

$\partial\bar{\partial}$ -PROBLEM ON WEAKLY 1-COMPLETE KÄHLER MANIFOLDS

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To the memory of Prof. Makoto Suzuki

Abstract. We consider a problem whether Kodaira's $\partial\bar{\partial}$ -Lemma holds on weakly 1-complete Kähler manifolds or not. This problem was proposed by S. Nakano. We prove that the Lemma holds for some class of complex quasi-tori \mathbb{C}^n/Γ , and it does not hold for the other class of them. Every complex quasi-tori is weakly 1-complete and complete Kähler. Then we get a negative answer for the above problem.

§1. Introduction

The following lemma proved by Kodaira is well-known and usually called “ $\partial\bar{\partial}$ -Lemma” ([9, Proposition 7.1]).

$\partial\bar{\partial}$ -LEMMA. *Let X be a compact Kähler manifold and φ a d -exact $(1, 1)$ -form on X . Then there exists a C^∞ -function Ψ on X such that*

$$\varphi = \partial\bar{\partial}\Psi$$

on X .

In [14] many problems concerning function theory of several complex variables are posed. There S. Nakano gives a problem concerning the above $\partial\bar{\partial}$ -Lemma as follows.

A complex manifold X is called *weakly 1-complete* if there exists a C^∞ -plurisubharmonic exhaustive function on X . Easily we can see that a compact complex manifold is weakly 1-complete, a strongly 1-convex manifold is weakly 1-complete and then every Stein manifold is weakly 1-complete.

PROBLEM 1.1. Can one show $\partial\bar{\partial}$ -Lemma on weakly 1-complete Kähler manifolds?

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We can give a very easy counterexample to this problem (Example 4.1); nevertheless, it is very interesting to consider this from the other aspects. We give reformed problems of it in §4.

A connected complex Lie group without global non-constant holomorphic function is called a *toroidal* group. Every complex n -dimensional toroidal group is isomorphic to \mathbb{C}^n/Γ for some discrete subgroup Γ ([8]). A complex torus is an example of a toroidal group.

It is shown that every toroidal group is always weakly 1-complete ([4], [11]). From the natural covering structure

$$\mathbb{C}^n \longrightarrow \mathbb{C}^n/\Gamma$$

it follows that every toroidal group \mathbb{C}^n/Γ is a complete Kähler manifold.

In this paper we will consider whether $\partial\bar{\partial}$ -Lemma holds on toroidal groups or not.

Every toroidal group \mathbb{C}^n/Γ satisfies either of the following statements (1) and (2) ([5], [12]):

- (1) $H^p(\mathbb{C}^n/\Gamma, \mathcal{O})$ is finite-dimensional for any p ;
- (2) $H^p(\mathbb{C}^n/\Gamma, \mathcal{O})$ is a non-Hausdorff and then infinite-dimensional locally convex space for any p with $1 \leq p \leq q$,

where \mathcal{O} denotes the structure sheaf of \mathbb{C}^n/Γ and $q := \text{rank } \Gamma - n$. From this result we can classify all toroidal groups. We say that a toroidal group is *of cohomologically finite type* if it satisfies the above property (1) and *non-Hausdorff type* if it satisfies the above property (2), respectively.

We will show that $\partial\bar{\partial}$ -Lemma holds for toroidal groups of cohomologically finite type and that it does not hold for toroidal groups of non-Hausdorff type.

This gives the negative answer for the above problem even if we consider it only for toroidal groups.

We wish to thank Prof. Koji Cho who gave a suggestion for us to generalize our former statements of Theorem 3.3.

§2. Toroidal groups

Throughout this section we consider a toroidal group \mathbb{C}^n/Γ , where Γ is a discrete subgroup of \mathbb{C}^n and of rank $n+q$ generated by \mathbb{R} -linearly independent vectors $\{e_1, e_2, \dots, e_n, v_1 = (v_{11}, \dots, v_{1n}), v_2 = (v_{21}, \dots, v_{2n}), \dots, v_q = (v_{q1}, \dots, v_{qn})\}$ over \mathbb{Z} and e_i denotes the i -th unit vector of \mathbb{C}^n . We take

$\operatorname{Re} v_i, \operatorname{Im} v_i \in \mathbb{R}^n$ with $v_i = \operatorname{Re} v_i + \sqrt{-1} \operatorname{Im} v_i$. Since $e_1, e_2, \dots, e_n, v_1, v_2, \dots, v_q$ are \mathbb{R} -linearly independent, $\operatorname{Im} v_1, \operatorname{Im} v_2, \dots, \operatorname{Im} v_q$ are \mathbb{R} -linearly independent. Then without loss of generality we may assume $\det [\operatorname{Im} v_{ij} ; 1 \leq i, j \leq q] \neq 0$ from now on. We set

$$(2.1) \quad K_{m,i} := \sum_{j=1}^n v_{ij} m_j - m_{n+i} \quad \text{and} \quad K_m := \max\{|K_{m,i}| ; 1 \leq i \leq q\}$$

for $m = (m_1, m_2, \dots, m_{n+q}) \in \mathbb{Z}^{n+q}$. From the result of [8] it follows that \mathbb{C}^n/Γ is toroidal if and only if

$$(2.2) \quad K_m > 0 \quad \text{for any } m \in \mathbb{Z}^{n+q} \setminus \{0\}.$$

