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ON A REGULARITY PROPERTY AND A PRIORI ESTIMATES FOR SOLUTIONS OF NONLINEAR PARABOLIC VARIATIONAL INEQUALITIES

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Abstract. In this paper we consider the following nonlinear parabolic variational inequality; $u(t) \in D(\Phi)$ for all $t \in I$, $(u_t(t), u(t)-v) + \langle \Delta_p u(t), u(t)-v\rangle + \Phi(u(t)) - \Phi(v) \leqq (f(t), u(t)-v)$ for all $v \in D(\Phi)$ a.e. $t \in I$, $u(x,0) = u_0(x)$, where Δ_p is the so-called p-Laplace operator and Φ is a proper, lower semicontinuous functional. We have obtained two results concerning to solutions of this problem. Firstly, we prove a few regularity properties of solutions. Secondly, we show the continuous dependence of solutions on given data u_0 and f.

§1. Introduction

Let Ω be a bounded domain in R^n with coordinates $x = (x_1, \dots, x_n)$. The boundary Γ of Ω is assumed to be of class C^1. For any positive number T we denote the open interval $(0,T)$ by I and the cylinder $\Omega \times I$ by G, i.e. $G = \{(x,t); x \in \Omega,\ t \in I\}$. The usual Sobolev space $W^{1,p}(\Omega)$ is defined as follows: $W^{1,p}(\Omega) = \{v \in L^p(\Omega); D_j v \in L^p(\Omega),\ j = 1, \dots, n\}$ with the norm $|v|_{1,p} = (|v|_p^p + |Dv|_p^p)^{1/p}$ $(1 \leqq p < \infty)$. Here $D_j v = \partial v/\partial x_j$, $Dv = (D_1 v, D_2 v, \dots, D_n v)$ and $|v|_p = \|v\|_{L^p(\Omega)}$.

Let S be a compact C^1 manifold of $(n-2)$-dimension contained in Γ. We assume that S divides Γ into two relatively open subsets Γ_1, Γ_2 such that $\Gamma_1 \neq \emptyset$, more precisely, $\Gamma = \Gamma_1 \cup \Gamma_2 \cup S$, $\Gamma_1 \cap \Gamma_2 = \emptyset$. Next let $C^\infty_{(0)}(\bar\Omega)$ be the family of all infinitely differentiable functions in $\bar\Omega$ vanishing in each neighborhood of $\overline{\Gamma_1}$. The completion of $C^\infty_{(0)}(\bar\Omega)$ with respect to the norm $|\ \ |_{1,p}$ is denoted by V. We denote the norm in V by $|\ \ |_V$, the dual space of V by V', the pairing between V and V' by $\langle\ ,\ \rangle$ and the inner product of $L^2(\Omega)$ by $(\ ,\ )$.

For any Banach space X and any s $(1 \leqq s < \infty)$ let us denote by $L^s(I, X)$ the space of equivalent classes of functions $v(t)$ from I to X, which are L^s-integrable on I. It is a Banach space with the norm $\|v\|_{L^s(I,X)} =$

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$(\int_I |v(t)|_X^s dt)^{1/s}$, where $|\quad|_X$ is the norm in X. In the case of $s = \infty$, $L^\infty(I, X)$ means the set of all measurable functions $v; I \to X$, satisfying $\|v\|_{L^\infty(I,X)} = \operatorname{ess.sup}_I |v(s)|_X < \infty$.

A functional $\Phi; X \to (-\infty, \infty]$ is said to be proper if $\Phi \not\equiv \infty$. And a proper functional Φ is said to be lower semicontinuous if

$$\Phi(v_0) \leqq \liminf_{v \to v_0} \Phi(v) \text{ for any } v_0 \in X.$$

For a proper lower semicontinuous functional Φ on X we set $D(\Phi) = \{v \in X; \Phi(v) < \infty\}$.

