P. B. Gilkey, J. V. Leahy and J. H. Park
Nagoya Math. J.
Vol. 156 (1999), 1–23

THE EIGENFORMS OF THE COMPLEX LAPLACIAN FOR A HERMITIAN SUBMERSION

PETER B. GILKEY[1], JOHN V. LEAHY AND JEONGHYEONG PARK[2]

Abstract. Let $\pi : Z \to Y$ be a Hermitian submersion. We study when the pull-back of an eigenform of the complex Laplacian on Y is an eigenform of the complex Laplacian on Z.

§1. Introduction

1.1. The real Laplacian

We introduce the following notational conventions. We assume that all manifolds are compact, connected, smooth, without boundary, and Riemannian. Let $\Delta_M^p := d_M\delta_M + \delta_M d_M$ be the *Laplace-Beltrami operator* on the space of smooth p forms $C^\infty(\Lambda^p M)$. Let $E(\lambda, \Delta_M^p) \subset C^\infty(\Lambda^p M)$ be the eigenspaces of Δ_M^p; the eigenvalues λ of Δ_M^p are non-negative. We may decompose $L^2\Lambda^p M$ as a direct sum $\oplus_{\lambda\geq 0} E(\lambda, \Delta_M^p)$. Let $\pi : Z \to Y$ be a *submersion*. This means that π is a smooth surjective map and that $\pi_* : T_z Z \to T_{\pi z} Y$ is surjective for all z. Let $m := \dim_{\mathbb{R}} Y$ and $n := \dim_{\mathbb{R}} Z$; we assume $n > m$. Let $\mathcal{V} := \ker(\pi_*)$ and $\mathcal{H} := \mathcal{V}^\perp$ be the vertical and horizontal distributions of π. We say π is a *Riemannian submersion* if π_* is an isometry from $\mathcal{H}_z$ to $T_{\pi z} Y$ for all z. We shall use capital letters for tensors on Y and lower case letters for tensors on Z. We shall use indices i, j, and k to index local orthonormal frames $\{e_i\}$, and $\{e^i\}$ for the vertical distributions and co-distributions $\mathcal{V}$ and $\mathcal{V}^*$ of π; we shall use indices a, b, and c to index local orthonormal frames $\{f_a\}$, $\{f^a\}$, $\{F_a\}$, and $\{F^a\}$ for the horizontal distributions and codistributions $\mathcal{H}$ and $\mathcal{H}^*$ of π as well as for the tangent and cotangent bundles TY and T^*Y of Y. We shall adopt the Ein-

Received November 4, 1997.

[1]Research partially supported by the NSF (USA) and MPIM (Germany)

[2]Research partially supported by KOSEF 971-0104-016-2 and BSRI 97-1425, the Korean Ministry of Education.

stein convention and sum over repeated indices. Let $\operatorname{ext}(\xi)$ and $\operatorname{int}(\xi)$ be exterior and interior multiplication by a covector ξ. Let Γ be the Christoffel symbols of the Levi-Civita connection. Let

$$\begin{aligned} &\theta := -g_Z([e_i, f_a], e_i) f^a = {}^Z\Gamma_{iia} f^a, \\ &\omega_{abi} := g_Z(e_i, [f_a, f_b])/2 = ({}^Z\Gamma_{abi} - {}^Z\Gamma_{bai})/2, \\ &\mathcal{E} := \omega_{abi} \operatorname{ext}_Z(e^i) \operatorname{int}_Z(f^a) \operatorname{int}_Z(f^b) \\ &\Xi := \operatorname{int}_Z(\theta) + \mathcal{E}; \end{aligned} \tag{1.2}$$

θ is the unnormalized mean curvature co-vector of the fibers of π, ω is the curvature of the horizontal distribution, and $\mathcal{E}$ is an endomorphism of the exterior algebra. The anti-symmetric $\mathcal{V}^*$ valued 2-tensor $\omega(f_a, f_b)$ is the metric dual of the projection $\pi_{\mathcal{V}}$ of $[f_a, f_b]/2$ on the vertical distribution $\mathcal{V}$. The fibers of π are minimal $\iff$ $\theta = 0$ $\iff$ π is a harmonic map. Let $1 \le p \le \dim_{\mathbb{R}} Y$. The horizontal distribution $\mathcal{H}$ is integrable $\iff$ $\omega = 0$; this implies $\mathcal{E} = 0$ on $\Lambda^p \mathcal{H}^*$. Pullback π^* defines a natural map from $C^\infty Y$ to $C^\infty Z$.

THEOREM 1.3. *Let $\pi : Z \to Y$ be a Riemannian submersion.*

(1) $\delta_Z \pi^* - \pi^* \delta_Y = \Xi \pi^*$ *and* $\Delta_Z \pi^* - \pi^* \Delta_Y = (d_Z \Xi + \Xi d_Z)\pi^*$ *on* $C^\infty(\Lambda^p Y)$.

(2) *If* $0 \ne \Phi \in E(\lambda, \Delta_Y^p)$ *and if* $\pi^*\Phi \in E(\mu, \Delta_Z^p)$, *then* $\lambda \le \mu$.

(3) *Fix p with $0 \le p \le \dim_{\mathbb{R}} Y$. The following conditions are equivalent:*

   i) $\Delta_Z^p \pi^* = \pi^* \Delta_Y^p$.

   ii) $\forall \lambda \ge 0, \exists \mu(\lambda) \ge 0$ *so* $\pi^* E(\lambda, \Delta_Y^p) \subset E(\mu(\lambda), \Delta_Z^p)$.

   iii) *The fibers of π are minimal and:*

      a) *if $p = 0$, there is no condition on ω.*

      b) *if $p > 0$, $\omega = 0$ so $\mathcal{H}$ is integrable.*

(4) *If* $0 \ne \Phi \in E(\lambda, \Delta_Y^0)$ *and if* $\pi^*\Phi \in E(\mu, \Delta_Z^0)$, *then* $\lambda = \mu$.

THEOREM 1.4. *Let $0 \le \lambda < \mu < \infty$ and let $p \ge 2$. There exists a Riemannian submersion $\pi : V \to U$ and there exists $0 \ne \Phi \in E(\lambda, \Delta_U^p)$ so that $\pi^*\Phi \in E(\mu, \Delta_V^p)$.*

For a generic Riemannian submersion, the pullback of an eigenform on Y will no longer be an eigenform on Z. We say that an eigenvalue changes if there exists $0 \neq \Phi \in E(\lambda, \Delta_Y^p)$ so $\pi^*\Phi \in E(\mu, \Delta_Z^p)$ with $\lambda \neq \mu$; this is a comparatively rare phenomena. Theorems 1.3 and 1.4 show that eigenvalues can change if $p \geq 2$ and that eigenvalues cannot change if $p = 0$; we do not know if eigenvalues can change if $p = 1$. Furthermore, if π^* preserves all the eigen p forms, then eigenvalues can not change.

Theorem 1.3 (1) for $p = 0$ and Theorem 1.3 (3) for $p = 0$ was proved by Watson [14]; Theorem 1.3 (1) for $p > 0$ and the equivalence of (i) and (iii) in Theorem 1.3 (3) for $p > 0$ was proved by Goldberg and Ishihara [8]. The remaining assertions of Theorem 1.3 and 1.4 were proved by Gilkey, Leahy, and Park [4], [5] and by Gilkey and Park [7]; see [6] for a similar discussion in the context of spinors. Bergery and Bourguignon [1] gave a careful discussion of the relationship between the complete spectrum of Δ_Y^0 and Δ_Z^0 if the fibers of π are totally geodesic. We also refer to Burstall [2] for related work on this subject; Gudmundsson [9] has compiled an excellent bibliography of harmonic morphisms which contains additional related references. We also refer to related work of Park [13].

1.5. The complex Laplacian

In this paper, we generalize Theorems 1.3 and 1.4 to the complex setting. Some of this generalization is straightforward, but many of the arguments given in the real case either need substantial modification or must be replaced entirely when passing to the complex case. We introduce some notational conventions. Let $w = (w_1, ..., w_{\bar{m}})$ for $w_i := u_i + \sqrt{-1}v_i$ be local holomorphic coordinates on a complex manifold M of complex dimension $\bar{m}$. The almost complex structure J is given by $J(\partial_i^u) := \partial_i^v$ and $J(\partial_i^v) := -\partial_i^u$. We say that a Riemannian metric g_M on M is *Hermitian* if $g_M(X, Y) = g_M(JX, JY)$ for all real tangent vectors; we restrict to such metrics henceforth. We complexify the exterior algebra to decompose $\Lambda M = \oplus_{p,q}\Lambda^{p,q}(M)$ into forms of bidegree (p, q). Let $\pi_M^{p,q}$ be the corresponding orthogonal projections. We decompose $d = \partial + \bar{\partial}$ and $\delta = \delta_1 + \delta_2$; δ_2 is the formal adjoint of $\bar{\partial}$. The complex or Dolbeault Laplacian is then defined on $C^\infty(\Lambda^{p,q}M)$ by $\Delta_M^{p,q} := \bar{\partial}\delta_2 + \delta_2\bar{\partial}$. This is a self-adjoint elliptic non-negative partial differential operator; $2\Delta_M^{p,q}$ is of Laplace type. If M is Kaehler, then we have $\Delta_M^n = 2 \oplus_{p+q=n} \Delta_M^{p,q}$.

We say that $\pi : Z \to Y$ is a *Hermitian submersion* if Z and Y are complex manifolds, if π is a complex analytic, if the metrics on Z and on

Y are Hermitian, and if π is a Riemannian submersion. We refer to work of Johnson [10] and Watson [15] for a discussion of some of the geometry which is involved; these authors also consider the almost complex and the Kaehler categories. We complexify π^* to define $\pi^* : C^\infty(\Lambda^{p,q}Y) \to C^\infty(\Lambda^{p,q}Z)$. We then have the relations $\pi^*\pi_Y^{p,q} = \pi_Z^{p,q}\pi^*$ and $\pi^*\bar\partial_Y = \bar\partial_Z\pi^*$. We extend interior multiplication, exterior multiplication, and ω to be complex linear. Note that $J\mathcal{H} \subset \mathcal{H}$. We define $J^*\omega(\xi_1,\xi_2) := \omega(J\xi_1, J\xi_2)$. This paper is devoted to the proof of the following two theorems which generalize Theorems 1.3 and 1.4 to the complex setting.

