when

$$\int_{X_{0,c}} |\xi|^n \omega^n < \infty .$$

The essential tool for obtaining the required estimate for existence and regularity of solutions will be the Weitzenböck identity, together with certain related formulae for the Laplace-Beltrami operator. Though the derivations of these formulae are a rather standard application of integration by parts, we will present them in outline here, both for the reader's convenience and for use in the next section. We begin with $\varphi \in C^\infty(X_{\varepsilon,c}, \Omega_X^{0,1}(E))$ satisfying the $\bar\partial$-Neumann/Dirichlet conditions (*), i.e.,

$$\sigma(\bar\partial_E^*, d\rho)\varphi \mid_{\{\rho=c\}} = \sigma(\bar\partial_E^*, d\rho)\bar\partial\varphi \mid_{\{\rho=c\}} = 0 ,$$

and $\varphi \mid_{\{\rho=\varepsilon\}} = 0$, where σ denotes the principal symbol (cf. [11]). If $g_{i\bar k}$ and $h_{\alpha\bar\beta}$ represent g and h in local holomorphic frames of $TX_{\varepsilon,c}$ and E respectively, then the volume form corresponds to

$$\omega^n = n!\left(\frac{\mathbf{i}}{2}\right)^n \det(g) d(z) \wedge d(\bar z) ,$$

and for any smooth section ψ we have

$$(\varphi, \bar\partial_E\psi) = \int_{X_{\varepsilon,c}} \varphi_{\bar i}^{\alpha} g^{k\bar i} h_{\alpha\bar\beta} \overline{\frac{\partial\psi^\beta}{\partial \bar z_k}} \det(g) .$$

Integrating by parts, and noting that the conditions (*) eliminate the boundary integral, it follows that

$$\bar\partial_E^*\varphi = -\frac{\partial\varphi_{\bar i}^{\alpha}}{\partial z_k} g^{k\bar i} + \varphi_{\bar a}^{\alpha} g^{b\bar a}\left(g^{k\bar i}\frac{\partial g_{b\bar i}}{\partial z_k}\right) - \varphi_{\bar i}^{\varepsilon} g^{k\bar i}\left(\frac{\partial h_{\varepsilon\bar\gamma}}{\partial z_k} h^{\alpha\bar\gamma}\right) - \varphi_{\bar i}^{\alpha} g^{k\bar i}\frac{\partial \log\det(g)}{\partial z_k}$$

and hence

$$(\bar\partial_E\bar\partial_E^*\varphi)_{\bar l} = -2\frac{\partial^2\varphi_{\bar i}^{\alpha}}{\partial\bar z_l\partial z_k} - \varphi_{\bar a}^{\alpha} g^{b\bar a} R_{bk\bar l}^{k} + \varphi_{\bar i}^{\varepsilon} g^{k\bar i} F_{\varepsilon k\bar l}^{\alpha} + \varphi_{\bar i}^{\alpha} g^{k\bar i} R_{bk\bar l}^{b} ,$$

where R and F denote the curvature forms of $TX_{\varepsilon,c}$ and E respectively. Here and in the following explicit use is made of the Kähler condition, and the fundamental fact that at any point of $X_{0,c}$ holomorphic normal frames may be chosen for TX and E.

Conversely

$$\|\bar{\partial}_E\varphi\|^2 = \int_{X_{\varepsilon,c}} \frac{\partial \varphi^\alpha_{\bar{i}}}{\partial \bar{z}_k}(g^{\mu\bar{k}}g^{\nu\bar{i}} - g^{\nu\bar{k}}g^{\mu\bar{i}})h_{\alpha\bar{\beta}}\overline{\frac{\partial \varphi^\beta_{\bar{\nu}}}{\partial \bar{z}_\mu}}\det(g)$$

from which we obtain

$$(\bar{\partial}^*_E\bar{\partial}_E\varphi)_{\bar{l}} = -2\frac{\partial^2\varphi_{\bar{l}}}{\partial z_i\partial\bar{z}_i} + 2\frac{\partial^2\varphi^\alpha_{\bar{i}}}{\partial z_i\partial\bar{z}_l}$$

and hence

$$(\Box''\varphi)_{\bar{l}} = -\frac{1}{2}\triangle\varphi_{\bar{l}} - \varphi^\alpha_{\bar{a}}g^{b\bar{a}}R^k_{bk\bar{l}} + \varphi^\alpha_{\bar{i}}g^{k\bar{i}}R^b_{bk\bar{l}} + \varphi^\varepsilon_{\bar{i}}g^{k\bar{i}}F^\alpha_{\varepsilon k\bar{l}}\ .$$

Given $\xi = \Box''\varphi$, we may rewrite this last expression in the simplified form

$$\triangle\varphi = 2F_h\cdot\varphi - 2\operatorname{tr}_k R_g\cdot\varphi + 2\operatorname{tr}_b R_g\cdot\varphi - 2\xi\ . \tag{1}$$

Now let $\nabla^{0,1} : \Omega^{0,1}_X(E) \to \Omega^{0,1}_X \otimes \Omega^{0,1}_X(E)$ denote the connection corresponding to the complex conjugate of the Chern connection on $\Omega^{1,0}_X$, hence in local coordinates

$$(\nabla^{0,1}\varphi)_{\bar{k}\bar{i}} = \frac{\partial\varphi^\alpha_{\bar{i}}}{\partial\bar{z}_k} + g_{j\bar{i}}\frac{\partial g^{j\bar{a}}}{\partial\bar{z}_k}\varphi^\alpha_{\bar{a}}\ .$$