We denote by π_q the projection $\mathbb{C}^n \ni (z_1, \dots, z_n) \mapsto (z_1, \dots, z_q) \in \mathbb{C}^q$. Since $\pi_q(e_i), \pi_q(v_i)$ ($1 \leq i \leq q$) are \mathbb{R} -linearly independent, π_q induces the \mathbb{C}^{*n-q} -principal bundle

$$(2.3) \quad \pi_q : \mathbb{C}^n/\Gamma \ni z + \Gamma \mapsto \pi_q(z) + \Gamma^* \in \mathbb{T}_C^q := \mathbb{C}^q/\Gamma^*$$

over the complex q -dimensional torus \mathbb{T}_C^q , where $\Gamma^* := \pi_q(\Gamma)$ ([5]). We put

$$\alpha_{ij} := \begin{cases} \operatorname{Re} v_{ij} & (1 \leq i \leq q, 1 \leq j \leq n) \\ 0 & (q+1 \leq i \leq n, 1 \leq j \leq n), \end{cases}$$

$$\beta_{ij} := \begin{cases} \operatorname{Im} v_{ij} & (1 \leq i \leq q, 1 \leq j \leq n) \\ \delta_{ij} & (q+1 \leq i \leq n, 1 \leq j \leq n), \end{cases}$$

$[\gamma_{ij} ; 1 \leq i, j \leq n] := [\beta_{ij} ; 1 \leq i, j \leq n]^{-1}$ and $v_i := \sqrt{-1} e_i$ for $q+1 \leq i \leq n$. Since $\{e_1, \dots, e_n, v_1, \dots, v_n\}$ are \mathbb{R} -linearly independent, we have an isomorphism

$$\phi : \mathbb{C}^n \ni (z_1, \dots, z_n) \mapsto (t_1, \dots, t_{2n}) \in \mathbb{R}^{2n}$$

as a real Lie group, where $(z_1, \dots, z_n) = \sum_{i=1}^n (t_i e_i + t_{n+i} v_i)$. Then we obtain the relations

$$(2.4) \quad t_j = x_j - \sum_{i,k=1}^n y_k \gamma_{ki} \alpha_{ij} \quad \text{and} \quad t_{n+j} = \sum_{i=1}^n y_i \gamma_{ij}$$

for $1 \leq j \leq n$, where $z_i = x_i + \sqrt{-1} y_i$. We put $t = (t', t'')$, $t' = (t_1, \dots, t_{n+q}) \in \mathbb{R}^{n+q}$ and $t'' = (t_{n+q+1}, \dots, t_{2n}) \in \mathbb{R}^{n-q}$. ϕ induces the isomorphism:

$\phi : \mathbb{C}^n/\Gamma \cong \mathbb{T}^{n+q} \times \mathbb{R}^{n-q}$ as a real Lie group, where \mathbb{T}^{n+q} is an $(n+q)$ -dimensional real torus. Sometimes we identify \mathbb{C}^n/Γ with the real Lie group $\mathbb{T}^{n+q} \times \mathbb{R}^{n-q}$ and use the real coordinate system (t_1, \dots, t_{2n}) instead of holomorphic coordinates.

We make the following change of holomorphic coordinates of \mathbb{C}^n :

$$\zeta_i = \sum_{j=1}^n z_j \gamma_{ji}.$$

Then we can regard $(\zeta_1, \dots, \zeta_n)$ as a local holomorphic coordinate system of \mathbb{C}^n/Γ and we have global vector fields and global 1-forms:

$$\begin{aligned} \frac{\partial}{\partial \bar{\zeta}_i} &= \sum_{j=1}^n \beta_{ij} \frac{\partial}{\partial \bar{z}_j}, & \frac{\partial}{\partial \zeta_i} &= \sum_{j=1}^n \beta_{ij} \frac{\partial}{\partial z_j}, \\ d\bar{\zeta}_i &= \sum_{j=1}^n \gamma_{ij} d\bar{z}_j, & d\zeta_i &= \sum_{j=1}^n \gamma_{ij} dz_j \end{aligned}$$

($1 \leq i \leq n$) on \mathbb{C}^n/Γ . It follows from (2.4) that

$$(2.5) \quad \begin{aligned} \frac{\partial}{\partial \bar{\zeta}_i} &= \frac{1}{2} \left(\sum_{j=1}^n \beta_{ij} \frac{\partial}{\partial t_j} - \sqrt{-1} \sum_{j=1}^n \alpha_{ij} \frac{\partial}{\partial t_j} + \sqrt{-1} \frac{\partial}{\partial t_{n+i}} \right), \\ \frac{\partial}{\partial \zeta_i} &= \frac{1}{2} \left(\sum_{j=1}^n \beta_{ij} \frac{\partial}{\partial t_j} + \sqrt{-1} \sum_{j=1}^n \alpha_{ij} \frac{\partial}{\partial t_j} - \sqrt{-1} \frac{\partial}{\partial t_{n+i}} \right). \end{aligned}$$

