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SOME NOTES ON THE MODULI OF STABLE SHEAVES ON ELLIPTIC SURFACES

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Abstract. In this paper, we shall consider the birational structure of moduli of stable sheaves on elliptic surfaces, which is a generalization of Friedman's results to higher rank cases. As applications, we show that some moduli spaces of stable sheaves on $\mathbb{P}^2$ are rational. We also compute the Picard groups of those on Abelian surfaces.

§0. Introduction

Let X be a smooth projective surface over $\mathbb{C}$ and H an ample divisor on X. Let $M_H(r, c_1, \Delta)$ be the moduli of stable sheaves E of rank r on X with $c_1(E) = c_1 \in \mathrm{NS}(X)$ and $\Delta(E) = \Delta$, where $\Delta(E) := c_2(E) - \{(\mathrm{rk}(E) - 1)/2\,\mathrm{rk}(E)\}(c_1(E)^2)$. In this note, we shall consider the moduli spaces on elliptic surfaces. Let $\pi : X \to B$ be an elliptic surface such that every singular fibre is irreducible. We denote the algebraic equivalence class of fibres by f. We assume that X is regular, the intersection number (c_1, f) is odd and H is sufficiently close to f. Then Friedman [F] showed that $M_H(2, c_1, \Delta)$ is birationally equivalent to $S^n(J^d X)$, where $n = \dim M_H(2, c_1, \Delta)/2$, $2d+1 = (c_1, f)$ and $J^d X$ is an elliptic surface over B whose generic fibre is the set of line bundles of degree d. In this note, we shall generalize it to the case where r and (c_1, f) are relatively prime.

As an application, we shall show that $M_H(r, kH, \Delta)$ is a rational variety if $(X, H) = (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$ and $(r, 3k) = 1$. We also consider moduli spaces on Abelian surfaces. In particular, we shall find a set of generators of $H^2(M_H(r, c_1, \Delta), \mathbb{Z})$. For general surfaces, Li [Li1], [Li2] considered the structure of $H^i(M_H(2, c_1, \Delta), \mathbb{Q})$, $i \le 2$ and $\mathrm{Pic}(M_H(2, c_1, \Delta)) \otimes \mathbb{Q}$ for $\Delta \gg 0$. For the integral cohomologies, Mukai [Mu3], [Mu5] and O'Grady [O] investigated the structure of $H^2(M_H(r, c_1, \Delta), \mathbb{Z})$ and the Picard group, if X is a K3 surface. By the same method as in [Y2], we get a set of generators of $H^2(M_H(r, c_1, \Delta), \mathbb{Z})$, if X is a ruled surface. Our results for Abelian

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surfaces are similar to these results.

In section 1, we shall consider the birational structure of $M_H(r, c_1, \Delta)$. Our method is the same as that in Friedman [F] and Maruyama [Ma2]. That is, we shall use elementary transformations. For simplicity, we assume that X is regular. Let E be a member of $M_H(r, c_1, \Delta)$. Since H is sufficiently close to the fibre, $E|_{\pi^{-1}(\eta)}$ is a stable vector bundle on the generic fibre $\pi^{-1}(\eta)$. Then there is a stable vector bundle E_1 on X such that $E_1|_l$ is semi-stable in the sense of Simpson [S] for all fibres l, and E is obtained from E_1 by successive elementary transformations along coherent sheaves of pure dimension 1 on fibres. Let E_2 be a stable vector bundle such that $E_2|_{\pi^{-1}(\eta)} \cong E_1|_{\pi^{-1}(\eta)}$, $E_2|_l$ is semi-stable in the sense of Simpson and $\det E_2|_l \cong \det E_1|_l$ for all fibres l. By using the irreducibility of l, we shall show that $E_2 \cong E_1 \otimes \pi^* L$, where $L \in \operatorname{Pic}(B)$. Then we can easily show that $S^n(J^d X)$ is birationally equivalent to an irreducible component of $M_H(r, c_1, \Delta)$, where $n = \dim M_H(r, c_1, \Delta)/2$ and d is an integer. By the dimension counting of non-locally free part (cf. [Y1, Thm. 0.4]), we see that every irreducible component contains vector bundles (In fact, the non-locally free part is of codimension $r-1$). Let E be a vector bundle of $M_H(r, c_1, \Delta)$. We note that $\operatorname{Ext}^2(E, E(-l))_0 \cong \operatorname{Hom}(E, E(K_X + l))_0^{\vee} = 0$ for all fibres l, where $\operatorname{Ext}^i(E, E(D))_0$ is the trace free part of $\operatorname{Ext}^i(E, E(D))$. Then $\operatorname{Ext}^1(E, E)_0 \to \operatorname{Ext}^1(E|_l, E|_l)_0$ is surjective. Considering the deformation space of $E|_l$, we shall show that $M_H(r, c_1, \Delta)$ is birationally equivalent to $S^n(J^d X)$.

In section 2, we shall treat the moduli spaces on $\mathbb{P}^2$. Let $V \subset H^0(\mathbb{P}^2, K_{\mathbb{P}^2}^{\vee})$ be a linear pencil which contains an elliptic curve C. Since $(K_{\mathbb{P}^2}, H) < 0$, we can deform $E \in M_H(r, c_1, \Delta)$ to a sheaf $E' \in M_H(r, c_1, \Delta)$ such that $E'|_C$ is semi-stable. If (c_1, H) and r are relatively prime, then $E'|_C$ is a stable vector bundle. Let $\mathbb{P}^2 \to \mathbb{P}^1$ be the rational map defined by V and $Y \to \mathbb{P}^2$ the blow-ups of $\mathbb{P}^2$ which defines the morphism $Y \to \mathbb{P}^1$. Then $M_H(r, c_1, \Delta)$ is birationally equivalent to a component of a moduli space $M_{H'}(r, c_1, \Delta)$, where H' is an ample divisor on Y which is sufficiently close to the fibre in $\operatorname{NS}(Y)$. Since $M_{H'}(r, c_1, \Delta)$ is birationally equivalent to a symmetric product of Y, we get that $M_H(r, c_1, \Delta)$ is rational. We also prove that the moduli of simple torsion free sheaves on Del Pezzo surfaces are irreducible.

In section 3, we shall consider the moduli spaces on an Abelian surface. We assume that $c_1 \bmod \operatorname{NS}(X)$ is a primitive element of $\operatorname{NS}(X)/r\operatorname{NS}(X)$. Mukai [Mu1] gave a complete description of $M_H(r, c_1, \Delta)$ in the case where

$\dim M_H(r, c_1, \Delta) = 2$. Hence we assume that $\dim M_H(r, c_1, \Delta) \geq 4$. By using a quasi-universal family [Mu3], we shall construct a set of generators of $H^i(M_H(r, c_1, \Delta), \mathbb{Z})$ for $i = 1, 2$, where H is a general polarization (Theorem 3.1). Our method is the same as in Göttsche and Huybrechts [G-H], that is, we shall deform X to a product of elliptic curves. Then $M_H(r, c_1, 0)$ is isomorphic to X and $M_H(r, c_1, \Delta)$ is birationally equivalent to $X \times \mathrm{Hilb}_X^{r\Delta}$. Since both spaces have trivial canonical bundles, there are closed subsets $Z_1 \subset M_H(r, c_1, \Delta)$ and $Z_2 \subset X \times \mathrm{Hilb}_X^{r\Delta}$ such that $\mathrm{codim}(Z_1) \geq 2$, $\mathrm{codim}(Z_2) \geq 2$ and $M_H(r, c_1, \Delta) \setminus Z_1 \cong (X \times \mathrm{Hilb}_X^{r\Delta}) \setminus Z_2$. Hence we get an isomorphism $H^i(M_H(r, c_1, \Delta), \mathbb{Z}) \cong H^i(X \times \mathrm{Hilb}_X^{r\Delta}, \mathbb{Z})$, $i = 1, 2$. Constructing a family of stable sheaves parametrized by $X \times \mathrm{Hilb}_X^{r\Delta} \setminus Z_2$ directly, we shall construct a set of generators of $H^i(M_H(r, c_1, \Delta), \mathbb{Z})$, $i = 1, 2$. By using deformation of X and the result in [Y4], we shall also show that the Betti numbers of $M_H(2, c_1, \Delta)$ are the same as those of $M_H(1, 0, 2\Delta)$ (Theorem 3.5). We next show that the morphism $M_H(r, c_1, \Delta) \to \mathrm{Pic}^0(X) \times X$ defined in [Y2, Sect. 5] is an Albanese map, if $\dim M_H(r, c_1, \Delta) \geq 4$. Combining all together, we also describe the Picard group of $M_H(r, c_1, \Delta)$ (Theorem 3.6).