Integrating by parts once again, and noting that with respect to holomorphic normal coordinates the first derivatives of g and g^{-1} may be neglected in the adjoint of $\nabla^{0,1}$, we have

$$\begin{aligned}\|\nabla^{0,1}\varphi\|^2 &= \int_{\partial X_{\varepsilon,c}}\langle\nabla^{0,1}\varphi, \sigma(\nabla^{0,1}, d\rho)\varphi\rangle \\ &\quad - \int_{X_{\varepsilon,c}}\left(\frac{\partial^2\varphi^\alpha_{\bar{i}}}{\partial z_\mu\partial\bar{z}_k} + \frac{\partial}{\partial z_\mu}(g_{j\bar{i}}\frac{\partial g^{j\bar{a}}}{\partial\bar{z}_k})\varphi^\alpha_{\bar{a}}\right)g^{\mu\bar{k}}g^{\nu\bar{i}}h_{\alpha\bar{\beta}}\overline{\varphi^\beta_{\bar{\nu}}}\det(g).\end{aligned} \tag{2}$$

Now

$$\begin{aligned}&\int_{\partial X_{\varepsilon,c}}\langle\nabla^{0,1}\varphi, \sigma(\nabla^{0,1}, d\rho)\varphi\rangle \\ &\qquad = \int_{\partial X_{\varepsilon,c}}\left(\frac{\partial\varphi^\alpha_{\bar{i}}}{\partial\bar{z}_k} + g_{j\bar{i}}\frac{\partial g^{j\bar{a}}}{\partial\bar{z}_k}\varphi^\alpha_{\bar{a}}\right)g^{\mu\bar{k}}g^{\nu\bar{i}}h_{\alpha\bar{\beta}}\overline{\frac{\partial\rho}{\partial\bar{z}_\mu}\varphi^\beta_{\bar{\nu}}} \\ &\qquad = \int_{\partial X_{\varepsilon,c}}\left(\frac{\partial\varphi^\alpha_{\bar{k}}}{\partial\bar{z}_i} + g_{j\bar{i}}\frac{\partial g^{j\bar{a}}}{\partial\bar{z}_k}\varphi^\alpha_{\bar{a}}\right)g^{\mu\bar{k}}g^{\nu\bar{i}}h_{\alpha\bar{\beta}}\overline{\frac{\partial\rho}{\partial\bar{z}_\mu}\varphi^\beta_{\bar{\nu}}}\ ,\end{aligned}$$

due to the fact that

$$0 =< \sigma(\bar\partial_E^*, d\rho)\bar\partial\varphi, \varphi >|_{\{\rho=\varepsilon\}\cup\{\rho=c\}} = \left(\frac{\partial\varphi^\alpha_{\bar i}}{\partial\bar z_k} - \frac{\partial\varphi^\alpha_{\bar k}}{\partial\bar z_i}\right)\frac{\partial\rho}{\partial z_\mu} g^{\mu\bar k} g^{\nu\bar i} h_{\alpha\bar\beta}\overline{\varphi^\beta_{\bar\nu}} \ .$$

Hence, recalling that $\omega = \mathbf{i}\partial\bar\partial\rho$, it follows that

$$\begin{aligned}
&\int_{\partial X_{\varepsilon,c}} \langle \nabla^{0,1}\varphi, \sigma(\nabla^{0,1}, d\rho)\varphi\rangle \\
&= -\int_{\partial X_{\varepsilon,c}} \varphi^\alpha_{\bar k} g^{\mu\bar k} g^{\nu\bar i} h_{\alpha\bar\beta}\overline{\left(\frac{\partial^2\rho}{\partial z_i\partial\bar z_\mu}\varphi^\beta_{\bar\nu} + \frac{\partial\rho}{\partial\bar z_\mu}\frac{\partial\varphi^\beta_{\bar\nu}}{\partial z_i}\right)} \\
&= -\int_{\partial X_{\varepsilon,c}} \varphi^\alpha_{\bar k}\delta_{ki} g^{\nu\bar i} h_{\alpha\bar\beta}\overline{\varphi^\beta_{\bar\nu}} \ ,
\end{aligned}$$

noting also that $\frac{\partial\rho}{\partial z_\mu}\varphi^\alpha_{\bar k} g^{\mu\bar k} = \sigma(\bar\partial_E^*, d\rho)\varphi$, which vanishes on the boundary due to (*). For the second term of (2), we have

$$\begin{aligned}
&\int_{X_{\varepsilon,c}} \left(\frac{\partial^2\varphi^\alpha_{\bar i}}{\partial z_\mu\partial\bar z_k} + \frac{\partial}{\partial z_\mu}(g_{j\bar i}\frac{\partial g^{j\bar a}}{\partial\bar z_k})\varphi^\alpha_{\bar a}\right) g^{\mu\bar k} g^{\nu\bar i} h_{\alpha\bar\beta}\overline{\varphi^\beta_{\bar\nu}}\det(g) \\
&\qquad = \int_{X_{\varepsilon,c}} \left(2\frac{\partial^2\varphi^\alpha_{\bar i}}{\partial z_k\partial\bar z_k} + \varphi^\alpha_{\bar a} g^{\mu\bar k}\overline{R^a_{ik\bar\mu}}\right) g^{\nu\bar i} h_{\alpha\bar\beta}\overline{\varphi^\beta_{\bar\nu}}\det(g) \ ,
\end{aligned}$$

