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COMPLETE SYSTEM OF FINITE ORDER FOR THE EMBEDDINGS OF PSEUDO-HERMITIAN MANIFOLDS INTO $\mathbb{C}^{N+1}$

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Abstract. Let $(M, \mathcal{V}, \theta)$ be a real analytic $(2n+1)$-dimensional pseudo-hermitian manifold with nondegenerate Levi form and F be a pseudo-hermitian embedding into $\mathbb{C}^{n+1}$. We show under certain generic conditions that F satisfies a complete system of finite order. We use a method of prolongation of the tangential Cauchy-Riemann equations and pseudo-hermitian embedding equation. Thus if $F \in C^k(M)$ for sufficiently large k, F is real analytic. As a corollary, if M is a real hypersurface in $\mathbb{C}^{n+1}$, then F extends holomorphically to a neighborhood of M provided that F is sufficiently smooth.

§0. Introduction

Let M be a smooth manifold of dimension $2n+1$. A CR structure $\mathcal{V}$ on M is a subbundle of the complexified tangent bundle $\mathbb{C}\,T(M)$ with the complex dimension n which satisfies

i) $\mathcal{V} \cap \overline{\mathcal{V}} = \{0\}$,

ii) $[\mathcal{V}, \mathcal{V}] \subset \mathcal{V}$ (integrability),

where $[\mathcal{V}, \mathcal{V}] \subset \mathcal{V}$ means that if X and Y are smooth sections of $\mathcal{V}$ then $[X, Y]$ is again a section of $\mathcal{V}$. $\mathcal{V}$ is said to be nondegenerate if the Levi form $\mathcal{L}$, defined by $\mathcal{L}(X, Y) := \sqrt{-1}\,[X, Y]$ modulo $\mathcal{V} + \overline{\mathcal{V}}$, is nondegenerate.

Let $\{Z_i\}_{i=1,\dots,n}$ be a basis of $\mathcal{V}$. Then $(M, \mathcal{V})$ is embeddable into $\mathbb{C}^{n+1}$ as a real hypersurface with induced CR structure $\mathcal{V}$ if and only if there exists $F = (f^1, \dots, f^{n+1}) : M \to \mathbb{C}^{n+1}$ such that

$$\overline{Z}_i f^j = 0 \quad \text{for all } i = 1, \dots, n,\ j = 1, \dots, n+1 \tag{0.1}$$

and

$$df^1 \wedge \cdots \wedge df^{n+1} \neq 0.$$

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(0.1) is called the tangential Cauchy-Riemann equations.

It is well known that any abstract real analytic (C^ω) CR manifold of dimension $2n+1$ is locally embeddable into $\mathbb{C}^{n+1}$ as a real hypersurface via a real analytic CR diffeomorphism ([B]). But, in general, a smooth CR embedding $F : M \to \mathbb{C}^{n+1}$ need not be C^ω even if M is C^ω as the following example shows:

Let $M = \mathbb{C} \times \mathbb{R} = \{(x + \sqrt{-1}\,y, t)\}$ and let $\gamma(t) = u(t) + \sqrt{-1}\,v(t)$ be a C^∞, but not C^ω, complex valued function. Then the mapping $F : (x + \sqrt{-1}\,y, t) \mapsto (x + \sqrt{-1}\,y, \gamma(t)) \in \mathbb{C}^2$ is a C^∞ CR embedding which is not C^ω.

On the other hand, if $F : M \to \mathbb{C}^{n+1}$ is a CR embedding and $\Phi : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ is a biholomorphic map, then $\Phi \circ F$ is also a CR embedding. Hence a CR embedding F can not be determined by a finite jet at a point.

If $F : M \to N$ is a CR embedding into another C^ω real hypersurface N in $\mathbb{C}^{m+1}$, $m \geq n$, then the unknown functions $F = (f^1, \ldots, f^{m+1})$ are analytically related by $r \circ F = 0$, where r is a C^ω defining function of N. In this case, Han ([H]) and Hayashimoto ([Ha1]) showed that a CR embedding $F : M \to N$ is C^ω and determined by a finite jet at a point under generic assumptions.

Their method is to construct a complete system (see Section 2 for definition) for $(f^1, \ldots, f^{n+1})$ by prolongation, which is a process of repeated differentiation of $r \circ F = 0$ and reduction of order of derivatives by using the tangential Cauchy-Riemann equations. In [H] and [Ha1], proofs mainly depend on the analytic relation among the unknown functions $F = (f^1, \ldots, f^{m+1})$ given by $r \circ F = 0$. However, we do not assume the analyticity of the target manifold. We show that a CR embedding $F : M \to \mathbb{C}^{n+1}$ satisfies a complete system of finite order under the assumption that F preserves the pseudo-hermitian structure.

For $(m+1)$-tuples of non-negative integers $A = (a_1, \ldots, a_{m+1})$ and $B = (b_1, \ldots, b_{m+1})$, let $\zeta^A \overline{\zeta}^B := \zeta_1^{a_1} \cdots \zeta_{m+1}^{a_{m+1}} \overline{\zeta}_1^{b_1} \cdots \overline{\zeta}_{m+1}^{b_{m+1}}$. The weight of $\zeta^A \overline{\zeta}^B := \sum_{j=1}^m (a_j + b_j) + 2(a_{m+1} + b_{m+1})$. If N is defined by

$$r(\zeta, \overline{\zeta}) = \zeta_{m+1} + \overline{\zeta}_{m+1} + \sum_{j=1}^m \lambda_j \zeta_j \overline{\zeta}_j + \sum_{A,B} c_{A\overline{B}} \zeta^A \overline{\zeta}^B = 0,$$

where λ_j is either 1 or -1 and weight of $\zeta^A \overline{\zeta}^B$ is greater than or equal to 3, then N is said to be in pre-normal form ([CM]).

