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SOME APPLICATIONS OF SERRE DUALITY IN CR MANIFOLDS

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Abstract. Applying the methods of Serre duality in the setting of CR manifolds we prove approximation theorems and we study the Hartogs-Bochner phenomenon in 1-concave CR generic manifolds.

In this paper we study Serre duality in CR manifolds to get some approximation theorem and a better understanding of the Hartogs-Bochner phenomenon in 1-concave CR manifolds.

Because of the lack of the Dolbeault isomorphism for the $\overline{\partial}_b$-cohomology in CR manifolds, Serre duality in this setting is not an exact copy of Serre duality in complex manifolds. The main point is that one has to be particularly careful and always to differentiate cohomology with smooth coefficients and current coefficients. Our main theorem in CR-manifolds (Theorem 3.2) is a consequence of rather classical abstract results on duality for complexes of topological vector spaces which we recall in Section 1 (cf. [3], [9], [21]). Note that in the case of compact CR manifolds, Serre duality was already studied by Hill and Nacinovich in [13].

Associating Serre duality with Malgrange's theorem on the vanishing of the $\overline{\partial}_b$-cohomology in top degree [19] and regularity for the tangential Cauchy- Riemann operator in bidegree $(0, 1)$ [5], we give several applications for 1-concave CR manifolds in Section 4. One of the most interesting is perhaps the study of the Hartogs-Bochner phenomenon. More precisely, we consider a connected non compact $\mathcal{C}^\infty$-smooth 1-concave CR manifold M and a relatively compact open subset D with $\mathcal{C}^\infty$-smooth boundary in M such that $M \setminus D$ is connected. The question is: does any smooth CR function f on ∂D extend to a CR function F in D. It turns out that, for the Hartogs-Bochner phenomenon, 1-concave CR manifolds are rather similar to non compact complex manifolds. In both cases the Hartogs-Bochner phenomenon holds if D is sufficiently small, by a result of Henkin [11] in

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the CR case and since it holds in $\mathbb{C}^n$ for the complex case; but it may be false in general, it is sufficient to consider a domain D which contains the zero set of a CR or a holomorphic function (some example is given in [12] for the CR case and in [16] for the complex case). Moreover, if we remove the connectedness of $M \setminus D$ and if we strengthen the assumption on f by some orthogonality condition to $\overline{\partial}$-closed forms due to Weinstock [23], the Hartogs-Bochner phenomenon holds in complex manifolds. In this paper we prove (Theorem 4.3) that under the same kind of orthogonality condition on f the Hartogs-Bochner phenomenon always holds in 1-concave CR manifolds.

We conclude our paper with a generalization of the Airapetjan-Henkin approximation theorem for $\overline{\partial}_b$-closed $(p, n-k-q)$-differential forms in q-concave CR manifolds of real codimension k in n-dimensional complex manifolds ([1], Theorem 7.2.3). Our theorem (Theorem 5.4) follows from Serre duality and from a theorem (Theorem 5.1) on global regularity for the tangential Cauchy-Riemann operator in bidegree (p, q) in q-convave CR manifolds.

§1. Abstract duality theorems

For sake of completeness, in this section we recall some well-known results about the duality for complexes of topological vector spaces which will be used in Section 3.

Definition 1.1. A *cohomological complex of topological vector spaces* is a pair $(E^\bullet, d)$ where $E^\bullet = (E^q)_{q\in\mathbb{Z}}$ is a sequence of topological vector spaces and $d = (d^q)_{q\in\mathbb{Z}}$ is a sequence of continuous linear maps d^q from E^q into E^{q+1} which satisfy $d^{q+1} \circ d^q = 0$.

A *homological complex of topological vector spaces* is a pair $(E_\bullet, d)$ where $E_\bullet = (E_q)_{q\in\mathbb{Z}}$ is a sequence of topological vector spaces and $d = (d_q)_{q\in\mathbb{Z}}$ is a sequence of continuous linear maps d_q from E_{q+1} into E_q which satisfy $d_q \circ d_{q+1} = 0$.

The *cohomology groups* $H^q(E^\bullet)$ of a cohomological complex $(E^\bullet, d)$ are the quotient spaces $\operatorname{Ker} d^q / \operatorname{Im} d^{q-1}$ endowed with the factor topology and the *homology groups* $H_q(E_\bullet)$ of a homological complex $(E_\bullet, d)$ are the quotient spaces $\operatorname{Ker} d_{q-1} / \operatorname{Im} d_q$ endowed with the factor topology.

The *dual complex* of a cohomological complex $(E^\bullet, d)$ of topological vector spaces is the homological complex $(E'_\bullet, d')$ where $E'_\bullet = (E'_q)_{q\in\mathbb{Z}}$ with E'_q the strong dual of E^q and $d' = (d'_q)_{q\in\mathbb{Z}}$ with d'_q the transpose map of d^q.

DEFINITION 1.2. A topological vector space E is *reflexive* if the natural map between E and the strong dual of its strong dual is a topological isomorphism.

DEFINITION 1.3. A topological vector space E is called a Fréchet-Schwartz space or simply an FS-*space* if its topology is defined by an increasing sequence of seminormes $(p_n)_{n\geq 0}$ such that for $n \geq 1$ the unit ball with respect to the seminorm p_n is relatively compact for the topology associated to the previous seminorm. The strong dual of a Fréchet-Schwartz space is called a *dual of Fréchet-Schwartz* or simply a DFS-space.

Remark. FS-spaces and DFS-spaces are reflexive (for more details on FS-spaces we refer the reader to [9], reflexive spaces are studied in [22]).

DEFINITION 1.4. If E is a topological vector space, then ${}^{\sigma}E$, the *associated separated space*, is the quotient space $E/\bar{O}$, where $\bar{O}$ denotes the closure of the origin in E.

The following theorem is proved in ([8], §3), (see also [21]).