so that

$$\begin{aligned}
(3)\quad (\Box''\varphi, \varphi) &= \|\bar\partial_E\varphi\|^2 + \|\bar\partial_E^*\varphi\|^2 \\
&= \|\nabla^{0,1}\varphi\|^2 + (\mathrm{tr}_\circ R_g\cdot\varphi, \varphi) + (F_h\cdot\varphi, \varphi) + \int_{\{\rho=c\}} |\varphi|^2,
\end{aligned}$$

where $\mathrm{tr}_\circ R_g\cdot\varphi = \mathrm{tr}_b R_g\cdot\varphi - \mathrm{tr}_k R_g\cdot\varphi + \varphi^\alpha_{\bar a} g^{\mu\bar k}\overline{R^a_{ik\bar\mu}}$. Finally let the connection $\nabla^{1,0}: \Omega^{0,1}_X(E) \to \Omega^{1,0}_X \otimes \Omega^{0,1}_X(E)$ be induced by the Chern connection ∂_E, and suppose $0 \le \eta \le 1$ is a smooth function with compact support in $\{\rho < 2\delta < c\}$, which is identically equal to one on $\{\rho < \delta\}$ for some $\delta > \varepsilon$. From the Bochner-Kodaira-Nakano identity, we see that

$$\begin{aligned}
\|\nabla^{1,0}(\eta\varphi)\|^2 &= \|\partial_E(\eta\varphi)\|^2 = (\Box'(\eta\varphi), \eta\varphi) \\
&= (\Box''(\eta\varphi), \eta\varphi) + \mathbf{i}([\Lambda, F_h](\eta\varphi), \eta\varphi) \ ,
\end{aligned}$$

where Λ denotes the adjoint of $\omega\wedge\cdot$. Thus from (3) we have

$$\begin{aligned}
\|\nabla^{1,0}(\eta\varphi)\|^2 = \|\nabla^{0,1}(\eta\varphi)\|^2 &+ \eta^2(\mathrm{tr}_\circ R_g\cdot\varphi, \varphi) + \eta^2(F_h\cdot\varphi, \varphi) \\
&+ \mathbf{i}(\Lambda(F_h\wedge(\eta\varphi)), \eta\varphi) \ .
\end{aligned}$$

Now, for arbitrary $\psi \in C^\infty(X_{\varepsilon,c}, \Omega_X^{0,1}(E))$,

$$\begin{aligned}\langle F_h \wedge \varphi, \omega \wedge \psi\rangle &= F^a_{\alpha c\bar{d}}\varphi^\alpha_{\bar{i}} g^{c\bar{\mu}}(g^{\nu\bar{d}}g^{\lambda\bar{i}} - g^{\nu\bar{i}}g^{\lambda\bar{d}})h_{\alpha\bar{\beta}}\overline{\frac{\mathbf{i}}{2}g_{\mu\bar{\nu}}\psi^\beta_{\bar{\lambda}}} \\ &= -\frac{\mathbf{i}}{2}g^{c\bar{d}}(F^a_{\alpha c\bar{d}}\varphi^\alpha_{\bar{i}} - F^a_{\alpha c\bar{i}}\varphi^\alpha_{\bar{d}})g^{\lambda\bar{i}}h_{\alpha\bar{\beta}}\overline{\psi^\beta_{\bar{\lambda}}}\ .\end{aligned}$$

Hence

$$\mathbf{i}\Lambda(F_h \wedge \eta\varphi) = \eta g^{c\bar{d}}(F^a_{\alpha c\bar{d}}\varphi^\alpha_{\bar{i}} - F^a_{\alpha c\bar{i}}\varphi^\alpha_{\bar{d}})$$

implies

$$(\eta g^{k\bar{l}}F^\alpha_{\varepsilon k\bar{l}}\varphi^\varepsilon_{\bar{i}}, \eta\varphi) + \mathbf{i}(\Lambda(F_h \wedge \eta\varphi), \eta\varphi) = (\eta^2 g^{c\bar{d}}F^a_{\alpha c\bar{d}}\varphi^\alpha, \varphi)\ ,$$

i.e.,

$$\|\nabla^{1,0}(\eta\varphi)\|^2 = \|\nabla^{0,1}(\eta\varphi)\|^2 + \eta^2(\mathrm{tr}_\circ\, R_g \cdot \varphi, \varphi) + \eta^2(\mathrm{tr}\, F_h \cdot \varphi, \varphi)\,. \tag{4}$$

In the following section these identities will be used to obtain the necessary estimates for solving both the Laplace and Cauchy-Riemann equations on $X_{0,c}$ via the method of Bando.

§3. Existence and regularity of solutions

As in [9], the following is simply an expanded version of Bando's argument [2], here taking account of additional curvature terms from the Kähler metric. From this point it will be assumed that

(i) $$\int_{X_{0,c}} |R_g|^n < \infty \ \text{ and} \int_{X_{0,c}} |F_h|^n < \infty\ ,$$

and that moreover the Sobolev inequality holds with respect to ω, i.e.,

(ii) $$\Big(\int_{X_{0,c}} |\psi|^{\frac{2n}{n-1}}\omega^n\Big)^{\frac{n-1}{n}} \le \mathrm{const}(n)\int_{X_{0,c}} |\nabla\psi|^2\omega^n$$

for any compactly supported, smooth E-valued $(0,1)$-form ψ, where $\nabla = \nabla^{1,0} + \nabla^{0,1}$. For the last part of the proof it will also be necessary to assume that there exists a constant $\Omega > 0$ such that

(iii) $$\int_{B(x_0,r)^*} \omega^n \le \Omega r^{2n} \ \text{ for all } \ 0 < r \le c' < c,$$

where $B(x_0, r)^* \subseteq X_{0,c}$ denotes the (punctured) geodesic ball of radius r of a metric distance function $\delta(x_0, x)$ on $X_{0,c}$. We note that existence of the Lipschitz 1-function $\delta(x_0, x)$ follows from the Hopf-Rinow theorem (cf. also [3]).