THEOREM 1.6. *Let $\pi : Z \to Y$ be a Hermitian submersion.*

(1) $\delta_{2,Z}\pi^* - \pi^*\delta_{2,Y} = \pi_Z^{p,q-1}\Xi\pi^*$ *and* $\Delta_Z^{p,q}\pi^* - \pi^*\Delta_Y^{p,q} = \pi_Z^{p,q}(\Xi\bar\partial_Z + \bar\partial_Z\Xi)\pi^*$ *on* $C^\infty(\Lambda^{p,q}Y)$.

(2) *If* $0 \neq \Phi \in E(\lambda, \Delta_Y^{p,q})$ *and if* $\pi^*\Phi \in E(\mu, \Delta_Z^{p,q})$, *then* $\lambda \le \mu$.

(3) *Fix* (p,q) *with* $0 \le p, q \le \dim_{\mathbb{C}} Y$. *The following conditions are equivalent:*

  i) $\Delta_Z^{p,q}\pi^* = \pi^*\Delta_Y^{p,q}$.

  ii) $\forall\lambda \ge 0, \exists\mu(\lambda) \ge 0$ *so* $\pi^*E(\lambda, \Delta_Y^{p,q}) \subset E(\mu(\lambda), \Delta_Z^{p,q})$.

  iii) *The fibers of* π *are minimal and:*

   a) *if* $p = 0$ *and if* $q = 0$, *there is no condition on* ω.

   b) *if* $p > 0$ *and if* $q = 0$, *then* $J^*\omega = -\omega$.

   c) *if* $p = 0$ *and if* $q > 0$, *then* $J^*\omega = \omega$ *i.e.* $\mathcal{H}_{1,0}$ *is integrable.*

   d) *if* $p > 0$ *and if* $q > 0$, *then* $\omega = 0$ *i.e.* $\mathcal{H}$ *is integrable.*

(4) *If* $0 \neq \Phi \in E(\lambda, \Delta_Y^{p,0})$ *and if* $\pi^*\Phi \in E(\mu, \Delta_Z^{p,0})$, *then* $\lambda = \mu$.

THEOREM 1.7. *Let* $0 \le \lambda < \mu < \infty$, *let* $q \ge 1$ *and let* $p + q \ge 2$. *There exists a Hermitian submersion* $\pi : V \to U$ *and there exists* $0 \neq \Phi \in E(\lambda, \Delta_U^{p,q})$ *so that* $\pi^*\Phi \in E(\mu, \Delta_V^{p,q})$.

In Theorem 1.4, we showed eigenvalues could change if the degree was at least 2; in Theorem 1.7, we deal with forms of total degree at least 2 and anti-holomorphic degree at least 1. In the real case, we do not know if a single eigenvalue on 1 forms can change; in the complex case, we do not know if a single eigenvalue on $(0,1)$ forms can change. Both theorems are incomplete in this respect.

Here is a brief outline to the paper:

§2 Equations of structure (proof of Thm. 1.6 (1)).

§3 Fiber products (proof of Thm. 1.6 (2)).

§4 Rigidity of eigenvalues (proof of Thm. 1.6 (3)).

§5 Forms of type $(p,0)$ (proof of Thm. 1.6 (4)).

§6 Hermitian submersions where eigenvalues change (proof of Thm. 1.7).

§7 Examples where $J^*\omega = \pm\omega$.

The material of §2 and §3 is a fairly straightforward extension of the corresponding results in the real case. Although Theorem 1.6 (3,4) looks quite similar to Theorem 1.3 (3,4), the proofs given in §4 and §5 are quite different as certain techniques do not generalize from the real to the complex setting. The examples given in §6 to prove Theorem 1.7 are, of course, quite different from those chosen in the real context. In §7, we give examples of Hermitian submersions where $J^*\omega = \pm\omega$ for ω non-trivial; this gives examples where eigen $(p,0)$ forms are preserved and where eigen $(0,q)$ forms are not preserved and similarly where eigen $(p,0)$ forms are not preserved but eigen $(0,q)$ forms are preserved for $p > 0$ and $q > 0$.

§2. Equations of structure

Proof of Theorem 1.6 (1). This is a straightforward application of Theorem 1.3 (1). Let $i_M^{p,q}$ denote the natural inclusion of $\Lambda^{p,q}(M)$ in $\Lambda(M)$. Dually let $\pi_M^{p,q} := (i_M^{p,q})^*$ denote orthogonal projection from $\Lambda(M)$ to $\Lambda^{p,q}(M)$. We have

$$\begin{aligned}\bar\partial_M &:= \pi_M^{p,q} \circ d_M \circ i_M^{p,q-1} \text{ on } C^\infty(\Lambda^{p,q-1}M)\\ \delta_{2,M} &:= (\bar\partial_M)^* = \pi_M^{p,q-1} \circ \delta_M \circ i_M^{p,q} \text{ on } C^\infty(\Lambda^{p,q}M).\end{aligned}$$

Since pullback commutes with both $i^{p,q}$ and $\pi^{p,q}$, we compute that

$$\begin{aligned}\delta_{2,Z}\pi^* - \pi^*\delta_{2,Y} &= \pi_Z^{p,q-1}\delta_Z i_Z^{p,q}\pi^* - \pi^*\pi_Y^{p,q-1}\delta_Y i_Y^{p,q}\\ &= \pi_Z^{p,q-1}(\delta_Z\pi^* - \pi^*\delta_Y)i_Y^{p,q} = \pi_Z^{p,q-1}\Xi\pi^* i_Y^{p,q}\end{aligned} \tag{2.1}$$

on $C^\infty(\Lambda^{p,q}M)$. We suppress the role of $i^{p,q}$ to complete the proof of the first identity; we use the identities $\bar\partial_Z\pi^* = \pi^*\bar\partial_Y$ and $\pi_Z^{p,q}\bar\partial_Z = \bar\partial_Z\pi_Z^{p,q-1}$ and equation (2.1) to complete the proof of Theorem 1.6 (1) by computing:

$$\begin{aligned}\Delta_Z^{p,q}\pi^*-\pi^*\Delta_Y^{p,q}&=\bar\partial_Z(\delta_{2,Z}\pi^*-\pi^*\delta_{2,Y})+(\delta_{2,Z}\pi^*-\pi^*\delta_{2,Y})\bar\partial_Y\\&=\bar\partial_Z\pi_Z^{p,q-1}\Xi\pi^*+\pi_Z^{p,q}\Xi\pi^*\bar\partial_Y=\pi_Z^{p,q}(\bar\partial_Z\Xi+\Xi\bar\partial_Z)\pi^*.\end{aligned}$$

□

§3. Fiber products

We adopt the following notational conventions. For $\alpha = 1, 2$, let $\pi_\alpha : U_\alpha \to Y$ be Riemannian submersions with horizontal and vertical distributions $\mathcal{H}_\alpha$ and $\mathcal{V}_\alpha$. Let W be the fiber product:

$$W := \{w = (u_1, u_2) \in U_1 \times U_2 : \pi_1(u_1) = \pi_2(u_2)\}. \tag{3.1}$$

Define a submersion π_W from W to Y by $\pi_W(w) := \pi_1(u_1) = \pi_2(u_2)$. The vertical space is $\mathcal{V}_W(w) = \mathcal{V}_1(u_1)\oplus\mathcal{V}_2(u_2)$ where we embed TW in $TU_1 \oplus TU_2$. Let

$$\mathcal{H}_W(w) := \{(\xi_1,\xi_2)\in\mathcal{H}_1(u_1)\oplus\mathcal{H}_2(u_2) : (\pi_1)_*\xi_1 = (\pi_2)_*\xi_2\}$$

define a complementary splitting; we define a new metric on W by requiring that $\mathcal{H}_W$, $\mathcal{V}_1$, and $\mathcal{V}_2$ are orthogonal, that the metrics on $\mathcal{V}_1$ and $\mathcal{V}_2$ are induced from the metrics on U_1 and U_2, and that $\pi_W(w)_* : \mathcal{H}_W(w) \to TY(\pi(w))$ is an isometry. Let $\sigma_\alpha(u_1, u_2) := u_\alpha$ define Riemannian submersions from W to U_α. If π_1 and π_2 are Hermitian submersions, then σ_1, σ_2, and π_W are Hermitian submersions.

Let $\{F_a\}$ be a local orthonormal frame for TY. Let f_a^α and f_a^W be the horizontal lift of F_a with respect to the submersions π_α and π_W. Note that f_a^W is also the horizontal lift of f_a^α with respect to the submersion σ_α. Let $\{e_i\}$ and $\{\hat e_j\}$ be local orthonormal frames for the vertical distributions of π_1 and π_2 and let $\{e_i^W\}$ and $\{\hat e_j^W\}$ be horizontal lifts to W with respect to the submersions σ_1 and σ_2. Then $\{e_i^W, \hat e_j^W\}$ is a local orthonormal frame for $\mathcal{V}(\pi_W)$, $\{e_i^W\}$ is a local orthonormal frame for $\mathcal{V}(\sigma_2)$, and $\{\hat e_j^W\}$ is a local orthonormal frame for $\mathcal{V}(\sigma_1)$.

Lemma 3.2.

(1) *Let $\pi : Z \to Y$ be a Riemannian submersion and let f_ν be the horizontal lift of vector fields F_ν from Y to Z. Then $\pi_*[f_1, f_2] = [F_1, F_2]$ and $g_Z([f_1, f_2], f_3) = \pi^*\{g_Y([F_1, F_2], F_3)\}$.*

(2) *Let* Ξ_α *be defined by Riemannian submersions* π_α *and let* Ξ_W *be defined by the fiber product* π_W*. Then* $\Xi_W\pi_W^* = \sigma_1^*\Xi_1\pi_1^* + \sigma_2^*\Xi_2\pi_2^*$.

(3) *Let* $\pi_\alpha : U_\alpha \to Y$ *be Hermitian submersions. If* $\Phi \in E(\lambda, \Delta_Y^{p,q})$ *and if* $\pi_\alpha^*\Phi \in E(\lambda + \varepsilon_\alpha, \Delta_{U_\alpha}^{p,q})$*, then* $\pi_W^*\Phi \in E(\lambda + \varepsilon_1 + \varepsilon_2, \Delta_W^{p,q})$.