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Notation. Let X be a smooth projective surface over $\mathbb{C}$ and H an ample divisor on X. For a scheme S, we denote the projection $S \times X \to S$ by p_S. We denote the Néron-Severi group of X by $\mathrm{NS}(X)$. For an $x \in \mathrm{NS}(X) \otimes \mathbb{Q}$, we set $P(x) := (x, x - K_X)/2 + \chi(\mathcal{O}_X)$.

For a torsion free sheaf E on X, we set

$$\Delta(E) := c_2(E) - \frac{\mathrm{rk}(E) - 1}{2\,\mathrm{rk}(E)}(c_1(E)^2).$$

We denote the traceless part of $\mathrm{Ext}^i(E, E(D))$ by $\mathrm{Ext}^i(E, E(D))_0$.

In this note, we use the notion of (semi-)stability of Mumford. Let $M_H(r, c_1, \Delta)$ be the moduli of stable sheaves E of rank r on X with $c_1(E) = c_1 \in \mathrm{NS}(X)$ and $\Delta(E) = \Delta$. We denote the open subscheme of $M_H(r, c_1, \Delta)$ consisting of stable vector bundles by $M_H(r, c_1, \Delta)_0$.

§1. Moduli spaces on elliptic surfaces

1.1. Preliminaries

Let $\pi : X \to B$ be an elliptic surface. We assume that every fibre is irreducible throughout this setion. We denote a fibre by f. Let η be the generic point of the base curve B. Let $J^d X \to B$ be the elliptic surface over B whose generic fibre is the set of line bundles of degree d on $X|_{\pi^{-1}(\eta)}$. For a coherent sheaf E on a fibre l, we set

$$\begin{aligned}\operatorname{rk}(E) &:= \operatorname{length}_{\mathcal{O}_{\eta_l}}(E \otimes \mathcal{O}_{\eta_l}),\\ \deg(E) &:= \chi(E),\end{aligned}$$

where η_l is the generic point of l.

A coherent sheaf E of pure dimension 1 on a fibre l is semi-stable if

$$\frac{\chi(F)}{\operatorname{rk}(F)} \leq \frac{\chi(E)}{\operatorname{rk}(E)}$$

for all subsheaves $F \neq 0$ of E.

Lemma 1.1. *Let L be a relatively ample divisor on X. Let D be a divisor on X such that $(D, f) \neq 0$ and $(D, L + kf) = 0$ for some positive number k. Then,*

$$(D^2) \leq \frac{-1}{(L, f)^2}((L^2) + 2k(L, f)). \tag{1.1}$$

Proof. We set $D = aL + bf + D'$, where $a, b \in \mathbb{Q}$ and $(D', L) = (D', f) = 0$. By the Hodge index theorem, $({D'}^2) \leq 0$. Hence $(D^2) = ((aL + bf)^2) + ({D'}^2) \leq ((aL + bf)^2) = a^2(L^2) + 2ab(L, f)$. Thus we may assume that $D = aL + bf$. $(D, L + kf) = 0$ implies that $b(L, f) = -a(L, L + kf)$. Hence $((aL + bf)^2) = -a^2((L^2) + 2k(L, f))$. Since $(L, f) \neq 0$, we get that $|a| \geq 1/|(L, f)|$. Hence (1.1) holds. ∎

Lemma 1.2. *Let r be a positive integer and c_1 an algebraically equivalence class on X such that (c_1, f) and r are relatively prime. Let L be an ample divisor on X. Then*

$$M_{L+nf}(r, c_1, \Delta) = \left\{ E \;\middle|\; \begin{array}{l} E \text{ is torsion free of rank } r \text{ with } (c_1(E), \Delta(E)) \\ = (c_1, \Delta) \text{ and } E|_{\pi^{-1}(\eta)} \text{ is stable.} \end{array} \right\}$$

for $n > (r^3(L, f)^2\Delta - 2(L^2))/4(L, f)^2$. We denote this space by $M(r, c_1, \Delta)$.

Proof. The proof is similar to that in [Y3, Prop. 6.2] (in [Y3], we used slightly different definition of Δ). ∎

Since $\operatorname{Ext}^2(E,E)_0 \cong \operatorname{Hom}(E,E)_0^\vee = 0$, $E \in M(r,c_1,\Delta)$, $M(r,c_1,\Delta)$ is smooth of dimension $2r\Delta-(r^2-1)\chi(\mathcal{O}_X)+\dim \operatorname{Pic}^0(X)$. For a stable sheaf $E \in M(r,c_1,\Delta)$, $\chi(E|_f)=(c_1,f)$ and $\chi(E\otimes k_x)=r$ are relatively prime, where E is locally free at $x \in X$ and k_x is the structure sheaf of x. Hence there is a universal family (cf. [Ma1, Thm. 6.11]).

LEMMA 1.3. *Let E be a vector bundle of rank r on X such that $(c_1(E),f)=d$, and let F be a coherent sheaf of pure dimension 1 on a fibre l with $\operatorname{rk}(F)=r_1$ and $\deg(F)=d_1$. Let $E\to F$ be a surjective homomorphism and E' the kernel. Then*

$$\Delta(E') = \Delta(E) + \frac{rd_1-r_1d}{r}. \tag{1.2}$$

Proof. For a coherent sheaf G on X, $\chi(G)=\operatorname{rk}(G)P(c_1(G)/\operatorname{rk}G)-\Delta(G)$. Since $\chi(E)=\chi(E')+\chi(F)$,

$$\begin{aligned}\Delta(E')-\Delta(E) &= d_1 - r(P(c_1(E)/\operatorname{rk}E)-P(c_1(E')/\operatorname{rk}E'))\\ &= d_1-\frac{r_1d}{r}.\end{aligned}$$

∎

The following is a special case of Maruyama [Ma2].

PROPOSITION 1.4. *Let E be a vector bundle on X such that $E|_{\pi^{-1}(\eta)}$ is a semi-stable vector bundle. Then there is a vector bundle E' on X such that $E'|_l$ is semi-stable for every fibre l and E is obtained from E' by successive elementary transformations along coherent sheaves of pure dimension 1 on fibres.*

Proof. We note that $\Delta(E)\geq 0$. We shall prove our claim by induction on $\Delta(E)$. We assume that there is a fibre l such that $E|_l$ is not semi-stable. Then there is a surjective homomorphism $E|_l \to F$ such that F is of pure dimension 1 and $\chi(E|_l)/\operatorname{rk}(E|_l) > \chi(F)/\operatorname{rk}F$. We shall consider the following elementary transformation along F:

$$0 \longrightarrow E_1 \longrightarrow E \longrightarrow F \longrightarrow 0.$$

Since $\operatorname{depth}_{\mathcal{O}_x} F = 1$, $x\in l$ and X is smooth, we see that $\operatorname{proj-dim}_{\mathcal{O}_x} F = \dim X - \operatorname{depth}_{\mathcal{O}_x} F = 1$. Hence E_1 is also locally free. By Lemma 1.3, we get that $\Delta(E_1)<\Delta(E)$. Hence we obtain our corollary.