THEOREM 1. *For any* $\xi \in L^n \cap C^\infty(X_{0,c}, \Omega_X^{0,1}(E))$ *such that* $\bar\partial\xi = 0$, *there exists* $u \in L^2 \cap C^\infty(X_{0,c}, E)$ *such that* $\bar\partial u = \xi$, *and* $\|u\|_{L^2} \le \|\xi\|_{L^2}$.

Proof. Consider $h_K = he^{-K\rho}$ for K fixed independently of ε in the following, noting that $F_{h_K} = F_h + K\partial\bar\partial\rho$.

LEMMA 1. (cf. [2, Lemma 2]) *There exist* $K > 0$, $c_0 > 0$ *and* $\delta > 0$ *independent of* $\varepsilon > 0$ *such that, for all smooth* $(0,1)$*-forms* φ *satisfying the Dirichlet-*$\bar\partial$*-Neumann condition, we have*

$$\begin{aligned}(\Box''\varphi,\varphi) &= ||\bar\partial\varphi||^2 + ||\bar\partial^*\varphi||^2 \\ &\ge c_0\left(||\varphi||^2 + \int_{\rho=c}|\varphi|^2 + ||\nabla^{0,1}\varphi||^2 + \int_{\varepsilon<\rho<\delta}|\nabla\varphi|^2\right).\end{aligned}$$

Proof. We split the term $\int|\nabla^{0,1}\varphi|^2 = \frac{1}{2}\int|\nabla^{0,1}\varphi|^2\times 2$ and subsequently omit $\frac{1}{2}\int|\nabla^{0,1}\varphi|^2 + \int_{\rho=c}|\varphi|^2$ from equation (3) . Thus,

$$(\Box''\varphi,\varphi) \ge \frac{1}{2}\int|\nabla^{0,1}\varphi|^2 + (F_{h_K}\cdot\varphi,\varphi) + (\mathrm{tr}_\circ R_g\cdot\varphi,\varphi).$$

Let $0 \le \eta \le 1$ be a smooth function with compact support in $\{\rho < 2\delta\}$ which is equal to 1 on $\{\rho \le \delta\}$. The radius $\delta > \varepsilon$ will be fixed in the following independently of ε. Now

$$\begin{aligned}\int|\nabla^{0,1}(\eta\varphi)|^2 &\le 2\int\left(|\bar\partial\eta|^2|\varphi|^2 + |\nabla^{0,1}\varphi|^2|\eta|^2\right) \\ &\le 2\int\left(|\bar\partial\eta|^2|\varphi|^2 + |\nabla^{0,1}\varphi|^2\right),\end{aligned}$$

hence

$$(\Box''\varphi,\varphi) \ge \frac{1}{4}\int(|\nabla^{0,1}(\varphi\eta)|^2 - 2|\bar\partial\eta|^2|\varphi|^2) + (F_{h_K}\varphi,\varphi) + (\mathrm{tr}_\circ R_g\cdot\varphi,\varphi).$$

From (4) it follows that

$$\int|\nabla^{0,1}(\eta\varphi)|^2 = \int|\nabla^{1,0}(\eta\varphi)|^2 - (\eta^2\,\mathrm{tr}\,F_{h_K}\varphi,\varphi) - (\eta^2\,\mathrm{tr}_\circ R_g\cdot\varphi,\varphi).$$

Thus from

$$\begin{aligned}\frac{1}{4}\int|\nabla^{0,1}(\eta\varphi)|^2 &\ge \frac{1}{2n}\int|\nabla^{0,1}(\eta\varphi)|^2 \\ &\ge \frac{1}{4n}\int|\nabla^{0,1}(\eta\varphi)|^2 + |\nabla^{1,0}(\eta\varphi)|^2 \\ &\qquad -\frac{1}{4n}((\eta^2\,\mathrm{tr}\,F_{h_K}\varphi,\varphi) + (\eta^2\,\mathrm{tr}_\circ R_g\varphi,\varphi)),\end{aligned}$$

we have

$$(\Box''\varphi,\varphi) \geq \frac{1}{4n}\int |\nabla(\eta\varphi)|^2 - \frac{1}{2}\int |\bar{\partial}\eta|^2|\varphi|^2 - \frac{1}{4n}((\eta^2 \operatorname{tr} F_{h_K}\varphi,\varphi) \\ +(\eta^2 \operatorname{tr}_\circ R_g\cdot\varphi,\varphi)) + (F_{h_K}\varphi,\varphi) + (\operatorname{tr}_\circ R_g\cdot\varphi,\varphi).$$