We consider the nonlinear parabolic variational inequality

$$\text{(1.1)} \quad \begin{cases} u(t) \in D(\Phi) \quad \text{for all} \quad t \in I, \\ (u_t(t), u(t) - v) + \langle \Delta_p u(t), u(t) - v \rangle + \Phi(u(t)) - \Phi(v) \\ \qquad\qquad \leqq (f(t), u(t) - v) \quad \text{for all } v \in D(\Phi) \text{ a.e. } t \in I, \\ u(x, 0) = u_0(x), \end{cases}$$

where $\langle \Delta_p u, v \rangle = \sum_{j=1}^n (|Du|^{p-2} D_j u, D_j v)$. In the above Φ is supposed to be proper, convex lower semicontinuous on V. Throughout this paper we assume that $1 < p \leqq 2$, if $n = 1, 2$, and $2n/(n+2) < p \leqq 2$, if $3 \leqq n$.

In this paper we denote by the same C any positive constant which does not depend on u, u_0 and f.

The first aim of this paper is to show the following

Theorem A. *It is assumed that the inequality*

$$\text{(1.2)} \qquad \langle \Delta_p u_0, u_0 - v \rangle + \Phi(u_0) - \Phi(v) \leqq (f(0), u_0 - v) \quad \textit{for any} \quad v \in D(\Phi)$$

holds, where $u_0 \in D(\Phi)$, $f \in L^2(I, L^2(\Omega))$ *and* $f_t \in L^2(I, L^2(\Omega))$. *Then, for any solution* u *of* (1.1) *it holds that* $(Du)_t \in L^2(I, L^p(\Omega))$ *and*

$$\|(Du)_t\|_{L^2(I, L^p(\Omega))} \leqq \sigma^{1/p}(u_0, f)$$

where $\sigma(u_0, f) = C(|Du_0|_p^p + \|f_t\|^2_{L^2(I, L^2(\Omega))})$.

It is easy to deduce the follwoing corollary from Theorem A:

Corollary. *Under the same assumptions as in Theorem A it holds that* $(Du)_t \in L^p(G)$ *and*

$$\|(Du)_t\|_{L^p(G)} \leqq \sigma(u_0, f).$$

Secondly, we give a priori estimates which assure the continuous dependence of solutions of (1.1) on the given data u_0 and f.

THEOREM B. *Let u_1 (resp. u_2) be any solution of* (1.1) *with $u_1(x,0) = u_{1,0}$ (resp. $u_2(x,0) = u_{2,0}$) and $f = f_1$ (resp. f_2). Let the condition* (1.2) *be satisfied for $u_{i,0}$ and f_i, $i = 1,2$. Under the condition that $u_{1,0}$, $u_{2,0} \in D(\Phi)$, f_1, $f_2 \in L^2(I, L^2(\Omega))$ and $(f_1)_t$, $(f_2)_t \in L^2(I, L^2(\Omega))$ the followings hold: for any $t \in I$*

$$|(u_1 - u_2)(t)|_2 \leqq C(|u_{1,0} - u_{2,0}|_2 + \|f_1 - f_2\|_{L^2(I,L^2(\Omega))}), \tag{1}$$

and

$$\|D(u_1 - u_2)\|_{L^2(I,L^p(\Omega))} \leqq (\sigma(u_{1,0}, f_1) + \sigma(u_{2,0}, f_2))^{(2-p)/2p} \cdot (|u_{1,0} - u_{2,0}|_2 + \|f_1 - f_2\|_{L^2(I,L^2(\Omega))}). \tag{2}$$

Under the same conditions on p as this paper we have proved some decay properties of solutions for (1.1) in [N5], where we have replaced Φ by the indicator function I_K of a closed onvex subset K of V and I by $R_1^+ = (0, \infty)$.

In [N2] we considered the nonlinear parabolic variational inequality

$$\begin{cases} u(t) \in D(\Phi) \quad \text{for all} \quad t \in I, \\ (u_t(t), u(t) - v) + \langle A(t)u(t), u(t) - v\rangle + \Phi(u(t)) - \Phi(v) \\ \qquad\qquad \leqq (f(t), u(t) - v) \quad \text{for all } v \in D(\Phi) \text{ a.e. } t \in I, \\ u(x,0) = u_0(x). \end{cases} \tag{1.3}$$

In the above the operator $A(t); V \to V'$ is defined in such a way that

$$\langle A(t)v, w\rangle = \sum_{j=1}^{n} (a_j(\,.\,, t, Dv), D_j w) \quad \text{for any} \quad v, w \in V. \tag{1.4}$$

Here the nonlinear functions $a_j(x,t,\eta)$, $j = 1, \dots, n$, satisfy the following