1.2. General element of $M(r, c_1, \Delta)$

If we fix the rank r and the equivalence class $c_1 \bmod \pi^* H^1(B, \mathbb{Z})$, then we may denote $M(r, c_1, \Delta)$ by $M(\Delta)$. In fact, $c_1 \bmod r\pi^* H^2(B, \mathbb{Z})$ is determined by $r\Delta$ and the isomorphic class of $M_H(r, c_1, \Delta)$ is determined by r, $c_1 \bmod r\pi^* H^2(B, \mathbb{Z})$ and Δ.

Let E be a general element of $M(\Delta)$. We shall consider the Harder-Narasimhan filtration of the restriction $E|_l$ of E to fibres l. In particular, we shall show that $E|_l$ is semi-stable for all singular fibres l.

LEMMA 1.5. *Let C be a projective curve and $\mathcal{O}_C(1)$ an ample divisor on C. Let L be a line bundle on C. Let Q be the subscheme of $Quot_{\mathcal{O}_C(-n)^{\oplus N}/C}$ parametrizing quotients $\mathcal{O}_C(-n)^{\oplus N} \to F$ such that* (i) *F is a locally free sheaf of rank r with* $\det F = L$ *and* (ii) $H^1(C, F(n)) = 0$. *Then Q is smooth and irreducible.*

Proof. Let $\lambda : \mathcal{O}_C(-n)^{\oplus N} \to F$ be a quotient which belongs to Q. Then we see that $\operatorname{Ext}^1(\ker \lambda, F) = 0$. Since $\operatorname{Hom}(\ker \lambda, F) \to \operatorname{Ext}^1(F, F) \overset{\mathrm{tr}}{\to} H^1(C, \mathcal{O}_C)$ is surjective, Q is smooth. For $k \geq n$, there is an exact sequence $0 \to \mathcal{O}_C^{\oplus(r-1)} \to F(k) \to L(rk) \to 0$. We set $\mathbb{P} := \mathbb{P}(\operatorname{Ext}^1(L(rk), \mathcal{O}_C^{\oplus(r-1)})^\vee)$. We shall consider the universal extension:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}\times C}^{\oplus(r-1)} \longrightarrow \mathcal{F} \longrightarrow L(rk) \otimes \mathcal{O}_{\mathbb{P}}(-1) \longrightarrow 0.$$

Let $\mathbb{P}'$ be the open subscheme of $\mathbb{P}$ of points y such that $H^1(C, \mathcal{F}_y) = 0$. Then $p_{\mathbb{P}'*}(\mathcal{F})$ is a locally free sheaf on $\mathbb{P}'$. Let $\phi : \mathbb{A} \to \mathbb{P}'$ be the vector bundle associated to the locally free sheaf $\mathcal{H}om(\mathcal{O}_{\mathbb{P}'}^{\oplus N}, p_{\mathbb{P}'*}(\mathcal{F}))$. Then there is a homomorphism $\Lambda : \mathcal{O}_{\mathbb{A}\times C}^{\oplus N} \to (\phi \times 1)^* \mathcal{F}$. Let $\mathbb{A}'$ be the open subscheme of $\mathbb{A}$ such that Λ is surjective. Then there is a surjective morphism $\mathbb{A}' \to Q$, and hence Q is irreducible.

PROPOSITION 1.6. *Let $M(\Delta)^0$ be the open subscheme of $M(\Delta)$ of elements E such that $E|_l$ is semi-stable for every singular fibre l. Then $M(\Delta)^0$ is a dense subscheme of $M(\Delta)$.*

Proof. We note that $M(\Delta)_0 := M(r, c_1, \Delta)_0$ is an open dense subscheme of $M(\Delta)$ (cf. [Y1, Thm. 0.4]). Hence it is sufficient to prove that $M(\Delta)^0 \cap M(\Delta)_0$ is an open dense subscheme of $M(\Delta)_0$. Let E be a locally free stable sheaf of $M(\Delta)$. Since $E|_{\pi^{-1}(\eta)}$ is stable, we see that $\operatorname{Ext}^2(E, E(-l))_0 \cong \operatorname{Hom}(E, E(l+K_X))_0^\vee = 0$. Hence we get that $\operatorname{Ext}^1(E, E)_0 \to \operatorname{Ext}^1(E|_l, E|_l)_0$

is surjective. Let m be the multiplicity of l and set $l = ml'$. By Proposition 1.4, there is a vector bundle E_1 on X such that $E_1|_l$ is semi-stable and $\det(E_1|_l) = \det(E|_l) \otimes \mathcal{O}_l(kl')$. Since $(c_1, f) = m(c_1, l')$, our assumption on r and c_1 implies that $(r, m) = 1$. Replacing E_1 by $E_1 \otimes \mathcal{O}_X(jl')$, we may assume that $\det(E_1|_l) = \det(E|_l)$. Let $\mathrm{Def}(E|_l)$ be the local deformation space of $E|_l$ of fixed determinant line bundle. We shall show that $\mathrm{Def}(E|_l)^u := \{F \in \mathrm{Def}(E|_l) \mid F \text{ is not semi-stable}\}$ is a proper closed subset of $\mathrm{Def}(E|_l)$. In the notation of Lemma 1.5, we assume that $L = \det(E_1|_l)$ and n is a sufficiently large integer such that there are quotients $\mathcal{O}_l(-n)^{\oplus N} \to E|_l$, $\mathcal{O}_l(-n)^{\oplus N} \to E_1|_l$ which belong to Q. We also assume that $H^0(l, \mathcal{O}_l^{\oplus N}) \to H^0(l, E|_l(n))$ and $H^0(l, \mathcal{O}_l^{\oplus N}) \to H^0(l, E_1|_l(n))$ are isomorphisms. Let $\mathcal{O}_{Q\times l}(-n)^{\oplus N} \to \mathcal{Q}$ be the universal quotient. Since $E_1|_l$ is semi-stable, Lemma 1.5 implies that $Q^u := \{y \in Q \mid \mathcal{Q}_y \text{ is not semi-stable}\}$ is a proper closed subset of Q. Since $\mathrm{Def}(E|_l)$ is a transversal slice of the $\mathrm{Aut}(\mathcal{O}_l^{\oplus N})$-orbit and Q^u is $\mathrm{Aut}(\mathcal{O}_l^{\oplus N})$-invariant, $\mathrm{Def}(E|_l)^u$ is also a proper closed subset of $\mathrm{Def}(E|_l)$. Combining the surjectivity of the homomorphism: $\mathrm{Ext}^1(E, E)_0 \to \mathrm{Ext}^1(E|_l, E|_l)_0$, we get that $M(\Delta)^0$ is an open dense subscheme of $M(\Delta)$. ∎

LEMMA 1.7. *Let l be a smooth fibre. Let $h := \{(r_1, d_1), \ldots, (r_s, d_s)\}$ be a sequence of pairs of integers such that $r_i > 0$, $1 \le i \le s$ and $d_1/r_1 > d_2/r_2 > \cdots > d_s/r_s$. Let D_h be the subset of $M(r, c_1, c_2)$ of elements E such that the Harder-Narasimhan filtration of $E|_l : 0 \subset F_1 \subset F_2 \subset \cdots \subset F_s = E|_l$ satisfies that $\mathrm{rk}(F_i/F_{i-1}) = r_i$ and $\deg(F_i/F_{i-1}) = d_i$, $1 \le i \le s$. Then $\mathrm{codim}(D_h) \ge \sum_{i<j} r_j d_i - r_i d_j$. In particular, if $\mathrm{codim}(D_h) = 1$, then $s = 2$ and $r_2 d_1 - r_1 d_2 = 1$.*

Proof. Let $\mathrm{Def}(E|_l)_h$ be the subset of $\mathrm{Def}(E|_l)$ of elements G such that the Harder-Narasimhan filtration of $G : 0 \subset F_1 \subset F_2 \subset \cdots \subset F_s = G$ satisfies that $\mathrm{rk}(F_i/F_{i-1}) = r_i$ and $\deg(F_i/F_{i-1}) = d_i$, $1 \le i \le s$. We assume that $\mathrm{Def}(E|_l)_h$ is not empty. We note that $\mathrm{Ext}^1(E, E)_0 \to \mathrm{Ext}^1(E|_l, E|_l)_0$ is surjective. It is known that $\mathrm{codim}(\mathrm{Def}(E|_l)_h) = \sum_{i<j} r_j d_i - r_i d_j$ (cf. [A-B, Thm. 7.14]). Hence we get our lemma. ∎

Let (r_1, d_1) be the pair of integers such that $0 < r_1 < r$ and $rd_1 - r_1 d = 1$. Let $M(\Delta)^1$ be the open subscheme of $M(\Delta)^0$ of elements E such that $E|_l$ is stable, or the Harder-Narasimhan filtration of $E|_l$ is $0 \subset F \subset E|_l$ for every smooth fibre l, where F is a stable vector bundle of rank r_1 on l with $\deg(F) = d_1$. Then $M(\Delta)^1$ is an open dense subscheme of $M(\Delta)^0$.