THEOREM 1.5. *Let $(E^\bullet, d)$ be a cohomological complex of reflexive topological vector spaces and $(E'_\bullet, d')$ its dual complex. Then the natural map between ${}^{\sigma}H_q(E'_\bullet)$ and $(H^q(E^\bullet))'$ is an algebraic isomorphism. If moreover $(E^\bullet, d)$ is a cohomological complex of* FS-*spaces and $(H^q(E^\bullet))'$ is endowed with the strong topology, then this isomorphism is also topological.*

In view of this theorem it is important to obtain conditions which ensure that for a given cohomological complex $(E^\bullet, d)$ the groups $H_q(E'_\bullet)$ are separated, *i.e.* ${}^{\sigma}H_q(E'_\bullet) = H_q(E'_\bullet)$.

THEOREM 1.6. *Let $(E^\bullet, d)$ be a cohomological complex of* FS-*spaces or of* DFS-*spaces and $(E'_\bullet, d)$ its dual complex. For each $q \in \mathbb{Z}$, the following assertions are equivalent:*

(i) $\operatorname{Im} d^q = \{g \in E^{q+1} \mid \langle g, f\rangle = 0,\ \forall f \in \operatorname{Ker} d'_q\}$;

(ii) *$H^{q+1}(E^\bullet)$ is separated;*

(iii) *d^q is a topological homomorphism;*

(iv) *d'_q is a topological homomorphism;*

(v) $H_q(E'_\bullet)$ *is separated;*

(vi) $\operatorname{Im} d'_q = \{f \in E'_q \mid \langle f, g\rangle = 0,\ \forall g \in \operatorname{Ker} d^q\}$.

Proof. First clearly (i) $\Rightarrow$ (ii) and (vi) $\Rightarrow$ (v) because $H^{q+1}(E^\bullet)$ (resp. $H_q(E'_\bullet)$) is separated if and only if $\operatorname{Im} d^q$ (resp. $\operatorname{Im} d'_q$) is closed. Moreover it follows from ([8], §0) that (ii), (iii), (iv) and (v) are equivalent.

It remains to prove that (iii) $\Rightarrow$ (vi) and (iv) $\Rightarrow$ (i). Since $(E^\bullet, d)$ is a complex of reflexive spaces the proof of (iv) $\Rightarrow$ (i) is a copy of the proof of (iii) $\Rightarrow$ (vi).

Assume that (iii) is satisfied. Since $\operatorname{Im} d'_q$ is always contained in $\{f \in E'_q \mid \langle f, g\rangle = 0,\ \forall g \in \operatorname{Ker} d^q\}$ by definition of the transpose map, we have only to prove the other inclusion. Let $f \in E'_q$ such that $\langle f, g\rangle = 0$ for all $g \in \operatorname{Ker} d^q$. Setting $L_f(g) = \langle f, h\rangle$ where $g = d^q h$, we define a linear map from $\operatorname{Im} d^q$ into $\mathbb{C}$, which is continuous by (iii). Applying the Hahn-Banach theorem, L_f extends to a continuous linear form $\widetilde{L}_f$ on E^{q+1} which satisfies $\langle \widetilde{L}_f, d^q h\rangle = \langle f, h\rangle$ for all $h \in E^q$. This proves that $f = d'_q \widetilde{L}_f$ by definition of d'_q, *i.e.*$f \in \operatorname{Im} d'_q$.

Corollary 1.7. *Let $(E^\bullet, d)$ be a cohomological complex of* FS*-spaces or of* DFS*-spaces, $(E'_\bullet, d')$ its dual complex and $q \in \mathbb{Z}$. If $\operatorname{Im} d^q = \{g \in E^{q+1} \mid \langle g, f\rangle = 0,\ \forall f \in \operatorname{Ker} d'_q\}$ then the natural map*

$$H_q(E') \to (H^q(E))'$$

is an algebraic isomorphism. If $(E^\bullet, d)$ is a complex of FS*-spaces, it is also topological when $(H^q(E))'$ is endowed with the strong topology.*

§2. The tangential Cauchy-Riemann complex

In this section we shall recall the main properties of CR manifolds and of the tangential Cauchy-Riemann complex which are useful in the further sections of this paper. For more details the reader may consult for example the book of Boggess [7].

Let X be a complex manifold of complex dimension n. If M is a $\mathcal{C}^\infty$-smooth real submanifold of real codimension k in X, we denote by $T^{\mathbb{C}}_\tau(M)$ the complex tangent space to M at $\tau \in M$.

Such a manifold M can be represented locally in the form

$$M = \{z \in \Omega \mid \rho_1(z) = \cdots = \rho_k(z) = 0\} \tag{2.1}$$

where the ρ_ν's, $1 \leq \nu \leq k$, are real $\mathcal{C}^\infty$ functions in an open subset Ω of X.

In this representation we have

$$(2.2) \qquad T_\tau^{\mathbb{C}}(M) = \left\{ \zeta \in \mathbb{C}^n \mid \sum_{j=1}^{n} \frac{\partial \rho_\nu}{\partial z_j}(\tau)\zeta_j = 0, \quad \nu = 1, \dots, k \right\}$$

and $\dim_{\mathbb{C}} T_\tau^{\mathbb{C}}(M) \geq n-k$, for $\tau \in M \cap \Omega$, where $(z_1, \dots, z_n)$ are local holomorphic coordinates in a neighborhood of τ.

Definition 2.1. The submanifold M is called CR if the number $\dim_{\mathbb{C}} T_\tau^{\mathbb{C}}(M)$ is independent of the point $\tau \in M$. If moreover $\dim_{\mathbb{C}} T_\tau^{\mathbb{C}}(M) = n-k$ for every $\tau \in M$, then M is called CR generic.

