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MODULI SPACES OF VECTOR BUNDLES OVER RULED SURFACES

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Abstract. We study moduli spaces $M(c_1, c_2, d, r)$ of isomorphism classes of algebraic 2-vector bundles with fixed numerical invariants c_1, c_2, d, r over a ruled surface. These moduli spaces are independent of any ample line bundle on the surface. The main result gives necessary and sufficient conditions for the non-emptiness of the space $M(c_1, c_2, d, r)$ and we apply this result to the moduli spaces $\mathcal{M}_L(c_1, c_2)$ of stable bundles, where L is an ample line bundle on the ruled surface.

Introduction

Let $\pi : X \to C$ be a ruled surface over a smooth algebraic curve C, defined over the complex number field $\mathbb{C}$. Let f be a fibre of π. Let $c_1 \in \mathrm{Num}(X)$ and $c_2 \in H^4(X, \mathbb{Z}) \cong \mathbb{Z}$ be fixed. For any polarization L, denote the moduli space of rank-2 vector bundles stable with respect to L in the sense of Mumford-Takemoto by $\mathcal{M}_L(c_1, c_2)$. Stable 2-vector bundles over a ruled surface have been investigated by many authors; see, for example [T1], [T2], [H-S], [Q1]. Let us mention that Takemoto [T1] showed that there is no rank-2 vector bundle (having $c_1.f$ even) stable with respect to every polarization L. In this paper we shall study algebraic 2-vector bundles over ruled surfaces, but we adopt another point of view: we shall study moduli spaces of (algebraic) 2-vector bundles over a ruled surface X, which are defined independent of any ample divisor (line bundle) on X, by taking into account the special geometry of a ruled surface (see [B], [B-St1], [B-St2] and also [Br1], [Br2], [W]).

In Section 1 (put for the convenience of the reader) we present (see [B]) two numerical invariants d and r for a 2-vector bundle with fixed Chern classes c_1 and c_2 and we define the set $M(c_1, c_2, d, r)$ of isomorphism classes of bundles with fixed invariants $c_1,\ c_2,\ d,\ r$. The integer d is given by the splitting of the bundle on the general fibre and the integer r is given by some normalization of the bundle. Recall that the set $M(c_1, c_2, d, r)$ carries

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a natural structure of an algebraic variety (see [B], [B-St1], [B-St2]). In Section 2 we study uniform vector bundles and we prove the existence of algebraic vector bundles given by extensions of line bundles and which are not uniform. In Section 3 the main result gives necessary and sufficient conditions for the non-emptiness of the space $M(c_1, c_2, d, r)$ and we apply this result to the moduli space of stable bundles $\mathcal{M}_L(c_1, c_2)$.

§1. Moduli spaces of rank-2 vector bundles

In this section we shall recall from ([B], [B-St1], [B-St2]) some basic notions and facts.

The notations and the terminology are those of Hartshorne's book [Ha]. Let C be a nonsingular curve of genus g over the complex number field and let $\pi : X \to C$ be a ruled surface over C. We shall write $X \cong \mathbb{P}(\mathcal{E})$ where $\mathcal{E}$ is normalized. Let us denote by $\mathbf{e}$ the divisor on C corresponding to $\bigwedge^2 \mathcal{E}$ and by $e = -\deg(\mathbf{e})$. We fix a point $p_0 \in C$ and a fibre $f_0 = \pi^{-1}(p_0)$ of X. Let C_0 be a section of π such that $\mathcal{O}_X(C_0) \cong \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$.

Any element of $\mathrm{Num}(X) \cong H^2(X, \mathbb{Z})$ can be written $aC_0 + bf_0$ with $a,\ b \in \mathbb{Z}$. We shall denote by $\mathcal{O}_C(1)$ the invertible sheaf associated to the divisor p_0 on C. If L is an element of $\mathrm{Pic}(C)$ we shall write $L = \mathcal{O}_C(k) \otimes L_0$, where $L_0 \in \mathrm{Pic}_0(C)$ and $k = \deg(L)$. We also denote by $F(aC_0 + bf_0) = F \otimes \mathcal{O}_X(a) \otimes \pi^* \mathcal{O}_C(b)$ for any sheaf F on X and any $a, b \in \mathbb{Z}$.