Proof. Let $\psi_i(t)$ and $\Psi_i(t)$ be the flows of the vector fields f_i, and F_i. By assumption $\pi_* f_i = F_i$ so $\pi\psi_i(t) = \Psi_i(t)\pi$. For $z_0 \in Z$, let

$$\begin{aligned}
&h(t) := \psi_1(-\sqrt{t})\psi_2(-\sqrt{t})\psi_1(\sqrt{t})\psi_2(\sqrt{t})(z_0), \text{ and} \\
&H(t) := \pi h(t) = \Psi_1(-\sqrt{t})\Psi_2(-\sqrt{t})\Psi_1(\sqrt{t})\Psi_2(\sqrt{t})(\pi z_0).
\end{aligned}$$

Then $\dot h(0) = [f_1, f_2](z_0)$ and $\pi_*\dot h(0) = \dot H(0) = [F_1, F_2](\pi z_0)$ so the first identity of assertion (1) follows. Let $\rho_{\mathcal{H}}$ be orthogonal projection on the horizontal space. Since f_3 is horizontal and since π_* is a Riemannian submersion,

$$\begin{aligned}
g_Z([f_1, f_2], f_3)(z_0) &= g_Z(\rho_{\mathcal{H}}[f_1, f_2], f_3)(z_0) = g_Y(\pi_*\rho_{\mathcal{H}}[f_1, f_2], F_3)(\pi z_0) \\
&= g_Y(\pi_*[f_1, f_2], F_3)(\pi z_0) = g_Y([F_1, F_2], F_3)(\pi z_0).
\end{aligned}$$

Note that f_a^W is the horizontal lift of f_a^1 and e_i^W is the horizontal lift of e_i with respect to σ_1. Similarly f_a^W is the horizontal lift of f_a^2 and $\hat e_j^W$ is the horizontal lift of $\hat e_j$ with respect to σ_2. Thus by assertion (1) and the definition in equation (1.2) we have

$$\begin{aligned}
\theta_W &= -\{g_W(e_i^W, [e_i^W, f_a^W]) + g_W(\hat e_j^W, [\hat e_j^W, f_a^W])\}\pi_W^*(F^a) \\
&= -\sigma_1^*\{g_1(e_i, [e_i, f_a^1])\pi_1^*(F^a)\} - \sigma_2^*\{g_2(\hat e_j, [\hat e_j, f_a^2])\pi_2^*(F^a)\} \\
&= \sigma_1^*\theta_1 + \sigma_2^*\theta_2.
\end{aligned}$$

Since f_a^W and e_i^W are the horizontal lifts of f_a^1 and e_i with respect to σ_1 and f_a^W and $\hat e_j^W$ are the horizontal lifts of f_a^2 and $\hat e_j$ with respect to σ_2, assertion (1) and the definition in equation (1.2) implies

$$\begin{aligned}
\omega_{abi}^W &= g_W(e_i^W, [f_a^W, f_b^W])/2 = \sigma_1^*\{g_1(e_i, [f_a^1, f_b^1])\}/2 = \sigma_1^*\omega_{abi}, \\
\hat\omega_{abj}^W &= g_W(\hat e_j^W, [f_a^W, f_b^W])/2 = \sigma_2^*\{g_2(\hat e_j, [f_a^2, f_b^2])\}/2 = \sigma_2^*\hat\omega_{abj}.
\end{aligned}$$

Since pullback commutes with ext and int, assertion (2) now follows from equation (1.2).

Pullback commutes with $\bar\partial$. We use assertion (2) and Theorem 1.6 (1) to prove assertion (3) by computing:

$$\begin{aligned}&\Delta_W^{p,q}\pi^*-\pi^*\Delta_Y^{p,q}=\pi_W^{p,q}(\bar\partial_W\Xi_W+\Xi_W\bar\partial_W)\pi^*\\ =&\Sigma_\alpha\sigma_\alpha^*\pi_{U_\alpha}^{p,q}(\bar\partial_{U_\alpha}\Xi_\alpha+\Xi_\alpha\bar\partial_{U_\alpha})\pi_\alpha^*=\Sigma_\alpha\sigma_\alpha^*(\Delta_{U_\alpha}^{p,q}\pi_\alpha^*-\pi_\alpha^*\Delta_Y^{p,q}).\end{aligned}$$

∎

Proof of Theorem 1.6 (2). Let $\pi : Z \to Y$ be a Hermitian submersion. Suppose given $0 \neq \Phi \in E(\lambda, \Delta_Y^{p,q})$ so that $\pi^*\Phi \in E(\lambda+\varepsilon, \Delta_Z^{p,q})$. Let $Z_0 = Z$ and inductively let $Z_n = W(Z_{n-1}, Z_{n-1})$ be the fiber product of Z_{n-1} with itself as defined in equation (3.1). Let $\pi_n : Z_n \to Y$ be the associated projection. By Lemma 3.2, $\pi_n^*\Phi \in E(\lambda + 2^n\varepsilon, \Delta_{Z_n}^{p,q})$. Since the Laplacian on Z_n is a non-negative operator, $\lambda + 2^n\varepsilon \geq 0$. Since this holds for all n, $\varepsilon \geq 0$ as desired. ∎

Remark 3.3. The proof of Theorem 1.6 (2) uses in an essential fashion the compactness of Y and Z through the assertion that $\Delta_Z^{p,q}$ is a non-negative operator. In fact, Theorem 1.6 (2) fails if this hypothesis is omitted. Let

$$Y := (0, \pi/2) \times (0, \pi/2) \subset \mathbb{C}$$

with the flat metric. Then $2\Delta_Y^{0,0} = -\partial_1^2 - \partial_2^2$. Let $F(y_1, y_2) = \cos(y_1)$; $2\Delta_Y^{0,0}F = F$ so $F \in E(1/2, \Delta_Y^{0,0})$. Let $Z := \mathbb{C} \times Y$ with the metric $ds_Z^2 := e^G ds_{\mathbb{C}}^2 + ds_Y^2$ where $G = G(y_1)$. It is then an easy exercise to compute $\theta = -d_Y G$; see, for example, [5]. Then

$$(\Delta_Z^{0,0}\pi^* - \pi^*\Delta_Y^{0,0})F = -\pi^*\{\operatorname{int}_Y(d_Y G)\bar\partial_1 F\} = \partial_1 G \sin(y_1)/2.$$

For *any* $\varepsilon \in \mathbb{R}$, we may choose $G_\varepsilon(y_1)$ so $\partial_1 G(y_1) = 2\varepsilon\cot(y_1)$. This gives a metric so that $\pi^*F \in E(1+\varepsilon, \Delta_Z^{0,0})$. Thus Theorem 1.6 (2) fails if Y is not compact; there is no local proof of Theorem 1.6 (2).

§4. Rigidity of eigenvalues

This section is devoted to the proof of Theorem 1.6 (3). The techniques used in [7] to prove Theorem 1.3 (3) do not generalize easily to the complex setting so we must use a quite different approach.

Let π be a Hermitian submersion. Then $\mathcal{H}$ and $\mathcal{V}$ are invariant under the almost complex structure J. The canonical decomposition of $TZ \otimes \mathbb{C} = TZ_{1,0} \oplus TZ_{0,1}$ therefore induces a decomposition $\mathcal{H} \otimes \mathbb{C} = \mathcal{H}_{1,0} \oplus \mathcal{H}_{0,1}$ and

$\mathcal{V}\otimes\mathbb{C}=\mathcal{V}_{1,0}\oplus\mathcal{V}_{0,1}$. Choose a local orthonormal frame field for $\mathcal{H}$ of the form $\{f_1,...,f_\nu,Jf_1,...,Jf_\nu\}$ where $\nu=\dim_{\mathbb{C}}Y$. The corresponding dual coframe field for $\mathcal{H}^*$ is then given by $\{f^1,...,f^\nu,-Jf^1,...,-Jf^\nu\}$. Let $\zeta_\alpha:=(f_\alpha-\sqrt{-1}Jf_\alpha)/2$ and dually $\zeta^\alpha:=(f^\alpha-\sqrt{-1}Jf^\alpha)$. The $\{\zeta_\alpha\}$ and $\{\bar\zeta_\alpha\}$ are frames for $\mathcal{H}_{1,0}$ and $\mathcal{H}_{0,1}$ and the $\{\zeta^\alpha\}$ and $\{\bar\zeta^\alpha\}$ are the corresponding dual frames for $\Lambda^{1,0}\mathcal{H}^*$ and $\Lambda^{0,1}\mathcal{H}^*$. Interior multiplication by ζ^α lowers the bi degree by $(0,1)$; interior multiplication by $\bar\zeta^\alpha$ lowers the bi degree by $(1,0)$. We let indices α and β range from 1 to $\dim_{\mathbb{C}}Y$ and sum over repeated indices.

Lemma 4.1. *Let $\pi:Z\to Y$ be a Hermitian submersion.*

(1) *We have $\omega(\zeta_\alpha,\zeta_\beta)\in\Lambda^{0,1}\mathcal{V}^*$.*

(2) *We have*
$$\pi_Z^{p,q}\mathcal{E}\pi^*=\operatorname{ext}_Z(\omega(\zeta_\alpha,\zeta_\beta))\operatorname{int}_Z(\zeta^\alpha)\operatorname{int}_Z(\zeta^\beta)$$
$$+2\operatorname{ext}_Z(\pi_Z^{1,0}\omega(\zeta_\alpha,\bar\zeta_\beta))\operatorname{int}_Z(\zeta^\alpha)\operatorname{int}_Z(\bar\zeta^\beta)\ \text{on}\ \Lambda^{p,q+1}Y.$$

(3) *We have $\omega(\zeta_\alpha,\zeta_\beta)=0$ for all α and β $\iff$ $J^*\omega=\omega$.*

(4) *We have $\omega(\zeta_\alpha,\bar\zeta_\beta)=0$ for all α and β $\iff$ $J^*\omega=-\omega$.*

Proof. Since Z is a complex manifold, the almost complex structure J is integrable and $[\zeta_\alpha,\zeta_\beta]\in TZ_{1,0}$. Since $\mathcal{H}$ and $\mathcal{V}$ are J invariant, $\rho_{\mathcal{V}}[\zeta_\alpha,\zeta_\beta]\in\mathcal{V}_{1,0}$. Let $\tilde g_Z$ be the extension of g_Z to be complex bilinear; ω is the dual of $\rho_{\mathcal{V}}[\cdot,\cdot]$ with respect to $\tilde g_Z$. The first assertion now follows since the dual of $\mathcal{V}_{1,0}$ with respect to $\tilde g_Z$ is $\Lambda^{0,1}\mathcal{V}^*$. To prove the second assertion, we compute:

$$\mathcal{E}=\operatorname{ext}_Z(\omega(\zeta_\alpha,\zeta_\beta))\operatorname{int}_Z(\zeta^\alpha)\operatorname{int}_Z(\zeta^\beta)\tag{4.2}$$
$$+\operatorname{ext}_Z(\omega(\bar\zeta_\alpha,\bar\zeta_\beta))\operatorname{int}_Z(\bar\zeta^\alpha)\operatorname{int}_Z(\bar\zeta^\beta)\tag{4.3}$$
$$+\operatorname{ext}_Z(\omega(\zeta_\alpha,\bar\zeta_\beta))\operatorname{int}_Z(\zeta^\alpha)\operatorname{int}_Z(\bar\zeta^\beta)\tag{4.4}$$
$$+\operatorname{ext}_Z(\omega(\bar\zeta_\alpha,\zeta_\beta))\operatorname{int}_Z(\bar\zeta^\alpha)\operatorname{int}_Z(\zeta^\beta).\tag{4.5}$$