Next, using the fact that $F_{h_K} = F_h + K\mathbf{I}_n\partial\bar{\partial}\rho$ and $\operatorname{tr} F_{h_K} = \operatorname{tr} F_h + nK\mathbf{I}_n$ at the origin of any holomorphic normal coordinate system, where the trace is taken over the form-indices, it follows that

$$(\Box''\varphi,\varphi) \geq \frac{1}{4n}\int |\nabla(\eta\varphi)|^2 + \int\left(\frac{3}{4}K - \frac{1}{2}|\bar{\partial}\eta|^2\right)|\varphi|^2 \\ -\frac{1}{4n}(\eta^2 \operatorname{tr} F_h\varphi,\varphi) + (F_h\cdot\varphi,\varphi) + \frac{4n-1}{4n}(\operatorname{tr}_\circ R_g\cdot\varphi,\varphi).$$

Since $\eta\varphi$ has compact support, the Sobolev inequality applies to $\eta\varphi$ and

$$|(\eta^2 \operatorname{tr} F_h\varphi,\varphi)| \leq c(n)\left(\int |\eta\varphi|^{\frac{2n}{n-1}}\right)^{\frac{n-1}{n}}\left(\int_{\operatorname{supp}\eta}|F_h|^n\right)^{\frac{1}{n}} \\ \leq c(n)\int|\nabla(\eta\varphi)|^2\left(\int_{\operatorname{supp}\eta}|F_h|^n\right)^{\frac{1}{n}}.$$

Moreover, writing $(F_h\cdot\varphi,\varphi) = ((1-\eta^2)F_h\cdot\varphi,\varphi) + (\eta^2F_h\varphi,\varphi)$ and $(\operatorname{tr}_\circ R_g\cdot\varphi,\varphi) = ((1-\eta^2)\operatorname{tr}_\circ R_g\cdot\varphi,\varphi) + (\eta^2\operatorname{tr}_\circ R_g\cdot\varphi,\varphi)$ we may apply the Sobolev inequality to the second term of each expression similarly. Thus

$$-\frac{1}{4n}(\eta^2 \operatorname{tr} F_h\varphi,\varphi) + (F_h\cdot\varphi,\varphi) + \frac{4n-1}{4n}(\operatorname{tr}_\circ R_g\cdot\varphi,\varphi) \\ \geq -\int|\nabla(\eta\varphi)|^2\left(2c(n)(\int_{\operatorname{supp}\eta}|F_h|^n)^{\frac{1}{n}} + c'(n)(\int_{\operatorname{supp}\eta}|R_g|^n)^{\frac{1}{n}}\right) \\ +((1-\eta^2)F_h\cdot\varphi,\varphi) + \frac{4n-1}{4n}((1-\eta^2)\operatorname{tr}_\circ R_g\cdot\varphi,\varphi).$$

We choose δ so that $2c(n)(\int_{\operatorname{supp}\eta}|F_h|^n)^{1/n} + c'(n)(\int_{\operatorname{supp}\eta}|R_g|^n)^{\frac{1}{n}} \leq \frac{1}{8n}$, thus

$$(\Box''\varphi,\varphi) \geq \\ \frac{1}{8n}\int|\nabla(\eta\varphi)|^2 + \int\left(\frac{3}{4}K - \frac{1}{2}|\bar{\partial}\eta|^2 - \max_{\{\rho\geq\delta\}}(|F_h|+|R_g|)\right)|\varphi|^2.$$

Choose K large so that the second term is larger than $\int|\varphi|^2$, and recalling that we omitted $\frac{1}{2}\int|\nabla^{0,1}\varphi|^2 + \int_{|z|=1}|\varphi|^2$, we obtain the desired estimate with $c_0 = \frac{1}{8n}$.

Lemma 1 shows that the kernel of $\Box''$ is trivial, hence there exists a unique $L^2(X_{\varepsilon,c})$-solution to the equation $\Box''\varphi_\varepsilon = \xi$ with $||\varphi_\varepsilon|| \leq ||\xi||$ due to the standard existence theorem for self-adjoint operators. Regularity of solutions follows from the basic estimate of the above lemma applied near $\rho = c$ and $\rho = \varepsilon$ respectively (cf. [11, theorem 2.1.7]). Now take the uniform limit of these solutions as $\varepsilon \to 0$. It is important to note that $\int_{X_{0,c}} |\varphi|^{\frac{2n}{n-1}} < \infty$ due to the estimate of Lemma 1 and the Sobolev inequality. Our aim is now to show that $\bar\partial^*\bar\partial\varphi \equiv 0$ on $X_{0,c}$, then we may set $u = \bar\partial^*\varphi$ and obtain the desired solution for the Cauchy-Riemann equation. First we need

Lemma 2. (cf. [3, lemmata 5.8, 5.9]) *Suppose that smooth nonnegative functions $f \in L^n(X_{0,c})$ and $u \in L^{\frac{sn}{n-1}}(X_{0,c})$ satisfy the equation $\Delta u \geq -fu$ on $X_{0,c}$ for some $s > 1$. Then, $u \in L^p(X_{0,c})$ for all $p > 1$. Also, one may replace the condition $u \in L^{\frac{sn}{n-1}}(X_{0,c})$ by $\int_{B_\delta(x_0,r)} u^s \leq o(r^2)$ for some $s > 1$.*

(Since an outline of the proof of this result is also discussed in [9], it will be omitted here.) Now

$$\triangle|\varphi|^2 = 2(|d|\varphi||^2 + |\varphi|\triangle|\varphi|) = 2(\mathrm{Re}\langle\triangle\varphi,\varphi\rangle + |\nabla\varphi|^2) \ ,$$

and hence it is a simple consequence of the Cauchy-Schwarz inequality that

$$\triangle|\varphi| \geq \frac{\mathrm{Re}\langle\triangle\varphi,\varphi\rangle}{|\varphi|} \ .$$