Then particularly for $q+1 \leq i \leq n$ we have

$$(2.6) \quad \frac{\partial}{\partial \bar{\zeta}_i} = \frac{1}{2} \left(\frac{\partial}{\partial t_i} + \sqrt{-1} \frac{\partial}{\partial t_{n+i}} \right).$$

§3. $\partial\bar{\partial}$ -Lemma

Let \mathcal{A} be the sheaf of germs of real analytic functions on \mathbb{C}^n/Γ and \mathcal{H} its subsheaf of germs of real analytic functions on \mathbb{C}^n/Γ that are holomorphic along each fiber of π_q of (2.3). We may consider $(\zeta_{q+1}, \dots, \zeta_n)$ is a holomorphic coordinate of each fiber of π_q . For $0 \leq p \leq q$ we denote by $\mathcal{H}^{r,p}$ the sheaf of germs of (r,p) -forms as follows

$$\begin{aligned} \varphi = \frac{1}{r!p!} \sum_{1 \leq j_1, \dots, j_r \leq n, 1 \leq i_1, \dots, i_p \leq q} \varphi_{j_1 \dots j_r, i_1 \dots i_p} d\zeta_{j_1} \wedge \dots \wedge d\zeta_{j_r} \\ \wedge d\bar{\zeta}_{i_1} \wedge \dots \wedge d\bar{\zeta}_{i_p}, \end{aligned}$$

where $\varphi_{j_1 \dots j_r, i_1 \dots i_p} \in \mathcal{H}$ is skew-symmetric in all indices. Henceforth all differential forms are denoted skew-symmetrically and we use the notations

$$\begin{aligned} J_r &= (j_1, \dots, j_r), & d\zeta_{J_r} &= d\zeta_{j_1} \wedge \dots \wedge d\zeta_{j_r}, \\ I_p &= (i_1, \dots, i_p), & d\bar{\zeta}_{I_p} &= d\bar{\zeta}_{i_1} \wedge \dots \wedge d\bar{\zeta}_{i_p}. \end{aligned}$$

Then we write $\varphi = 1/(r!p!) \sum_{J_r, I_p} \varphi_{J_r, I_p} d\zeta_{J_r} \wedge d\bar{\zeta}_{I_p}$.

Let Ω^r be the sheaf of germs of holomorphic $(r, 0)$ -forms on \mathbb{C}^n/Γ . We have the following lemma.

LEMMA 3.1. *The sequence*

$$0 \longrightarrow \Omega^r \longrightarrow \mathcal{H}^{r,0} \xrightarrow{\bar{\partial}} \mathcal{H}^{r,1} \xrightarrow{\bar{\partial}} \dots \xrightarrow{\bar{\partial}} \mathcal{H}^{r,q} \longrightarrow 0$$

is exact on \mathbb{C}^n/Γ and one obtain a kind of Dolbeault isomorphism

$$H^p(\mathbb{C}^n/\Gamma, \Omega^r) = \frac{\{\varphi \in H^0(\mathbb{C}^n/\Gamma, \mathcal{H}^{r,p}) \mid \bar{\partial}\varphi = 0\}}{\bar{\partial}H^0(\mathbb{C}^n/\Gamma, \mathcal{H}^{r,p-1})}$$

for $p \geq 1$.

Proof. If $r = 0$, then $\Omega^r = \mathcal{O}$. We obtain the exact sequence:

$$(3.1) \quad 0 \longrightarrow \Omega^0 \longrightarrow \mathcal{H}^{0,0} \longrightarrow \mathcal{H}^{0,1} \longrightarrow \dots \longrightarrow \mathcal{H}^{0,q} \longrightarrow 0$$

by [5, Proposition 3.4]. We can take a basis

$$\{d\zeta_{J_r} \mid 1 \leq j_1 < \dots < j_r \leq n\}$$

of $H^0(\mathbb{C}^n/\Gamma, \Omega^r)$. For every points $[z] \in \mathbb{C}^n/\Gamma$ we have the isomorphisms

$$\begin{aligned} \Omega_{[z]}^r &\cong \bigoplus_{J_r} \Omega_{[z]}^0(d\zeta_{J_r})_{[z]}, \\ \mathcal{H}_{[z]}^{r,p} &\cong \bigoplus_{J_r} \mathcal{H}_{[z]}^{0,p}(d\zeta_{J_r})_{[z]} \end{aligned}$$

of each stalk of sheaves. Observing coefficients of each $(d\zeta_{J_r})_{[z]}$, we can divide the sheaf complex of the statement of the lemma to $\binom{n}{r}$ complexes so that each complex can be identified with (3.1). This argument shows also the latter half of the lemma. \square