Now let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be an n-tuple of non-negative integers. Define $Z^\alpha := (Z_1)^{\alpha_1} \cdots (Z_n)^{\alpha_n}$ and $|\alpha| := \alpha_1 + \cdots + \alpha_n$. Then

THEOREM 0.1. ([H]) *Let M^{2n+1} be a C^ω CR manifold of nondegenerate Levi form. Let $\{Z_1, \ldots, Z_n\}$ be C^ω independent sections of the CR structure bundle $\mathcal{V}$. Let N be a C^ω real hypersurface in $\mathbb{C}^{m+1}$, $m \geq n$, which is in pre-normal form. Let $F : M \to N$ be a CR mapping. Suppose that for some positive integer k, the vectors $\{Z^\alpha F : |\alpha| \leq k\}$ evaluated at the reference point together with $(0, \ldots, 0, 1)$ span $\mathbb{C}^{m+1}$ over $\mathbb{C}$. Then F satisfies a complete system of order $2k+1$. Thus F is determined by $2k$-jet at a point and F is C^ω provided that $F \in C^{2k+1}$.*

A CR function f on a C^ω real hypersurface M extends to a holomorphic function of a neighborhood of M if and only if f is C^ω ([T]). Then by Theorem 0.1, F extends holomorphically to a neighborhood of M.

We say that a CR mapping $F : M \to \widetilde{M}$ satisfies the Hopf lemma property at $p \in M$ if the component of F normal to $\widetilde{M}$ has a nonzero derivative at p in the normal direction to M ([BHR]). Let $\mathcal{I}$ be an ideal generated by $z_1, \ldots, z_n, \overline{z}_1, \cdots, \overline{z}_n, \operatorname{Im} z_{n+1}$. For CR functions $f^1, \ldots, f^n$ of class C^m, the symbol $sp\langle f^1, \ldots, f^n\rangle \not\equiv 0 \pmod{\mathcal{I}^{m+1}}$ means that there does not exist $(a_1, \ldots, a_n) \in \mathbb{C}^n \setminus (0, \ldots, 0)$ such that $a_1 f^1 + \cdots + a_n f^n \equiv 0 \pmod{\mathcal{I}^{m+1}}$.

THEOREM 0.2. ([Ha1]) *Let M and $\widetilde{M}$ be C^ω real hypersurfaces in $\mathbb{C}^{n+1}$ and let $F : M \to \widetilde{M}$ be a CR mapping. Suppose that $\widetilde{M}$ has a nondegenerate Levi form at the origin and that the origin in M is a point of finite type $l < \infty$ in the sense of Bloom-Graham. Consider the following three cases*:

i) *M has a nondegenerate Levi form $(l = 2)$.*

ii) *M has a degenerate Levi form and $n = 1$.*

iii) *M has a degenerate Levi form and $n \geq 2$.*

In case i) *or* ii), *if $F \in C^{l+1}$ satisfies the Hopf lemma property at the origin, then it satisfies a complete system of order $l+1$.*

In case iii), *if $F = (f^1, \ldots, f^{n+1}) \in C^m$ satisfies $sp\langle f^1, \ldots, f^n\rangle \not\equiv 0 \pmod{\mathcal{I}^{m+1}}$, then it satisfies a complete system of finite order.*

In this paper, we impose a relation among the partial derivatives of $\{f^1, \ldots, f^{n+1}\}$ instead of a relation among the unknown functions

$\{f^1, \ldots, f^{n+1}\}$. We show that a CR embedding F of a C^ω CR manifold M into $\mathbb{C}^{n+1}$ is C^ω and determined by a finite jet at a point under the additional condition that F preserves the pseudo-hermitian structure on M.

A contact form θ is a real valued nonvanishing 1-form which annihilates $\mathcal{V} \oplus \overline{\mathcal{V}}$. It is determined only up to a conformal factor. A CR manifold with a specified choice of contact form θ is called a pseudo-hermitian manifold. A CR diffeomorphism F which preserves the pseudo-hermitian structure $(M, \mathcal{V}, \theta)$ is called a pseudo-hermitian embedding. In this case, F satisfies an additional first order differential equation

$$F^*(\widetilde{\theta}) = \theta,$$

where $\widetilde{\theta}$ is a contact form of $F(M)$ in $\mathbb{C}^{n+1}$ such that $\|\widetilde{\theta}\| \equiv 1$, where $\|\cdot\|$ is the Euclidean norm for 1-forms.

More generally, we consider

$$F^*(\widetilde{\theta}) = \lambda\theta, \tag{0.2}$$

where λ is a given nonvanishing C^ω function defined on M.

We differentiate (0.2) repeatedly and reduce the order of derivatives using the tangential Cauchy-Riemann equations to construct a complete system for F.

If M is C^ω near $p \in M$, then there exist Moser's normal coordinates $(z, v) = (z_1, \ldots, z_n, v)$ at p and a basis $\{Z_1, \ldots, Z_n\}$ of $\mathcal{V}$ such that for each j,

$$Z_j = \frac{\partial}{\partial z_j} + \sum_{k=1}^{n} \overline{z}_k X_j^k + v X_j^{n+1},$$

where X_j^k, $k = 1, \ldots, n+1$, are C^ω vector fields on M.

Assume that $F(p) = (0, \ldots, 0)$ and $F(M) \subset \mathbb{C}^{n+1}$ is in pre-normal form. Let $\alpha = (a_1, \ldots, a_n)$ be an n-tuple of non-negative integers. Define $I_k(\alpha) = a_k$, $k = 1, \ldots, n$. Then our results are

Theorem 0.3. *Let $(M, \mathcal{V}, \theta)$ be a germ of C^ω pseudo-hermitian manifold with nondegenerate Levi form at the reference point p and let $F := (f^1, \ldots, f^{n+1}) : M \to \mathbb{C}^{n+1}$ be a CR diffeomorphism which satisfies the condition (0.2). Let $\{Z_i\}_{i=1,\ldots,n}$ be C^ω sections of $\mathcal{V}$ as above such that $Z_j f^k(p) = \delta_j^k$, $j, k = 1, \ldots, n$. Suppose that for all $j = 1, \ldots, n$, there exist multi-indices α_j with $|\alpha_j| \leq \sigma$ for some positive integer σ which have the following property*:

The matrix $A = (\mathrm{A}^i_j)_{i,j=1,\dots,n}$ *of size* $n(n+1)\times n(n+1)$ *is non-singular, where each block* A^i_j *is an* $(n+1)\times(n+1)$ *matrix*

$$\mathrm{A}^i_j = \begin{pmatrix} Z^{\alpha_j}k_i, & I_1(\alpha_j)Z^{\tilde\alpha_{j,1}}k_i, & \cdots & I_n(\alpha_j)Z^{\tilde\alpha_{j,n}}k_i \\ Z_1Z^{\alpha_j}k_i, & I_1(\alpha_j+e_1)Z_1Z^{\tilde\alpha_{j,1}}k_i, & \cdots & I_n(\alpha_j+e_1)Z_1Z^{\tilde\alpha_{j,n}}k_i \\ \vdots & \vdots & & \vdots \\ Z_nZ^{\alpha_j}k_i, & I_1(\alpha_j+e_n)Z_nZ^{\tilde\alpha_{j,1}}k_i, & \cdots & I_n(\alpha_j+e_n)Z_nZ^{\tilde\alpha_{j,n}}k_i \end{pmatrix},$$

where

$$k_i = \sum_{j=1}^n a^j_i Z_j f^{n+1}, \quad (a^j_i) = (Z_i f^j)^{-1}$$

and

$$Z^{\tilde\alpha_{j,l}}k_i = \begin{cases} Z^{\alpha_j - e_l}k_i & \text{if } I_l(\alpha_j)\neq 0, \\ 0 & \text{if } I_l(\alpha_j) = 0. \end{cases}$$

Then F *satisfies a complete system of order* $2\sigma+4$*. Thus* F *is determined by* $(2\sigma+3)$*-jet at a point and* F *is* C^ω *provided that* $F\in C^{2\sigma+4}$.

COROLLARY 0.4. *Let* M *be a* C^ω *real hypersurface in* $\mathbb{C}^{n+1}$ *with nondegenerate Levi form. Then every CR diffeomorphism satisfying the conditions of Theorem* 0.3 *is real analytic and hence extends holomorphically to an open neighborhood of* M.

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§1. Pseudo-hermitian structure and pseudo-hermitian embedding

Let $(M,\mathcal{V},\theta)$ be a pseudo-hermitian manifold with nondegenerate Levi form. In this section we denote $\mathcal{V}$ by $H^{1,0}$ and $\overline{\mathcal{V}}$ by $H^{0,1}$. As in [W], we can choose a coframe $\{\theta^i,\theta^{\bar i}\}$ of $H^{1,0}\oplus H^{0,1}$ by requiring $d\theta = \sqrt{-1}\sum_{i,j=1}^n g_{i\bar j}\, \theta^i\wedge\theta^{\bar j}$ and define the connection form (w^i_j) as well as the torsion form (τ^i) via the structure equations

$$\begin{aligned} & d\theta^i = \sum_{k=1}^n \theta^k\wedge w^i_k + \theta\wedge\tau^i, \\ (1.1)\qquad & \tau^i \equiv 0 \mod \theta^{\bar k}, \\ & dg_{i\bar j} - \sum_{k=1}^n w^k_i g_{k\bar j} - \sum_{k=1}^n g_{i\bar k} w^{\bar k}_{\bar j} = 0. \end{aligned}$$

The collection of one forms $\{\theta, \theta^i, \theta^{\bar{i}}, w_j^i, w_{\bar{j}}^{\bar{i}}\}$ forms an intrinsic basis of a given pseudo-hermitian structure.

Let $\{Z_i\}_{i=1,\ldots,n}$ be the dual frame of $\{\theta^i\}_{i=1,\ldots,n}$ for $H^{1,0}$ and T be the unique real vector field such that $\theta(\mathrm{T}) = 1$, $\mathrm{T} \rfloor d\theta = 0$. Then (1.1) implies

$$\begin{aligned}
[\overline{Z}_j, Z_i] &= \sqrt{-1}\, g_{i\bar{j}} \mathrm{T} + \sum_{k=1}^{n} w_i^k(\overline{Z}_j) Z_k - \sum_{k=1}^{n} w_{\bar{j}}^{\bar{k}}(Z_i) \overline{Z}_k, \\
[Z_j, Z_i] &= \sum_{k=1}^{n} w_i^k(Z_j) Z_k - \sum_{k=1}^{n} w_j^k(Z_i) Z_k, \\
[Z_i, \mathrm{T}] &= \sum_{k=1}^{n} \tau^{\bar{k}}(Z_i) \overline{Z}_k - \sum_{k=1}^{n} w_i^k(\mathrm{T}) Z_k.
\end{aligned} \tag{1.2}$$

If M is a germ of C^ω CR manifold, then we may regard M as a C^ω real hypersurface in $\mathbb{C}^{n+1}$. Now we introduce a special coordinate system on M which is called Moser's normal coordinates. Let $z = (z', w) \in \mathbb{C}^{n+1}$, $w = u + iv$.

Definition 1.1. M is said to be in Moser's normal form if M is defined by $\rho(z, \bar{z}) = 2u - \langle z', z' \rangle - F_A(z', \bar{z}', v)$, where

$$F_A(z', \bar{z}', v) = \sum_{\substack{|\alpha|, |\beta| \geq 2 \\ l \geq 0}} A^l_{\alpha\beta} z'^{\alpha} \bar{z}'^{\beta} v^l$$

with the trace condition

$$\mathrm{tr}\, A^l_{2\bar{2}} = \mathrm{tr}^2 A^l_{2\bar{3}} = \mathrm{tr}^3 A^l_{3\bar{3}} = 0$$

for all $l \geq 0$.

We have.