The terms in (4.2) lower the horizontal bi degree by (0,2), the terms in (4.3) lower the horizontal bi degree by (2,0), and the terms in (4.4) and (4.5) lower the horizontal bi degree by (1,1); the symmetries involved permit us to combine these two terms. Thus in (4.2) we must use exterior multiplication by $\pi_Z^{0,1}\omega(\zeta_\alpha,\zeta_\beta)$; in (4.4) and in (4.5) we must use exterior multiplication by $\pi_Z^{1,0}\omega(\zeta_\alpha\bar\zeta_\beta)$; (4.3) plays no role. By the first assertion, we may replace $\pi_Z^{0,1}\omega(\zeta_\alpha,\zeta_\beta)$ by $\omega(\zeta_\alpha,\zeta_\beta)$; this proves the second assertion. The final assertions are immediate consequences of the definition. ∎

Suppose that assertion (3-iii) of Theorem 1.6 holds. We apply assertion (1) of Theorem 1.6. Since $\theta = 0$, Ξ is determined by $\mathcal{E}$. If $p = 0$ and if $q = 0$, $\mathcal{E}$ acts trivially on $\Lambda^{0,1}Z$ so (3-i) follows. If $p > 0$ and if $q > 0$, we assume $\omega = 0$ and $\mathcal{E} = 0$. If $p > 0$ and if $q = 0$, we need only consider the action of $\pi_Z^{p,0}\mathcal{E}\pi^*$ on $\Lambda^{p,1}Y$. Thus only the terms in (4.4) and (4.5) above are relevant and these vanish since we assumed $J^*\omega = -\omega$. If $p = 0$ and $q > 0$, then we need only consider the action of $\pi_Z^{0,q}\mathcal{E}\pi^*$ on $\Lambda^{0,q+1}Y$. Thus only the terms in (4.2) are relevant. These vanish since we assumed $J^*\omega = \omega$. This shows assertion (3-iii) implies assertion (3-i). It is immediate that assertion (3-i) implies assertion (3-ii). The remainder of this section is devoted to the proof that (3-ii) implies (3-iii); Theorem 1.6 (2) will play a crucial role in the proof.

We begin with a technical Lemma in the theory of PDE's.

Lemma 4.6.

(1) *Let R be any linear operator on $C^\infty(\Lambda^{p,q}Y)$ so that $(R\Phi, \Phi)_{L^2} = 0$ for all Φ in $C^\infty(\Lambda^{p,q}Y)$. Then $R = 0$.*

(2) *Let P be a 1^{th} order partial differential operator on $C^\infty(\Lambda^{p,q}Y)$. Suppose that P is non-negative, i.e. $(P\Phi, \Phi)_{L^2} \geq 0$ for all $\Phi \in C^\infty(\Lambda^{p,q}Y)$. Then P is a 0^{th} order operator, i.e. if $\Phi(y_0) = 0$, then $P\Phi(y_0) = 0$.*

Proof. Let ε be a real parameter. Since $(R(\Phi_1+\varepsilon\Phi_2), \Phi_1+\varepsilon\Phi_2) = 0$ for all ε, $(R\Phi_1, \Phi_2) + (R\Phi_2, \Phi_1) = 0$; replacing ε by $\sqrt{-1}\varepsilon$ yields $(R\Phi_1, \Phi_2) - (R\Phi_2, \Phi_1) = 0$. Thus $(R\Phi_1, \Phi_2) = 0$ for all Φ_i; we take $\Phi_2 = R\Phi_1$ to see $R = 0$.

We use the method of stationary phase to prove the second assertion. Decompose $P = \Sigma_a P^a \partial_a^y + Q$. We must show $P^a = 0$ for all a. Let $\Psi \in C^\infty(Y)$ and let $\Phi_0 \in C^\infty(\Lambda^{p,q}Y)$. Let $\Phi(t) := e^{\sqrt{-1}t\Psi}\Phi_0$. Let $R(\Psi) := \Sigma_a \partial_a^y(\Psi)P^a$. Then

$$\begin{aligned}
&P\Phi(t) = e^{\sqrt{-1}t\Psi}P\Phi_0 + \Sigma_a\sqrt{-1}t\partial_a^y(\Psi)P^a e^{\sqrt{-1}t\Psi}\Phi_0 \text{ so}\\
&(P\Phi(t), \Phi(t))_{L^2} = (P\Phi_0, \Phi_0)_{L^2} + t\sqrt{-1}(R(\Psi)\Phi_0, \Phi_0)_{L^2} \geq 0.
\end{aligned}$$

This inequality holds for all t so $(R(\Psi)\Phi_0, \Phi_0)_{L^2} = 0$ and thus $R(\Psi) = 0$ for all Ψ. This implies $P^a = 0$ for all a. $\square$

Remark 4.7. This Lemma fails in the real setting. Let $M = S^1$ and let $P = \partial_\theta$. Then $(Pf, f)_{L^2} = 0$ for any real smooth function f on S^1.

4.8. Integration over the fiber and pushforward

We adopt the following notational conventions. Let $X(y) := \pi^{-1}y$ be the fiber of π over a point $y \in Y$, let $m := \dim_{\mathbb{R}} Y$, let $n := \dim_{\mathbb{R}} Z$, let $X(y) := \pi^{-1}(y)$, and let $\nu_X := e^1 \wedge ... \wedge e^{n-m}$. Then $dvol_Z = \nu_X \wedge \pi^* dvol_Y$ and the restriction of ν_X to $X(y)$ is the Riemannian volume element of the fiber. Let $V(y) := \int_{x \in X(y)} \nu_X(x)$ be the volume of $X(y)$. We average over the fibers to define push forward

$$\pi_* : C^\infty(\Lambda^p Z) \to C^\infty(\Lambda^p Y)$$

as follows. Let $\phi \in C^\infty(\Lambda^p Z)$ and let $F_1, ..., F_p$ be tangent vectors at $y \in Y$. Let f_1,..., f_p be the corresponding horizontal lifts. We define

$$(\pi_*\phi)(F_1, ..., F_p) := V(y)^{-1} \textstyle\int_{x \in X(y)} \phi(f_1, ..., f_p)(x, y)\nu_X(x).$$

Alternatively, let $\pi^{\mathcal{H}}$ be orthogonal projection of $\Lambda^p Z$ on $\pi^*\Lambda^p Y$. Decompose $\pi^{\mathcal{H}}\phi = \Sigma_{|A|=p} c_A(x, y)\pi^* dy^A$. Then

$$\pi_*\phi = \Sigma_{|A|=p}\{V(y)^{-1} \textstyle\int_{x \in X(y)} c_A(x, y)\nu_X(x)\}dy^A.$$

It is immediate from the definition that $\pi_*\pi^*$ is the identity on $C^\infty(\Lambda^p Y)$.

We may decompose any real covector ξ into complex covectors of degrees $(1, 0)$ and $(0, 1)$ to express $\xi = \xi^{1,0} + \xi^{0,1}$; $\bar{\xi}^{1,0} = \xi^{0,1}$.

LEMMA 4.9. *Let $\pi : Z \to Y$ be a Hermitian submersion. Fix (p, q) and assume that for all λ, $\pi^* E(\lambda, \Delta_Y^{p,q}) \subseteq E(\lambda + \varepsilon(\lambda), \Delta_Z^{p,q})$. Then for any $\xi \in \mathcal{H}^*$ and for any $\Phi \in \Lambda^{p,q} Y$, we have*

$$0 = \pi_Z^{p,q}(\operatorname{ext}_Z(\xi^{0,1})\mathcal{E} + \mathcal{E}\operatorname{ext}_Z(\xi^{0,1}))\pi^*\Phi, \text{ and}$$
$$0 = \pi_Z^{p,q}(\operatorname{ext}_Z(\xi^{0,1})\operatorname{int}_Z(\theta) + \operatorname{int}_Z(\theta)\operatorname{ext}_Z(\xi^{0,1}))\pi^*\Phi.$$

Proof. We define a 1^{th} order differential operator on $C^\infty(\Lambda^{p,q} Y)$ by:

$$P := \pi_*\pi_Z^{p,q}\{\bar\partial_Z \Xi + \Xi\bar\partial_Z\}\pi^*.$$

Let $\Phi_\lambda \in E(\lambda, \Delta_Y^{p,q})$. By Theorem 1.6 (1),

$$\varepsilon(\lambda)\pi^*\Phi_\lambda = \pi_Z^{p,q}\{\bar\partial_Z \Xi + \Xi\bar\partial_Z\}\pi^*\Phi_\lambda.$$

Since $\pi_*\pi^*$ is the identity, we see $P\Phi_\lambda = \varepsilon(\lambda)\Phi_\lambda$. Thus $\{E(\lambda, \Delta_Y^{p,q})\}$ are eigenspaces of P. Since these eigenspaces are orthogonal and the eigenvalues are real, we see that P is self-adjoint. By Theorem 1.6 (2), $\varepsilon(\lambda) \geq 0$ so P is a non-negative first order self-adjoint differential operator. Thus P has order 0. If $\Phi \in C^\infty(\Lambda^{p,q}Y)$, we may expand $\Phi = \Sigma_\lambda \Phi_\lambda$ for $\Phi_\lambda \in E(\lambda, \Delta_Y^{p,q})$. This series converges in the C^∞ topology; see Gilkey [3] for example. Then $P\Phi = \Sigma_\lambda \varepsilon(\lambda)\Phi_\lambda$ so

$$(\Delta_Z^{p,q}\pi^* - \pi^*\Delta_Y^{p,q})\Phi = \pi_Z^{p,q}(\bar\partial_Z \Xi + \Xi\bar\partial_Z)\pi^*\Phi = \Sigma_\lambda \pi^*\varepsilon(\lambda)\Phi_\lambda = \pi^* P(\Phi).$$