Now we recall the argument of §4 of [5]. For $\varphi \in H^0(\mathbb{C}^n/\Gamma, \mathcal{H}^{r,p})$. We can write

$$\varphi = \frac{1}{r!p!} \sum_{J_r, I_p} \varphi_{J_r, I_p} d\zeta_{J_r} \wedge d\bar{\zeta}_{I_p},$$

where $\varphi_{J_r, I_p} \in H^0(\mathbb{C}^n/\Gamma, \mathcal{H}^{0,0})$. The function φ_{J_r, I_p} has the Fourier expansion on \mathbb{C}^n/Γ :

$$\varphi_{J_r, I_p} = \sum_{m \in \mathbb{Z}^{n+q}} C_{J_r, I_p}^m(t'') \exp(2\pi\sqrt{-1} \langle m, t' \rangle),$$

where $C_{J_r, I_p}^m(t'')$'s are C^∞ functions on t'' and $\langle m, t' \rangle := \sum_{i=1}^{n+q} m_i t_i$. Since the function φ_{J_r, I_p} is holomorphic along the fibers of the map of (2.3), then for $q+1 \leq i \leq n$

$$\frac{\partial C_{J_r, I_p}^m(t'')}{\partial \bar{\zeta}_i} = 0.$$

From (2.6) we have the following Fourier series:

$$\varphi_{J_r, I_p} = \sum_{m \in \mathbb{Z}^{n+q}} c_{J_r, I_p}^m \exp\left(-2\pi \sum_{i=q+1}^n m_i t_{n+i}\right) \exp(2\pi\sqrt{-1} \langle m, t' \rangle),$$

where c_{J_r, I_p}^m 's are constants.

We put

$$(3.2) \quad \varphi_{J_r, I_p}^m = c_{J_r, I_p}^m \exp\left(-2\pi \sum_{i=q+1}^n m_i t_{n+i}\right) \exp(2\pi\sqrt{-1} \langle m, t' \rangle)$$

and

$$\varphi^m = \frac{1}{r!p!} \sum_{J_r, I_p} \varphi_{J_r, I_p}^m d\zeta_{J_r} \wedge d\bar{\zeta}_{I_p}.$$

Then $\varphi = \sum_{m \in \mathbb{Z}^{n+q}} \varphi^m$. It follows from (2.1), (2.5) and (3.2) that for $1 \leq \ell \leq q$

$$(3.3) \quad \frac{\partial \varphi_{J_r, I_p}^m}{\partial \bar{\zeta}_\ell} = \pi K_{m, \ell} \varphi_{J_r, I_p}^m, \quad \frac{\partial \varphi_{J_r, I_p}^m}{\partial \zeta_\ell} = \pi \bar{K}_{m, \ell} \varphi_{J_r, I_p}^m.$$

Now we suppose φ is $\bar{\partial}$ -closed, that is, $\bar{\partial}\varphi^m = 0$ for any $m \in \mathbb{Z}^{n+q}$. The compatibility condition for φ to be $\bar{\partial}$ -closed is expressed by the Fourier coefficients such that

$$(3.4) \quad \sum_{\ell=1}^{p+1} (-1)^\ell K_{m, i_\ell} c_{J_r, i_1 \dots \hat{i}_\ell \dots i_{p+1}}^m = 0$$

for any $J_r, I_{p+1} = (i_1, \dots, i_{p+1})$, and $m \in \mathbb{Z}^{n+q}$. For $m \in \mathbb{Z}^{n+q} \setminus \{0\}$ we put $i(m) := \min\{i \mid |K_{m,i}| = K_m, 1 \leq i \leq q\}$. Replacing $I_{p+1} = (i_1, \dots, i_{p+1})$ of (3.4) by $(i(m), i_1, \dots, i_p)$, then we have

$$(3.5) \quad K_{m,i(m)} c_{J_r, i_1 \dots i_p}^m = \sum_{\ell=1}^p (-1)^{\ell+1} K_{m, i_\ell} c_{J_r, i(m) i_1 \dots \hat{i}_\ell \dots i_p}^m = 0.$$

For $m \neq 0$ we have, by (2.2), $K_{m,i(m)} \neq 0$ and then we can put

$$\begin{aligned} \psi^m := & \frac{(-1)^r}{\pi^r!(p-1)!} \sum_{J_r, I_{p-1}} \frac{c_{J_r, i(m) i_1 \dots i_{p-1}}^m}{K_{m,i(m)}} \exp\left(-2\pi \sum_{i=q+1}^n m_i t_{n+i}\right) \\ & \times \exp(2\pi\sqrt{-1} \langle m, t' \rangle) d\zeta_{J_r} \wedge d\bar{\zeta}_{I_{p-1}}, \end{aligned}$$

where $I_{p-1} := (i_1, \dots, i_{p-1})$. Then by (3.3) and (3.5) we obtain

$$\bar{\partial}\psi^m = \varphi^m$$

for $m \neq 0$. This means that any $\bar{\partial}$ -closed form $\varphi = \sum_{m \in \mathbb{Z}^{n+q}} \varphi^m$ has a formal solution $\sum_{m \neq 0} \psi^m$ of the $\bar{\partial}$ -equation:

$$\bar{\partial} \sum_{m \neq 0} \psi^m = \sum_{m \neq 0} \varphi^m.$$

Hence it is determined by the behavior of the lower limit of the sequence of positive numbers:

$$\{K_m \mid m \in \mathbb{Z}^{n+q}\}$$

whether the formal solution is a real solution or not.