1.3. Vector bundles on elliptic curves

The following is due to Atiyah [A].

LEMMA 1.8. *Let C be a smooth elliptic curve. Let r be a positive integer and d an integer such that $(r,d)=1$. Then,*
(1) There is a stable vector bundle of rank r and degree d.
(2) Let (r_1,d_1) be the pair of integers such that $r_1d-rd_1=1$ and $0<r_1<r$. Let E_1 be a stable vector bundle of rank r_1 and degree d_1. Then every stable vector bundle E of rank r and degree d is defined by an exact sequence

$$0 \longrightarrow E_1 \longrightarrow E \longrightarrow E_2 \longrightarrow 0, \tag{1.3}$$

where E_2 is a stable vector bundle of rank $r_2:=r-r_1$ and degree $d_2:=d-d_1$.
(3) Let $0\subset F_1\subset F_2\subset\cdots\subset F_s=E$ be the Harder-Narasimhan filtration of a vector bundle E. Then $E\cong\bigoplus_{i=1}^s E_i$, where $E_i:=F_i/F_{i-1}$.

Proof. (1) We shall prove our claim by induction on r. If $r=1$, then our claim obviously holds. Let (r_1,d_1) be the pair of integers such that $r_1d-rd_1=1$ and $0<r_1<r$. We set $r_2:=r-r_1$ and $d_2:=d-d_1$. By induction hypothesis, there are stable vector bundles E_i of rank r_i and degree d_i, $i=1,2$. Since $d_1/r_1<d_2/r_2$, $\operatorname{Hom}(E_2,E_1)=0$. By using the Riemann-Roch theorem, we get that $\operatorname{Ext}^1(E_2,E_1)\cong\mathbb{C}$. Let $0\to E_1\to E\to E_2\to 0$ be a non-trivial extension. We shall show that E is stable. If E is not stable, then there is a semi-stable subsheaf G of E such that $\deg G/\operatorname{rk}G>d/r$. Since G and E_2 are semi-stable and $G\to E\to E_2$ is not zero, $\deg G/\operatorname{rk}G\le d_2/r_2$. We assume that $\deg G/\operatorname{rk}G<d_2/r_2$. Then we see that $1/rr_2=d_2/r_2-d/r>d_2/r_2-\deg G/\operatorname{rk}G\ge 1/r_2\operatorname{rk}G$, which is a contradiction. Hence $\deg G/\operatorname{rk}G=d_2/r_2$. Then we get that $\operatorname{rk}G=r_2$ and $\deg G=d_2$. Hence $G\cong E_2$, which is a contradiction.

(2) Let E be a stable vector bundle of rank r and degree d. Then $\operatorname{Ext}^1(E_1,E)\cong\operatorname{Hom}(E,E_1)^\vee=0$. By the Riemann-Roch theorem, there is a non-zero homomorphism $\varphi:E_1\to E$. We shall show that φ is injective and $\operatorname{coker}\varphi$ is stable. Since E_1 and E are stable, $d_1/r_1\le\deg\varphi(E_1)/\operatorname{rk}\varphi(E_1)<d/r$. In the same way as in the proof of (1), we see that $\operatorname{rk}\varphi(E_1)=r_1$ and $\deg\varphi(E_1)=d_1$. Hence we get that $E_1\cong\varphi(E_1)$. We set $E_2:=\operatorname{coker}\varphi$. We assume that there is a quotient G of E_2 such that G is semi-stable and $d_2/r_2>\deg G/\operatorname{rk}G$. Since G is a quotient of E, we get that $d/r<\deg G/\operatorname{rk}G$. Hence we get that $d/r<\deg G/\operatorname{rk}G<d_2/r_2$. Then $1/rr_2=$

$d_2/r_2 - d/r > d_2/r_2 - \deg G/\operatorname{rk} G \geq 1/r_2 \operatorname{rk} G$, which is a contradiction. Hence E_2 is a stable vector bundle.

(3) Since $\deg E_i/\operatorname{rk} E_i > \deg E_j/\operatorname{rk} E_j$, $i < j$, the Serre duality implies that $\operatorname{Ext}^1(E_j, E_i) = 0$, $i < j$. By the induction on s, we see that $E \cong \bigoplus_i E_i$. ∎

Lemma 1.9. *Let* (r,d) *(resp.* (r_1,d_1), (r_2,d_2)*) be the pair in Lemma* 1.8. *Let* E *be a vector bundle of rank* r *on an elliptic curve* C *with degree* d *and* E_2 *a stable vector bundle of rank* r_2 *on* C *with degree* d_2.
(1) *If* E *is stable, then* $\operatorname{Hom}(E, E_2) \cong \mathbb{C}$ *and a non-zero homomorphism is surjective.*
(2) *Let* F_1 *(resp.* F_2*) be a stable vector bundle of rank* r_1 *and degree* d_1 *(resp. rank* r_2 *and degree* d_2*). We assume that* $E \cong F_1 \oplus F_2$ *and there is a surjective homomorphism* $\varphi : E \to E_2$ *such that* $\ker \varphi$ *is also stable. Then* $E_2 \cong F_2$ *and* $\operatorname{Hom}(E, E_2) \cong \mathbb{C}^{\oplus 2}$.

Proof. (1) Since E is stable, $\operatorname{Ext}^1(E, E_2) \cong \operatorname{Hom}(E_2, E)^\vee = 0$. By the Riemann-Roch theorem, we see that $\dim \operatorname{Hom}(E, E_2) = 1$. In the same way as in the proof of Lemma 1.8, we see that a non-zero homomorphism $E \to E_2$ is surjective.

(2) If $E_2 \not\cong F_2$, then $\ker \varphi \cong \ker(\varphi|_{F_1}) \oplus F_2$. Since $\varphi|_{F_1} : F_1 \to E_2$ is surjective, $\ker(\varphi|_{F_1}) \neq 0$. Hence $E_2 \cong F_2$. By the Riemann-Roch theorem, $\operatorname{Hom}(F_1, E_2) \cong \mathbb{C}$. Therefore $\operatorname{Hom}(E, E_2) \cong \mathbb{C}^{\oplus 2}$.

1.4.

Let B_0 be the open subscheme of B such that $\pi : X_0 := X \times_B B_0 \to B_0$ is smooth. We assume that π has a section σ. We denote the relative moduli space of stable vector bundles of rank r on fibres with degree d by $\mathcal{M}_{X_0/B_0}(r,d) \to B_0$. We assume that $(r,d) = 1$. We shall construct a family of stable vector bundles $\mathcal{E}_{r,d}$ on $X_0 \times_{B_0} X_0$ and show that $\mathcal{M}_{X_0/B_0}(r,d) \cong X_0$ as a B_0-scheme, by using induction on r. If $r = 1$, then $\mathcal{E}_{1,d} := \mathcal{O}_{X_0 \times_{B_0} X_0}((d+1)\sigma - \Delta)$ is a universal family, where Δ is the diagonal of $X_0 \times_{B_0} X_0$. Let (r_1, d_1) be the pair of integers such that $r_1 d - r d_1 = 1$ and $0 < r_1 < r$. We set $r_2 = r - r_1$ and $d_2 = d - d_1$. Let E be a vector bundle on X_0 such that $E|_l$ is a stable vector bundle of rank r_2 and $\det E|_l \cong \mathcal{O}_l(d_2\sigma)$ for every fibre l. By using Lemma 1.8, we see that $\mathcal{L} := \operatorname{Ext}^1_{p_{X_0}}(E, \mathcal{E}_{r_1,d_1})$ is a line bundle on X_0. Then there is the universal extension

$$0 \longrightarrow \mathcal{E}_{r_1,d_1} \longrightarrow \mathcal{E}_{r,d} \longrightarrow E \otimes p_{X_0}^*(\mathcal{L}) \longrightarrow 0, \tag{1.4}$$

which parametrizes stable vector bundles of rank r on fibres with degree d. Hence there is a morphism $X_0 \to \mathcal{M}_{X_0/B_0}(r,d)$. By our construction, this morphism is injective. By ZMT, it is an isomorphism.