ASSUMPTION (I). For $(x,t) \in G$, $\eta \in R^n - \{0\}$, $\xi \in R^n$ and j, $1 \leqq j \leqq n$,

1-1 $\qquad a_j(x,t,\eta) \in C^0(\Omega \times I \times R^n) \cap C^1(\Omega \times I \times (R^n - \{0\}))$,

$$\text{1-2} \qquad \sum_{i,j=1}^{n} (\partial/\partial\eta_i) a_j(x,t,\eta)\xi_i\xi_j \geqq \gamma|\eta|^{p-2}|\xi|^2,$$

$$\text{1-3} \qquad |(\partial/\partial t)a_j(x,t,\eta)| \leqq \Lambda|\eta|^{p-1}.$$

Here γ, Λ are some positive constants.

It is easy to see that the operator Δ_p satisfies the Assumption (I). However, it is assumed that $2 \leqq p$ in [N2]. In such a case we considered the existence and the regularity of solutions of (1.3) under the following assumptions in [N2]:

Assumption (II). The function $u_0(x)$ in (1.3) belongs to $D(\Phi)$ and there exists an element $z_0(x)$ in $L^2(\Omega)$ such that the inequality

$$(1.5) \quad (z_0, v-u_0) + \langle A(0)u_0, v-u_0\rangle + \Phi(v) - \Phi(u_0) \geqq (f(0), v-u_0)$$

holds for all $v \in D(\Phi)$.

Assumption (III). There exists v_0 in $D(\Phi)$ such that for any $t \in I$

$$(1.6) \quad \{\langle A(t)v, v-v_0\rangle + \Phi(v)\}/|v|_V \to \infty \text{ uniformly as } |v|_V \to \infty.$$

Concerning with the regularity of solutions of (1.3), we obtained the following results in [N2, p.276]: let the Assumptions (I), (II) and (III) be satisfied. If $f,\ f_t \in L^2(I, L^2(\Omega))$, it holds that $\|(|Du|^{(p-2)/2}Du)_t\|^2_{L^2(G)}$, $\|(a_j(\,.\,,\,.\,,Du))_t\|^{p'}_{L^{p'}(G)}$ and $\|(|Du|^{(p-2)}Du)_t\|^{p'}_{L^{p'}(G)} \leqq \gamma(u_0, z_0, f)$ for any solution u of (1.3). Here p' is the adjoint number of p, i.e., $p' = p/(p-1)$, and $\gamma(u_0, z_0, f) = C(1+|z_0|_2^2+|u_0|_2^2+|Du_0|_p^p+\|f\|^2_{L^2(I,L^2(\Omega))}+\|f_t\|^2_{L^2(I,L^2(\Omega))})$. In [N2] we have treated (1.3) when the operator A has a nonlinear perturbed term of lower order.

In [N4] we considerd the nonlinear parabolic equation

$$(1.7) \quad \begin{cases} u \in L^\infty(I, W_0^{1,p}(\Omega)) \cap C(I, L^2(\Omega)),\ u_t \in L^2(I, L^2(\Omega)), \\ (u_t(t), v) + \langle A(t)u(t), v\rangle = \langle f(t), v\rangle \\ \qquad\qquad\qquad\qquad \text{for all } v \in W_0^{1,p}(\Omega) \ \text{ a.e. } \ t \in I, \\ u(x,0) = u_0. \end{cases}$$

under the following assumptions:

ASSUMPTION (IV). $f \in L^{p'}(I, W^{1,p'}(\Omega))$ and $f_t \in L^2(I, L^2(\Omega))$.

ASSUMPTION (V). $u_0 \in W^{1,p}(\Omega)$ and $Du_0 \in L^2(\Omega)$. Further, there exists a function $z_0 \in L^2(\Omega)$ such that the equality

$$(z_0, v) + \langle A(0)u_0, v\rangle = \langle f(0), v\rangle \quad \text{for all} \ \ v \in W_0^{1,p}(\Omega)$$