The following theorem characterizes toroidal groups of cohomologically finite type.

THEOREM 3.2. ([5], [13]) *Let \mathbb{C}^n/Γ be a toroidal group. Then the following statements (1), (2), (3) and (4) are equivalent.*

- (1) \mathbb{C}^n/Γ is of cohomologically finite type.
- (2) There exists a $a > 0$ such that

$$\sup_{m \neq 0} \exp(-a\|m^*\|)/K_m < \infty,$$

where $\|m^*\| = \max\{|m_i|; 1 \leq i \leq n\}$.

(3)

$$\dim H^p(\mathbb{C}^n/\Gamma, \Omega^r) = \begin{cases} \binom{n}{r} \binom{q}{p} & \text{if } 1 \leq p \leq q \text{ and } 0 \leq r \leq n \\ 0 & \text{if } p > q \text{ or } r > n. \end{cases}$$

(4) *Every C^∞ $\bar{\partial}$ -closed (r, p) -form on \mathbb{C}^n/Γ is $\bar{\partial}$ -cohomologous to a constant form*

$$\frac{1}{r!p!} \sum_{J_r, I_p} c_{J_r, I_p} d\zeta_{J_r} \wedge d\bar{\zeta}_{I_p},$$

where c_{J_r, I_p} 's are constants, $r \geq 0$ and $p \geq 0$.

Let $r, p \geq 1$ and let φ be a d -exact C^∞ (r, p) -form on \mathbb{C}^n/Γ . Then there exists $(r + p - 1)$ -form $\psi = \psi_{(r-1, p)} + \psi_{(r, p-1)}$ such that

$$\varphi = d\psi = \partial\psi_{(r-1, p)} + \bar{\partial}\psi_{(r-1, p)} + \partial\psi_{(r, p-1)} + \bar{\partial}\psi_{(r, p-1)},$$

where $\psi_{(i, j)}$ denotes the component of type (i, j) of ψ . Since φ is (r, p) -form, then $\partial\psi_{(r, p-1)} = 0$ and $\bar{\partial}\psi_{(r-1, p)} = 0$. Then $\bar{\psi}_{(r, p-1)}$ and $\psi_{(r-1, p)}$ are a $\bar{\partial}$ -closed form of type $(p-1, r)$ and a $\bar{\partial}$ -closed form of type $(r-1, p)$ on \mathbb{C}^n/Γ , respectively. Now suppose that \mathbb{C}^n/Γ is of cohomologically finite type. Then by Theorem 3.2, $\bar{\psi}_{(r, p-1)}$ and $\psi_{(r-1, p)}$ are $\bar{\partial}$ -cohomologue to some constant forms, that is, there exist a $(r-1, p-1)$ -form $\Psi^{(1)}$ and a $(p-1, r-1)$ -form $\Psi^{(2)}$ such that

$$\begin{aligned} \psi_{(r-1, p)} &= \frac{1}{(r-1)!p!} \sum_{J_{r-1}, I_p} c_{J_{r-1}, I_p}^{(1)} d\zeta_{J_{r-1}} \wedge d\bar{\zeta}_{I_p} + \bar{\partial}\Psi^{(1)}, \\ \bar{\psi}_{(r, p-1)} &= \frac{1}{r!(p-1)!} \sum_{J_{p-1}, I_r} c_{J_{p-1}, I_r}^{(2)} d\zeta_{J_{p-1}} \wedge d\bar{\zeta}_{I_r} + \bar{\partial}\Psi^{(2)}. \end{aligned}$$

Since the constant forms are ∂ -, $\bar{\partial}$ -closed, we have

$$\begin{aligned} \varphi &= \partial\psi_{(r-1, p)} + \bar{\partial}\psi_{(r, p-1)} \\ &= \partial\bar{\partial}\Psi^{(1)} + \bar{\partial}\partial\Psi^{(2)} \\ &= \partial\bar{\partial}(\Psi^{(1)} - \bar{\Psi}^{(2)}). \end{aligned}$$

This shows $\partial\bar{\partial}$ -Lemma holds on toroidal groups of cohomologically finite type. We have the following theorem.