LEMMA 1.10. *Let E and E' be semi-stable vector bundles on a multiple fibre $l = ml'$ such that $\operatorname{rk} E = \operatorname{rk} E'$, $\det E \cong \det E'$, and $\chi(E) = \chi(E') = d$. Then,*

$$\operatorname{Hom}(E,E') = \begin{cases} \mathbb{C}, & \text{if } E \cong E', \\ 0, & \text{otherwise.} \end{cases} \tag{1.5}$$

Proof. We set $L := \mathcal{O}_X(-l')|_{l'}$. We note that $\operatorname{rk}(E \otimes L^{\otimes k}) = r$ and $\chi(E \otimes L^{\otimes k}) = d/m$ for $0 \le k \le m-1$. Since $(r,d) = 1$ and E is semi-stable, $E \otimes L^{\otimes k}$ is a stable sheaf on ml'. Thus $0 \subset E(-(m-1)l') \subset E(-(m-2)l') \subset \cdots \subset E(-l') \subset E$ is a Jordan-Hölder filtration of E. Since the order of $L \in \operatorname{Pic}^0(l')$ is m and $(m,r) = 1$, $\det E \cong \det E'$ and the stabilities of $E|_{l'}$ and $E|_{l'}$ imply that $\operatorname{Hom}(E|_l, E' \otimes L^{\otimes k}) = 0$ for $1 \le k \le m-1$. Let $\varphi : E \to E'$ be a non-zero homomorphism. We shall show that φ is an isomorphism. Since $\operatorname{Hom}(E|_l, E' \otimes L^{\otimes k}) = 0$ for $1 \le k \le m-1$, we see that $\varphi|_{l'} : E|_{l'} \to E'|_{l'}$ is not zero, which implies that $E|_{l'} \cong E'|_{l'}$. By Nakayama's lemma, φ is an isomorphism. Then it is easy to see that $\operatorname{Hom}(E,E') \cong \mathbb{C}$. ∎

LEMMA 1.11. *Let E, E' be vector bundles of rank r on X such that $E|_l$ and $E'|_l$ are semi-stable for all fibres l and $\det E \cong \det E'$. Then there is a line bundle L on B such that $E \cong E' \otimes \pi^*(L)$.*

Proof. We note that $E|_{\pi^{-1}(\eta)} \cong E'|_{\pi^{-1}(\eta)}$. By the upper semi-continuity of $h^0(l, E'^{\vee} \otimes E|_l)$, there is a non-zero homomorphism $E'|_l \to E|_l$ for every fibre l. Since $E|_l$ and $E'|_l$ are semi-stable, Lemma 1.10 implies that $E|_l \cong E'|_l$ and $H^0(l, E'^{\vee} \otimes E|_l) \cong \mathbb{C}$. By the base change theorem, we get that $L := \pi_*(E'^{\vee} \otimes E)$ is a line on B and $\pi^*(L) \otimes E' \to E$ is an isomorphism. ∎

COROLLARY 1.12. *$M(\Delta)$ is not empty if and only if $\Delta \ge \Delta_0 := \frac{(r^2-1)}{2r}\chi(\mathcal{O}_X)$.*

Proof. We set $\Delta' := \min\{\Delta \mid M(\Delta) \ne \emptyset\}$. Lemma 1.11 implies that $\dim \operatorname{Pic}^0(X) = \dim M(\Delta') = 2r\Delta' - (r^2-1)\chi(\mathcal{O}_X) + \dim \operatorname{Pic}^0(X)$. Hence we get our claim. ∎

Remark 1.1. Let E be an element of $M(\Delta_0)$. By Lemma 1.11, there is a surjective morphism $\mathrm{Pic}^0(X) \to M(\Delta_0)$ sending $L \in \mathrm{Pic}^0(X)$ to $E \otimes L$. Hence we get that $M(\Delta_0) = \mathrm{Pic}^0(X)/\Phi(E)$, where $\Phi(E) := \{L \in \mathrm{Pic}^0(X) \mid E \otimes L \cong E\}$. In particular, if $\mathrm{Pic}^0(X) = \mathrm{Pic}^0(B)$, then $M(\Delta_0) = \mathrm{Pic}^0(X)$.

1.5. Construction of a family

We assume that $\pi : X \to B$ has a section and show that $M(\Delta)$ is birational to $M(\Delta_0) \times S^n X$, where $n := r(\Delta - \Delta_0)$. Let $\mathcal{E}$ be a universal family on $M(\Delta_0) \times X$. Let (r_1, d_1) be the pair of integers such that $r_1 d - r d_1 = -1$ and $0 < r_1 < r$, and let $\mathcal{E}_{r_1,d_1}$ be the vector bundle on $X_0 \times_{B_0} X_0$. Let $j : X_0 \times_{B_0} X_0 \to X_0 \times X$ be the immersion. We denote the projection $M(\Delta_0) \times X_0 \to M(\Delta_0)$ by q_1 and $M(\Delta_0) \times X_0 \to X_0$ by q_2. By Lemma 1.9, $\mathcal{L} := \mathrm{Hom}_{p_{M(\Delta_0)\times X_0}}((q_1 \times 1_X)^*\mathcal{E}, (q_2 \times 1_X)^* j_* \mathcal{E}_{r_1,d_1})$ is a line bundle on $M(\Delta_0) \times X_0$, and there is a surjective homomorphism: $(q_1 \times 1_X)^*\mathcal{E} \to (q_2 \times 1_X)^* j_* \mathcal{E}_{r_1,d_1} \otimes p^*_{M(\Delta_0)\times X_0}(\mathcal{L})^\vee$. Let $p_i : X_0^n := X_0 \times X_0 \times \cdots \times X_0 \to X_0$ be the i-th projection, $1 \le i \le n$. Then there is a homomorphism

$$\Lambda : \widetilde{\mathcal{E}} \longrightarrow \bigoplus_{i=1}^{n} (q_2 \circ (1_{M(\Delta_0)} \times p_i) \times 1_X)^* j_* \mathcal{E}_{r_1,d_1} \otimes \mathcal{L}_i, \tag{1.6}$$

where $\widetilde{\mathcal{E}}$ is the pull-back of $\mathcal{E}$ to $M(\Delta_0) \times X_0^n \times X$ and $\mathcal{L}_i = (1_{M(\Delta_0)} \times p_i \times 1_X)^* p^*_{M(\Delta_0)\times X_0}(\mathcal{L})^\vee$. We set $\Gamma := \{(x_1, x_1, \ldots, x_n) \in X_0^n \mid \pi(x_i) = \pi(x_j) \text{ for some } i \ne j\}$. Then $\Lambda_1 := \Lambda|_{M(\Delta_0)\times(X_0^n\setminus\Gamma)\times X}$ is a surjective homomorphism. We set $\mathcal{F} := \ker \Lambda_1$. By Lemma 1.3, $\mathcal{F}$ is a family of stable vector bundles on X. Hence there is a morphism $M(\Delta_0) \times (X_0^n \setminus \Gamma) \to M(\Delta)$. By our construction, this morphism is $\mathfrak{S}_n$-invariant, and hence we get a morphism $\nu : M(\Delta_0) \times (X_0^n/\mathfrak{S}_n) \to M(\Delta)$. By our construction, it is injective. Since $\dim S^n X = 2n = \dim M(\Delta) - \dim M(\Delta_0)$, ZMT implies that $M(\Delta_0) \times (X_0^n/\mathfrak{S}_n) \to M(\Delta)$ is an immersion. We set $M(\Delta)^2 := \nu(M(\Delta_0) \times (X_0^n/\mathfrak{S}_n))$. We shall show that $M(\Delta)^2$ is dense. For this purpose, we shall estimate the dimension of $M(\Delta)^1 \setminus M(\Delta)^2$.