holds

In [N4, p.51, p.62] we obtained the following results concerned to the regularity of any solution u of (1.7): under the Assumptions (I), (IV) and (V) it holds that (1) if $2(n+1)/(n+3) < p \leqq 2$, then $D^2u \in L^{k(p_\infty)}(I', L^{k(p_\infty)}_{\mathrm{loc}}(\Omega))$, $(Du)_t \in L^{k(p_\infty)}(\Omega \times I')$ and $D^2u \in L^{m(p_\infty)}(\Omega \times I')$, (2) if $2n/(n+2) < p \leqq 2(n+1)/(n+3)$, then $D^2u \in L^p(I, L^p_{\mathrm{loc}}(\Omega))$, $(Du)_t \in L^p(\Omega \times I')$ and $D^2u \in L^{2p/(4-p)}(G)$. Here $D^2u = D_iD_ju$, $1 \leqq i, j \leqq n$, $I' = (a, b)$ with any a, b, $0 < a < b < T$, $k(p_\infty) = 4(n+1)(p-1)/((n+3)p-4)$ and $m(p_\infty) = 2(n+1)(p-1)/(2p+n-3)$. It is easy to see that $p < k(p_\infty)$ (resp. $p > k(p_\infty)$), if $2(n+1)/(n+3) < p \leqq 2$ (resp. $1 < p \leqq 2(n+1)/(n+3)$).

The other results on the regularity of solutions for nonlinear parabolic variational inequalities and nonlinear parabolic differential equations are referred to [N2] and [N4].

In the abstract framework of a Hilbert triple $\{V, H, V'\}$ G. Savaré has considered (1.3) when $A(t)$, $t \in I$, is a family of linear continuous coercive operators from V to V' and Φ is a proper convex lower semicontinuous functional on V. In [Sa] he obtained the estimate $\|u_1 - u_2\|^2_{i(I)} \leqq C(|u_{1,0} - u_{2,0}|^2_H + (\|f_1\|_{S(I)} + \|f_2\|_{S(I)})\|f_1 - f_2\|_{S(I)})$, where u_1 (resp. u_2) is any solution of (1.3) with $f = f_1$ (resp. f_2) and $u_0 = u_{1,0}$ (resp. $u_{2,0}$). Here $i(I) = L^2(I, V) \cap L^\infty(I, H)$ and $S(I) = L^2(I, V') + L^1(I, H) + B^{-1/2}_{21}(I, H)$ with the norm $\|v\|_{i(I)} = \|v\|_{L^2(I,V)} + \|v\|_{L^\infty(I,H)}$ and $\|v\|_{S(I)} = \inf(\|v_1\|_{L^2(I,V')} + \|v_2\|_{L^1(I,H)} + \|v_3\|_{B^{-1/2}_{21}(I,H)})$, where the infimum is taken for $v = v_1 + v_2 + v_3$ such that $v_1 \in L^2(I, V')$, $v_2 \in L^1(I, H)$ and $v_3 \in B^{-1/2}_{21}(I, H)$. The definition and the properties of the space $B^{-1/2}_{21}(I, H)$ are referred to [Sa].

At last we refer to the results in [C]. In [C] Y. Cheng considered the nonlinear elliptic equation with the Dirichlet boundary condition; $-\mathrm{div}(|Du|^{p-2}Du) = f$ in Ω and $u = 0$ on Γ in the weak sense that

$$\langle \Delta_p u, \phi\rangle = (f, \phi) \text{ for all } \phi \in C_0^\infty(\Omega). \tag{1.8}$$

Y. Cheng obtained the followings for solutions of (1.8) in [C]: (1) when $2 \leqq p$, it holds that $|u_1 - u_2|_{1,p} \leqq C|f_1 - f_2|_{-1,p'}^{1/(p-1)}$, and (2) when $1 < p < 2$, it holds that $|u_1 - u_2|_{1,p} \leqq C(|f_1|_{-1,p'} + |f_2|_{-1,p'})^{(2-p)/(p-1)}|f_1 - f_2|_{-1,p'}$. Here u_1 (resp. u_2) is any solution of (1.8) for $f = f_1$ (resp. f_2).

This paper is constructed as follows: in Section 2 we prepare some lemmas and a proposition, which play important roles in the proof of our theorems. In Section 3 we give the proofs of our theorems.

§2. Lemmas

LEMMA 2.1. ([C, p.736, Theorem 3]) *There exists a positive constant γ_0 depending only on p such that the following inequality holds:*

$$\sum_{j=1}^{n}(|\xi|^{p-2}\xi_j - |\eta|^{p-2}\eta_j)(\xi_j - \eta_j) \geqq \gamma_0(|\xi| + |\eta|)^{p-2}|\xi - \eta|^2 \tag{2.1}$$

for any ξ and $\eta \in R^n$, where the right-hand side is defined to be 0, if $\xi = \eta = 0$.