Let E be an algebraic rank-2 vector bundle on X with fixed numerical Chern classes $c_1 = (\alpha, \beta) \in H^2(X, \mathbb{Z}) \cong \mathbb{Z} \times \mathbb{Z},\ c_2 = \gamma \in H^4(X, \mathbb{Z}) \cong \mathbb{Z}$, where $\alpha, \beta, \gamma \in \mathbb{Z}$.

Since the fibres of π are isomorphic to $\mathbb{P}^1$ we can speak about the generic splitting type of E and we have $E|_f \cong \mathcal{O}_f(d) \oplus \mathcal{O}_f(d')$ for a general fibre f, where $d' \leq d,\ d + d' = \alpha$. The integer d is the first numerical invariant of E.

The second numerical invariant is obtained by the following normalization:

$$-r = \inf\{l \mid \exists L \in \mathrm{Pic}(C), \deg(L) = l, \ \text{ s.t. } H^0(X, E(-dC_0) \otimes \pi^* L) \neq 0\}.$$

We shall denote by $M(\alpha, \beta, \gamma, d, r)$ or $M(c_1, c_2, d, r)$ or M the set of isomorphism classes of algebraic rank-2 vector bundles on X with fixed Chern classes $c_1,\ c_2$ and invariants d and r.

With these notations we have the following result (see [B]):

THEOREM 1. *For every vector bundle $E \in M(c_1, c_2, d, r)$ there exist $L_1,\ L_2 \in \mathrm{Pic}_0(C)$ and $Y \subset X$ a locally complete intersection of codimension 2 in X, or the empty set, such that E is given by an extension*

$$(1) \quad 0 \to \mathcal{O}_X(dC_0 + rf_0) \otimes \pi^* L_2 \to E \to \mathcal{O}_X(d^{'} C_0 + sf_0) \otimes \pi^* L_1 \otimes I_Y \to 0,$$

where $c_1 = (\alpha, \beta) \in \mathbb{Z} \times \mathbb{Z}$, $c_2 = \gamma \in \mathbb{Z}$, $d + d^{'} = \alpha$, $d \geq d^{'}$, $r + s = \beta$, $l(c_1, c_2, d, r) := \gamma + \alpha(de - r) - \beta d + 2dr - d^2 e = \deg(Y) \geq 0$.

Remark. By applying Theorem 1 we can obtain the canonical extensions used in [Br1], [Br2].
Indeed, let us suppose first that $d > d^{'}$. From the exact sequence (1) it follows that

$$\mathcal{O}_C(r) \otimes L_2 \cong \pi_* E(-dC_0)$$

so

$$\mathcal{O}_X(rf_0) \otimes \pi^* L_2 \cong \pi^* \pi_* E(-dC_0)$$

and

$$\mathcal{O}_X(dC_0 + rf_0) \otimes \pi^* L_2 \cong (\pi^* \pi_* E(-dC_0))(dC_0).$$

If $d = d^{'}$ then, by applying π_* to the short exact sequence

$$0 \to \mathcal{O}_X(rf_0) \otimes \pi^* L_2 \to E(-dC_0) \to \mathcal{O}_X(sf_0) \otimes \pi^* L_1 \otimes I_Y \to 0$$

it follows the exact sequence

$$0 \to \mathcal{O}_C(r) \otimes L_2 \to \pi_* E(-dC_0) \to \mathcal{O}_C(s) \otimes L_1 \otimes \mathcal{O}_C(-Z_1) \to 0,$$

where Z_1 is an effective divisor on C with the support $\pi(Y)$. With the notation $Z = \pi^{-1}(Z_1)$, by applying π^* (π is a flat morphism) we obtain the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \to & \mathcal{O}_X(rf_0) \otimes \pi^* L_2 & \to & E(-dC_0) & \to & \mathcal{O}_X(sf_0) \otimes \pi^* L_1 \otimes I_Y & \to & 0 \\
 & & \wr\uparrow id & & \uparrow \varphi & & \uparrow \psi & & \\
0 & \to & \mathcal{O}_X(rf_0) \otimes \pi^* L_2 & \to & \pi^* \pi_* E(-dC_0) & \to & \mathcal{O}_X(sf_0) \otimes \pi^* L_1 \otimes I_Z & \to & 0
\end{array}$$

From the injectivity of ψ we obtain the injectivity of φ. Because of

$$\mathcal{O}_X(sf_0) \otimes \pi^* L_1 \otimes I_{Y \subset Z} \cong \text{Coker } \psi \cong \text{Coker } \varphi$$

we conclude.