LEMMA 1.13. $\dim(M(\Delta)^1 \setminus M(\Delta)^2) = 2n - 1 + \dim M(\Delta_0)$.

Proof. Assume that the restriction $E|_l$ of $E \in M(\Delta)^1$ to a smooth fibre l is not stable. By the definition of $M(\Delta)^1$, we see that $E|_l \cong E_1 \oplus E_2$, where E_1 (resp. E_2) is a stable vector bundle of rank r_1 and degree d_1 (resp. rank r_2 and degree d_2). We set $E' := \ker(E \to E_1)$. Then there is an exact sequence

$$0 \longrightarrow E_1 \longrightarrow E'|_l \longrightarrow E_2 \longrightarrow 0. \tag{1.7}$$

Then E is obtained by the inverse transform from E':

$$0 \longrightarrow E \longrightarrow E'(l) \longrightarrow E_2 \longrightarrow 0. \tag{1.8}$$

By (1.7), $E'|_l$ is stable or $E'|_l \cong E_1 \oplus E_2$. By Lemma 1.3, $\Delta(E') = \Delta(E) - 1/r$. Conversely, for $E' \in M(\Delta - 1/r)^1$, we shall consider a surjective homomorphism $\psi : E' \to F_2$ such that the kernel of $E'|_l \to F_2$ is stable, where F_2 is a stable vector bundle of rank r_2 on a smooth fibre l with degree d_2. If $\ker\psi \otimes \mathcal{O}_X(l)$ belongs to $M(\Delta)^1 \setminus M(\Delta)^2$, then (i) $E'|_l$ is stable and E' belongs to $M(\Delta - 1/r)^1 \setminus M(\Delta - 1/r)^2$, or (ii) $E'|_l$ is not stable and F_2 is a direct summand of $E'|_l$. Since $\#\{l \mid E'|_l \text{ is not stable}\} \leq n-1$, by using Lemma 1.9, we see that

$$\begin{aligned}
&\dim(M(\Delta)^1 \setminus M(\Delta)^2) \\
&\quad = \max\{\dim(M(\Delta - 1/r)^1 \setminus M(\Delta - 1/r)^2) + 2, \dim M(\Delta - 1/r)^1 + 1\} \\
&\quad = 2n - 1 + \dim M(\Delta_0).
\end{aligned}$$

∎

Theorem 1.14. *$M(\Delta)$ is irreducible and birational to $M(\Delta_0) \times S^n(J^{d_1}X)$, where $n := r(\Delta - \Delta_0)$.*

Proof. If $\pi : X \to B$ has a section, we have proved our theorem. For general cases, we shall consider a Galois covering $\gamma : B' \to B$ such that $\pi' : X \times_B B' \to B'$ has a section σ'. Let B_1 be an open subscheme of B_0 such that $\gamma^{-1}(B_1) \to B_1$ is etale. We set $X_1' := \pi^{-1}(B_1) \times_B B'$. Let $\mathcal{E}'_{r_1,d_1}$ be the vector bundle on $X_1' \times_{\gamma^{-1}(B_1)} X_1'$ and $j' : X_1' \times_{\gamma^{-1}(B_1)} X_1' \cong X_1' \times_{B_1} X_1 \hookrightarrow X_1' \times X_1$ the inclusion. Let $X_1' \to J^{d_1}X$ be the morphism induced by $\mathcal{E}'_{r_1,d_1}$. For a $g \in \mathrm{Gal}(B'/B)$, let $\tilde{g} : X_1' \to X_1'$ be the automorphism of X_1' sending $(x, y) \in \pi^{-1}(B_1) \times_B B'$ to $(x + (d_1 - 1)(\sigma'(g(y)) - \sigma'(y)), g(y))$. Then it defines an action of $\mathrm{Gal}(B'/B)$ to X_1'. By the construction of $\mathcal{E}'_{r_1,d_1}$, we see that $\det(\mathcal{E}'_{r_1,d_1})|_{\tilde{g}((x,y))} \cong \det(\mathcal{E}'_{r_1,d_1})|_{(x,y)}$. Hence $(\mathcal{E}'_{r_1,d_1})|_{\tilde{g}((x,y))} \cong (\mathcal{E}'_{r_1,d_1})|_{(x,y)}$. Thus the morphism $X_1' \to J^{d_1}X$ is $\mathrm{Gal}(B'/B)$-invariant. Then we get that $X_1'/\mathrm{Gal}(B'/B) \to J^{d_1}X$ is an immersion. Replacing $j_*\mathcal{E}_{r_1,d_1}$ by $j'_*\mathcal{E}'_{r_1,d_1}$, we can construct a family of stable vector bundles $\mathcal{F}$ parametrized by $M(\Delta_0) \times ((X_1')^n \setminus \Gamma')$, where Γ' is the pull-back of Γ to $(X_1')^n$. Hence we get a morphism $M(\Delta_0) \times ((X_1')^n \setminus \Gamma') \to M(\Delta)$. By the construction, $\mathrm{Gal}(B'/B) \times \mathfrak{S}_n$ acts on $((X_1')^n \setminus \Gamma')$, and this morphism is $\mathrm{Gal}(B'/B) \times \mathfrak{S}_n$-invariant. Hence we get a morphism $M(\Delta_0) \times ((J^{d_1}X_1)^n \setminus \Gamma)/\mathfrak{S}_n \to M(\Delta)$. Then we see that $M(\Delta)$ is birationally equivalent to $M(\Delta_0) \times S^n(J^{d_1}X)$.

∎

§2. Moduli spaces on Del Pezzo surfaces

2.1.

We shall apply Theorem 1.14 to moduli spaces on Del Pezzo surfaces.

THEOREM 2.1. *We assume that* $X = \mathbb{P}^2$ *and set* $H := \mathcal{O}_{\mathbb{P}^2}(1)$. *Then* $M_H(r, kH, \Delta)$ *is a rational variety if* $(r, 3k) = 1$.

Proof. Let $V \subset H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3))$ be a pencil such that every member $D \in V$ is irreducible and $\#\{P \mid P \in \bigcap_{D\in V} D\} = 9$. Let $\phi : Y \to \mathbb{P}^2$ be the blow-ups of $\mathbb{P}^2$ at base points of V. Then there is an elliptic fibration $\pi : Y \to \mathbb{P}^1$ such that every fibre is isomorphic to a member D of V. We set

$$N := \{E \in M_H(r, kH, \Delta)_0 \mid \phi^*E|_{\pi^{-1}(\eta)} \text{ is stable}\}, \tag{2.1}$$

where η is the generic point of $\mathbb{P}^1$. Let E be a stable vector bundle of rank r on $\mathbb{P}^2$ with $c_1(E) = kH$. Then $\operatorname{Ext}^2(E, E(-3))_0 \cong \operatorname{Hom}(E, E)_0^\vee = 0$. Let $D \in V$ be a smooth elliptic curve. Then we get the surjective homomorphism $\operatorname{Ext}^1(E, E)_0 \to \operatorname{Ext}^1(E|_D, E|_D)_0$. Hence $\operatorname{Def}(E) \to \operatorname{Def}(E|_D)$ is submersive. Since $(r, \deg(E|_D)) = (r, 3k) = 1$, we can deform E to a stable sheaf F such that $F|_D$ is a stable vector bundle on D. By the openness of stability, $F|_{\pi^{-1}(\eta)}$ is a stable vector bundle. Hence N is an open dense subscheme of $M_H(r, kH, \Delta)$ and there is an open immersion $\phi^* : N \to M(r, k\phi^*H, \Delta)$. By Theorem 1.14, N is bitarional to S^nY, where $n = r\Delta - (r^2 - 1)/2$. Since S^nY is a rational variety, we get our theorem. ∎

DEFINITION 2.1. $\operatorname{Spl}(r, c_1, \Delta)$ *is the moduli space of simple torsion free sheaves* E *of rank* r *with* $c_1(E) = c_1$ *and* $\Delta(E) = \Delta$.