We can show easily the following lemma by Hölder's inequality:

LEMMA 2.2. *Let $0 < r < 1$ and $r' = r/(r-1)$. If $F(x) \in L^r(Q)$, $F(x)H(x) \in L^1(Q)$ and $\int_Q |H(x)|^{r'} dx < \infty$, then it holds that*

$$\left(\int_Q |F(x)|^r dx\right)^{1/r} \leqq \left(\int_Q |F(x)H(x)| dx\right)\left(\int_Q |H(x)|^{r'} dx\right)^{-1/r'}. \tag{2.2}$$

Here Q is any bounded domain in R^m.

By the same way as in [N1, p.77, Lemma 1.4] we can show the following

LEMMA 2.3. *Let v be a distribution in Q and let $\{v_j\}_{j=1}^{\infty}$ be a sequence in a reflexive Banach space X, where $C_0^\infty(Q)$ is dense. Let norms $|v_j|_X$ be uniformly bounded. If $(v_j, \psi) \to (v, \psi)$ as $j \to \infty$ for any $\psi \in C_0^\infty(Q)$, then v belongs to X and the sequence v_j converges weakly to v in X as $j \to \infty$.*

Next, we prepare a priori estimates for solutions of (1.1).

PROPOSITION 2.1. *Under the assumptions in Theorem A there exists a unique solution u of (1.1) with the following properties:*

(1) $\quad u \in L^\infty(I, V) \cap C(I, L^2(\Omega)), \quad u_t \in L^\infty(I, L^2(\Omega))$

(2) $\quad |u_t(t)|_2^2,\ |u(t)|_V^p \leqq \sigma(u_0, f)$ *uniformly in $t \in I$.*

For solutions of (1.3) the assertions of the above proposition were proved in [N2, p.275, Theorem 1] under the Assumptions (I), (II) and (III) when $2 \leqq p$. In the cases that $1 < p \leqq 2$, if $n = 1, 2$, and $2n/(n+2) < p \leqq 2$, if $3 \leqq n$ we can show the same results as [N2] by the similar way to [N2] with slight modifications. Moreover, the above proposition was proved by J. Kacur in [K3, p.116, Theorem 5.2.3]. In [K3] the operator Δ_p was replaced by a genral nonlinear elliptic operator which was independent of t.

Remark 2.1. We extend $u(t)$ and $f(t)$ outside I in such a way that

$$\begin{aligned} u(t) &= u(T), \qquad f(t) = f(T) \quad \text{for} \quad t > T; \\ u(t) &= u_0, \qquad\quad f(t) = f(0) \quad \text{for} \quad t < 0. \end{aligned} \tag{2.3}$$

LEMMA 2.4. *Under the assumptions in Theorem A the inequality*

$$|(u(h) - u_0)/h|_2 \leqq C\|f_t\|_{L^2(I, L^2(\Omega))} \qquad (h \neq 0) \tag{2.4}$$

holds for any solution u of (1.1). *Here the positive constant C depends only on T.*

Proof. By Remark 2.1 it is easy to see that the estimate (2.4) is valid for $h < 0$. Therefore, we may assume that $0 < h$ from now on.

For any solution u of (1.1) and a.e. $t \in I$ let us take $v = u_0$ in (1.1). After this we take $v = u(t)$ in (1.2). Adding these two inequalities, we get

$$\begin{aligned} (u_t(t) - (u_0)_t, u(t) - u_0) &+ \langle \Delta_p u(t) - \Delta_p u_0, u(t) - u_0 \rangle \\ &\leqq (f(t) - f(0), u(t) - u_0) \quad \text{a.e. } t \in I. \end{aligned} \tag{2.5}$$

Using the fact that $0 \leqq \langle \Delta_p u(t) - \Delta_p u_0, u(t) - u_0 \rangle$, we get

$$\frac{1}{2}\frac{d}{dt}|u(t) - u_0|_2^2 \leqq (f(t) - f(0), u(t) - u_0) \quad \text{a.e. } t \in I. \tag{2.6}$$