Recall that a set M of vector bundles on a $\mathbb{C}-$scheme X is called *bounded* if there exists an algebraic $\mathbb{C}$-scheme T and a vector bundle V on $T \times X$ such that every $E \in M$ is isomorphic with $V_t = V|_{t \times X}$ for some closed point $t \in T$ (see [K]).

For the next result see [B]:

THEOREM 2. *The set $M(c_1, c_2, d, r)$ is bounded.*

§2. Uniform bundles

In what follows, we keep the notations from Section 1.

DEFINITION 3. A 2-vector bundle E is called an *uniform bundle* if the splitting type is preserved on all fibres of X.

Theorem 1 allows us to give a criterion for uniformness.

LEMMA 4. *Let f be a fibre of X and let us suppose that $I_{Y \cap f \subset f} \cong \mathcal{O}_f(-n)$. Then $E|_f \cong \mathcal{O}_f(d+n) \oplus \mathcal{O}_f(d^{'} - n)$.*

Proof. We suppose that $E|_f \cong \mathcal{O}_f(a) \oplus \mathcal{O}_f(a^{'})$, where $a \geq a^{'}$. Then we have a surjective morphism

$$E|_f \rightarrow \mathcal{O}_f(d^{'}) \otimes I_Y \otimes \mathcal{O}_f$$

in virtue of Theorem 1. On the other hand, the restriction of the sequence

$$0 \rightarrow I_Y \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_Y \rightarrow 0$$

to f gives a surjective morphism

$$I_Y \otimes \mathcal{O}_f \rightarrow I_{Y \cap f \subset f} \cong \mathcal{O}_f(-n).$$

So, we obtain another surjective morphism

$$\mathcal{O}_f(a) \oplus \mathcal{O}_f(a^{'}) \rightarrow \mathcal{O}_f(d^{'} - n).$$

By using the inequalities $a \geq a^{'}$, $d \geq d^{'} \geq d^{'} - n$ and the equality $a + a^{'} = d + d^{'} = \alpha$ it follows that $a^{'} = d^{'} - n$ and $a = d + n$.

COROLLARY 5. *E is an uniform bundle if and only if* $l(c_1, c_2, d, r) = 0$.

By means of Corollary 5 the uniform bundles are given by extensions of line bundles. It is naturally to ask if the converse is true. Unfortunately, this question has a negative answer, as proved by the following

PROPOSITION 6. *On the rational ruled surface* $\mathbb{F}_e$ *with* $e \geq 1$ *there exist non-uniform bundles given by extensions of line bundles.*

For the proof we need some preparations.

Let E be a 2-vector bundle given by an extension

$$0 \to F \to E \to G \to 0, \tag{2}$$

where $F = \mathcal{O}_X(aC_0 + r' f_0) \otimes \pi^* L_2'$, $G = \mathcal{O}_X(a' C_0 + s' f_0) \otimes \pi^* L_1'$ ($L_1', L_2' \in \mathrm{Pic}_0(C)$) are line bundles on X. By means of Theorem 1, E sits also in a canonical extension (1). If $a \geq a'$ then E is obviously uniform. Then, we shall suppose that $a < a'$.

LEMMA 7. *With the above notations we have* $d \leq a'$.

Proof. Indeed, by the restriction of the sequence (2) to a general fibre f we obtain a surjective morphism

$$\mathcal{O}_f(d) \oplus \mathcal{O}_f(d') \to \mathcal{O}_f(a').$$

If $d > a'$, then it follows that $d' = a'$ which contradicts the inequalities $a < a'$, $d \geq d'$ $(a + a' = d + d')$.