Since P is a 0^{th} order operator, $P(F\Phi) = FP(\Phi)$ for any $F \in C^\infty(Y)$ so the derivatives of F do not enter into this equation. This implies

$$\pi_Z^{p,q}(\operatorname{ext}_Z(\pi^*\bar\partial_Y F)\Xi + \Xi \operatorname{ext}_Z(\pi^*\bar\partial_Y F))\pi^*\Phi = 0. \tag{4.10}$$

The definition given in equation (1.2) permits us to decompose $\Xi = \operatorname{int}_Z(\theta) + \mathcal{E}$ where $\operatorname{int}_Z(\theta)$ does not involve any vertical covectors and where $\mathcal{E}$ does involve vertical covectors. Thus equation (4.10) decouples into two separate equations involving $\operatorname{int}_Z(\theta)$ and $\mathcal{E}$ separately. If ξ is a horizontal covector at z_0, we can choose F so $\pi^* d_Y F(z_0) = \xi$. Then $\pi^*\bar\partial_Y F(z_0) = \xi^{0,1}$ and the Lemma follows. ∎

Proof of Theorem 1.6 (3). We must show (3-ii) implies (3-iii). Recall that

$$\begin{aligned}
&\operatorname{int}_Z(\xi_1)\operatorname{int}_Z(\xi_2) + \operatorname{int}_Z(\xi_2)\operatorname{int}_Z(\xi_1) = 0,\\
&\operatorname{ext}_Z(\xi_1)\operatorname{ext}_Z(\xi_2) + \operatorname{ext}_Z(\xi_2)\operatorname{ext}_Z(\xi_1) = 0, \text{ and}\\
&\operatorname{int}_Z(\xi_1)\operatorname{ext}_Z(\xi_2) + \operatorname{ext}_Z(\xi_2)\operatorname{int}_Z(\xi_1) = g(\xi_1, \xi_2).
\end{aligned}$$

Recall that $\tilde g_Z$ is the extension of g_Z to be complex bilinear. Suppose the hypothesis of Theorem 1.6 (3-ii) hold. To simplify notation, let $\eta = \xi^{0,1}$, let $E^\eta := \operatorname{ext}_Z(\eta)$, let $I^\theta := \operatorname{int}_Z(\theta)$, let $E^i := \operatorname{ext}_Z(e^i)$, and let $I^a := \operatorname{int}_Z(f^a)$. By Lemma 4.9,

$$0 = \pi_Z^{p,q}(E^\eta I^\theta + I^\theta E^\eta) = \pi_Z^{p,q}\tilde g_Z(\theta, \eta);$$

This implies $\theta = 0$ since θ is horizontal. We also compute:

$$\begin{aligned}
0 &= \omega_{abi}\pi_Z^{p,q}\{E^\eta E^i I^a I^b + E^i I^a I^b E^\eta\}\pi^*\\
&= \omega_{abi}\pi_Z^{p,q}\{-E^i E^\eta I^a I^b + E^i I^a I^b E^\eta\}\pi^*\\
&= \omega_{abi}\pi_Z^{p,q}\{E^i I^a E^\eta I^b + E^i I^a I^b E^\eta - \tilde g_Z(\eta, f^a)E^i I^b\}\pi^*\\
&= \omega_{abi}\pi_Z^{p,q}\{-\tilde g_Z(\eta, f^a)E^i I^b + \tilde g_Z(\eta, f^b)E^i I^a\}\pi^*
\end{aligned} \tag{4.11}$$

on $\Lambda^{p,q}Y$. We take $\eta = \bar\zeta^\alpha$. The dual of $\bar\zeta^\alpha$ with respect to $\tilde g_Z$ is ζ_α. Thus by equation (4.11),

$$0 = \operatorname{ext}_Z(\pi^{0,1}\omega(\zeta_\alpha,\zeta_\beta))\operatorname{int}_Z(\zeta^\beta) + \operatorname{ext}_Z(\pi^{1,0}\omega(\zeta_\alpha,\bar\zeta_\beta))\operatorname{int}_Z(\bar\zeta^\beta)$$

on $\Lambda^{p,q}\mathcal{H}^*$ for all α and β. These equations decouple and we have:

$$0 = \operatorname{ext}_Z(\pi^{0,1}\omega(\zeta_\alpha,\zeta_\beta))\operatorname{int}_Z(\zeta^\beta) \tag{4.12}$$

$$0 = \operatorname{ext}_Z(\pi^{1,0}\omega(\zeta_\alpha,\bar\zeta_\beta))\operatorname{int}_Z(\bar\zeta^\beta). \tag{4.13}$$

If $p = 0$, we can draw no conclusion from equation (4.13); if $q = 0$, we can draw no conclusion from equation (4.12). If $p > 0$, equation (4.13) shows $\pi^{1,0}\omega(\zeta_\alpha,\bar\zeta_\beta) = 0$ for all α and β; by Lemma 4.1, this implies $J^*\omega = -\omega$. If $q > 0$, equation (4.12) shows $\pi^{0,1}\omega(\zeta_\alpha,\zeta_\beta) = 0$ and hence $\omega(\zeta_\alpha,\zeta_\beta) = 0$ for all α and β; by Lemma 4.1, this implies $J^*\omega = \omega$. If $p > 0$ and $q > 0$, we combine these two identities to see $\omega = 0$. This shows the conditions of (3-iii) are satisfied. ∎

§5. Forms of type $(p, 0)$

Suppose that there exists $0 \neq \Phi \in E(\lambda,\Delta_Y^0)$ with $\pi^*\Phi \in E(\lambda+\varepsilon,\Delta_Z^0)$. Since Δ_Y^0 is a real operator, we may assume that Φ is real. We apply Theorem 1.3 (1) to see that $\varepsilon\pi^*\Phi = \operatorname{int}_Z(\theta)\pi^* d_Y\Phi$. Choose y_0 so $\Phi(y_0)$ is maximum; by replacing Φ by $-\Phi$ if necessary, we may assume $\Phi(y_0) > 0$. Then $d_Y\Phi(y_0) = 0$ so $\varepsilon\pi^*\Phi(y_0) = 0$ implies $\varepsilon = 0$. This argument shows that a single eigenvalue can not change in the real context for the scalar Laplacian. In the complex case, we can not use this argument since the operator in question is not real; instead, we use the push-forward defined in §4.8 and apply Theorem 1.6 (2).

Give the fiber $X(y) := \pi^{-1}(y)$ of π over y the orientation induced from the complex structures. Let $\nu_X := e^1 \wedge ... \wedge e^{n-m}$. Then $V(y) := \int_{X(y)} \nu_X$ is the volume of the fiber. We begin our discussion with the following technical Lemma.

LEMMA 5.1. *Let $\pi : Z \to Y$ be a Riemannian submersion*

(1) *We have $d\nu_X = -\theta \wedge \nu_X - \omega_{abi} ext_Z(f^a) ext_Z(f^b) int_Z(e^i)\nu_X$.*

(2) *Let $\mathcal{O}$ be a neighborhood of y_0 in Y. We can find a local diffeomorphism T from $X \times \mathcal{O}$ to Z so that $T(x, 0) = x$, so that $\pi(T(x, y)) = y$, and so that $T_*(\partial_a^y)(x, 0) = \mathcal{H}(\partial_a^y)$.*

(3) *We have* $\Theta := \pi_*\theta = -d_Y \log(V)$.

Proof. We compute:

$$d\nu_X = \Gamma_{ijk}\,\mathrm{ext}_Z(e^i)\,\mathrm{ext}_Z(e^k)\,\mathrm{int}_Z(e^j)\nu_X \tag{5.2}$$

$$+(\Gamma_{ija}-\Gamma_{aji})\,\mathrm{ext}_Z(e^i)\,\mathrm{ext}_Z(f^a)\,\mathrm{int}_Z(e^j)\nu_X \tag{5.3}$$

$$+\Gamma_{ajb}\,\mathrm{ext}_Z(f^a)\,\mathrm{ext}_Z(f^b)\,\mathrm{int}_Z(e^j)\nu_X. \tag{5.4}$$

The terms in (5.2) yield 0. In (5.3), we must set $i = j$. Since $\Gamma_{iia}-\Gamma_{aii} = \Gamma_{iia}$, equation (5.3) yields $-\theta\wedge\nu_X$. The terms in (5.4) yield $-(\Gamma_{abj}-\Gamma_{baj})/2 = -\omega_{abi}$. This proves the first identity.

We choose coordinates on a neighborhood $\mathcal{O}$ of $y_0 \in Y$ to identify $\mathcal{O}$ with $\mathbb{R}^m$ and y_0 with zero. If $y \in \mathbb{R}^m$, y determines a vector field on $\mathcal{O}$ and the horizontal lift $\mathcal{H}(y)$ determines a vector field on $\pi^{-1}\mathcal{O}$. Let $\phi(t, y, x)$ be the corresponding flow from a point $x \in X$. There is a constant C so that ϕ is a smooth map defined for $|ty| \le C$. Note that $\phi(ts, y, x) = \phi(t, sy, x)$; we set $T(x, y) = \phi(1, y, x)$ and restrict to $|y| \le C^{-1}$ to prove the second assertion.