Inserting the expression for $\triangle\varphi$ from equation (1) and applying once again the Cauchy-Schwarz inequality to $\langle\triangle\varphi,\varphi\rangle$ we obtain

$$\triangle|\varphi| \geq -2(|F_h| + n|R_g|)|\varphi| - 2|\xi|$$

on $X_{0,c}$. Using the lemma above with $u = |\varphi| + 1$, $f = 2(|F_h| + n|R_g| + |\xi|)$ and $s = 2$, we conclude that $|\varphi| \in L^p(X_{0,c})$ for all $p > 1$. Next, with $\eta \in C^1_o(X_{0,c})$, we write

$$\begin{aligned}\int_{\mathrm{supp}\,\eta} |\nabla(\eta\varphi)|^2 &= \int \left(5|d\eta\otimes\varphi|^2 + \frac{5}{4}\eta^2|\nabla\varphi|^2 - |2d\eta\otimes\varphi - \frac{1}{2}\eta\nabla\varphi|^2\right) \\ &\leq 5\int |d\eta|^2|\varphi|^2 + \frac{5}{4}\int |<\eta^2\varphi,\triangle\varphi>| \\ &\leq 5\int |d\eta|^2|\varphi|^2 + c(n)\int (|F_h| + n|R_g| + |\xi|)\eta^2|\varphi|^2\end{aligned}$$

$$\leq \left(\int |d\eta|^{2n}\right)^{\frac{1}{n}} \cdot \left(\int_{\operatorname{supp}\eta} |\varphi|^{\frac{2n}{n-1}}\right)^{\frac{n-1}{n}}$$
$$+ c(n)\left(\int_{\operatorname{supp}\eta} (|F| + n|R_g| + |\xi|)^n\right)^{\frac{1}{n}} \left(\int |\eta\varphi|^{\frac{2n}{n-1}}\right)^{\frac{n-1}{n}}.$$

For $c' > 2r > 2r' > 0$, choose η such that

(a) $\eta(x) = \eta(\delta(x_0, x))$,

(b) $\eta = 0$ for $\delta(x_0, x) > 2r$ and $\delta(x_0, x) < r'$, and $\eta = 1$ on $r > \delta(x_0, x) > 2r'$,

(c) $|d\eta| \leq 2/r$ on $2r > \delta(x_0, x) > r$ and $|d\eta| \leq 2/r'$ on $2r' > \delta(x_0, x) > r'$.

We may let $r' \to 0$ since $\int_{B_\delta(x_0, 2r')} |d\eta|^{2n} \leq \mathrm{const}(n)$. Since $|\varphi| \in L^p(X_{0,c})$ for any $p > 1$, the Hölder inequality together with the basic volume assumption (iii) above implies $\int_{\operatorname{supp}\eta} |\varphi|^{\frac{2n}{n-1}} \leq o(r^{2n-\varepsilon})$ for any $\varepsilon > 0$. Thus, the above inequality shows that $\int_{B_\delta(x_0,r)} |\nabla\varphi|^2 \leq o(r^{2(n-1)-\varepsilon})$ for any $\varepsilon > 0$. For $n \geq 3$, we see $\bar\partial^*\bar\partial\varphi = 0$ as follows. Let ζ be a smooth function with $\zeta = 1$ on $\delta(x_0, x) > 2r$, $\zeta = 0$ on $\delta(x_0, x) < r$. Since $0 = \bar\partial\xi = \bar\partial\square''\varphi = \bar\partial\bar\partial^*\bar\partial\varphi$,

$$\int_{X_{0,c}} |\zeta\bar\partial^*\bar\partial\varphi|^2 = (\bar\partial^*\bar\partial\varphi, \zeta^2\bar\partial^*\bar\partial\varphi) = (\bar\partial\varphi, \bar\partial(\zeta^2\bar\partial^*\bar\partial\varphi))$$
$$= 2(\bar\partial\varphi, \zeta\bar\partial\zeta \wedge \bar\partial^*\bar\partial\varphi) \leq 4\int |\bar\partial\varphi|^2|d\zeta|^2 + \frac{1}{2}\int |\zeta\bar\partial^*\bar\partial\varphi|^2.$$

Since $|d\zeta| \leq 2o(r^{-1})$ and $\int |\nabla\varphi|^2 \leq o(r^3)$ for $n \geq 3$, by letting $r \to 0$ we may conclude the proof. For $n = 2$, the equation $\Delta|\bar\partial\varphi| \geq -c(n)(|F_h| + |R_g|)|\bar\partial\varphi|$ and $\int_{B_\delta(x_0,r)} |\nabla\varphi|^{1.5} \leq o(r^2)$ implies $|\bar\partial\varphi| \in L^p(X_{0,c})$ for all $p > 1$ by Lemma 2, thus we may proceed in a manner similar to the case $n \geq 3$ to obtain $\bar\partial^*\bar\partial\varphi = 0$. This completes the proof of the theorem.