LEMMA 8. *If* $d = a'$ *then E is uniform.*

Proof. Let f be a fibre of X such that the splitting type of $E|_f$ is different from the generic splitting type of E. According to Lemma 4

$$E|_f \cong \mathcal{O}_f(d + n) \oplus \mathcal{O}_f(d' - n),$$

where $n > 0$.

By the restriction of (2) to f we obtain a surjective morphism

$$\mathcal{O}_f(d + n) \oplus \mathcal{O}_f(d' - n) \to \mathcal{O}_f(d).$$

Because of $d + n > d$ it follows $d' - n = d$, contradiction.

LEMMA 9. *In the above hypotheses, if $d = a'$, then $E \cong F \oplus G$.*

Proof. Let us observe that we can suppose, without loss of generality, that $a = 0$ and $r' = 0$ (by twisting the sequences (1) and (2) with $\mathcal{O}_X(-aC_0 - r'f_0)$). Then, it follows that $d = a' = \alpha > 0$, $s' = \beta$ and $d' = 0$.

Therefore, the sequences (1) and (2) become:

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & & \uparrow & & & & \\ & & & & \mathcal{O}_X(\alpha C_0 + \beta f_0) \otimes \pi^* L_1' & & & & \\ & & & \overset{\chi}{\nearrow} & \uparrow \varphi & & & & \\ (1') \quad 0 & \longrightarrow & \mathcal{O}_X(\alpha C_0 + r f_0) \otimes \pi^* L_2 & \xrightarrow{\psi} & E & \longrightarrow & \mathcal{O}_X(s f_0) \otimes \pi^* L_1 \otimes I_Y & \longrightarrow & 0 \\ & & & & \uparrow & & & & \\ & & & & \pi^* L_2' & & & & \\ & & & & \uparrow & & & & \\ & & & & 0 & & & & \end{array}$$

The computation of $c_2(E)$ in $(1')$ gives $\deg(Y) = -\alpha s$. Moreover, by means of Lemma 8, $\deg(Y) = 0$, so $s = 0$ (we supposed $\alpha > 0$).

The homomorphism $\chi = \varphi\psi$ is non-zero, otherwise $\mathcal{O}_X(\alpha C_0 + \beta f_0) \subset \pi^*(L_2')$ (which would contradict the condition $\alpha > 0$), so $L_2 = L_1'$ and χ is the multiplication by a $\lambda \in \mathbb{C}^*$, and the assertion follows.

In this moment, we are able to give the counter-example announced in Proposition 6.

Proof of Proposition 6. Let G be $\mathcal{O}_X(2C_0)$ and let F be $\mathcal{O}_X$. Then:

$$\dim H^1(G^{-1}) = e + 1 \neq 0.$$

For E given by an extension $\xi \in \mathrm{Ext}^1(G, \mathcal{O}_X)$, keeping the notations from Section 1, we have $d \leq 2$ (Lemma 7) , $d \geq d'$, $d + d' = 2$ and $r + s = 0$.

There are only two possibilities:

(a) $d = 2$, $d' = 0$, which implies $E \cong \mathcal{O}_X \oplus \mathcal{O}_X(2C_0)$ (Lemma 9).

(b) $d = d' = 1$ and, in this case, in the canonical extension (1) of E, we have

$$\deg(Y) = dd^{'}e - ds - d^{'}r = e \geq 1.$$

By applying Corollary 5, all vector bundles given by non-zero extensions from $\mathrm{Ext}^1(G, \mathcal{O}_X)$ are non-uniform.

§3. Non-emptiness of moduli spaces

For a rank-2 vector bundle E, we shall denote by d_E and r_E the invariants of E, when confusions may appear.

THEOREM 10. *$M(c_1, c_2, d, r)$ is non-empty if and only if $l := l(c_1, c_2, d, r) \geq 0$ and one of the following conditions holds:*
(I) $2d > \alpha$ *or,*
(II) $2d = \alpha, \ \ \beta - 2r \leq g + l.$

Proof. We observe that if $M \neq \emptyset$ then, by means of Theorem 1, the elements of M lie among 2-vector bundles given by extensions of type (1). Therefore, we conclude that $M \neq \emptyset$ if and only if in the extensions of type (1) there are 2-vector bundles with $d_E = d$ and $r_E = r$.