In the local representation, M is CR generic if and only if

$$\overline{\partial}\rho_1 \wedge \cdots \wedge \overline{\partial}\rho_k \neq 0 \ \text{ on } \ M.$$

Definition 2.2. Let M be a $\mathcal{C}^\infty$-smooth CR generic submanifold of X. M is r-concave, $0 \leq r \leq n-k$, if for each $\tau \in M$, each local representation of M of type 2.1 in a neighborhood of τ in X and each $x \in \mathbb{R}^k \setminus \{0\}$, the quadratic form on $T_\tau^{\mathbb{C}}(M)$ defined by $\sum_{\alpha,\beta} \frac{\partial^2 \rho_x}{\partial z_\alpha \partial \bar{z}_\beta}(\tau)\zeta_\alpha \bar{\zeta}_\beta$, where $\rho_x = x_1\rho_1 + \cdots + x_k\rho_k$ and $\zeta \in T_\tau^{\mathbb{C}}(M)$, has at least r negative eigenvalues.

Our choice to deal with embedded CR generic manifolds in this paper is not a true restriction. In fact for all the results we give in Sections 4 and 5, we need to be in an r-concave abstract CR manifold, $r \geq 1$, which is locally embeddable at each point. By a theorem of Hill and Nacinovich ([12], Proposition 3.1) it is known that an r-concave abstract CR manifold, $r \geq 1$, which is locally embeddable at each point can always be embedded as a CR generic manifold in a complex manifold.

Let X be an n-dimensional complex manifold, M a CR generic submanifold of X of real codimension k locally defined by 2.1 and $p, q \in \mathbb{N}$. We denote by $\Lambda_X^{p,q}$ the vector bundle over X of (p,q)-forms in X and by $\Lambda_M^{p,q}$ the vector bundle over M of (p,q)-forms in M (see e.g. [7]). $\mathcal{E}^{p,q}(M)$ is the space of $\mathcal{C}^\infty$-smooth sections of $\Lambda_M^{p,q}$ over M and $\mathcal{D}^{p,q}(M)$ the space of compactly supported elements of $\mathcal{E}^{p,q}(M)$. Note that $\Lambda_M^{p,q} = 0$ if either $p > n$ or $q > n-k$ and consequently $\mathcal{E}^{p,q}(M) = \mathcal{D}^{p,q}(M) = 0$ for such p and q.

We put on $\mathcal{E}^{p,q}(M)$ the topology of uniform convergence on compact sets of the sections and all their derivatives. Endowed with this topology $\mathcal{E}^{p,q}(M)$ is a Fréchet-Schwartz space.

Let K be a compact subset of M, let $\mathcal{D}^{p,q}_K(M)$ the closed subspace of $\mathcal{E}^{p,q}(M)$ of forms with support in K endowed with the induced topology. Choose $(K_n)_{n\in\mathbb{N}}$ an exhausting sequence of compact subsets of M. Then $\mathcal{D}^{p,q}(M) = \bigcup_{n\geq 0} \mathcal{D}^{p,q}_{K_n}(M)$. We put on $\mathcal{D}^{p,q}(M)$ the strict inductive limit topology defined by the FS-spaces $\mathcal{D}^{p,q}_{K_n}(M)$.

The natural projection $\Lambda^{p,q}_X|_M \to \Lambda^{p,q}_M$ induces a projection t_M from $\mathcal{E}^{p,q}(X)$ onto $\mathcal{E}^{p,q}(M)$.

For $f \in \mathcal{E}^{p,q}(M)$, let $\tilde{f} \in \mathcal{E}^{p,q}(X)$ with $t_M(\tilde{f}) = f$, then $\overline{\partial}_M f = t_M(\overline{\partial}\tilde{f})$ is independent of the choice of the extension $\tilde{f}$ of f (cf. [7], Chap. 8, Lemma 1). In this way we have defined a continuous linear map from $\mathcal{E}^{p,q}(M)$ into $\mathcal{E}^{p,q+1}(M)$. Moreover if $f \in \mathcal{E}^{p,q}(M)$ and $g \in \mathcal{E}^{p,q+1}(M)$, the equation $\overline{\partial}_M f = g$ is equivalent to

$$\int_M g \wedge \varphi = (-1)^{p+q-1} \int_M f \wedge \overline{\partial}\varphi \tag{2.3}$$

for each form $\varphi \in \mathcal{D}^{n-p,n-k-q-1}(X)$ (cf. [7], Chap. 8, Lemma 6).

Definition 2.3. Let $p \in \mathbb{N}$, $0 \leq p \leq n$. The *tangential Cauchy-Riemann complexes* on M, $(\mathcal{E}^{p,\bullet}(M), \overline{\partial}_M)$ and $(\mathcal{D}^{p,\bullet}(M), \overline{\partial}_M)$ are the cohomological complexes of topological vector spaces defined as follows:

(i) for $q < 0$, $E^q = 0$ and $d^q \equiv 0$;

(ii) for $q \geq 0$, $E^q = \mathcal{E}^{p,q}(M)$, respectively $\mathcal{D}^{p,q}(M)$, and $d^q = \overline{\partial}_M$.

The space of currents on M of bidimension (p,q) or bidegree $(n-p, n-k-q)$ is the dual of the space $\mathcal{D}^{p,q}(M)$ and is denoted either by $\mathcal{D}'_{p,q}(M)$, if we use the graduation given by the bidimension, or by $\mathcal{D}'^{n-p,n-k-q}(M)$, if we use the graduation given by the bidegree. An element of $\mathcal{D}'^{n-p,n-k-q}(M)$ can be identified with a distribution section of $\Lambda_M^{n-p,n-k-q}$. The dual of $\mathcal{E}^{p,q}(M)$ denoted by either $\mathcal{E}'_{p,q}(M)$ or $\mathcal{E}'^{n-p,n-k-q}(M)$ is the space of currents of bidimension (p,q), respectively bidegree $(n-p, n-k-q)$ with compact supports. The dual complexes of $(\mathcal{E}^{p,\bullet}(M), \overline{\partial}_M)$ and $(\mathcal{D}^{p,\bullet}(M), \overline{\partial}_M)$ are the homological complexes $(\mathcal{E}'_{p,\bullet}(M), \overline{\partial}_M)$ and $(\mathcal{D}'_{p,\bullet}(M), \overline{\partial}_M)$ if we use the bidimension graduation and where $\overline{\partial}_M$ also denotes the transpose of $\overline{\partial}_M$.