We shall next consider the irreducibility of $\operatorname{Spl}(r, c_1, \Delta)$ for Del Pezzo surfaces.

PROPOSITION 2.2. *Let* $\pi : X \to \mathbb{P}^1$ *be a rational elliptic surface with a section* σ. *For a divisor class* $c_1 \in \operatorname{NS}(X)$ *such that* (c_1, f) *and* r *are relatively prime, we shall consider the moduli space* $M(\Delta) = M(r, c_1, \Delta)$. *Then* $M(\Delta)$ *is irreducible and rational.*

Proof. Since π has a section σ, the canonical bundle formula implies that $K_X \cong \pi^*\mathcal{O}_{\mathbb{P}^1}(-1)$. Hence σ is an exceptional curve of the first kind by the adjunction formula $(K_X + \sigma, \sigma) = -2$. Since $R^1\pi_*\mathcal{O}_X$ is locally free of rank 1, the exact sequence $0 \to \mathcal{O}_X \to \mathcal{O}_X(\sigma) \to \mathcal{O}_\sigma(-1) \to 0$ implies that $\pi_*\mathcal{O}_\sigma(-1) \cong R^1\pi_*\mathcal{O}_X$, and hence we get that $\pi_*K_X^\vee(\sigma) \cong \pi_*K_X^\vee$. Let $\phi : X \to Y$ be the contraction of σ. Then we get that $H^0(Y, K_Y^\vee) \cong H^0(X, K_X^\vee(\sigma)) \cong H^0(X, K_X^\vee) \cong \mathbb{C}^{\oplus 2}$. By the Riemann-Roch theorem, $H^1(Y, K_Y^\vee) = 0$. Let $\delta : \mathcal{Y} \to S$ be a smooth family of 8-points blow-ups of $\mathbb{P}^2$ such that $H^1(\mathcal{Y}_s, K_{\mathcal{Y}_s}^\vee) = 0$ for all $s \in S$ and $\mathcal{Y}_{s_0} = Y$ for some $s_0 \in S$. Let ξ be the generic point of S. By the base change theorem, $\delta_*(K_{\mathcal{Y}/S}^\vee)$ is a locally free sheaf of rank 2 and $\delta_*(K_{\mathcal{Y}/S}^\vee) \otimes k(s) \to H^0(K_{Y_s}^\vee), s \in S$ is an isomorphism. We set $\mathcal{O}_\mathcal{Z} := \operatorname{coker}(\delta^*\delta_*(K_{\mathcal{Y}/S}^\vee) \to K_{\mathcal{Y}/S}^\vee) \otimes K_{\mathcal{Y}/S}$. Then $\mathcal{O}_\mathcal{Z} \otimes k(s)$ defines a reduced one point of Y_s. Thus $\mathcal{Z}$ defines a section of δ. Let $\phi_S : \mathcal{X} \to \mathcal{Y}$ be the blow-up of $\mathcal{Y}$ along $\mathcal{Z}$ and set $\epsilon := \delta \circ \phi$. Then there is a morphism $\pi_S : \mathcal{X} \to \mathbb{P} := \mathbb{P}(\epsilon_*(K_{\mathcal{Y}/S}^\vee))$, which defines a family of elliptic fibrations. Choosing a sufficiently general family, we may assume that $\pi_S|_\xi : \mathcal{X}_\xi \to \mathbb{P}^1_{k(\xi)}$ is an elliptic surface such that every fibre is irreducible. Let $\mathcal{O}_\mathcal{X}(1)$ be a relative ample line bundle on $\mathcal{X}$ which is sufficiently close to the pull-back of an ample line bundle on $\mathbb{P}$. For a line bundle $\mathcal{L}$ on $\mathcal{X}$ such that $c_1(\mathcal{L}_{s_0}) = c_1$, we shall consider the relative moduli space $\mathcal{M}(r, \mathcal{L}, \Delta) \to S$ of stable sheaves E of rank r on $\mathcal{X}_s, s \in S$ such that $c_1(E) = \mathcal{L}_s$ and $\Delta(E) = \Delta$. By Maruyama [Ma1, Cor. 5.9.1, Prop. 6.7], $\mathcal{M}(r, \mathcal{L}, \Delta)$ is smooth and projective over S. By Theorem 1.14, the generic fibre is irreducible, and hence every fibre is irreducible. Thus $M(\Delta)$ is irreducible. Since $M(\Delta)$ contain an irreducible component which is birational to S^nX for some n (see the proof of Theorem 1.14), $M(\Delta)$ is a rational variety. □

LEMMA 2.3. *Let $\phi : \widetilde{X} \to X$ be a one point blow-up of a surface X and E a simple torsion free sheaf of rank r on X which is locally free at the center of the blow-up. Let C_1 be the exceptional divisor of ϕ and $\phi^*E \to \mathcal{O}_{C_1}^{\oplus k}$, $0 < k < r$ a surjective homomorphism. We set $E' := \ker(\phi^*E \to \mathcal{O}_{C_1}^{\oplus k})$. Then E' is also a simple torsion free sheaf.*

Proof. We note that $\operatorname{Ext}^1(\mathcal{O}_{C_1}^{\oplus k}, E) \cong H^1(C_1, E^\vee \otimes \mathcal{O}_{C_1}(K_{\widetilde{X}})^{\oplus k}) \cong H^1(C_1, \mathcal{O}_{C_1}(-1)^{\oplus rk}) = 0$. By the exact sequence $0 \to E' \to E \to \mathcal{O}_{C_1}^{\oplus k} \to 0$, we see that $\operatorname{Hom}(E, E) \cong \operatorname{Hom}(E', E)$. Since $\operatorname{Hom}(E', E') \to \operatorname{Hom}(E', E)$ is injective, we get that $\operatorname{Hom}(E', E') = \mathbb{C}$. □

COROLLARY 2.4. *Let E be a simple torsion free sheaf of rank r on X with $c_1(E) = c_1$ and $\Delta(E) = (\Delta)$ which is locally free at the center* *of a blow-up* $\phi : \widetilde{X} \to X$, *and* E' *the kernel of a surjective homomorphism* $\phi^* E \to \mathcal{O}_{C_1}^{\oplus k}$, $0 \le k < r$. *We set* $\Delta(E') = \Delta'$. *Then, if* $\mathrm{Spl}(r, \phi^* c_1 - kC_1, \Delta')$ *is irreducible,* $\mathrm{Spl}(r, c_1, \Delta)$ *is also irreducible.*