It is clear that all the vector bundles given by an extension of type (1) have $d_E = d$ so we shall look for bundles with $r_E = r$.

We fix $L_1, L_2 \in \mathrm{Pic}_0(C)$ and $Y \subset X$ a locally complete intersection (or the empty set) and we denote

$$N_1 = \mathcal{O}_X(d^{'}C_0 + sf_0) \otimes \pi^* L_1$$

$$N_2 = \mathcal{O}_X(dC_0 + rf_0) \otimes \pi^* L_2$$

and $l = \deg(Y)$.

Consider the spectral sequence of terms

$$E_2^{p,q} = H^p(X, \mathcal{E}xt^q(I_Y \otimes N_1, N_2))$$

which converges to

$$\mathrm{Ext}^{p+q}(I_Y \otimes N_1, N_2).$$

We have

$$\mathcal{E}xt^0(I_Y \otimes N_1, N_2) \cong N_2 \otimes N_1^{-1} \text{ and } \mathcal{E}xt^1(I_Y \otimes N_1, N_2) \cong \mathcal{O}_Y.$$

But $H^2(X, N_2 \otimes N_1^{-1}) = 0$ so the exact sequence of lower terms becomes

$$0 \to H^1(X, N_2 \otimes N_1^{-1}) \to \mathrm{Ext}^1(I_Y \otimes N_1, N_2) \to H^0(Y, \mathcal{O}_Y) \to 0.$$

Now, by a result due to Serre (see [O-S-S], Chap.I, 5, [Se]), any element in the group $\mathrm{Ext}^1(I_Y \otimes N_1, N_2)$ which has an invertible image in $H^0(Y, \mathcal{O}_Y)$ defines an extension of the desired form with E a 2-vector bundle.

We write the sequence (1) under the equivalent form

$$(3) \qquad 0 \to \mathcal{O}_X \to E(-dC_0) \otimes \pi^* L^{''} \to \mathcal{O}_X((d^{'} - d)C_0 + (s-r)f_0) \otimes \pi^*(\tilde{L}) \otimes I_Y \to 0$$

where $\tilde{L} = L_1 \otimes L_2^{-1}$, $L^{''} = \mathcal{O}_C(-r) \otimes L_2^{-1}$ and $\deg(L^{''}) = -r$.

From the definition, it follows $r \leq r_E$ for every bundle E given by an extension (1). We distinguish three cases:
(I) $d > d^{'}$. In this case we shall prove that M is non-empty if and only if $l \geq 0$. To do this we prove that *all* vector bundles from extension (1) have $r_E = r$.

We verify that for all $L^{'} \in \mathrm{Pic}(C)$ with $\deg(L^{'}) < 0$ we have

$$H^0(E(-dC_0) \otimes \pi^*(L^{''} \otimes L^{'})) = 0,$$

which is true because $H^0(L^{'}) = 0$ and

$$H^0(\mathcal{O}_X((d^{'} - d)C_0 + (s-r)f_0) \otimes \pi^*(L_1 \otimes L_2^{-1} \otimes L^{'}) \otimes I_Y) = 0.$$

(II) a°. $d = d^{'}, r \geq s$. Then M is non-empty if and only if $l \geq 0$. The proof runs like in the first case with the remark $\deg(\mathcal{O}_C(s-r) \otimes L_1 \otimes L_2^{-1} \otimes L^{'}) < 0$.
(II) b°. $d = d^{'}, r < s$. Then M is non-empty if and only if $l \geq 0$ and $\beta - 2r \leq g + l$.

Let us see first that the natural isomorphism

$$M(2d, \beta, \gamma, d, r) \longrightarrow M(0, \beta, l, 0, r)$$

$$E \longrightarrow E(-dC_0)$$

allows us to suppose $d = d^{'} = 0$.