It follows from 2.3 that we have the following commutative diagram

$$\begin{array}{ccc} \mathcal{E}^{p,q}(M) & \xrightarrow{\overline{\partial}_M} & \mathcal{E}^{p,q+1}(M) \\ \big\downarrow & & \big\downarrow \\ \mathcal{D}'^{p,q}(M) & \xrightarrow{\overline{\partial}_M} & \mathcal{D}'^{p,q+1}(M) \end{array}$$

where the vertical arrows are the natural injections induced by the injection of $\mathcal{E}(M)$ into $\mathcal{D}'(M)$. This remark shows that for most of the applications, it is more convenient to use the bidegree graduation. In this case the dual complexes of $(\mathcal{E}^{p,\bullet}(M), \overline{\partial}_M)$ and $(\mathcal{D}^{p,\bullet}(M), \overline{\partial}_M)$ will be the cohomological complexes $(\mathcal{E}'^{n-p,\bullet}(M), \overline{\partial}_M)$ and $(\mathcal{D}'^{n-p,\bullet}(M), \overline{\partial}_M)$.

Since the spaces $\mathcal{E}^{p,q}(M)$ are FS-spaces and the spaces $\mathcal{D}^{p,q}(M)$ are strict inductive limits of FS-spaces, all the complexes $(\mathcal{E}^{p,\bullet}(M), \overline{\partial}_M)$, $(\mathcal{E}'^{p,\bullet}(M), \overline{\partial}_M)$, $(\mathcal{D}^{p,\bullet}(M), \overline{\partial}_M)$ and $(\mathcal{D}'^{p,\bullet}(M), \overline{\partial}_M)$ are complexes of reflexive spaces.

In the next section we shall use the following notations:

$$\begin{array}{ll} Z^{p,q}_{\infty}(M) = \mathcal{E}^{p,q}(M) \cap \operatorname{Ker} \overline{\partial}_M, & E^{p,q}_{\infty}(M) = \mathcal{E}^{p,q}(M) \cap \operatorname{Im} \overline{\partial}_M \\ Z^{p,q}_{c,\infty}(M) = \mathcal{D}^{p,q}(M) \cap \operatorname{Ker} \overline{\partial}_M, & E^{p,q}_{c,\infty}(M) = \mathcal{D}^{p,q}(M) \cap \operatorname{Im} \overline{\partial}_M \\ Z^{p,q}_{\mathrm{cur}}(M) = \mathcal{D}'^{p,q}(M) \cap \operatorname{Ker} \overline{\partial}_M, & E^{p,q}_{\mathrm{cur}}(M) = \mathcal{D}'^{p,q}(M) \cap \operatorname{Im} \overline{\partial}_M \\ Z^{p,q}_{c,\mathrm{cur}}(M) = \mathcal{E}'^{p,q}(M) \cap \operatorname{Ker} \overline{\partial}_M, & E^{p,q}_{c,\mathrm{cur}}(M) = \mathcal{E}'^{p,q}(M) \cap \operatorname{Im} \overline{\partial}_M \\ H^{p,q}(\mathcal{E}(M)) = H^q(\mathcal{E}^{p,\bullet}(M)), & H^{p,q}(\mathcal{D}(M)) = H^q(\mathcal{D}^{p,\bullet}(M)) \\ H^{p,q}(\mathcal{D}'(M)) = H_{n-k-q}(\mathcal{D}'_{n-p,\bullet}(M)), & H^{p,q}(\mathcal{E}'(M)) = H_{n-k-q}(\mathcal{E}'_{n-p,\bullet}(M)). \end{array}$$

§3. Serre duality in CR manifolds

Let M be a CR generic submanifold of real codimension k in an n-dimensional complex manifold X and p an integer. As we have seen in Section 2 the complexes $(\mathcal{D}^{p,\bullet}(M), \overline{\partial}_M)$, $(\mathcal{E}^{p,\bullet}(M), \overline{\partial}_M)$ and their dual complexes $(\mathcal{D}'^{n-p,\bullet}(M), \overline{\partial}_M)$ and $(\mathcal{E}'^{n-p,\bullet}(M), \overline{\partial}_M)$ are complexes of reflexive topological vector spaces. Moreover for all (p,q), $0 \le p \le n$ and $0 \le q \le n-k$, $\mathcal{E}^{p,q}(M)$ is a FS-space and its dual $\mathcal{E}'^{n-p,n-k-q}(M)$ is a DFS-space. Consequently we can apply all the results of Section 1 to these complexes and we get the following theorems:

THEOREM 3.1. *Let M be a CR generic submanifold of real codimension k in an n-dimensional complex manifold X. Let $p, q \in \mathbb{N}$ with $0 \le p \le n$*

and $0 \leq q \leq n-k$, then the natural linear maps

$$ {}^{\sigma}H^{n-p,n-k-q}(\mathcal{D}'(M)) \longrightarrow (H^{p,q}(\mathcal{D}(M)))' \tag{3.1} $$

$$ {}^{\sigma}H^{n-p,n-k-q}(\mathcal{E}'(M)) \longrightarrow (H^{p,q}(\mathcal{E}(M)))' \tag{3.2} $$

$$ {}^{\sigma}H^{p,q}(\mathcal{D}(M)) \longrightarrow (H^{n-p,n-k-q}(\mathcal{D}'(M)))' \tag{3.3} $$

$$ {}^{\sigma}H^{p,q}(\mathcal{E}(M)) \longrightarrow (H^{n-p,n-k-q}(\mathcal{E}'(M)))' \tag{3.4} $$

are algebraic isomorphisms.