We decompose $\Lambda Z = \oplus_{p,q}\Lambda^p\mathcal{H}^*\otimes\Lambda^q\mathcal{V}^*$ and let $\rho_{p,q}$ be the corresponding orthogonal projections. Fix $y_0 \in Y$. We use assertion (2) to assume that $Z = X \times \mathcal{O}$ and that $\mathcal{H}(x, y_0) = \mathrm{span}\{\partial^y_a\}$. We choose local coordinates $x_\alpha = (x^s_\alpha)$ on X and let ϕ_α be a partition of unity subordinate to this cover. Let

$$\begin{aligned} X^s_\alpha &:= \rho_{0,1}dx^s_\alpha = dx^s_\alpha - C^s_{\alpha,a}dy^a, \text{ and}\\ \nu_X &= g_\alpha(x, y)X^1\wedge\ldots\wedge X^{n-m}. \end{aligned}$$

We use assertion (1) to see that $\rho_{1,n-m}d\nu_X = -\theta\wedge\nu_X$ We evaluate at a point $(x, 0)$ and use the fact $C^i_a(x, 0) = 0$ to compute $\rho_{1,n-m}d\nu_X = (g^{-1}_\alpha\partial^y_a g_\alpha)dy^a\wedge\nu_X$. Consequently $-\theta(\partial^y_a)(x, 0) = g^{-1}_\alpha\partial^y_a g_\alpha$. We compute:

$$\partial^y_a V(Y) = \Sigma_\alpha \textstyle\int_X \phi_\alpha g^{-1}_\alpha(\partial^y_a g_\alpha)\nu_X = -\Sigma_\alpha \int_X \phi_\alpha\theta(\partial^y_a)\nu_X = -V(y)\Theta(\partial^y_a).$$

$\square$

Proof of Theorem 1.6 (4). Let $0 \ne \Phi \in E(\lambda, \Delta^{p,0}_Y)$ and let $\pi^*\Phi \in E(\lambda+\varepsilon, \Delta^{p,0}_Z)$. Since $\Delta^{p,0} = \delta_2\bar\partial$, we use Theorem 1.6 (1) to see $\varepsilon\pi^*\Phi = \pi^{p,0}_Z\Xi\pi^*\bar\partial_Y\Phi$. Since $\mathcal{E}$ has a non-trivial vertical component, $0 = \pi^{p,0}_Z\mathcal{E}\pi^*\bar\partial_Y\Phi$ so $\varepsilon\pi^*\Phi = \pi^{p,0}_Z\,\mathrm{int}_Z(\theta)\pi^*\bar\partial_Y\Phi$. We apply π_* to see that

$$\varepsilon\Phi = \pi^{p,0}_Y\,\mathrm{int}_Y(\Theta)\bar\partial_Y\Phi. \tag{5.5}$$

Let $g(t)_Z := V^{2t}ds^2_{\mathcal{V}} + ds^2_{\mathcal{H}}$ define a conformal variation of the metric on the vertical distribution and leave the metric on the horizontal distribution unchanged. Then $\pi : Z(t) \to Y$ is a Hermitian submersion and $\mathcal{E}$ transforms conformally. We use Lemma 5.1 to see $\theta(t) = (1 + t\dim(X))\theta$ and $\pi^*\Phi \in E(\lambda + (1 + t\dim(X))\varepsilon, \Delta^{p,0}_{Z(t)})$. Thus by Theorem 1.6 (2), $\lambda + (1 + t\dim(X))\varepsilon \geq 0$ for all $t \in \mathbb{R}$. This shows $\varepsilon = 0$. ∎

§6. Hermitian submersions where eigenvalues change

We begin by reducing the proof of Theorem 1.7 to the special case $\lambda = 0$ and $(p, q) = (1, 1)$ or $(p, q) = (0, 2)$:

LEMMA 6.1. *Suppose there exists a Hermitian submersion $\pi_1 : Z_1 \to Y_1$ and there exists $0 \neq \Phi_1 \in E(0, \Delta^{r,s}_{Y_1})$ so that $\pi_1^*\Phi_1 \in E(\sigma, \Delta^{r,s}_{Z_1})$ for some $\sigma > 0$. Let $0 \leq \lambda < \mu$, let $r \leq p$, and let $s \leq q$ be given. Then there exists a Hermitian submersion $\pi : Z \to Y$ and $0 \neq \Phi \in E(\lambda, \Delta^{p,q}_Y)$ so that $\pi^*\Phi \in E(\mu, \Delta^{p,q}_Z)$.*

Proof. If M is a complex manifold with a Hermitian metric, let $M(c)$ denote M with the scaled metric $c^{-2}ds^2_M$. Since $\Delta^{p,q}_{M(c)} = c^2\Delta^{p,q}_M$,

$$E(\lambda, \Delta^{p,q}_M) = E(c^2\lambda, \Delta^{p,q}_{M(c)}).$$

Give $M := M_1 \times M_2$ the product metric and product holomorphic structure. Then

$$E(\lambda_1, \Delta^{p_1,q_1}_{M_1}) \wedge E(\lambda_2, \Delta^{p_2,q_2}_{M_2}) \subset E(\lambda_1 + \lambda_2, \Delta^{(p_1+p_2,q_1+q_2)}_M).$$

Assume the conditions of the Lemma hold. Choose $c > 0$ so that $\mu = c^2\sigma + \lambda$. Let W be a holomorphic flat torus of complex dimension at least $\max(p - r, q - s)$. By rescaling the metric on W, we may choose $0 \neq \Phi_2 \in E(\lambda, \Delta^{p-r,q-s}_W)$. Let $Y := Y_1(c) \times W$, let $Z := Z_1(c) \times W$, and let $\pi(z_1, w) = (\pi_1(z_1), w)$. Then π is a Hermitian submersion. Let $\Phi := \Phi_1 \wedge \Phi_2$. Then $0 \neq \Phi \in E(\lambda, \Delta^{p,q}_Y)$ and $\pi^*\Phi = (\pi_1^*\Phi_1) \wedge \Phi_2 \in E(c^2\sigma + \lambda = \mu, \Delta^{p,q}_Z)$. ∎

6.2. The geometry of principal S^1 bundles

Let L be a complex line bundle over Y. We suppose that L is equipped with a smooth fiber metric and a unitary connection ${}^L\nabla$. Let $\pi : S(L) \to Y$. Then π defines a Riemannian principal S^1 bundle; this is also the circle bundle of the underlying real 2-plane bundle.

LEMMA 6.3. *Let s be a local orthonormal section to L. Let ${}^L\nabla s = \sqrt{-1}\mathcal{A}_s s$ define the normalized connection 1-form $\mathcal{A}_s$. Let $(t,y)\mapsto e^{\sqrt{-1}t}s(y)$ give local coordinates (t,y) to $S=S(L)$.*

1. *The fibers of π are totally geodesic.*
2. *We have ∂_t is an invariantly defined unit tangent vector spanning $\mathcal{V}$.*
3. *If $\tilde{s}=e^{\sqrt{-1}\Phi}s$, then $\partial_t=\partial_{\tilde{t}}$, $\partial_a^y=\partial_a^{\tilde{y}}-\partial_a^y\Phi\partial_t$, and $\tilde{\mathcal{A}}_s=\mathcal{A}_s+d_Y\Phi$.*
4. *The horizontal lift of a vector field Ψ on Y is given by $\mathcal{H}\Psi:=\Psi-\mathcal{A}_s(\Psi)\partial_t$.*
5. *We have $e^1:=dt+\pi^*\mathcal{A}_s$ is dual to ∂_t and spans $\mathcal{V}^*$.*
6. *The normalized curvature $\mathcal{F}:=d_Y\mathcal{A}_s$ is invariantly defined.*
7. *We have $de^1=\pi^*\mathcal{F}$ and $\mathcal{E}=-\operatorname{ext}_S(e^1)\pi^*\operatorname{int}_Y(\mathcal{F})$.*

Proof. The flow $v\to e^{\sqrt{-1}t}v$ for $v\in S(L)$ and $t\in\mathbb{R}$ is invariantly defined; ∂_t is the associated unit vertical Killing vector field. Assertions (1) and (2) now follow. Since ${}^L\nabla$ is unitary, $\mathcal{A}_s$ is a real 1-form. If $\tilde{s}=e^{\sqrt{-1}\Phi(y)}s$, then $(\tilde{y},\tilde{t})=(y,t-\Phi)$; assertion (3) now follows. We show that $\mathcal{H}$ is invariantly defined by computing:

$$\partial_a^y-\mathcal{A}_s(\partial_a^y)\partial_t=\partial_a^{\tilde{y}}-\partial_a^y\Phi\partial_t-\mathcal{A}_s(\partial_a^y)\partial_t=\partial_a^{\tilde{y}}-\mathcal{A}_{\tilde{s}}(\partial_a^y)\partial_t.$$

Fix $y_0\in Y$ and choose Φ so $(\mathcal{A}_s+d_Y\Phi)(y_0)=0$. Since $\mathcal{A}_{\tilde{s}}(y_0)=0$, the $\partial_a^{\tilde{y}}$ are horizontal. Thus at y_0, $\mathcal{H}\partial_a^y=\partial_a^{\tilde{y}}$ is horizontal. Since $\mathcal{H}\Psi$ is invariantly defined, $\mathcal{H}\Psi$ is the horizontal lift. Since $e^1(\mathcal{H}\Psi)=0$ for all Ψ and since $e^1(dt)=1$, e^1 is the vertical projection of dt and is invariantly defined. By (3), $d_Y\mathcal{A}_{\tilde{s}}=d_Y\mathcal{A}_s$ so the curvature $\mathcal{F}$ is invariantly defined. Clearly $de^1=\pi^*\mathcal{F}$. We compute:

$$\begin{aligned}\mathcal{E}:=&\ \operatorname{ext}_S(e^1)g_S(\partial_t,[\mathcal{H}\partial_a^y,\mathcal{H}\partial_b^y])\pi^*\operatorname{int}_Y(dy^a)\operatorname{int}_Y(dy^b)/2\\ =&\ \operatorname{ext}_S(e^1)\pi^*\{-\partial_a^y\mathcal{A}_b+\partial_b^y\mathcal{A}_a\}\operatorname{int}_Y(dy^a)\operatorname{int}_Y(dy^b)/2\\ =&\ -\operatorname{ext}_S(e^1)\pi^*\operatorname{int}_Y(\mathcal{F}).\end{aligned}$$ □

At this point, we shall digress briefly. Muto [11], [12] gave examples of Riemannian principal S^1 bundles where eigenvalues change. The following Lemma follows from his calculations and forms the basis for the proof of Theorem 1.4.

LEMMA 6.4. *Let ${}^L\nabla$ be a unitary connection on a complex line bundle over Y with associated curvature 2-form $\mathcal{F} = \mathcal{F}({}^L\nabla)$ and associated principal circle bundle $S = S(L)$. Let $\Phi \in E(\lambda, \Delta_Y^p)$. Assume that $d_Y\Phi = 0$, that $d_Y \operatorname{int}_Y(\mathcal{F})\Phi = 0$, and that $-\operatorname{ext}_Y(\mathcal{F})\operatorname{int}_Y(\mathcal{F})\Phi = \varepsilon\Phi$ for ε constant. Then $\pi^*\Phi \in E(\lambda+\varepsilon, \Delta_S^p)$.*

Proof. We apply Theorem 1.3 (1) and Lemma 6.3. Since $\theta = 0$ and since $d_Y\Phi = 0$,

$$\begin{aligned}\Delta_S^p\pi^*\Phi - \pi^*\Delta_Y^p\Phi &= d_S\mathcal{E}\pi^*\Phi = -d_S(e^1 \wedge \pi^* \operatorname{int}_Y(\mathcal{F})\Phi)\\ &= -\pi^* \operatorname{ext}_Y(\mathcal{F})\operatorname{int}_Y(\mathcal{F})\Phi = \varepsilon\pi^*\Phi.\end{aligned}$$

□

In the following Lemma, we construct line bundles with non-trivial curvature over the flat k dimensional torus.