Integrating the above over $(0, h)$ for any $h > 0$, we have

$$|u(h) - u_0|_2^2 \leqq 2\int_0^h (f(t) - f(0), u(t) - u_0)dt. \tag{2.7}$$

Dividing (2.7) by h^2 and using Hölder's inequality, we get

$$|(u(h) - u_0)/h|_2^2 \leqq \frac{2}{h^2}\int_0^h |(f(t) - f(0), u(t) - u_0)|dt \tag{2.8}$$

$$\leqq 2\int_0^h |(f(t)-f(0))/h|_2 |(u(t)-u_0)/t|_2 dt$$
$$\leqq \int_0^h |(f(t)-f(0))/h|_2^2 dt + \int_0^h |(u(t)-u_0)/t|_2^2 dt,$$

because $0 < 1/h < 1/t$ for $0 < t < h$.

Here, we estimate the first term on the right-hand side of (2.8) as follows: at first,

$$\tag{2.9} \begin{aligned} |(f(t)-f(0))/h|_2^2 = |\frac{1}{h}\int_0^t f_t(s)ds|_2^2 = \int_\Omega \left(\frac{1}{h}\int_0^t f_t(s,x)ds\right)^2 dx \\ \leqq \int_\Omega \frac{t}{h^2}\int_0^t f_t^2(s,x)dsdx = \frac{t}{h^2}\int_0^t |f_t(s)|_2^2 ds. \end{aligned}$$

Then, we have

$$\tag{2.10} \begin{aligned} \int_0^h |(f(t)-f(0))/h|_2^2 dt \leqq \int_0^h \left(\frac{t}{h^2}\int_0^t |f_t(s)|_2^2 ds\right) dt \\ \leqq \frac{1}{2}\int_0^h |f_t(s)|_2^2 ds \leqq \frac{1}{2}\|f_t\|^2_{L^2(I,L^2(\Omega))}. \end{aligned}$$

Therefore, from (2.8) and (2.10) we get

$$\tag{2.11} |(u(h)-u_0)/h|_2^2 \leqq \frac{1}{2}\|f_t\|^2_{L^2(I,L^2(\Omega))} + \int_0^h |(u(t)-u_0)/t|_2^2 dt.$$

Let us use Gronwall's inequality to obtain the estimate (2.4). In this way we finish the proof of this lemma. □

§3. Proofs of our theorems

At first we give the proof of Theorem A.

Proof of Theorem A. For any solution u of (1.1) and a.e. $t \in I$, we take $v = u(t+h)$ in (1.1), where $|h| < \min(t, T-t)$. After this we take $v = u(t)$ in (1.1) for $t = t+h$. Adding these two inequalities, we have

$$\tag{3.1} \begin{aligned} (u_t(t+h) - u_t(t), u(t+h)-u(t)) \\ +\langle \Delta_p u(t+h) - \Delta_p u(t), u(t+h)-u(t)\rangle \\ \leqq (f(t+h)-f(t), u(t+h)-u(t)) \quad \text{a.e. } t \in I. \end{aligned}$$

Dividing (3.1) by h^2, we get

$$\begin{aligned}(3.2)\qquad & \frac{1}{2}\frac{d}{dt}|u(t+h)-u(t)|_2^2/h^2 \\ &\qquad +\langle \Delta_p u(t+h)-\Delta_p u(t), u(t+h)-u(t)\rangle/h^2 \\ &\leqq (\delta_h f(t), \delta_h u(t)) \qquad \text{a.e. } t\in I.\end{aligned}$$

where $\delta_h f(t) = (f(t+h)-f(t))/h$ and $\delta_h u(t) = (u(t+h)-u(t))/h$.

Let us integrate (3.2) over I to get

$$\begin{aligned}(3.3)\qquad & \frac{1}{2}|u(T+h)-u(T)|_2^2/h^2 - \frac{1}{2}|u(h)-u_0|_2^2/h^2 \\ &\qquad + \int_I \langle \Delta_p u(t+h)-\Delta_p u(t), u(t+h)-u(t)\rangle/h^2 dt \\ &\leqq \int_I (\delta_h f(t), \delta_h u(t))dt.\end{aligned}$$