§4. Applications to holomorphic extension

Let s be a holomorphic section of E in a small coordinate neighbourhood U around $x_1 \in X_{0,c}$, and consider ϑ a cut-off function with compact support in U, which is identically equal to one near x_1. ϑs may then be regarded as a smooth section of E on $X_{0,c}$, with $\xi = \bar\partial(\vartheta s)$ corresponding to a $\bar\partial$-closed $(0,1)$-form which vanishes identically near x_1. Now $\psi = 2n\vartheta\log|z|$ is a compactly supported function on U which is plurisubharmonic near $z(x_1) = 0$, and extends smoothly to $X_{0,c} \setminus \{x_1\}$. It follows that $\mathbf{i}\partial\bar\partial\psi$ is locally bounded below, and hence that $K\rho + \psi$ is plurisubharmonic for K sufficiently large. Now repeat the solution of the $\bar\partial$-Neumann problem for $\bar\partial u = \xi$ with respect to $h_K = he^{-(K\rho+\psi)}$, noting that the potentially difficult term $-\eta^2 \operatorname{tr}(\partial\bar\partial\psi)$ will vanish if we define $\operatorname{supp}\eta \cap U = \emptyset$. Moreover, the fact that $\xi \equiv 0$ near x_1 implies that $\|\xi\|_{L^n} < \infty$, while the basic estimate above implies that $\|u\|_{L^2} \leq \|\xi\|_{L^2}$ (cf. [11]). Hence $u(x_1) = 0$ and $\vartheta s - u$ is a holomorphic section of E on $X_{0,c}$ which agrees with s at x_1. An argument identical to the one given by Bando in lemmata 7 and 8 of [2] may then be used to embed $E\,|_{X_{0,c}} \hookrightarrow \underline{\mathbb{C}}^N$ for sufficiently large N, and thus to obtain a reflexive sheaf extension $\mathcal{F} \to X$ via Hartogs' theorem. In summary,

Corollary 1. *Let X^n be a reduced complex space with normal isolated singularity at $x_0 \in X$, and $\rho : X \to [0, \infty)$ a smooth, strongly plurisubharmonic exhaustion function centred at x_0 which satisfies the conditions* (i)–(iii) *above. If $E \to X \setminus \{x_0\}$ is a Hermitian-holomorphic vector bundle with L^n_{loc}-curvature, then there exists a reflexive sheaf $\mathcal{F} \to X$ such that $\mathcal{F}\,|_{X\setminus\{x_0\}} \cong \mathcal{O}(E)$.*

A further consequence of the solubility of the $\bar\partial$-Neumann problem on $X_{0,c}$ is the solvability of the equation $\bar\partial u = F_h$, which implies existence of a holomorphic connection on $E \to X_{0,c}$ (cf. [9]). The extension argument of [4], Theorem 2.2 will then automatically imply the following

Corollary 2. *Let X^n be a reduced analytic space with isolated singularity $x_0 \in X$ and strongly plurisubharmonic function $\rho : X \to [0, \infty)$ satisfying conditions* (i)–(iii)*. Consider $\pi : Y \to X$ to be a surjective holomorphic map from a complex manifold Y such that $\pi^{-1}(x_0)$ has codimension greater than one. If $E \to X\setminus\{x_0\}$ is a Hermitian-holomorphic vector bundle with L^n_{loc}-curvature, then there exists a unique holomorphic vector bundle $V \to Y$ such that $V\,|_{Y\setminus\pi^{-1}(x_0)} \cong \pi^* E$.*

A natural instance of this result occurs when π corresponds to a quotient of $\mathbb{C}^n$ under the action of a finite group $G \subset \mathbb{SU}(n)$, i.e., X has an orbifold singularity at x_0. Another potential class of examples corresponds to isolated singularities with “small resolution”. Explicit examples of such singularities, with X a hypersurface in $\mathbb{C}^4$ and $\pi^{-1}(x_0) \cong \mathbb{CP}_1$ were presented in [13]. At present an example which admits a strongly plurisubharmonic function ρ of the required type is not known to the authors, however.

Our final application concerns the problem of holomorphic extension from the strictly pseudoconvex CR-hypersurface $S_c \subset X$ corresponding to $\rho = c$. Let $\sigma : S_c \to E$ be a section of $E \to X \setminus \{x_0\}$, such that σ is closed with respect to the tangential Cauchy-Riemann operator on $E \mid_{S_c}$, i.e., $\bar{\partial}_b \sigma = 0$ (cf. [11]). Let $s : X_{0,c} \to E$ be a smooth extension of σ, with support in an arbitrarily small neighbourhood of S_c. From the standard theory of the tangential Cauchy-Riemann complex for S_c, we note that $\bar{\partial}_b \sigma = 0$ if and only if the $(n, n-1)$-form $\xi = \bar{\partial}^* \star \bar{s}$ satisfies the $\bar{\partial}$-Neumann conditions (*), where $\star$ denotes the Hodge operator on forms ([11], Proposition 5.2.2). The essential idea of the extension technique of Kohn and Rossi [12] is to obtain a solution to the equation $\bar{\partial}^* u = \xi$ such that u again satisfies the conditions (*), and this is done in a manner entirely analogous to the theory of the equation $\bar{\partial} u = \xi$ for a $(0,1)$-form ξ. Moreover, u satisfies $(a) \bar{\partial}(s - \star\bar{u}) = 0$ and $(b) \star \bar{u} \mid_{S_c} = 0$, hence $s - \star\bar{u}$ is a holomorphic extension of σ. For the case of $X_{0,c}$ corresponding to the punctured neighbourhood of an isolated singularity our adaptation of Bando's method is applied to this end. The argument here also is essentially a dualised version of the method of the previous sections, and goes through with only minor alterations which will be outlined below.