Theorem 1.2. ([CM], [M]) *For any C^ω CR hypersurface M with non-degenerate Levi form, there exists a holomorphic change of coordinates $\zeta = \Phi(z, w)$ such that $\Phi(M)$ is in Moser's normal form.*

Thus we may regard $M = \{\rho = 0\}$ is in Moser's normal form and $\theta = \mu\sqrt{-1}\, \partial\rho$ for some nonvanishing C^ω function μ. Let

$$Z_j = \frac{\partial}{\partial z_j} - \frac{\rho_j}{\rho_w} \frac{\partial}{\partial w}, \quad j = 1, \ldots, n$$

and

$$\mathrm{T} = -\sqrt{-1}\sum_{j=1}^{n}\eta^j\frac{\partial}{\partial z^j} + \sqrt{-1}\sum_{j=1}^{n}\overline{\eta}^j\frac{\partial}{\partial \overline{z}^j}$$
$$-\sqrt{-1}\,\frac{1}{\rho_w}\Big(1-\sum_{j=1}^{n}\rho_j\eta^j\Big)\frac{\partial}{\partial w} + \sqrt{-1}\,\frac{1}{\rho_{\overline{w}}}\Big(1-\sum_{j=1}^{n}\rho_{\overline{j}}\overline{\eta}^j\Big)\frac{\partial}{\partial \overline{w}},$$

where

$$\rho_j = \rho_{z_j},$$
$$g_{j\overline{k}} = -\rho_{j\overline{k}} + \frac{\rho_{j\overline{w}}}{\rho_{\overline{w}}}\rho_{\overline{k}} + \frac{\rho_{w\overline{k}}}{\rho_w}\rho_j - \frac{\rho_{w\overline{w}}}{\rho_w\rho_{\overline{w}}}\rho_j\rho_{\overline{k}},$$
$$\eta_j = \frac{\rho_{j\overline{w}}}{\rho_{\overline{w}}} - \frac{\rho_{w\overline{w}}}{\rho_w\rho_{\overline{w}}}\rho_j$$

and

$$\eta^k = \sum_{j=1}^{n} g^{k\overline{j}}\overline{\eta}_j\,, \quad (g^{k\overline{j}}) = (g_{i\overline{j}})^{-1}.$$

Then T is the unique real vector field such that $\sqrt{-1}\,\partial\rho(\mathrm{T}) = 1$ and $\mathrm{T}\rfloor\sqrt{-1}\,\overline{\partial}\partial\rho = 0$. By (1.2), we have $\overline{Z}^{\alpha}(g_{i\overline{j}})(0) = 0$ for all $1 \le |\alpha|$ and $\overline{Z}^{\beta}\big(\omega_j^i(\overline{Z}_k)\big)(0) = \overline{Z}^{\beta}\big(\tau^i(\overline{Z}_j)\big)(0) = 0$ for all $0 \le |\beta|$.

Now let N be a real hypersurface in $\mathbb{C}^{n+1}$. Suppose $N = \{r = 0\}$ for some smooth real valued function r such that $dr \neq 0$ on N, $\sqrt{-1}\,\partial\overline{\partial}r$ is nondegenerate. Then N inherits a nondegenerate CR structure from $\mathbb{C}^{n+1}$ by choosing $H^{1,0} = \mathbb{C}\,T(N) \cap T^{1,0}(\mathbb{C}^{n+1})$.

Definition 1.3. Let $(M, \mathcal{V}, \theta)$ be a CR manifold with a specified contact form θ with nondegenerate Levi form. Then a CR embedding $F : M \to \mathbb{C}^{n+1}$ is called a pseudo-hermitian embedding if $F^*(\sqrt{-1}\,\partial r) = \theta$, where $N = F(M) = \{r = 0\}$ and $\|\nabla r\| \equiv 1$.

§2. E. Cartan's equivalence problem and the complete systems

In this section, we explain E. Cartan's equivalence problem and the concept of complete system. We refer to [HY] and [H] as references.

Let M be a C^∞ manifold of dimension n and G be a linear subgroup of $GL(n, \mathbb{R})$. A G-structure on M is the reduction of coframe bundle of M to a subbundle with the structure group G.

Now let M and $\widetilde{M}$ be manifolds of dimension n with G-structures and fix $\theta = (\theta^1, \dots, \theta^n)^t$, $\widetilde{\theta} = (\widetilde{\theta}^1, \dots, \widetilde{\theta}^n)^t$, sections of the G-structure bundles of M and $\widetilde{M}$ respectively. Then E. Cartan's equivalence problem is to find necessary and sufficient conditions that there exists a diffeomorphism $f : M \to \widetilde{M}$ such that $f^*(\widetilde{\theta}) = g_0\theta$ where g_0 is a G-valued function defined on M.

Locally, the G-structure bundles are equivalent to the product space $U \times G$ and $V \times G$, where U and V are open subsets of M and $\widetilde{M}$ respectively. Define the left G action on $U \times G$ by $h(x, g) = (x, hg)$ for all $x \in U$ and g, $h \in G$ and consider a tautological 1-form $\Theta = g\theta$ on $U \times G$. Then the equivalence problem is lifted to G-structure bundles as follows.

Proposition 2.1. *There exists a diffeomorphism $f : U \to V$ satisfying $f^*(\widetilde{\theta}) = g_0\theta$ with $g_0 : U \to G$ if and only if there exists a diffeomorphism $F : U \times G \to V \times G$ satisfying*

i) $F^*(\widetilde{\Theta}) = \Theta$

ii) *the following diagram commutes*:

$$\begin{array}{ccc} U \times G & \xrightarrow{F} & V \times G \\ {\scriptstyle \pi_U}\downarrow & & {\scriptstyle \pi_V}\downarrow \\ U & \xrightarrow{f} & V \end{array}$$

iii) $F(x, gh) = gF(x, h)$ *for all $x \in U$ and g, $h \in G$.*

Proof. Suppose f satisfies $f^*(\widetilde{\theta}) = g_0\theta$, where g_0 is a G-valued function defined on U. Define $F : U \times G \to V \times G$ by $F(x, g) = (f(x), gg_0^{-1}(x))$. Then F satisfies ii) and iii). Moreover,

$$F^*(\widetilde{\Theta}) = F^*(\widetilde{g}\,\widetilde{\theta}) = gg_0^{-1} f^*(\widetilde{\theta}) = gg_0^{-1} g_0\theta = g\theta = \Theta.$$