LEMMA 6.5. *Let $(x^1, ..., x^k)$ for $0 \le x^i \le 2\pi$ be the usual periodic parameters on the flat k dimensional torus $Y_k := S^1 \times ... \times S^1$. Let $1 \le i < j \le k$ be given. Then there exists a unitary connection ${}^L\nabla$ on a complex line bundle L over Y_k so that $\mathcal{F}(L) = (dx^i \wedge dx^j)/2\pi$.*

Proof. To simplify the notation, we may assume $i = 1$ and $j = 2$. Let $W = Y_{k-1}$ and let $w := (x^2, ..., x^k)$. We decompose $Y_k = [0, 2\pi] \times W/\cong$ where the identification is given by $(0, w) \cong (2\pi, w)$. Let $L := [0, 2\pi] \times W \times \mathbb{C}/\cong$ where the identification is given by $(0, w, z) \cong (2\pi, w, e^{-\sqrt{-1}x^2}z)$. Then L is a complex line bundle over Y_k which has a natural fiber metric since the clutching function $e^{-\sqrt{-1}x^2}$ is unitary. Let $A(x) = (x^1dx^2)/2\pi$ be the connection 1-form. The clutching or transition function, which describes how fiber at $x_1 = 0$ is glued to the fiber at $x_1 = 2\pi$, is defined by $\Phi = -x^2$. Since $\mathcal{A}(0, w) = \mathcal{A}(2\pi, w) + d\Phi$, equation (3) in Lemma 6.3 is satisfied so $\mathcal{A}$ defines a Riemannian connection ∇ on L with associated normalized curvature $(dx^1 \wedge dx^2)/2\pi$. □

Remark 6.6. Let $Y = Y_2$ and choose L so $\mathcal{F} = (dx \wedge dy)/2\pi$. We use Lemma 6.3 to see that $de^1 = \pi^*(dx \wedge dy)/2\pi$ and thus $\mathcal{E} = -\operatorname{ext}_S(e^1)\pi^* \cdot \operatorname{int}_Y(dx \wedge dy)/2\pi$. It then follows $dx \wedge dy \in E(0, \Delta_Y^2)$ and $\pi^*(dx \wedge dy) \in E(1/4\pi^2, \Delta_S^2)$ so this provides an example where an eigenvalue of the real Laplacian on 2-forms changes.

6.7. Forms of type $(1,1)$

Over $Y = Y_2$, let $S_0 = Y \times S^1$ be the trivial circle bundle and with $\mathcal{F}_0 = 0$. Let S_1 be a circle bundle with $\mathcal{F}_1 = (dx \wedge dy)/2\pi$. Let $Z = W(S_0, S_1)$ be the fiber product discussed in §3. Let e^0 and e^1 be the corresponding dual vertical covectors; $d_Z e^0 = 0$ and $d_Z e^1 = \pi^*(dx \wedge dy)/2\pi$. Define an almost complex structure J on Z by requiring that g_Z is Hermitian, that π^* preserves J, that $J(e^0) = -e^1$, and that $J(e^1) = e^0$. Let $\xi^1 := e^0 - \sqrt{-1}e^1$ and $\xi^2 := \pi^*(dx - \sqrt{-1}dy)$ be a frame for $\Lambda^{0,1}Z$. Then

$$d_Z\xi^1 = -\sqrt{-1}\pi^*(dx \wedge dy)/2\pi \in C^\infty(\Lambda^{1,1}Z) \text{ and } d_Z\xi^2 = 0$$

so the Nirenberg-Neulander theorem shows J is an integrable almost complex structure. As the metric g_Y is flat, $dx \wedge dy \in E(0, \Delta_Y^{1,1})$. Note $\delta_{2,Y}(dx \wedge dy) = 0$. We use Theorem 1.6 (1), Lemma 3.2 (2), and Lemma 6.3 (7) to see

$$\begin{aligned}
&\delta_{2,Z}\pi^*(dx \wedge dy) = -\pi_Z^{1,0}\operatorname{ext}_Z(e^1)\pi^*\operatorname{int}_Y(\mathcal{F}_1)dx \wedge dy\\
&\quad = \pi_Z^{1,0}e^1/2\pi = -\sqrt{-1}(e^0 + \sqrt{-1}e^1)/4\pi\\
&\Delta_Z^{1,1}\pi^*(dx \wedge dy) = \bar\partial_Z\delta_{2,Z}\pi^*(dx \wedge dy) = \pi_Z^{1,1}d_Z e^1/4\pi\\
&\quad = \pi^*(dx \wedge dy)/8\pi^2.
\end{aligned}$$

This shows $dx \wedge dy \in E(0, \Delta_Y^{1,1})$ and $\pi^*(dx \wedge dy) \in E(1/8\pi^2, \Delta_Z^{1,1})$. This provides an example where a harmonic form of type (1,1) pulls back to an eigen form corresponding to a non-zero eigenvalue.

6.8. Forms of type $(0,2)$

Let $z = (z^1, z^2)$ for $z^i = x^i + \sqrt{-1}y^i$ be complex coordinates on $Y = Y_4$. Let

$$\Phi := (dx^1 - \sqrt{-1}dy^1) \wedge (dx^2 - \sqrt{-1}dy^2).$$

Then $\Phi \in E(0, \Delta_Y^{0,2})$ and Φ generates the line bundle $\Lambda^{0,2}(Y)$. Use Lemma 6.5 to construct line bundles L_i over Y so $\mathcal{F}_1 = (dx^1 \wedge dx^2)/2\pi$ and $\mathcal{F}_2 = (dx^1 \wedge dy^2)/2\pi$. These are not holomorphic line bundles since the curvatures are not (1,1) forms. Let $Z := W(S(L_1), S(L_2))$ be the fiber product of the associated unit circle bundles. Let e^i be the associated vertical covectors. We define an almost complex structure J on Z by requiring g_Z is Hermitian, that π^* preserves J, and that $J(e^1) = -e^2$ and $J(e^2) = e^1$. Let ξ^i span $\Lambda^{0,1}Z$ for $\xi^1 := \pi^* d\bar z^1$, for $\xi^2 := \pi^* d\bar z^2$, and for $\xi^3 := e^1 - \sqrt{-1}e^2$. Then $d\xi^1 = 0$,

$d\xi^2 = 0$, and

$$
\begin{aligned}
d_Z\xi^3 &= \pi^*(dx^1 \wedge (dx^2 - \sqrt{-1}dy^2))/2\pi \\
&= \pi^*\{(dx^1 + \sqrt{-1}dy^1) \wedge (dx^2 - \sqrt{-1}dy^2) \\
&\quad +(dx^1 - \sqrt{-1}dy^1) \wedge (dx^2 - \sqrt{-1}dy^2)\}/4\pi
\end{aligned} \tag{6.9}
$$

This decomposes $d_Z\xi^3$ as the sum of forms of type (1,1) and (0,2) so $d_Z\xi^3$ has no (2,0) component. Thus the almost complex structure J is integrable and by the Nirenberg Neulander theorem defines a complex structure on Z. We compute $\mathcal{E}\pi^*\Phi = (e^1 - \sqrt{-1}e^2)/2\pi$. This is of type $(0,1)$ and

$$d_Z\mathcal{E}\pi^*\Phi = \pi^*(dx^1 \wedge dx^2 - \sqrt{-1}dx^1 \wedge dy^2)/4\pi^2.$$

We use equation (6.9) to see $\pi_Z^{0,2}d_Z\mathcal{E}\pi^*\Phi = \pi^*\Phi/8\pi^2$ so $\pi^*\Phi \in E(1/8\pi^2, \Delta_Z^{0,2})$. This provides an example where a harmonic form of type (0,2) pulls back to an eigen form corresponding to a non-zero eigenvalue.

Proof of Theorem 1.7. By Lemma 6.1, it suffices to prove Theorem 1.7 in the special cases $(p,q) = (1,1)$ and $(p,q) = (0,2)$ with $\lambda = 0$. The first case is handled in §6.7 and the second case is handled in §6.8. ∎

6.10. Forms of type $(0,1)$

If we suppose that $\theta = 0$, that $\Phi \in E(\lambda, \Delta_Y^{0,1})$, and that $\pi^*\Phi \in E(\lambda + \epsilon, \Delta_Z^{0,1})$, we see that $\epsilon\pi^*\Phi = \pi^{0,1}\mathcal{E}\pi^*\bar\partial_Y\Phi$. The left hand side is a horizontal $(0,1)$ form; the right hand side is a vertical $(0,1)$ form. Consequently $\epsilon = 0$. Thus to construct an example where an eigenvalue changes for a $(0,1)$ form, we must consider Hermitian submersions where the fibers are not minimal. We know of no examples where eigenvalues can change but are unable to prove that they can not.

6.11. Holomorphic line bundles

The examples of §6.7 and §6.8 involved manifolds Y with flat metrics. We conclude this section by constructing other families of examples where eigenvalues change where the metric on the base is not flat. We restrict to the case $p = q \geq 1$ for the sake of simplicity. Let L be a holomorphic line bundle over Y. Let $\langle , \rangle$ be a fiber metric on L. If s_h is a local non-vanishing holomorphic section to L, let $\nabla_L s_h := \partial_Y \log\langle s_h, s_h\rangle \cdot s_h$. Let $\tilde s_h = e^F s_h$ be another local non-vanishing holomorphic section to L where F is a locally defined holomorphic function on Y. Since $\bar\partial_Y F = 0$, we have

$$\begin{aligned}\nabla_L \tilde s_h &= (dF + \partial_Y \log\langle s_h, s_h\rangle)e^F s_h = (\partial_Y F + \partial_Y \log\langle s_h, s_h\rangle)\tilde s_h\\ &= \partial_Y \log\langle \tilde s_h, \tilde s_h\rangle \cdot \tilde s_h.\end{aligned}$$

Thus ∇_L is invariantly defined and $\mathcal{F} = -\sqrt{-1}\bar\partial_Y\partial_Y \log\langle s_h, s_h\rangle$. We see that ∇_L is Riemannian since $\langle\nabla_L s_h, s_h\rangle + \langle s_h, \nabla_L s_h\rangle = d\langle s_h, s_h\rangle$.