If moreover $(H^{p,q}(\mathcal{E}(M))'$ is endowed with the strong topology then the isomorphism 3.2 *is also topological.*

THEOREM 3.2. *Let M be a CR generic submanifold of real codimension k in an n-dimensional complex manifold X. Let $p, q \in \mathbb{N}$ with $0 \leq p \leq n$ and $0 \leq q \leq n-k$, then the following assertions are equivalent:*

(i) $E_{c,\mathrm{cur}}^{n-p,n-k-q}(M) = \Big\{T \in \mathcal{E}'^{n-p,n-k-q}(M) \,|\, \langle T, \varphi\rangle = 0,\ \forall \varphi \in Z_{\infty}^{p,q}(M)\Big\}$

(ii) $H^{n-p,n-k-q}(\mathcal{E}'(M))$ *is separated;*

(iii) $E_{\infty}^{p,q+1}(M) = \Big\{f \in \mathcal{E}^{p,q+1}(M) \,|\, \langle T, f\rangle = 0,\ \forall T \in Z_{c,\mathrm{cur}}^{n-p,n-k-q-1}(M)\Big\}$

(iv) $H^{p,q+1}(\mathcal{E}(M))$ *is separated.*

Moreover, if these assertions hold, then

a) *the natural linear map $H^{n-p,n-k-q}(\mathcal{E}'(M)) \longrightarrow (H^{p,q}(\mathcal{E}(M))'$ is a topological isomorphism;*

b) *the natural linear map $H^{p,q+1}(\mathcal{E}(M)) \longrightarrow (H^{n-p,n-k-q-1}(\mathcal{E}'(M))'$ is an algebraic isomorphism.*

Let us recall some well known results on sufficient conditions on the CR manifold M which ensure the separation of the cohomology groups $H^{p,q}(\mathcal{E}(M))$.

First let us consider the case when M is supposed to be compact, which implies that $\mathcal{E}^{p,q}(M) = \mathcal{D}^{p,q}(M)$ and $\mathcal{E}'^{p,q}(M) = \mathcal{D}'^{p,q}(M)$ for all $p, q \in \mathbb{N}$. In 1981, Henkin [10] has proven that if M is compact and r-concave then the groups $H^{p,q}(\mathcal{E}(M))$ are finite dimensional if $0 \leq p \leq n$ and $q \leq r-1$ or $q \geq n-k-r+1$. Moreover Hill and Nacinovich [12] have shown that $H^{p,r}(\mathcal{E}(M))$ is separated.

Another situation, where separation theorems are known, is the case when M admits an exhausting function with pseudoconvexity or pseudoconcavity properties (see [12] and [1] for a more particular case). Let us give here only some results in a special situation which occurs rather often. If X is a pseudoconcave manifold and if M is r-concave then $H^{p,q}(\mathcal{E}(M))$ is finite dimensional and hence separated for $0 \leq p \leq n$ and $q \leq r-1$. If X is a pseudoconvex manifold and if M is r-concave then $H^{p,q}(\mathcal{E}(M))$ is finite dimensional and hence separated for $0 \leq p \leq n$ and $q \geq n-k-r+1$.

If we associate these results with the theorems 3.1 and 3.2, we get separation results on $H^{p,q}(\mathcal{E}'(M))$, duality isomorphisms and characterizations of exact forms or currents. In the compact case such separation theorems for current cohomology and duality isomorphisms were already obtained by Hill and Nacinovich [13] even for abstract CR manifolds.

§4. Application of duality in 1-concave CR manifolds

In this section M will always denote a 1-concave CR generic manifold of real codimension k embedded in a complex manifold X of complex dimension n, $n \geq 3$.

Approximation theorem

A consequence of duality and Malgrange's vanishing theorem [19] for $\mathcal{C}^\infty$-smooth forms is the following approximation theorem which is a version of Theorem 3.2 in [19] for currents.

THEOREM 4.1. *Let M be a connected, non compact, $\mathcal{C}^\infty$-smooth, 1-concave, CR generic manifold of real codimension k in a complex manifold X of complex dimension n, $n \geq 3$, and p an integer, $0 \leq p \leq n$, then the space $Z^{p,n-k-1}_\infty(M)$ is dense in the space $Z^{p,n-k-1}_{\mathrm{cur}}(M)$ for the strong topology of $\mathcal{D}'^{p,n-k-1}(M)$.*

Proof. By the Hahn-Banach theorem and the reflexivity of $\mathcal{D}^{p,\bullet}(M)$, it is sufficient to prove that for any $f \in \mathcal{D}^{n-p,1}(M)$ such that $\langle f, \varphi \rangle = 0$ for all $\varphi \in Z^{p,n-k-1}_\infty(M)$ we have $\langle T, f \rangle = 0$ for all $T \in Z^{p,n-k-1}_{\mathrm{cur}}(M)$. Let

$$f \in \mathcal{D}^{n-p,1}(M) \cap \left\{ T \in \mathcal{E}'^{n-p,1}(M) \mid \langle T, \varphi \rangle = 0, \ \ \forall \varphi \in Z^{p,n-k-1}_\infty(M) \right\} .$$

Malgrange's theorem [19] claims that, under the hypothesis of the theorem, $H^{p,n-k}(\mathcal{E}(M)) = 0$ and hence is separated, then using Theorem 3.2 we get that $f \in \mathcal{D}^{n-p,1}(M) \cap E^{n-p,1}_{c,\mathrm{cur}}(M)$. From Corollary 0.2 in [5] on regularity of

$\overline{\partial}_M$ in 1-concave CR manifolds we deduce that there exists $g \in \mathcal{D}^{n-p,0}(M)$ such that $f = \overline{\partial}_M g$. Let $T \in Z^{p,n-k-1}_{\mathrm{cur}}(M)$, then $\langle T, f\rangle = \langle T, \overline{\partial}_M g\rangle = \langle \overline{\partial}_M T, g\rangle = 0$.

Hartogs-Bochner phenomenon

A second application of duality and Malgrange's vanishing theorem [19] is a generalization of Weinstock's theorem [23] on the Hartogs-Bochner phenomenon in $\mathbb{C}^n$ to 1-concave CR generic manifolds.