Conversely, suppose that $F : U \times G \to V \times G$ satisfies i)−iii). Define $f : U \to V$ and $g_0 : U \to G$ by $F(x, e) = (f(x), g_0^{-1})$ where e is the identity of G. Then $F(x, g) = gF(x, e) = (f(x), gg_0^{-1})$ and i) implies that

$$g\theta = F^*(\widetilde{\theta}) = (gg_0^{-1}) f^*(\widetilde{\theta}),$$

therefore $f^*(\widetilde{\theta}) = g_0\theta$. ∎

Now apply d to $\Theta = g\theta$. Then we get

$$d\Theta = dg \wedge \theta + g\, d\theta.$$

Substituting $\theta = g^{-1}\Theta$ to the above equation, we obtain

$$d\Theta = dgg^{-1} \wedge +g\, d\theta.$$

We only consider the case that there exists unique 1-forms ω^i_j, $i, j = 1, \ldots, n$, such that

$$d\theta^i = -\sum_{j=1}^{n} \omega^i_j \wedge \theta^j$$

and

$$[\omega^i_j(x)] \in \mathcal{G}$$

for all $x \in U$, where $\mathcal{G}$ is the Lie algebra of G. This Lie algebra valued 1-form $\omega = [\omega^i_j]$ is called a torsion-free connection. Then we get

$$d\Theta = dgg^{-1} \wedge \Theta - g\omega \wedge g^{-1}\Theta = (dgg^{-1} - g\omega g^{-1}) \wedge \Theta.$$

Let

$$\Omega = -(dgg^{-1} - g\omega g^{-1}),$$

then Ω is a $\mathcal{G}$-valued 1-form on $U \times G$ and we have

$$d\Theta = -\Omega \wedge \Theta.$$

Then it is easy to show

PROPOSITION 2.2. *Let Θ and Ω be the 1-forms as before. Then Θ^i, Ω^i_j, $i, j = 1, \ldots, n$, span the cotangent space at each point $U \times G$. Furthermore, if $\widetilde{\Theta}^i, \widetilde{\Omega}^i_j$ are the corresponding 1-forms on $V \times G$ and*

$$F : U \times G \longrightarrow V \times G$$

is the mapping in Proposition 2.1, then

$$F^*(\widetilde{\Omega}^i_j) = \Omega^i_j.$$

The set $\{\Theta^i, \Omega^i_j\}$ is called a complete set of invariants for the equivalence problem. Let f be the solution of equivalence problem. Then the lift of f satisfies the equation

$$\begin{aligned} F^*(\widetilde{\Theta}^i) &= \Theta^i, \\ F^*(\widetilde{\Omega}^i_j) &= \Omega^i_j, \quad i,j = 1,\dots,n. \end{aligned} \tag{2.1}$$

Since $\{\Theta^i, \Omega^i_j\}$ span the cotangent space of $U \times G$, (2.1) determine all the first derivatives of F, hence all the second derivatives of f. In fact, f satisfies

$$\frac{\partial^2 f^a}{\partial x^i \partial x^j} = h^a_{ij}\left(x, f, \frac{\partial f^b}{\partial x^k} : b,k = 1,\dots,n\right), \tag{2.2}$$

where h^a_{ij} is a C^∞ function in its arguments.

The concept of complete system is the generalization of the equation (2.2). We explain it in jet theoretical manner. We use the notation in [O].

Let $J^q(M, \mathbb{R}^N)$ be the q-th order jet space of $M \times \mathbb{R}^N$. Consider a system of differential equations of order q for unknown functions $f = (f^1, \dots, f^N)$ of independent variables $x = (x^1, \dots, x^n)$

$$\Delta_\lambda(x, f^{(q)}) = 0, \quad \lambda = 1,\dots,l. \tag{2.3}$$

Then complete system of order k is defined as follows.

Definition 2.3. A C^k ($k \geq q$) solution of (2.3) satisfies a complete system of order k if there exist C^∞ functions $H^a_J(x, f^{(p)} : p < k)$ in their arguments such that

$$f^a_J = H^a_J(x, f^{(p)} : p < k)$$

for all $a = 1,\dots,N$ and for all multi-indices J with $|J| = k$.

Let $\phi^a_I = df^a_I - \sum_{j=1}^n f^a_{I,j}\, dx^j$, $a = 1,\dots,N$, $|I| \leq k-2$ be the contact 1-forms defined on $J^{k-1}(M, \mathbb{R}^N)$ and $\mathcal{S}_\Delta \subseteqq J^{k-1}(M, \mathbb{R}^N)$ be the prolongation of the set $\{\Delta_\lambda = 0\} \subseteqq J^q(M, \mathbb{R}^N)$. Assume $dx^1 \wedge \cdots \wedge dx^n \neq 0$ on $\mathcal{S}_\Delta$. Then, if a solution f of (2.3) satisfies a complete system of order k, f is an integral manifold of the distribution

$$\phi^a_I = 0, \quad a = 1,\dots,N,\ |I| \leq k-2$$

and

$$df_I^a - \sum_{j=1}^{n} H_{I,j}^a \, dx^j = 0, \quad |I| = k-1,$$

where $H_{I,j}^a = D_j H_I^a$.