THEOREM 3.3. *Let \mathbb{C}^n/Γ be a toroidal group. Then*

- (1) *If \mathbb{C}^n/Γ is of cohomologically finite type and $r, p \geq 1$, then for any d -exact (r, p) -form φ there exists $(r - 1, p - 1)$ -form Ψ such that $\varphi = \partial\bar{\partial}\Psi$ on \mathbb{C}^n/Γ . Further if $r = p$ and φ is a real form, one can choose the above Ψ so that $\sqrt{-1}\Psi$ is also real.*
- (2) *If \mathbb{C}^n/Γ is of non-Hausdorff type and $1 \leq r, p \leq q$, for some d -exact (r, p) -form φ there is no solution Ψ satisfying the $\partial\bar{\partial}$ -equation $\varphi = \partial\bar{\partial}\Psi$ on \mathbb{C}^n/Γ .*

Proof. It remains only to prove the latter half of (1) and (2). Suppose $\varphi = \partial\bar{\partial}\Psi$ and φ is real. Then $\varphi = \bar{\varphi} = \bar{\partial}\partial\bar{\Psi} = \partial\bar{\partial}(-\bar{\Psi})$. We obtain

$$\varphi = \partial\bar{\partial}\left(\frac{\Psi - \bar{\Psi}}{2}\right).$$

Since $\sqrt{-1}(\Psi - \bar{\Psi})/2$ is real, we obtain the assertion of the latter half of (1).

Next to prove (2) we assume that \mathbb{C}^n/Γ is of non-Hausdorff type. By Theorem 3.2 we have

$$(3.6) \quad \sup_{m \neq 0} \exp(-a\|m^*\|)/K_m = \infty$$

for any $a > 0$. For $m = (m_1, m_2, \dots, m_{n+q}) \in \mathbb{Z}^{n+q}$ we put $\|m'\| := \max\{|m_i|, |m_{n+i}| \mid 1 \leq i \leq q\}$ and $\|m''\| := \max\{|m_j| \mid q+1 \leq j \leq n\}$. By (3.6) there exists $\varepsilon > 0$ such that we can choose a sequence $\{m_\mu \mid \mu \in \mathbb{N}\} \in \mathbb{Z}^{n+q} \setminus \{0\}$ satisfying $\exp(-\varepsilon\|m'_\mu\| - \mu\|m''_\mu\|)/K_{m_\mu} > \mu$ for any $\mu \in \mathbb{N}$ ([5, Lemma 4.2]). Put

$$\delta^m := \begin{cases} \exp(-\varepsilon\|m'_\mu\| - \mu\|m''_\mu\|)/K_{m_\mu} & m = m_\mu \text{ for some } \mu \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

We can find i_0 so that $1 \leq i_0 \leq q$ and $\sup\{\mu \mid K_m = |K_{m_\mu, i_0}|\} = \infty$. We may assume $i_0 = q$ without loss of generality. We take a $(r - 1, p - 1)$ -form

$$\begin{aligned} \psi^m := & \delta^m \exp\left(-2\pi \sum_{i=q+1}^n m_i t_{n+i}\right) \exp(2\pi\sqrt{-1}\langle m, t' \rangle) \\ & \times d\zeta_1 \wedge \cdots \wedge d\zeta_{r-1} \wedge \bar{d}\bar{\zeta}_1 \wedge \cdots \wedge \bar{d}\bar{\zeta}_{p-1}. \end{aligned}$$

By the choice of the sequence $\{m_\mu \mid \mu \in \mathbb{N}\}$ the formal series $\sum_m \psi^m$ cannot converge to any form. On the other hand

$$\begin{aligned} \bar{\partial}\psi^m &= \sum_{\ell=1}^q K_{m,\ell} \delta^{m_\mu} \exp\left(-2\pi \sum_{i=q+1}^n m_i t_{n+i}\right) \exp(2\pi\sqrt{-1} \langle m, t' \rangle) \\ &\quad \times d\bar{\zeta}_\ell \wedge d\zeta_1 \wedge \cdots \wedge d\zeta_{r-1} \wedge d\bar{\zeta}_1 \wedge \cdots \wedge d\bar{\zeta}_{p-1}. \end{aligned}$$

Since

$$K_{m_\mu,\ell} \delta^{m_\mu} = \frac{K_{m_\mu,\ell}}{K_{m_\mu}} \exp(-\varepsilon \|m'_\mu\| - \mu \|m''_\mu\|)$$

and $|K_{m_\mu,\ell}/K_{m_\mu}| \leq 1$, $\sum_m \bar{\partial}\psi^m$ converges to a $\bar{\partial}$ -closed $(r-1, p)$ -form η ([5, Lemma 4.1]). We put $\varphi = d\eta = \partial\eta$. We suppose that there exists a $C^\infty(r-1, p-1)$ -form λ satisfying

$$\partial\bar{\partial}\lambda = \varphi.$$

We can write

$$\begin{aligned} \lambda^m &:= \frac{1}{\pi(r-1)!(p-1)!} \sum_{J_{r-1}, I_{p-1}} b_{J_{r-1}, I_{p-1}}^m(t'') \\ &\quad \times \exp(2\pi\sqrt{-1} \langle m, t' \rangle) d\zeta_{J_{r-1}} \wedge d\bar{\zeta}_{I_{p-1}}, \end{aligned}$$

where $b_{J_{r-1}, I_{p-1}}^m(t'')$'s are C^∞ functions in $t'' \in \mathbb{R}^{n-q}$ and $\lambda = \sum_m \lambda^m$. Then we have $\partial\bar{\partial}\lambda^m = \partial\bar{\partial}\psi^m$. Comparing the term of the left form to the right form of this equation involving only the differential $d\zeta_1 \wedge \cdots \wedge d\zeta_{r-1} \wedge d\zeta_q \wedge d\bar{\zeta}_1 \wedge \cdots \wedge d\bar{\zeta}_{p-1} \wedge d\bar{\zeta}_q$. We can obtain the same formula as (3.3) for C^∞ forms λ^m . Then we obtain