By Lemma 2.1 it holds that

$$\begin{aligned}(3.4)\qquad & \gamma_0(|Du(t+h)|+|Du(t)|)^{p-2}|Du(t+h)-Du(t)|^2/h^2 \\ &\leqq \sum_{j=1}^{n}\big(|Du(t+h)|^{p-2}D_j u(t+h) - |Du(t)|^{p-2}D_j u(t)\big)\cdot \\ &\qquad\qquad (D_j u(t+h)-D_j u(t))/h^2.\end{aligned}$$

Integrating (3.4) over Ω, we get

$$\begin{aligned}(3.5)\qquad & \gamma_0\int_\Omega (|Du(t+h)|+|Du(t)|)^{p-2}|Du(t+h)-Du(t)|^2/h^2 dx \\ &\leqq \langle \Delta_p u(t+h)-\Delta_p u(t), u(t+h)-u(t)\rangle/h^2.\end{aligned}$$

Next let us set $F = |Du(t+h)-Du(t)|^2/h^2$, $H = (|Du(t+h)| + |Du(t)|)^{p-2}$, $r = p/2$ and $Q = \Omega$ in Lemma 2.2. Then, from (2.2), (3.5) and Proposition 2.1 we get

$$\begin{aligned}(3.6)\qquad & \left(\int_\Omega |(Du(t+h)-Du(t))/h|^p dx\right)^{2/p} \\ &\leqq \frac{1}{h^2}\left(\int_\Omega (|Du(t+h)|+|Du(t)|)^{p-2}|Du(t+h)-Du(t)|^2 dx\right)\cdot \\ &\qquad\qquad \left(\int_\Omega (|Du(t+h)|+|Du(t)|)^p dx\right)^{(2-p)/p} \\ &\leqq \sigma^{(2-p)/p}(u_0,f)(1/\gamma)^{-1}\langle \Delta_p u(t+h)-\Delta_p u(t), u(t+h)-u(t)\rangle/h^2.\end{aligned}$$

Let us integrate (3.6) over I to have

$$(3.7)\qquad \int_I \left(\int_\Omega |(Du(t+h) - Du(t))/h|^p dx \right)^{2/p} dt \\ \leqq \sigma^{(2-p)/p}(u_0, f) \int_I \langle \Delta_p u(t+h) - \Delta_p u(t), u(t+h) - u(t) \rangle / h^2 dt.$$

Combining (3.7) with (3.3) multiplied by $\sigma^{(2-p)/p}(u_0, f)$, we obtain

$$(3.8)\quad \int_I \left(\int_\Omega |(Du(t+h) - Du(t))/h|^p dx \right)^{2/p} dt \\ \leqq \sigma^{(2-p)/p}(u_0, f)(|(u(h) - u_0)/h|_2^2 + \int_I |(\delta_h f(t), \delta_h u(t))| dt) \\ \leqq \sigma^{(2-p)/p}(u_0, f)(|(u(h) - u_0)/h|_2^2 + \int_I |\delta_h f(t)|_2 |\delta_h u(t)|_2 dt) \\ \leqq \sigma^{(2-p)/p}(u_0, f)(|(u(h) - u_0)/h|_2^2 + \int_I |\delta_h u(t)|_2^2 dt + \int_I |\delta_h f(t))|_2^2 dt)$$

By the similar calculations to (2.9) and (2.10) the last two terms in the blackts on the right-hand side of (3.8) are estimated as follows:

$$\int_I |\delta_h f(t)|_2^2 dt \leqq \|f_t\|^2_{L^2(I, L^2(\Omega))},$$
$$\int_I |\delta_h u(t)|_2^2 dt \leqq \|u_t\|^2_{L^2(I, L^2(\Omega))} \leqq \sigma(u_0, f).$$

Here we have used Proposition 2.1 in the last inequality. In (3.8) let us use the above two estimates and Lemma 2.4. Then, we get

$$(3.9)\qquad \int_I \left(\int_\Omega |(Du(t+h) - Du(t))/h|^p dx \right)^{2/p} dt \leqq \sigma^{2/p}(u_0, f).$$

By virtue of Lemma 2.3 we finish the proof of our theorem, see [N3, p.184] in detail. ∎

Proof of Theorem B. For $i = 1$ and $i = 2$ let u_i be any solution of the inequality

$$(3.10)_i \qquad \begin{cases} ((u_i)_t(s), u_i(s) - v) + \langle \Delta_p u_i(s), u_i(s) - v \rangle + \Phi(u_i(s)) - \Phi(v) \\ \qquad\qquad \leqq (f_i(s), u_i(s) - v) \quad \text{for all } v \in D(\Phi) \text{ a.e. } s \in I, \\ u_i(x, 0) = u_{i,0} \end{cases}$$