In particular, we have

PROPOSITION 2.4. *Let $f \in C^k$ be a solution of* (2.3)*. Suppose f satisfies a complete system of order k, then f is determined by $(k-1)$-jet at a point and f is C^∞. Furthermore, if* (2.3) *is real analytic and each H_J^a is real analytic then f is real analytic.*

§3. Proof of Theorem 0.3

Let $(M, \mathcal{V}, \theta)$ and $\{Z_1, \ldots, Z_n, \mathrm{T}\}$ be as in Section 1 and let $F : M \to \mathbb{C}^{n+1}$ be a CR diffeomorphism which satisfies the condition of Theorem 0.3. Then by the hypotheses on the normalization we have for all $i, j = 1, \ldots, n$,

$$\begin{aligned} &Z_i f^j(0) = \delta_i^j, \\ &\mathrm{T} f^j(0) = 0, \\ &Z_i f^{n+1}(0) = 0 \end{aligned}$$

and

$$\mathrm{T} f^{n+1}(0) = \sqrt{-1}.$$

Now let $N = F(M) = \{r = 0\}$, where $\|\nabla r\| \equiv 1$ and $F(0) = 0$. Then $F^*(\sqrt{-1}\,\partial r) = \lambda\theta = \lambda\mu\sqrt{-1}\,\partial\rho$ implies

$$\sqrt{-1}\left(\sum_{l=1}^{n+1} r_l \mathrm{T} f^l\right) = \lambda\mu = \widetilde{\lambda}, \tag{3.1}$$

where $r_l = \partial r / \partial \zeta_l$, $l = 1, \ldots, n+1$ and $\widetilde{\lambda}(0) = 1$. To differentiate (3.1), we have to express the derivatives of r in terms of the derivatives of F. By applying Z_j, $\overline{Z}_j$ and T to $r \circ F = 0$, we have

$$\begin{aligned} &\sum_{l=1}^{n+1} r_l Z_j f^l = 0, \\ &\sum_{l=1}^{n+1} r_{\bar{l}} \overline{Z}_j \overline{f}^{\,l} = 0, \\ &\sum_{l=1}^{n+1} r_l \mathrm{T} f^l + \sum_{l=1}^{n+1} r_{\bar{l}} \mathrm{T} \overline{f}^{\,l} = 0. \end{aligned} \tag{3.2}$$

Furthermore, on N

$$\|\nabla r\|^2 = \sum_{l=1}^{n+1} r_l r_{\bar{l}} \equiv 1. \tag{3.3}$$

We solve (3.2) and (3.3) for r_l, $l = 1, \ldots, n+1$, and their conjugates in terms of the derivatives of F and $\overline{F}$. Substituting for r_l, $l = 1, \ldots, n+1$, in (3.1) we get

$$\begin{aligned} h := \left(\sum_{j=1}^{n} k_j \mathrm{T} f^j + \mathrm{T} f^{n+1} \right) \left(\sum_{j=1}^{n} k_{\bar{j}} \mathrm{T} \overline{f}^j + \mathrm{T} \overline{f}^{n+1} \right) & \\ - \widetilde{\lambda}^2 \left(\sum_{j=1}^{n} k_j k_{\bar{j}} + 1 \right) = 0, & \end{aligned} \tag{3.4}$$

where $k_j = -\sum_{i=1}^{n} a^i_j Z_i f^{n+1}$, $(a^i_j) = (Z_j f^k)^{-1}_{j,k=1,\ldots,n}$ and $k_{\bar{j}} = \overline{k}_j$.

Now we apply $\overline{Z}^{\alpha}$, $|\alpha| \le \sigma + 1$, to (3.4) and reduce the order of derivatives of F by using

$$\begin{aligned} \overline{Z}_k Z_j F &= [\overline{Z}_k, Z_j] F + Z_j \overline{Z}_k F \\ &= \sqrt{-1}\, g_{j\bar{k}} \mathrm{T} F + \sum_{i=1}^{n} \omega^i_j (\overline{Z}_k) Z_i F, \\ \overline{Z}_k \mathrm{T} F &= [\overline{Z}_k, \mathrm{T}] F + \mathrm{T} \overline{Z}_k F \\ &= \sum_{i=1}^{n} \tau^i (\overline{Z}_k) Z_i F. \end{aligned} \tag{3.5}$$

We regard $\overline{Z}^{\alpha} h$ as a function on the jet space $\{(x, F, \overline{F}, ZF, \mathrm{T}F, \overline{Z}^{\gamma} (\overline{ZF}, \mathrm{T}\overline{F}) : x \in M, |\gamma| \le \sigma + 1\}$ of order $\sigma + 2$.

Lemma 3.1. *There exist smooth functions P_{il}, Q_l, $i = 1, \ldots, n$ and $l = 1, \ldots, n+1$ such that*

$$\begin{aligned} Z_i f^l &= P_{il} \big(\overline{Z}^{\alpha} (\overline{ZF}, \mathrm{T}\overline{F}), |\alpha| \le \sigma + 1 \big), \\ \mathrm{T} f^l &= Q_l \big(\overline{Z}^{\alpha} (\overline{ZF}, \mathrm{T}\overline{F}), |\alpha| \le \sigma + 1 \big). \end{aligned} \tag{3.6}$$

Proof. Let $\mathrm{A} = \sum_{j=1}^{n} k_j \, \mathrm{T} f^j + \mathrm{T} f^{n+1}$ and $\mathrm{B} = \sum_{j=1}^{n} k_j k_{\bar{j}} + 1$. Then

$$\frac{\partial(h)}{\partial(Z_i f^l)}(0) = 0, \quad i = 1, \ldots, n,\ l = 1, \ldots, n+1$$

and

$$\frac{\partial(h)}{\partial(\mathrm{T}f^l)}(0) = \frac{\partial(A)}{\partial(\mathrm{T}f^l)}\overline{A}(0) \neq 0 \quad \text{if and only if } l = n+1.$$

Let $\langle z', z'\rangle = \sum_{j=1}^n \lambda_j z_j \overline{z}_j$, where $\lambda_j = \pm 1$. By the condition that $F(M)$ is in pre-normal form, we can show that $\big(\partial(\overline{Z}_j\,\mathrm{B})/\partial(Z_i f^l)\big)(0) = 0$ for all $i, j = 1, \dots, n$, and $l = 1, \dots, n+1$. Hence

$$\frac{\partial(\overline{Z}_j h)}{\partial(Z_i f^l)}(0) = -\frac{\partial(\overline{Z}_j\,\mathrm{B})}{\partial(Z_i f^l)}(0) = 0$$

and

$$\begin{aligned}\frac{\partial(\overline{Z}_j h)}{\partial(\mathrm{T}f^i)}(0) &= \frac{\partial(\overline{Z}_j\,\mathrm{A})}{\partial(\mathrm{T}f^i)}(0)\overline{\mathrm{A}}(0)\\ &= \overline{Z}_j k_i(0)\overline{A}(0)\\ &= i\lambda_j \delta_i^j \mathrm{T}f^{n+1}(0)\mathrm{T}\overline{f}^{\,n+1}(0)\end{aligned}$$

for all $i, j = 1, \dots, n$ and $l = 1, \dots, n+1$.