In this case, the sequence (3) becomes

$$0 \to \mathcal{O}_X \to E \otimes \mathcal{O}_X(-rf_0) \otimes \pi^* L_2^{-1} \to \mathcal{O}_X((s-r)f_0) \otimes \pi^*(L_1 \otimes L_2^{-1}) \otimes I_Y \to 0.$$

The definition of the second invariant implies that $r_E = r$ if and only if $E^{'} := \pi_* E \otimes \mathcal{O}_C(-rp_0) \otimes L_2^{-1}$ is normalised. $E^{'}$ belong to an extension

$$(4) \qquad 0 \to \mathcal{O}_C \to E^{'} \to L \to 0$$

where $L = \mathcal{O}_C((s-r)p_0) \otimes L_1 \otimes L_2^{-1} \otimes \mathcal{O}_C(-Z_1)$ with Z_1 an effective divisor on C with support $\pi(Y)$ and $\mathrm{card}(Y) \leq \deg(Z_1) \leq l = \deg(Y)$.

According to a result of Nagata ([N] or [Ha] Ex.V.2.5) , if $E^{'}$ is normalised, then

$$-\deg(E') = r - s + \deg(Z_1) \geq -g$$

which proves "only if" part of (II) b°.

For "if" part we choose Y reduced, obtained by intersection between C_0 and l distinct fibres of X. In this case, we have the following short exact sequence

(5) $$0 \to I_Z \to I_Y \to I_{Y \subset Z} \to 0$$

where $Z_1 = \pi(Y) = p_1 + \cdots + p_l$, $Y \subset Z = \pi^{-1}(Z_1) = f_1 + \cdots + f_l$ with f_i distinct fibres, $\mathcal{O}_Z = \mathcal{O}_{f_1} \oplus \cdots \oplus \mathcal{O}_{f_l}$, $I_{Y \subset Z} = \mathcal{O}_{f_1}(-1) \oplus \cdots \oplus \mathcal{O}_{f_l}(-1)$.

So, the sequence (5) becomes

$$0 \to I_Z \to I_Y \to \mathcal{O}_{f_1}(-1) \oplus \cdots \oplus \mathcal{O}_{f_l}(-1) \to 0.$$

Tensoring by $K_X \otimes N_2^{-1} \otimes N_1$ and taking the long cohomology sequence we obtain an injective map:

$$H^1(K_X \otimes N_2^{-1} \otimes N_1 \otimes I_Z) \longrightarrow H^1(K_X \otimes N_2^{-1} \otimes N_1 \otimes I_Y).$$

By dualizing, it follows that the natural map

$$\mathrm{Ext}^1(I_Y \otimes N_1, N_2) \xrightarrow{\varphi} \mathrm{Ext}^1(I_Z \otimes N_1, N_2) \cong \mathrm{Ext}^1(L, \mathcal{O}_C)$$

is surjective, which shows that all bundles in (4) are coming from (1) by applying π_*.

According to [Ha] (Ex. V.2.5), there is a *non-empty* open set $V \subset \mathrm{Ext}^1(L, \mathcal{O}_C)$ (don't forget the condition $s - r \leq g + l$!) such that all $\xi \in V$ define normalised vector bundles on C.

Now, in $\mathrm{Ext}^1(I_Y \otimes N_1, N_2)$ the set of vector bundles is a non-empty open set U. It is clear that $\varphi^{-1}(V) \cap U \neq \emptyset$ (being open sets in Zariski topology), so we conclude.

§4. Moduli of stable bundles

There is an interesting relation between the moduli spaces $M(c_1, c_2, d, r)$ and the Qin's sets $E_\zeta(c_1, c_2)$ (see [Q1], [Q2] for precised definitions).

As in the proof of Theorem 10, case (I) we conclude that if ζ is a normalized class reprezenting a non-empty wall of type (c_1, c_2) such that $l_\zeta(c_1, c_2) > 0$ then, for $(2d - \alpha, 2r - \beta) = \zeta$, $E_\zeta(c_1, c_2)$ and $M(c_1, c_2, d, r)$ are coincident modulo a factor of $\mathrm{Pic}_0(C)$ (Qin workes with first Chern class c_1 as an element in $\mathrm{Pic}(X)$).