$$\begin{aligned} &(-1)^{r+p} |K_{m_\mu, q}|^2 \delta^{m_\mu} \exp\left(-2\pi \sum_{i=q+1}^n m_i t_{n+i}\right) \\ &= (-1)^{r+p} |K_{m_\mu, q}|^2 b_{1 \dots r-1, 1 \dots p-1}^{m_\mu}(t'') \\ &\quad + \sum_{k=1}^{p-1} \sum_{\ell=1}^{r-1} (-1)^{r+k+\ell} K_{m_\mu, \ell} \bar{K}_{m_\mu, k} b_{1 \dots \hat{\ell} \dots r-1, 1 \dots \hat{k} \dots p-1, q}^{m_\mu}(t'') \\ &\quad + \sum_{\ell=1}^{r-1} (-1)^{r+p+\ell} K_{m_\mu, \ell} \bar{K}_{m_\mu, q} b_{1 \dots \hat{\ell} \dots r-1, q, 1 \dots p-1}^{m_\mu}(t'') \\ &\quad + \sum_{k=1}^{p-1} (-1)^{k+1} K_{m_\mu, q} \bar{K}_{m_\mu, k} b_{1 \dots r-1, 1 \dots \hat{k} \dots p-1, q}^{m_\mu}(t''). \end{aligned}$$

Since we can choose a subsequence $\{m_{\mu_s}\}$ so that

$$|K_{m_{\mu_s}, q}| = K_m,$$

we have, for $t'' = 0$,

$$\begin{aligned} \alpha |\delta^{m_{\mu_s}}| &\leq |b_{1 \dots r-1, 1 \dots p-1}^{m_{\mu_s}}(0)| + |b_{1 \dots \hat{\ell} \dots r-1 q, 1 \dots \hat{k} \dots p-1 q}^{m_{\mu_s}}(0)| \\ &\quad + |b_{1 \dots \hat{\ell} \dots r-1 q, 1 \dots p-1}^{m_{\mu_s}}(0)| + |b_{1 \dots r-1, 1 \dots \hat{k} \dots p-1 q}^{m_{\mu_s}}(0)|, \end{aligned}$$

for some positive constant α . Since the coefficients $b_{I_{r-1}, I_{p-1}}^m(t'')$ of Fourier series converge to 0 ([2, Proposition 6]), this contradicts that $\lim_{\mu \rightarrow \infty} \delta^{m_{\mu}} = \infty$. \square

Remark. In the statement (2) of Theorem 3.3 if $r = p$ and $1 \leq p \leq q$, one can choose φ as a real form. Under the assumption of Theorem 3.3, take any (p, p) -form φ satisfying (2) of Theorem 3.3 and put $\varphi_1 := (\varphi + \bar{\varphi})/2$, $\varphi_2 := (\varphi - \bar{\varphi})/(2\sqrt{-1})$. φ_1 and φ_2 are also d -exact and real (p, p) -forms. Suppose $\varphi_i = \partial\bar{\partial}\Psi_i$ for some $(p-1, p-1)$ -form Ψ_i ($i = 1, 2$). Then $\varphi = \partial\bar{\partial}(\Psi_1 + \sqrt{-1}\Psi_2)$. This is a contradiction. Hence at least one of φ_1 and φ_2 satisfies the statement (2) of Theorem 3.3.

§4. Examples and related problems

We can show a very easy counter-example to the problem of the introduction of this paper.

EXAMPLE 4.1. Let $\mathbb{T}_C := \mathbb{C}/\mathbb{Z}\{1, \sqrt{-1}\}$ be a complex torus of complex dimension 1. We put $X := \mathbb{T}_C \times \mathbb{C}$. Trivially X is weakly 1-complete and complete Kähler. Let z be a holomorphic local coordinate induced by the projection: $\mathbb{C} \rightarrow \mathbb{T}_C := \mathbb{C}/\mathbb{Z}\{1, \sqrt{-1}\}$ and w be a global coordinate of \mathbb{C} . We consider a $(0, 1)$ -form $\psi := w d\bar{z}$ and $\varphi := d\psi$. Suppose there exists a C^∞ function Ψ on X such that

$$(4.1) \quad \partial\bar{\partial}\Psi = \varphi.$$

Then $\partial(\bar{\partial}\Psi - \psi) = 0$. This means $\partial\bar{\Psi} - \bar{\psi} = (\partial\bar{\Psi}/\partial z - \bar{w}) dz + (\partial\bar{\Psi}/\partial w) dw$ is $\bar{\partial}$ -closed and then a holomorphic 1-form. Then $\partial\bar{\Psi}/\partial z - \bar{w}$ and $\partial\bar{\Psi}/\partial w$ are holomorphic on X . We have an entire holomorphic function $G(w) := \partial\bar{\Psi}/\partial z - \bar{w}$. We put $x := \operatorname{Re} z$, $y := \operatorname{Im} z$, $u := \operatorname{Re} w$ and $v := \operatorname{Im} w$. We can expand $\bar{\Psi}$ to Fourier series:

$$\bar{\Psi} := \sum_{m \in \mathbb{Z}^2} a^m(u, v) \exp(2\pi\sqrt{-1}(m_1 x + m_2 y)).$$