Let $\mathcal{O}$ be the set of analytic functions $\mathcal{G}(x, F, \overline{F}, ZF, \mathrm{T}F, \overline{Z}^{\gamma}(\overline{ZF}, \mathrm{T}\overline{F}) : |\gamma| \leq N < \infty)$ in their arguments such that for any multi-index $0 \leq |\beta|$, $\big(\partial(\overline{Z}^{\beta}\mathcal{G})/\partial(Z_i f^l)\big)(0) = 0$ for all $i = 1, \dots, n$ and $l = 1, \dots, n+1$. Then by assumption on $\{Z_1, \dots, Z_n, \mathrm{T}\}$, we can show that $\mathrm{A}, \overline{Z}^{\alpha} k_j \in \mathcal{O}$ for all $2 \leq |\alpha|$ and $j = 1, \dots, n$.

Now choose $\{\alpha_1, \dots, \alpha_n\}$ which satisfy the condition of Theorem 0.3. Let $\widetilde{h} := \widetilde{\lambda}^{-2} h = \widetilde{\lambda}^{-2}\,\mathrm{A}\,\overline{\mathrm{A}} - \mathrm{B}$. Then

$$\begin{aligned}\overline{Z}^{\alpha_j}\widetilde{h} &= -\overline{Z}^{\alpha_j}\,\mathrm{B} + \mathcal{O}\\ &= -\sum_{s=1}^n k_s \overline{Z}^{\alpha_j} k_{\overline{s}} - \sum_{s=1}^n \sum_{\substack{\beta+\gamma=\alpha_j\\ |\beta|=1}} \overline{Z}^{\beta} k_s \overline{Z}^{\gamma} k_{\overline{s}} + \mathcal{O}\\ &= -\sum_{s=1}^n k_s \overline{Z}^{\alpha_j} k_{\overline{s}} - \sum_{s=1}^n \sum_{t=1}^n \sqrt{-1}\,\lambda_t I_t(\alpha_j) a_s^t \mathrm{T}f^{n+1} \overline{Z}^{\tilde{\alpha}_{j,t}} k_{\overline{s}} + \mathcal{O}\end{aligned}$$

and

$$\begin{aligned}\overline{Z}_i\overline{Z}^{\alpha_j}\widetilde{h} = &-\sum_{s=1}^n k_s \overline{Z}_i \overline{Z}^{\alpha_j} k_{\overline{s}} - \sum_{s=1}^n \overline{Z}_i k_s \overline{Z}^{\alpha_j} k_{\overline{s}}\\ &-\sum_{s=1}^n \sum_{\substack{\beta+\gamma=\alpha_j\\ |\beta|=1}} \overline{Z}^{\beta} k_s \overline{Z}_i \overline{Z}^{\gamma} k_{\overline{s}} + \mathcal{O}\end{aligned}$$

$$= -\sum_{s=1}^{n} k_s \overline{Z}_i \overline{Z}^{\alpha_j} k_{\overline{s}}$$
$$-\sum_{s=1}^{n}\sum_{t=1}^{n} \sqrt{-1}\,\lambda_t\big(I_t(\alpha_j)+\delta_i^t\big)a_s^t \mathrm{T} f^{n+1}\overline{Z}_i \overline{Z}^{\tilde{\alpha}_{j,t}} k_{\overline{s}} + \mathcal{O},$$

where

$$\overline{Z}^{\tilde{\alpha}_{j,t}} k_{\overline{s}} = \begin{cases} \overline{Z}^{\alpha_j - e_t} k_{\overline{s}} & \text{if } I_t(\alpha_j) \neq 0 \\ 0 & \text{if } I_t(\alpha_j) = 0 \end{cases}.$$

This implies that for each $i, j = 1, \dots, n$,

$$\frac{\partial(\overline{Z}^{\alpha_j}\widetilde{h})}{\partial(Z_s f^{n+1})} = -\overline{Z}^{\alpha_j} k_{\overline{s}} + \{\text{the terms which vanish at } 0\},$$
$$\frac{\partial(\overline{Z}^{\alpha_j}\widetilde{h})}{\partial(a_s^t)} = -\sqrt{-1}\,\lambda_t I_t(\alpha_j)\mathrm{T} f^{n+1}\overline{Z}^{\tilde{\alpha}_{j,t}} k_{\overline{s}}$$
$$+\{\text{the terms which vanish at } 0\},$$
$$\frac{\partial(\overline{Z}_i\overline{Z}^{\alpha_j}\widetilde{h})}{\partial(Z_s f^{n+1})} = -\overline{Z}_i\overline{Z}^{\alpha_j} k_{\overline{s}} + \{\text{the terms which vanish at } 0\}$$

and

$$\frac{\partial(\overline{Z}_i\overline{Z}^{\alpha_j}\widetilde{h})}{\partial(a_s^t)} = -\sqrt{-1}\,\lambda_t I_t(\alpha_j + e_i)\mathrm{T} f^{n+1}\overline{Z}_i\overline{Z}^{\tilde{\alpha}_{j,t}} k_{\overline{s}}$$
$$+\{\text{the terms which vanish at } 0\}$$

for all $s, t = 1, \dots, n$. Thus, after changing of rows and columns and multiplying nonzero constants, we get