The problem we consider can be set in the following terms: let M be a connected, non compact $\mathcal{C}^\infty$-smooth, 1-concave CR generic manifold and D a relatively compact open subset with $\mathcal{C}^\infty$-smooth boundary contained in M; we are looking for a characterization of the trace on ∂D of the $\mathcal{C}^\infty$-smooth functions in $\overline{D}$ which are CR on D.

Definition 4.2. Let M be a CR manifold of class $\mathcal{C}^\infty$ and V a $\mathcal{C}^\infty$-smooth submanifold of M. A function f of class $\mathcal{C}^\infty$ on V will be called a *CR-smooth function* if there exists a function $\tilde{f}$ of class $\mathcal{C}^\infty$ in M such that $\tilde{f}|_V = f$ and $\overline{\partial}_M \tilde{f}$ vanishes to infinite order on V.

Note that in this definition V is not supposed to be a CR manifold. If V is CR, then this definition coincides with the classical one (cf. [15]).

Let $D \subset\subset M$ be a domain with $\mathcal{C}^\infty$-smooth boundary and $f \in \mathcal{C}^\infty(\partial D)$ the trace of a $\mathcal{C}^\infty$-smooth function in $\overline{D}$ which is CR in D. It is clear that f is a CR-smooth function on ∂D but this condition is not sufficient to get the CR extension of f as it was noticed by Hill and Nacinovich (cf. [12], Section 5 or [18], Section II.8). This condition becomes sufficient if ∂D is connected and if either D is sufficiently small (see [11]) or some pseudoconvexity properties are fulfilled (see [17]).

By analogy with Weinstock's theorem, we get

Theorem 4.3. *Let M be a connected, non compact, $\mathcal{C}^\infty$-smooth, 1-concave, CR generic manifold of real codimension k in a complex manifold X of complex dimension n, $n \geq 3$, and $D \subset\subset M$ a domain with $\mathcal{C}^\infty$-smooth boundary.*

For a function $f \in \mathcal{C}^\infty(\partial D)$ the following properties are equivalent:

(i) *There exists a function $F \in \mathcal{C}^\infty(\overline{D})$ which is CR in D and such that $F|_{\partial D} = f$.*

(ii) *f is CR-smooth and $\int_{\partial D} f\varphi = 0$ for any CR $(n, n-k-1)$-form φ of class $\mathcal{C}^\infty$ in a neighborhood of $\overline{D}$.*

Proof. We assume first that (i) is fulfilled. It is clear that f is CR-smooth and moreover if φ is a CR $(n, n-k-1)$-form of class $\mathcal{C}^\infty$ in a neighborhood of $\overline{D}$ then

$$\int_{\partial D} f\varphi = \int_{\partial D} F\varphi = \int_D d(F\varphi) = \int_D F\overline{\partial}_M\varphi = 0$$

by Stokes theorem since F and φ are CR.

Assume now that condition (ii) is satisfied. Let $U_1, \dots, U_N$ be the relatively compact connected components of $M \setminus \overline{D}$. Take points $z_j \in U_j$, $1 \leq j \leq N$, and set $\widetilde{M} = M \setminus \{z_1, \dots, z_N\}$. Then $\widetilde{M}$ is a connected, non compact, $\mathcal{C}^\infty$-smooth, 1-concave CR generic manifold and consequently, by Malgrange's theorem [19], $H^{n,n-k}(\mathcal{E}(\widetilde{M}))$ vanishes and hence is separated. We deduce from Theorem 3.2 that

$$(4.1)\quad E^{0,1}_{c,\mathrm{cur}}(M) = \Big\{T \in \mathcal{E}'^{0,1}(M) \mid \langle T, \varphi\rangle = 0,\ \forall\varphi \in Z^{n,n-k-1}_\infty(M)\Big\}.$$

As f is CR-smooth, let $\tilde{f}$ be a $\mathcal{C}^\infty$-smooth extension of f to M such that $\overline{\partial}\tilde{f}$ vanishes to infinite order on ∂D. Set $g = \chi_{\overline{D}}\overline{\partial}_M\tilde{f}$ where $\chi_{\overline{D}}$ is the characteristic function of $\overline{D}$. The $(0,1)$-form g is of class $\mathcal{C}^\infty$ and if $\varphi \in Z^{n,n-k-1}_\infty(M)$

$$\langle g, \varphi\rangle = \int_D \overline{\partial}_M\tilde{f} \wedge \varphi = \int_D \overline{\partial}_M(\tilde{f} \wedge \varphi) = \int_D d(\tilde{f} \wedge \varphi) = \int_{\partial D} f \wedge \varphi = 0$$

by Stokes theorem. Therefore by 4.1, $g \in E^{0,1}_{c,\mathrm{cur}}(M)$, *i.e.*there is a distribution S with compact support on $\widetilde{M}$ such that $g = \overline{\partial}_{\widetilde{M}}S$.

But by regularity of the $\overline{\partial}_M$ operator in bidegree (0.1) ([5], Corollary 0.2), S is defined by a function h of class $\mathcal{C}^\infty$ with compact support in $\widetilde{M}$. Setting $h(z_j) = 0$, $1 \leq j \leq N$, we get a function h of class $\mathcal{C}^\infty$ in M, which vanishes on an open subset of each connected component of $M \setminus \overline{D}$, which is CR on $M \setminus \overline{D}$ since $\operatorname{supp} g \in \overline{D}$. Then by analytic continuation of CR functions in 1-concave CR manifolds, h vanishes on $M \setminus \overline{D}$. Setting $F = \tilde{f} - h$, we obtain the required extension.

§5. Approximation theorem in q-concave CR manifolds

We begin this section with a theorem on the global regularity of the $\overline{\partial}_b$-operator in bidegree (p,q) in q-concave CR manifolds, $0 \le p \le n$, $1 \le q \le n$. Such a result is of particular interest since $\overline{\partial}_b$ is not locally solvable in bidegree (p,q) in q-concave CR manifolds (cf. [4]). For $q=1$, the regularity of $\overline{\partial}_b$ is proven in [5], (see also [2]).