For $(n, n-1)$-forms $\varphi = \varphi_{\bar{i}} dz \wedge d\bar{z}[i]$ the inner product in any local frame will be made up of cofactors of the matrix $\bar{g}^{-1}$ where g represents the metric on TX as usual. Hence we may write

$$\langle dz \wedge d\bar{z}[i], dz \wedge d\bar{z}[j] \rangle = \det(g)^{-2} g_{i,\bar{j}} \ .$$

If it is assumed that φ satisfies the $\bar{\partial}$-Neumann/Dirichlet conditions (*), then via integration by parts we obtain

$$(\Box'' \varphi)_{\bar{i}} = -\frac{1}{4} \triangle \varphi_{\bar{i}} - \varphi_{\bar{a}} R^i_{ak\bar{k}} + \varphi_{\bar{i}} R^b_{bk\bar{k}} + \varphi^\nu F^\alpha_{\nu k\bar{k}} + (-1)^{i+k+1} \varphi^\nu_{\bar{k}} F^\alpha_{\nu i\bar{k}} \ .$$

Similarly

$$\nabla^{0,1} \varphi = \Sigma_{i,k} \left(\frac{\partial \varphi^\alpha_{\bar{i}}}{\partial \bar{z}_k} + g^{j\bar{i}} \frac{\partial g_{j\bar{a}}}{\partial \bar{z}_k} \varphi^\alpha_{\bar{a}} - \frac{\partial \log \det(g)}{\partial \bar{z}_k} \varphi^\alpha_{\bar{i}} \right) dz \wedge d\bar{z}[i] \otimes d\bar{z}_k \ ,$$

and hence

$$\|\nabla^{0,1} \varphi\|^2 = -\Big(\frac{1}{4} \triangle \varphi - \varphi_{\bar{a}} \overline{R^i_{ak\bar{k}}} - \varphi_{\bar{i}} R^b_{bk\bar{k}}, \varphi \Big) - \int_{\{\rho = c\}} |\varphi|^2 \ .$$

From here we proceed to solve the Laplace equation for ξ as before. The remainder of the argument is almost identical, apart from the obvious requirement that it is the term $\bar{\partial}\bar{\partial}^*\varphi$ which must be shown to vanish in this context. Note that

$$\begin{aligned}\int |\zeta\bar{\partial}\bar{\partial}^*\varphi|^2 &= (\bar{\partial}\bar{\partial}^*\varphi, \zeta^2\xi) - (\bar{\partial}\bar{\partial}^*\varphi, \zeta^2\bar{\partial}^*\bar{\partial}\varphi) \\ &= (\bar{\partial}^*\varphi, \bar{\partial}^*(\zeta^2\xi)) - (\bar{\partial}(\zeta^2\bar{\partial}\bar{\partial}^*\varphi), \bar{\partial}\varphi) \ ,\end{aligned}$$

since the Neumann conditions imply that $\sigma(\bar{\partial}^*, d\rho)\zeta^2\xi$ and $\sigma(\bar{\partial}^*, d\rho)\bar{\partial}\varphi$ must vanish on S_c. Moreover, if it is assumed that

$$\operatorname{supp}(s) \cap \operatorname{supp}(1-\zeta) = \emptyset \ ,$$

then $\bar{\partial}^*(\zeta^2\xi) = 0$ implies

$$\int |\zeta\bar{\partial}\bar{\partial}^*\varphi|^2 = -2(\zeta\bar{\partial}\zeta \wedge \bar{\partial}\bar{\partial}^*\varphi, \bar{\partial}\varphi) \leq 4\int |d\zeta|^2|\nabla\varphi|^2 + \frac{1}{4}\int |\zeta\bar{\partial}\bar{\partial}^*\varphi|^2 \ .$$

The argument may then be completed as before, with $u = \bar{\partial}\varphi$. In conclusion

COROLLARY 3. *Let X^n be a reduced analytic space with isolated singularity x_0, and let $\rho : X \to [0,\infty)$ be a strongly plurisubharmonic function satisfying the conditions* (i)–(iii). *If $E \to X \setminus \{x_0\}$ is a Hermitian-holomorphic vector bundle with L^n_{loc}-curvature, and σ a $\bar{\partial}_b$-closed section of $E\,|_{S_c}$, then there exists a unique holomorphic extension of σ as a section of E on $X_{0,c}$, and hence as a section of the reflexive sheaf $\mathcal{F}$ on X_c.*

Remark. The uniqueness follows from an idea suggested to us by Ohsawa. Let s and s' be two such holomorphic extensions of σ, so that $s - s'$ corresponds to a holomorphic section on $X_{0,c}$ which vanishes on S_c. In an open neighbourhood of any point $x \in X$ which also lies on S_c, $s - s'$ may be viewed as a vector-valued holomorphic function, such that the restriction of $s - s'$ to any holomorphic curve which cuts S_c transversely at x may be Schwarz-reflected to a local holomorphic extension on both sides of the boundary. It follows that $s - s'$ must vanish identically along the curve, since by assumption it vanishes on the subset of real codimension one corresponding to the intersection of the curve with S_c. Hence $s - s'$ must vanish identically on $X_{0,c}$.

Ohsawa has also pointed out to us that the problem of holomorphic extension may be approached directly through the solution of the Cauchy-Riemann equation, rather than through the dual problem. With respect to a modified metric of the form $\hat{g} = g - \partial\bar{\partial}\log(c-\rho)$, which effectively pushes S_c to infinity, one may expect to find a solution of the equation $\bar{\partial}u = \bar{\partial}s$, for a smooth compactly supported extension s via the method of [15]. Provided the curvature of E is sufficiently regular (eg., L^n_{loc}), u should be in L^2 with respect to $\hat{g}$, and will consequently vanish on S_c.

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