This is a consequence of the following facts:

(a) $l_\zeta(c_1, c_2) = l(c_1, c_2, d, r)$

(b) condition $\zeta^2 < 0$ implies $2d > \alpha$

(c) in the case $2d > \alpha$ the bundles L_1, L_2 and the set Y from the sequence (1) are uniquely determined by E.

(d) if $l(c_1, c_2, d, r) > 0$ then in the sequence (1) the bundles are given only by non-trivial extensions.

In fact it is not hard to see that $M(c_1, c_2, d, r) \neq \emptyset$ iff $E_\zeta(c_1, c_2) \neq \emptyset$ so, by means of Theorem 10, $E_\zeta(c_1, c_2) \neq \emptyset$ if $l_\zeta(c_1, c_2) > 0$. But we have even more:

Corollary 11. *Let X be a ruled surface different from $\mathbb{P}^1 \times \mathbb{P}^1$ and let $\mathcal{C}$ be a chamber of type (c_1, c_2) different from $\mathcal{C}_{f_0}$. Then the moduli space $\mathcal{M}_\mathcal{C}(c_1, c_2) \neq \emptyset$.*

Proof. From Theorem 1.3.3 in [Q2] it follows that

$$\mathcal{M}_\mathcal{C}(c_1, c_2) = (\mathcal{M}_{\mathcal{C}_1}(c_1, c_2) - \bigsqcup_\zeta E_{(-\zeta)}(c_1, c_2)) \bigsqcup_\zeta E_\zeta(c_1, c_2) \ ,$$

where $\mathcal{C}_1$ is the chamber lying above $\mathcal{C}$ and sharing with $\mathcal{C}$ a non-empty common wall W and ζ runs over all normalised classes representing W. By the above considerations, it follows that $E_\zeta(c_1, c_2) \neq \emptyset$ if $l(c_1, c_2, d, r) > 0$. It remains the case $l(c_1, c_2, d, r) = 0$ and it will be sufficient to prove that

$$h^1(X, N_2 \otimes N_1^{-1}) := \dim\ H^1(X, N_2 \otimes N_1^{-1}) > 0$$

(see the proof of Theorem 10).

We have

$$N_2 \otimes N_1^{-1} = \mathcal{O}_X((d - d')C_0 + (r - s)f_0) \otimes \pi^*(L_2 \otimes L_1^{-1}) \ ,$$

where $d - d' = 2d - \alpha = u$ and $r - s = 2r - \beta = v$. But $\zeta = uC_0 + vf_0$ is a normalized class and this implies that $u > 0$ and $v < 0$ (see [Q1]).

Because $H^2(X, N_2 \otimes N_1^{-1}) = 0$, the Riemann-Roch Theorem gives the equality:

$$\chi = h^0(X, N_2 \otimes N_1^{-1}) - h^1(X, N_2 \otimes N_1^{-1}) = 1 - g + (1/2)((u+1)(2v - ue) + u(2 - 2g)).$$

But $\zeta^2 < 0$ gives $u(2v - ue) < 0$; it follows $2v - ue < 0$.

If $g \geq 1$, then obviously $\chi < 0$. If $g = 0$, then $e \geq 0$ and

$$\chi = 1 + v + (u/2)(2(v + 1) - e(u + 1)).$$

If $e \geq 1$, then $\chi < 0$. For $e = 0$ we get $X = \mathbb{P}^1 \times \mathbb{P}^1$, which we excluded. Thus, in all cases $\chi < 0$; it follows $h^1(X, N_2 \otimes N_1^{-1}) > 0$ and the proof is over.

Remark. Let us suppose that $X = \mathbb{P}^1 \times \mathbb{P}^1$ and that $\mathcal{C}$ is a chamber of type (c_1, c_2) lying below a non-empty wall defined by a normalized class $\zeta = uC_0 + vf_0$ with $v \leq -2$. Then the same conclusion as in the above corollary holds.

Indeed, in this case we have $\chi = (1+v)(1+u)$. Since $v < -1$, then again $\chi < 0$.

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