THEOREM 5.1. *Let M be a q-concave CR generic submanifold of class $\mathcal{C}^\infty$ and real codimension k of a complex manifold X of complex dimension n, $1 \le q \le n-k$, $n \ge 3$, and T a $(p,q-1)$-current, $0 \le p \le n$. If $\overline{\partial}_M T$ is defined by a $\mathcal{C}^\infty$-smooth (p,q)-form on M then for any neighborhood U of* $\operatorname{supp} T$, *there exists a $(p,q-1)$-form u of class $\mathcal{C}^\infty$ on M such that*

$$\overline{\partial}_M u = \overline{\partial}_M T \quad \textit{and} \quad \operatorname{supp} u \subset U \ .$$

This theorem is a consequence of a theorem on local regularity for $\overline{\partial}_b$ in bidegree (p,q) in q-concave manifolds proved by Barkatou [6] (see also [2]) and of the De Rham-Weil isomorphism.

We first recall Barkatou's result

PROPOSITION 5.2. *Let M be as in Theorem* 5.1 *and p an integer with $0 \le p \le n$. Let $z_0 \in M$ then there exists a neighborhood $M_0 \subset M$ of z_0 such that if T is a $(p,q-1)$ current on M with $\overline{\partial}_M T$ defined by a $\mathcal{C}^\infty$-smooth (p,q)-form on M then their exists a $(p,q-1)$ form u of class $\mathcal{C}^\infty$ on M_0 with $\overline{\partial}_M u = \overline{\partial}_M T$.*

Let M be a q-concave CR generic manifold in X, we denote by Ω^p_M the sheaf of CR p-forms on M, $0 \le p \le n$. Consider the following diagram

$$(5.1)\qquad \begin{array}{ccccccccccc} & \overset{\bar\partial_M}{\nearrow} & \mathcal{E}^{p,1} & \xrightarrow{\bar\partial_M} & \mathcal{E}^{p,2} & \longrightarrow \cdots \xrightarrow{\bar\partial_M} & \mathcal{E}^{p,q-1} & \xrightarrow{\bar\partial_M} & \mathcal{E}^{p,q} \\ \Omega^p_M & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ & \underset{\bar\partial_M}{\searrow} & \mathcal{D}'^{p,1} & \xrightarrow{\bar\partial_M} & \mathcal{D}'^{p,2} & \longrightarrow \cdots \xrightarrow{\bar\partial_M} & \mathcal{D}'^{p,q-1} & \xrightarrow{\bar\partial_M} & \mathcal{D}'^{p,q} \end{array}$$

where the vertical arrows are the natural injections.

This diagram is commutative and by the Poincaré lemma for the $\overline{\partial}_M$ operator in q-concave CR manifolds [20], both horizontal lines are exact. Moreover the sheaves $\mathcal{D}'^{p,j}$ and $\mathcal{E}^{p,j}$ are sheaves of $\mathcal{E}$-modules and consequently acyclic.

Let us denote by $\widetilde{\mathcal{D}}'^{p,q}$ the image by $\overline{\partial}_M$ of the sheaf $\mathcal{D}^{p,q-1}$, this is the sheaf of locally $\overline{\partial}_M$-exact currents, and by $\widetilde{\mathcal{E}}^{p,q}$ the image by $\overline{\partial}_M$ of the sheaf $\mathcal{E}^{p,q-1}$, this is the sheaf of locally $\overline{\partial}_M$-exact $\mathcal{C}^\infty$-smooth forms.

Lemma 5.3. *Under the hypothesis of Theorem* 5.1 *and using the previous notations the natural map*

$$\begin{aligned} H^k(M,\mathcal{E}^{p,k}) &\xrightarrow{\theta_k} H^k(M,\mathcal{D}'^{p,k}), \quad 0\le k\le q-1\\ H^q(M,\widetilde{\mathcal{E}}^{p,q}) &\xrightarrow{\widetilde{\theta}_q} H^q(M,\widetilde{\mathcal{D}}'^{p,q}) \end{aligned}$$

induced by the inclusion of $\mathcal{E}^{p,k}$ into $\mathcal{D}'^{p,k}$, $0\le k\le q$ are isomorphisms.

Proof. Following the proof of the de Rham-Weil isomorphism (cf. [14], Chap. 7) we get isomorphisms

$$\begin{aligned} \delta^k_\infty : H^k(M,\mathcal{E}^{p,k}) &\longrightarrow H^k(M,\Omega^p_M) \ \text{ and}\\ \delta^k_{\text{cur}} : H^k(M,\mathcal{D}'^{p,k}) &\longrightarrow H^k(M,\Omega^p_M), \quad 0\le k\le q-1 \end{aligned}$$

$$\begin{aligned} \tilde{\delta}^q_\infty : H^q(M,\widetilde{\mathcal{E}}^{p,q}) &\longrightarrow H^q(M,\Omega^p_M) \ \text{ and}\\ \tilde{\delta}^q_{\text{cur}} : H^q(M,\widetilde{\mathcal{D}}'^{p,q}) &\longrightarrow H^q(M,\Omega^p_M). \end{aligned}$$

such that the following diagrams are commutative

$$\begin{array}{ccc} H^k(M,\mathcal{D}'^{p,k}) & \overset{\delta^k_{\text{cur}}}{\searrow} & \\ \theta_k \uparrow & & H^k(M,\Omega^p_M), \quad 0\le k\le q-1\\ H^k(M,\mathcal{E}^{p,k}) & \underset{\delta^k_\infty}{\nearrow} & \end{array}$$

$$\begin{array}{ccc} H^q(M,\widetilde{\mathcal{D}}'^{p,q}) & \overset{\tilde{\delta}^q_{\text{cur}}}{\searrow} & \\ \tilde{\theta}_k \uparrow & & H^q(M,\Omega^p_M),\\ H^q(M,\widetilde{\mathcal{E}}^{p,q}) & \underset{\tilde{\delta}^q_\infty}{\nearrow} & \end{array}$$

where θ_k, $0\le k\le q-1$ et $\widetilde{\theta}_q$ are the natural maps induced by the inclusion of $\mathcal{E}^{p,k}$ into $\mathcal{D}'^{p,k}$, $0\le k\le q$. Consequently the maps θ_k, $0\le k\le q-1$ and $\widetilde{\theta}_q$ are isomorphisms.