In $(3.10)_1$ (resp. $(3.10)_2$) we take $v = u_2$ (resp. u_1). Then, let us add these two inequalities in order to get

$$(3.11)\qquad \begin{aligned}&(((u_1)_t - (u_2)_t)(s), u_1(s) - u_2(s))\\ &\qquad\qquad + \langle \Delta_p u_1(s) - \Delta_p u_2(s), u_1(s) - u_2(s)\rangle\\ &\leqq (f_1(s) - f_2(s), u_1(s) - u_2(s)) \text{ a.e. } s \in I.\end{aligned}$$

Then, we have

$$(3.12)\qquad \begin{aligned}&\frac{1}{2}\frac{d}{ds}|u_1(s) - u_2(s)|_2^2 + \langle \Delta_p u_1(s) - \Delta_p u_2(s), u_1(s) - u_2(s)\rangle\\ &\leqq (f_1(s) - f_2(s), u_1(s) - u_2(s)) \quad \text{a.e. } \ s \in I.\end{aligned}$$

Let us apply Hölder's inequality to the right-hand side of (3.12) and integrate it over $(0, t)$ for any $t \in I$ to obtain the inequality

$$(3.13)\qquad \begin{aligned}&|u_1(t) - u_2(t)|_2^2 + 2\int_0^t \langle \Delta_p u_1(s) - \Delta_p u_2(s), u_1(s) - u_2(s)\rangle ds\\ &\leqq |u_{1,0} - u_{2,0}|_2^2 + \int_0^t |f_1(s) - f_2(s)|_2^2 ds + \int_0^t |u_1(s) - u_2(s)|_2^2 ds.\end{aligned}$$

After using the fact that $0 \leqq \langle \Delta_p u_1(s) - \Delta_p u_2(s), u_1(s) - u_2(s)\rangle$ for any s let us apply Gronwall's inequality again to have

$$(3.14)\qquad |u_1(t) - u_2(t)|_2^2 \leqq C(|u_{1,0} - u_{2,0}|_2^2 + \int_0^T |f_1(t) - f_2(t)|_2^2 dt).$$

In this way we finish the proof for (1) in Theorem B.

Secondly, as in (3.6) we have the following: from Proposition 2.1

$$(3.15)\qquad \begin{aligned}&\left(\int_\Omega |Du_1(t) - Du_2(t)|^p dx\right)^{2/p}\\ &\leqq (\sigma(u_{1,0}, f_1) + \sigma(u_{2,0}, f_2))^{(2-p)/p} \langle \Delta_p u_1(t) - \Delta_p u_1(t), u_1(t) - u_2(t)\rangle.\end{aligned}$$

By virtue of (3.13)–(3.15) we have

$$(3.16)\qquad \begin{aligned}&\int_I \left(\int_\Omega |Du_1(t) - Du_2(t)|^p dx\right)^{2/p} dt\\ &\leqq (\sigma(u_{1,0}, f_1) + \sigma(u_{2,0}, f_2))^{(2-p)/p} \cdot\\ &\qquad\qquad (|u_{1,0} - u_{2,0}|_2^2 + \|f_1 - f_2\|_{L^2(I, L^2(\Omega))}^2).\end{aligned}$$

Thus, we finish the proof. $\square$

Remark 3.1. When $2 \leqq p$, it holds that in Theorem B the estimate (1) is valid and the estimate (2) is replaced by the following form:

$$\|u_1 - u_2\|^p_{L^p(I,V)} \leqq C(|u_{1,0} - u_{2,0}|_2^2 + \|f_1 - f_2\|^{p'}_{L^{p'}(I,V')})$$

with some positive constant C. The above assertion is deduced from (3.13), (3.14), Hölder's and Poincaré's inequalities and the inequality

$$\sum_{j=1}^{n}(|\xi|^{p-2}\xi_j - |\eta|^{p-2}\eta_j)(\xi_j - \eta_j) \geqq C_0|\xi - \eta|^p \quad \text{for any} \quad \xi, \eta \in R^n,$$

where C_0 depends only on p and n, see [C].

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