6.12. Hodge manifolds

We say that L is a *positive* line bundle over Y if the curvature $\mathcal{F}(L)$ is the Kaehler form of a Hermitian metric on Y; there is a possible sign convention which plays no role in our development. If Y admits a positive line bundle, then Y is said to be *Hodge*. For example, the hyperplane bundle H is a positive line bundle over complex projective space $\mathbb{CP}^\nu$ and the associated metric is the Fubini-Study metric. More generally, if Y is any holomorphic submanifold of $\mathbb{CP}^\nu$, then the restriction of the hyperplane bundle to Y is a positive line bundle over Y and the metric on Y is the restriction of the Fubini-Study metric to Y. Conversely, if Y admits a positive line bundle L, then there exists a holomorphic embedding $\alpha : Y \to \mathbb{CP}^\nu$ for some ν and a positive integer k so that $L^{\otimes k} = \alpha^*(H)$. Thus we may identify the set of Hodge manifolds with the set of smooth algebraic varieties.

6.13. Other examples where eigenvalues change

Other examples where eigenvalues change Let L be a positive line bundle over Y and let $Z := Z(j,k) := W(S(L^{\otimes j}), S(L^{\otimes k}))$ be the fiber product of the circle bundles defined by the circle bundles of the tensor powers of L. Let e^j be the corresponding vertical covectors. We extend the almost complex structure from Y to Z by defining $J(e^j) = -e^k$ and $J(e^k) := e^j$. We use the Nirenberg-Neulander theorem to see that J is integrable; the integrability condition on horizontal covectors is immediate so we must only check the vertical component;

$$d(e^j - \sqrt{-1}e^k) = (j - \sqrt{-1}k)\pi^*\mathcal{F}.$$

THEOREM 6.14. *Let $1 \le p \le \bar m$ and let $\mu := (j^2 + k^2)p(\bar m + 1 - p)$. Then $\mathcal{F}^p \in E(0, \Delta_Y^{2p}) \cap E(0, \Delta_Y^{p,p})$ and $\pi^*(\mathcal{F}^P) \in E(\mu, \Delta_Z^{2p}) \cap E(\mu/2, \Delta_Z^{p,p})$.*

Proof. We have $\mathcal{F}^p$ is a harmonic form of type (p,p). Since Y is Kaehler, we have that $E(0, \Delta_Y^{2p}) \cap C^\infty(\Lambda^{p,p}Y) = E(0, \Delta_Y^{p,p})$. so $\mathcal{F}^p \in E(0, \Delta_Y^{p,p})$. We compute

$$\pi_Z^{p,p-1}\mathcal{E}\pi^*\mathcal{F}^p = (j - \sqrt{-1}k)p(\bar m + 1 - p)(e^j + \sqrt{-1}e^k) \wedge \pi^*\mathcal{F}^{p-1}/2$$

so $d\pi_Z^{p,p-1}\mathcal{E}\pi^*\mathcal{F}^p = \mu\pi^*\mathcal{F}^p/2$. Thus $\pi^*\mathcal{F}^p \in E(\mu/2, \Delta_Z^{p,p})$; the proof of the corresponding assertion in the real case is similar. ∎

Remark 6.15. Note that the manifold Z constructed in Theorem 6.14 is in general not Kaehler. For example, if L is the Hopf line bundle over the Riemann sphere S^2 and if $(j,k) = (0,1)$, then $Z = S^1 \times S^3$ is the Hopf manifold and $\pi : S^1 \times S^3 \to S^2$ is essentially just the Hopf fibration where we normalize the metrics suitably.

§7. Manifolds where $J^*\omega = \pm\omega$

Let $\pi : Z \to Y$ be a Hermitian submersion with minimal fibers. Let $p > 0$ and let $q > 0$. In Theorem 1.6 (3), we saw that π^* preserves the eigenspaces on forms of type $(p,0)$ if and only if $J^*\omega = -\omega$ and that π^* preserves the eigenspaces of forms of type $(0,q)$ if and only if $J^*\omega = \omega$. In this section, we give examples to illustrate these two cases. The case $J^*\omega = \omega$ is relatively easy; the case $J^*\omega = -\omega$ requires more work.

7.1. Hermitian submersions with $J^*\omega = \omega$

Let Y be a Riemann surface so $\dim_{\mathbb{C}} Y = 1$. Then $\mathcal{H}_{1,0}$ is a 1 dimensional complex foliation and hence $\mathcal{H}_{1,0}$ is necessarily integrable. Thus $J^*\omega = \omega$. The submersion constructed in §6.7 gives an example $\pi : W(S_0, S_1) \to S^1 \times S^1$ with non-trivial curvature tensor ω satisfying $J^*\omega = \omega$.

7.2. Hermitian submersions with $J^*\omega = -\omega$

We have $J = -1$ on $\Lambda^{2,0} \oplus \Lambda^{0,2}$ and $J = +1$ on $\Lambda^{1,1}$. Let S_i be circle bundles over a torus Y_k with curvatures $\mathcal{F}^i$ and corresponding dual vertical covectors e^i. We assume that $J^*\mathcal{F}^i = -\mathcal{F}^i$ or equivalently that we may decompose $\mathcal{F}^i = \xi^i + \bar{\xi}^i$ for $\xi^i \in \Lambda^{2,0}$. We define an almost complex structure on $Z = W(S_0, S_1)$ by requiring that g_Z is Hermitian, that π^* preserves J, that $J(e^0) = -e^1$, and that $J(e^1) = e^0$. We compute

$$\begin{aligned} d(e^0 - \sqrt{-1}e^1) &= \pi^*(\mathcal{F}^0 - \sqrt{-1}\mathcal{F}^1) \\ &= \pi^*(\xi^0 - \sqrt{-1}\xi^1) + \pi^*(\bar{\xi}^0 - \sqrt{-1}\bar{\xi}^1). \end{aligned}$$

The Nirenberg-Neulander integrability condition is satisfied if and only if

$$\xi^0 = \sqrt{-1}\xi^1. \tag{7.3}$$

Define horizontal 2-forms ω^i by the evaluation: $\omega^i(f_a, f_b) = g_Z(e_i, [f_a, f_b])/2$;

$$-\omega^i(f_a, f_b) = -e^i([f_a, f_b])/2 = de^i(f_a, f_b)/2 = \pi^*\mathcal{F}^i(f_a, f_b)/2.$$

Thus $\omega^i = -\pi^*\mathcal{F}^i/2$ and $J^*\omega^i = -\omega^i$. Thus it suffices to give an example where equation (7.3) is satisfied.

Let $\xi^0 := (dx^1 + \sqrt{-1}dy^1) \wedge (dx^2 + \sqrt{-1}dy^2)/4\pi$. Then

$$\mathcal{F}^0 = (dx^1 \wedge dx^2 - dy^1 \wedge dy^2)/2\pi, \text{ and}$$
$$\mathcal{F}^1 = (-dx^1 \wedge dy^2 + dx^2 \wedge dy^1)/2\pi.$$

We use Lemma 6.5 to construct bundles L_i over the torus with

$$\mathcal{F}^2 = (dx^1 \wedge dx^2)/2\pi, \ \mathcal{F}^3 = (dy^1 \wedge dy^2)/2\pi,$$
$$\mathcal{F}^4 = (dx^1 \wedge dy^2)/2\pi, \ \mathcal{F}^5 = (dx^2 \wedge dy^1)/2\pi.$$

Since $\mathcal{F}(L_i^*) = -\mathcal{F}(L_i)$ and $\mathcal{F}(L_i \otimes L_j) = \mathcal{F}(L_i) + \mathcal{F}(L_j)$, $L_0 := L_2 \otimes L_3^*$ and $L_1 := L_4^* \otimes L_5$ define circle bundles over the torus with the desired curvatures.

References

[1] L. Berard Bergery and J .P. Bourguignon, *Laplacians and Riemannian submersions with totally geodesic fibers*, Illinois J Math, **26** (1982), 181–200.

[2] F. Burstall, *Non-linear functional analysis and harmonic maps*, Ph D Thesis (Warwick).

[3] P. B. Gilkey, Invariance Theory, the Heat Equation, and the Atiyah-Singer Index theorem (2^{nd} edition), ISBN 0-8493-7874-4, CRC Press, Boca Raton, Florida, 1994.

[4] P. B. Gilkey, J. V. Leahy, and J. H. Park, *The spectral geometry of the Hopf fibration*, Journal Physics A, **29** (1996), 5645–5656.

[5] ———, *Eigenvalues of the form valued Laplacian for Riemannian submersions.*, Proc. AMS, **126** (1998), 1845-1850.

[6] ———, *Eigenforms of the spin Laplacian and projectable spinors for principal bundles.*, J. Nucl. Phys. B., **514 [PM]** (1998), 740–752.

[7] P. B. Gilkey and J. H. Park, *Riemannian submersions which preserve the eigenforms of the Laplacian*, Illinois J Math, **40** (1996), 194–201.

[8] S. I. Goldberg and T. Ishihara, *Riemannian submersions commuting with the Laplacian*, J. Diff. Geo., **13** (1978), 139–144.

[9] S. Gudmundsson, *The Bibliography of Harmonic Morphisms*, (Available on the internet at http://www.maths.lth.se/matematiklu/personal/sigma/harmonic /bibliography.html).

[10] D. L. Johnson, *Kaehler submersions and holomorphic connections*, J. Diff. Geo., **15** (1980), 71–79.

[11] Y. Muto, *Some eigenforms of the Laplace-Beltrami operators in a Riemannian submersion*, J. Korean Math. Soc., **15** (1978), 39–57.

[12] ______, *Riemannian submersion and the Laplace-Beltrami operator*, Kodai Math J., **1** (1978), 329–338.

[13] J. H. Park, *The Laplace-Beltrami operator and Riemannian submersion with minimal and not totally geodesic fibers*, Bull. Korean Math. Soc, **27** (1990), 39–47.

[14] B. Watson, *Manifold maps commuting with the Laplacian*, J. Diff. Geo., **8** (1973), 85–94.

[15] ______, *Almost Hermitian submersions*, J. Diff. Geo., **11** (1976), 147–165.

Peter B. Gilkey
Mathematics Department
University of Oregon
Eugene, OR 97403, USA
gilkey@darkwing.uoregon.edu

John V. Leahy
Mathematics Department
University of Oregon
Eugene, OR 97403, USA
leahy@darkwing.uoregon.edu

JeongHyeong Park
Department of Mathematics
Honam University
Seobongdong 59, Kwangsanku
Kwangju, 506-090, South Korea
jhpark@honam.honam.ac.kr