Proof of Theorem 5.1. Let $T \in \mathcal{D}'^{p,q-1}(M)$ such that $\overline{\partial}_M T$ is defined by a $\mathcal{C}^\infty$-smooth (p,q)-form on M. Proposition 5.2 implies that $\overline{\partial}_M T \in \widetilde{\mathcal{E}}^{p,q}(M) \subset \widetilde{\mathcal{D}}'^{p,q}(M)$. Let $\widetilde{\theta}_q$ be the isomorphism of Lemma 5.3, then the image by $\widetilde{\theta}_q$ of the class of $\overline{\partial}_M T$ in $H^q(M, \widetilde{\mathcal{E}}^{p,q})$ is the zero point of $H^q(M, \widetilde{\mathcal{D}}'^{p,q})$. We deduce from the injectivity of $\widetilde{\theta}_q$ that the class of $\overline{\partial}_M T$ in $H^q(M, \widetilde{\mathcal{E}}^{p,q})$ is equal to zero, which says that there exists $g \in \mathcal{E}^{p,q-1}(M)$ with $\overline{\partial}_M g = \overline{\partial}_M T$.

Now we shall modify g to get a form $u \in \mathcal{E}^{p,q-1}(M)$ with $\overline{\partial}_M u = \overline{\partial}_M T$ and $\operatorname{supp} u \subset U$ where U is a given neighborhood of $\operatorname{supp} T$.

Let U be a neighborhood of the support of T. Let us consider the $(p,q-1)$-current $T-g$ on M, it is $\overline{\partial}_M$-closed and using the isomorphism θ_{q-1} of Lemma 5.3 we get a $(p,q-2)$-current S on M and a $(p,q-1)$-form $h \in \mathcal{E}^{p,q-1}(M)$ such that $T-g+\overline{\partial}_M S = h$. Then $\overline{\partial}_M S = g+h$ on $M \setminus \operatorname{supp} T$. Using once more the isomorphism θ_{q-1} we can find a $(p,q-2)$-form $v \in \mathcal{E}^{p,q-2}(M)$ such that

$$\overline{\partial}_M v = g + h \quad \text{on} \quad M \setminus \operatorname{supp} T \ .$$

Let χ be a $\mathcal{C}^\infty$-smooth function such that $\chi = 0$ on a neighborhood of $\operatorname{supp} T$ and $\chi = 1$ on a neighborhood of $M \setminus U$. Setting $u = g + h - \overline{\partial}_M \chi v$ we get a $\mathcal{C}^\infty$-smooth $(p,q-1)$-form on M such that $\overline{\partial}_M u = \overline{\partial}_M T$ and $\operatorname{supp} u \subset U$.

Using Theorem 5.1 we can prove a generalization of Theorem 4.1 to the case of q-concave CR manifolds, $q \geq 1$.

THEOREM 5.4. *Let M be a $\mathcal{C}^\infty$-smooth q-concave CR generic manifold of real codimension k in a complex manifold X of complex dimension n, $1 \leq q \leq n-k$, $n \geq 3$, and p an integer, $0 \leq p \leq n$. Assume that $H^{p,n-k-q+1}_\infty(M)$ is separated, then the space $Z^{p,n-k-q}_\infty(M)$ is dense in the space $Z^{p,n-k-q}_{\mathrm{cur}}(M)$ for the strong topology of $\mathcal{D}'^{p,n-k-q}(M)$.*

Proof. It is sufficient to repeat the proof of Theorem 4.1 replacing Malgrange's theorem by the assumption $H^{p,n-k-q+1}_\infty(M)$ is separated and Corollary 0.2 in [5] by Theorem 5.1.

Remark 5.5. 1) If $q = 1$ and if M is connected and non compact, by Malgrange's theorem [19] the assumption $H^{p,n-k-q+1}_\infty(M)$ separated is automatically fulfilled and Theorem 5.4 is nothing else than Theorem 4.1.

2) If M is compact, by Theorem 4 in [10] or Theorem 7.1 in [12] $H^{p,n-k-q+1}_\infty(M)$ is finite dimensional and hence separated and consequently

it follows from Theorem 5.1 and Proposition 3.1 in [12] that in a $\mathcal{C}^\infty$-smooth compact q-concave locally embeddable CR manifold M, $q \geq 1$, of real dimension $2n-k$ and CR dimension $n-k$, the space $Z_\infty^{p,n-k-q}(M)$, is dense in $Z_{\mathrm{cur}}^{p,n-k-q}(M)$ for $0 \leq p \leq n$.

3) Note that the cohomology groups $H_\infty^{p,n-k-q+1}(M)$ are separated as soon as M admits an exhausting function with good pseudoconvexity properties since, usually in this case, these groups are finite dimensional (see for example Theorem 6.1 in [12]). As a particular case of this situation we get a version for currents of the approximation Theorem 7.2.3 of [1] as a corollary of Theorem 5.4.

Corollary 5.6. *Let M be a $\mathcal{C}^\infty$-smooth q-concave CR generic manifold of real codimention k in an n-dimensional pseudoconvex complex manifold, $q \geq 1$, then for all p, $0 \leq p \leq n$, the space $Z_\infty^{p,n-k-q}(M)$ is dense in the space $Z_{\mathrm{cur}}^{p,n-k-q}(M)$ for the strong topology of $\mathcal{D}'^{p,n-k-q}(M)$*

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