$$\text{(3.7)}\quad -d_{(a_s^t, Z_s f^{n+1}, \mathrm{T} f^t, \mathrm{T} f^{n+1})_{(s,t=1,\dots,n)}}(h, \overline{Z}_j h, \overline{Z}^{\alpha_j}\widetilde{h}, \overline{Z}_i\overline{Z}^{\alpha_j}\widetilde{h} : i, j = 1, \dots, n)$$
$$= \begin{pmatrix} 0\cdots 0 & A_0 \\ A_j^i & * \end{pmatrix}_{i,j=1,\dots,n},$$

where

$$A_0 := \begin{pmatrix} 0\cdots 0 & 1 \\ \mathrm{Id}_n & * \end{pmatrix}$$

and

$$A_j^i := \begin{pmatrix} \overline{Z}^{\alpha_j} k_{\overline{i}}, & I_1(\alpha_j)\overline{Z}^{\tilde{\alpha}_{j,1}} k_{\overline{i}}, & \cdots & I_n(\alpha_j)\overline{Z}^{\tilde{\alpha}_{j,n}} k_{\overline{i}} \\ \overline{Z}_1\overline{Z}^{\alpha_j} k_{\overline{i}}, & I_1(\alpha_j+e_1)\overline{Z}_1\overline{Z}^{\tilde{\alpha}_{j,1}} k_{\overline{i}}, & \cdots & I_n(\alpha_j+e_1)\overline{Z}_1\overline{Z}^{\tilde{\alpha}_{j,n}} k_{\overline{i}} \\ \vdots & \vdots & & \vdots \\ \overline{Z}_n\overline{Z}^{\alpha_j} k_{\overline{i}}, & I_1(\alpha_j+e_n)\overline{Z}_n\overline{Z}^{\tilde{\alpha}_{j,1}} k_{\overline{i}}, & \cdots & I_n(\alpha_j+e_n)\overline{Z}_n\overline{Z}^{\tilde{\alpha}_{j,n}} k_{\overline{i}} \end{pmatrix}.$$

Let $H := (h, \overline{Z}_j h, \overline{Z}^{\alpha_j}\widetilde{h}, \overline{Z}_i\overline{Z}^{\alpha_j}\widetilde{h}; i,j = 1,\dots,n)$. Then $H : J^{\sigma+2}(M, \mathbb{C}^{n+1}) \to \mathbb{C}^m$ for sufficiently m satisfies

$$H(x, F, \overline{F}, ZF, \mathrm{T}F, \overline{Z}^{\alpha}(\overline{ZF}, \mathrm{T}\overline{F}) : |\alpha| \le \sigma + 1) = 0. \tag{3.8}$$

Then by the implicit function theorem and (3.7), we can solve (3.8) for $Z_i f^l$ and $\mathrm{T}f^l$ in terms of $\overline{Z}^{\alpha}(\overline{ZF}, \mathrm{T}\overline{F})$, $|\alpha| \le \sigma+1$, for all $i = 1,\dots,n$ and $l = 1,\dots,n+1$. ∎

Next we show that equations (3.6) admit a prolongation to a complete system of order $2\sigma + 4$ using the same method as in [H] and [Ha1].

Let $\beta = (\beta_1,\dots,\beta_n)$ be any multi-index. Apply Z^β to (3.6). Then we have

$$\begin{aligned} Z^\beta Z_i f^l &= Z^\beta P_{il}\big(\overline{Z}^{\alpha}(\overline{ZF}, \mathrm{T}\overline{F}) : |\alpha| \le \sigma+1\big),\\ Z^\beta \mathrm{T} f^l &= Z^\beta Q_l\big(\overline{Z}^{\alpha}(\overline{ZF}, \mathrm{T}\overline{F}) : |\alpha| \le \sigma+1\big). \end{aligned} \tag{3.9}$$

By (3.5), the order of derivatives of $\overline{F}$ reduces to $\sigma + 2$.

Now let C_p be the set of C^ω functions in arguments

$$\mathrm{T}^t Z^\alpha f^l : t + |\alpha| \le p$$

and $C_{p,q}$ be the subset of C_p of C^ω functions in arguments

$$\mathrm{T}^t Z^\alpha f^j : t + |\alpha| \le p,\ t \le q$$

and $\overline{C}_p$, $\overline{C}_{p,q}$ be the complex conjugates of C_p and $C_{p,q}$, respectively. Then by (3.9) we have

$$Z^\beta Z_i f^l,\ Z^\beta \mathrm{T} f^l \in \overline{C}_{\sigma+2}. \tag{3.10}$$

Apply $\overline{Z}_k$ to (3.10) to have

$$\overline{Z}_k Z^\beta \mathrm{T} f^l \in \overline{C}_{\sigma+3,\sigma+2}.$$

This gives

$$Z^{\beta'}\mathrm{T}^2 f^l \in \overline{C}_{\sigma+3,\sigma+2}, \quad |\beta'| = |\beta| - 1.$$

By applying $\overline{Z}$ repeatedly, we have

$$Z^\beta \mathrm{T}^q f^l \in \overline{C}_{\sigma+q+1,\sigma+2}$$

for all multi-indices β and $q \ge 1$, which shows that

$$C_{p,q} \subset \overline{C}_{\sigma+q+1,\sigma+2} \tag{3.11}$$

for all pair (p,q) with $p \geq q$.

Taking the complex conjugate of (3.11), we have

$$\overline{C}_{p,q} \subset C_{\sigma+q+1,\sigma+2}.$$

In particular, if $q = \sigma + 2$,

$$\overline{C}_{p,\sigma+2} \subset C_{2\sigma+3,\sigma+2}. \tag{3.12}$$

Substitute (3.12) in (3.11), to get

$$C_{p,q} \subset C_{2\sigma+3,\sigma+2}$$

for any pair (p,q) with $p \geq q$. This gives

$$C_{2\sigma+4} \subset C_{2\sigma+3},$$

which completes the proof of Theorem 0.3.

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