$\partial\bar{\Psi}/\partial z = \pi \sum_m (m_1\sqrt{-1} - m_2)a^m(u, v) \exp(2\pi\sqrt{-1}(m_1x + m_2y))$. Since $\partial\bar{\Psi}/\partial z = \bar{w} + G(w)$ is constant on variables x and y , $a^m = 0$ if $m \neq 0$. Then $\bar{\Psi} = a^0(u, v)$ and $\partial\bar{\Psi}/\partial z = 0$. This contradicts (4.1). By the same reason of Remark in §3 we can select a real $(1, 1)$ -form that has no solution of the $\partial\bar{\partial}$ -equation on X .

Considering the fact that $H^1(X, \mathcal{O})$ is an infinite-dimensional Fréchet space and $\partial\bar{\partial}$ -Lemma holds for toroidal groups of cohomologically finite type, we can give the following

PROBLEM 4.2. Can one show $\partial\bar{\partial}$ -Lemma on a weakly 1-complete Kähler manifold X with $\dim H^1(X, \mathcal{O}) < \infty$?

If X is strongly 1-convex in the sense of Andreotti and Grauert [2], then $\dim H^1(X, \mathcal{O}) < \infty$. Miyajima [7] considers another type of the $\partial\bar{\partial}$ -equation on strongly pseudoconvex Kähler manifolds. In general case of strongly 1-convex Kähler manifolds the above problem still remains unsolved.

Further a weakly reformed problem of Problem 1.1 is posed in [10].

PROBLEM 4.3. Let L be a holomorphic line bundle on a weakly 1-complete manifold X . We assume that the first Chern class $c_1(L)$ has a positive form. Then does L have a Hermitian metric with a positive curvature form?

We remark here that if $\partial\bar{\partial}$ -Lemma holds on X , then Problem 4.3 can be solved affirmatively for X . The following example shows that $\partial\bar{\partial}$ -Lemma does not hold even if \mathbb{C}^n/Γ is a quasi-abelian variety.

EXAMPLE 4.4. We consider a toroidal group of §2 in the case of $n = 2$ and $q = 1$.

Let Γ be the discrete subgroup generated by $\{e_1, e_2, v_1 := (\sqrt{-1}, \beta)\}$ over \mathbb{Z} , where β is an irrational real number. From (2.1) we have $K_m = \sqrt{(\beta m_2 - m_3)^2 + m_1^2}$ and $K_m > 0$ for $m \neq 0$. Then \mathbb{C}^2/Γ is toroidal. We put $v_2 := (\beta, \sqrt{-1})$ and consider a complex torus $\mathbb{C}^2/\mathbb{Z}\{e_1, e_2, v_1, v_2\}$. Any such torus is an abelian variety ([3, §2.6 The Riemann Conditions]). We have the covering projection:

$$\mathbb{C}^2/\Gamma \longrightarrow \mathbb{C}^2/\mathbb{Z}\{e_1, e_2, v_1, v_2\}.$$

This means every \mathbb{C}^2/Γ is a quasi-abelian variety for any β ([1, Theorem 4.6]). We obtain the following (1) and (2).

- (1) If β is an algebraic number, then by Liouville's criterion there exists a positive number M and a positive integer ℓ such that $|\beta - m_3/m_2| > M/|m_2|^\ell$ for any integer m_3 and $m_2 \neq 0$. Since $K_m \geq |\beta m_2 - m_3| > M/|m_2|^{\ell-1}$ ($m_2 \neq 0$),

$$\sup \left\{ \frac{\exp(-\sqrt{m_1^2 + m_2^2})}{K_m} \mid m \in \mathbb{Z}^3 \setminus \{0\} \right\} < \infty.$$

By Theorem 3.2 \mathbb{C}^2/Γ is of cohomologically finite type and then $\partial\bar{\partial}$ -Lemma holds on it.

- (2) If β is approximated by rational numbers very well, namely, satisfying for any $a > 0$

$$\sup \left\{ \frac{\exp(-a|m|)}{|\beta - n/m|} \mid m, n \in \mathbb{Z}, m \neq 0 \right\} = \infty,$$

(We find examples of such β in [6] and [12]), by Liouville's criterion such β must be a transcendental number and \mathbb{C}^2/Γ is of non-Hausdorff type. Then $\partial\bar{\partial}$ -Lemma does not hold on it.

Added in proof. After this paper was submitted, we obtained an answer to Problem 4.2 in the following form: There exists a 1-convex Kähler manifold on which the $\partial\bar{\partial}$ -Lemma does not hold. This result will appear in our forthcoming paper.

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