Proof. Let $\mathrm{Spl}(r, \phi^* c_1, \Delta)^0$ be the open subscheme of $\mathrm{Spl}(r, \phi^* c_1, \Delta)$ of elements E such that $E|_{C_1} \cong \mathcal{O}_{C_1}^{\oplus r}$. Then $\phi^* : \mathrm{Spl}(r, c_1, \Delta)' \to \mathrm{Spl}(r, \phi^* c_1, \Delta)^0$ is an isomorphism, where $\mathrm{Spl}(r, c_1, \Delta)'$ is the open dense subspace of $\mathrm{Spl}(r, c_1, \Delta)$ consisting of E such that E is locally free at the center of the blow-up. For an $E \in \mathrm{Spl}(r, \phi^* c_1, \Delta)^0$, the quotients $\phi^* E \to \mathcal{O}_{C_1}^{\oplus k}$ is parametrized by the Grassmannian variety $G(H^0(C_1, E|_{C_1}^{\vee}), k)$. Let $\mathrm{Spl}(r, \phi^* c_1 - kC_1, \Delta')^0$ be the open subscheme of $\mathrm{Spl}(r, \phi^* c_1 - kC_1, \Delta')$ of elements E' such that $E'|_{C_1} \cong \mathcal{O}_{C_1}(1)^{\oplus k} \oplus \mathcal{O}_{C_1}^{\oplus (r-k)}$. By using Lemma 2.3, we can show that there is an open subscheme U of $\mathrm{Spl}(r, \phi^* c_1 - kC_1, \Delta')^0$ and a surjective morphism $U \to \mathrm{Spl}(r, \phi^* c_1, \Delta)^0$ such that every fibre is a Grassmannian variety. Hence, the irreducibility of $\mathrm{Spl}(r, \phi^* c_1 - kC_1, \Delta')$ implies that of $\mathrm{Spl}(r, c_1, \Delta)$. ∎

Proposition 2.5. *Let* X *be a Del Pezzo surface and* c_1 *an element of* $\mathrm{NS}(X)$. *Then* $\mathrm{Spl}(r, c_1, \Delta)$ *is irreducible.*

Proof. Let X be a Del Pezzo surface which is an n-points blow-ups of $\mathbb{P}^2$. If $n < 8$, then we shall take a blow-ups X' of X at general $8 - n$ points. Then $|K_{X'}^{\vee}|$ is a linear pencil with a base point. Let X'' be the blow-up of X' at the base point. Then X'' is an elliptic surface with a section. By using Corollary 2.4 and Proposition 2.2, we get our claim. ∎

§3. Moduli spaces on Abelian surfaces

3.1.

For a manifold V and $\alpha \in H^*(V, \mathbb{Z})$, $[\alpha]_i \in H^i(V, \mathbb{Z})$ denotes the i-th component of α. Let $K(V)$ be the Grothendieck group of V. Let $p : X \to \mathrm{Spec}(\mathbb{C})$ be an Abelian surface over $\mathbb{C}$. We set

$$\begin{cases} H^{ev}(X, \mathbb{Z}) := H^0(X, \mathbb{Z}) \oplus H^2(X, \mathbb{Z}) \oplus H^4(X, \mathbb{Z}) \\ H^{odd}(X, \mathbb{Z}) := H^1(X, \mathbb{Z}) \oplus H^3(X, \mathbb{Z}). \end{cases} \tag{3.1}$$

Let E_0 be an element of $M_H(r, c_1, \Delta)$. We set

$$H(r, c_1, \Delta) := \{\alpha \in H^{ev}(X, \mathbb{Z}) \mid [p_*((\mathrm{ch}\, E_0)\alpha)]_0 = 0\}. \tag{3.2}$$

Let $\mathcal{F}$ be a quasi-universal family of similitude ρ on $M_H(r, c_1, \Delta) \times X$ [Mu3, Thm. A.5]. Then Mukai [Mu3], [Mu5] and Drezet [D], [D-N] defines a homomorphism

$$\kappa_2 : H(r, c_1, \Delta) \longrightarrow H^2(M_H(r, c_1, \Delta), \mathbb{Z}) \tag{3.3}$$

such that

$$\kappa_2(\alpha) = \frac{1}{\rho}\left[p_{M_H(r,c_1,\Delta)*}(\mathrm{ch}(\mathcal{F})\alpha)\right]_2. \tag{3.4}$$

Remark 3.1. In the notation of Mukai [Mu5, Sect. 5], $\kappa_2(\alpha) = -\theta_v(\alpha^\vee)$ and $H(r, c_1, \Delta) = v^\perp$, where $v := (r, c_1, (c_1^2)/2r - \Delta) \in H^{ev}(X, \mathbb{Z})$ is the Chern character of E_0 and ${}^\vee : H^{ev}(X, \mathbb{Z}) \to H^{ev}(X, \mathbb{Z})$ is the automorphism sending $\alpha = \alpha_0 + \alpha_2 + \alpha_4$, $\alpha_i \in H^{2i}(X, \mathbb{Z})$ to $\alpha^\vee = \alpha_0 - \alpha_2 + \alpha_4$. Since we used Drezet's notation in [Y2], [Y3], we shall use Drezet's homomorphism in this note.

We also consider the homomorphism:

$$\kappa_1 : H^{odd}(X, \mathbb{Z}) \longrightarrow H^1(M_H(r, c_1, \Delta), \mathbb{Z}) \tag{3.5}$$

such that

$$\kappa_1(\alpha) = \frac{1}{\rho}\left[p_{M_H(r,c_1,\Delta)*}(\mathrm{ch}(\mathcal{F})\alpha)\right]_1. \tag{3.6}$$

We note that κ_1 and κ_2 do not depend on the choice of $\mathcal{F}$. In this section, we shall prove the following theorem.

Theorem 3.1. *Let c_1 be an element of $\mathrm{NS}(X)$ such that $c_1 \bmod rH^2(X, \mathbb{Z})$ is a primitive element of $H^2(X, \mathbb{Z}/r\mathbb{Z})$ and H a general ample divisor. We assume that $\dim M_H(r, c_1, \Delta) = 2r\Delta + 2 \geq 6$. Let $\mathfrak{a} : M_H(r, c_1, \Delta) \to \mathrm{Alb}(M_H(r, c_1, \Delta))$ be an Albanese map. Then the following holds.*

(1) *κ_1 is an isomorphism and κ_2 is injective.*

(2)

$$\begin{aligned} &H^2(M_H(r, c_1, \Delta), \mathbb{Z}) \\ &\quad = \kappa_2(H(r, c_1, \Delta)) \oplus \mathfrak{a}^* H^2(\mathrm{Alb}(M_H(r, c_1, \Delta), \mathbb{Z}) \\ &\quad = \kappa_2(H(r, c_1, \Delta)) \oplus \bigwedge^2 \kappa_1(H^{odd}(X, \mathbb{Z})). \end{aligned} \tag{3.7}$$

(3)

$$\mathrm{NS}(M_H(r, c_1, \Delta)) = \kappa_2(H(r, c_1, \Delta)_{alg}) \oplus \mathfrak{a}^* \, \mathrm{NS}(\mathrm{Alb}(M_H(r, c_1, \Delta)), \tag{3.8}$$

where $H(r, c_1, \Delta)_{alg} := (H^0(X, \mathbb{Z}) \oplus \mathrm{NS}(X) \oplus H^4(X, \mathbb{Z})) \cap H(r, c_1, \Delta)$.

3.2.

We first assume that X is a product of elliptic curves. Let C_1 and C_2 be elliptic curves and set $X = C_1 \times C_2$. Since we use products such as $X \times X \times \cdots \times X$ and $C_1 \times C_1 \times \cdots \times C_1$, for convenience sake, we shall introduce indices of C_1, C_2 and X. We set $C_k^i := C_k$ and $X^i := C_1^i \times C_2^i$ for $i = -1, 0, 1, \dots, n$ (≥ 2), and $k = 1, 2$. We set $a := -1$. We shall construct a family of stable sheaves on $X^a (= X)$. Let $\Delta_k^{i,j}$ be the diagonal of $C_k^i \times C_k^j = C_k \times C_k$. Let p_k^i be a point of C_k^i. We also denote $c_1(\mathcal{O}(p_k^i))$ by p_k^i. For simplicity, we denote the pull-backs of p_k^i and $\Delta_k^{i,j}$ to $X^0 \times (X^1 \times X^2 \times \cdots \times X^n) \times X^a$ by p_k^i and $\Delta_k^{i,j}$ respectively. Let $\Delta_X^{i,j,k}$ be the pull-back of the diagonal of $X^i \times X^j \times X^k$ to $X^1 \times X^2 \times \cdots \times X^n$ and $\Delta_X^{i,j}$ that of $X^i \times X^j$ to $X^1 \times X^2 \times \cdots \times X^n$. We set $Z := \bigcup_{i<j<k} \Delta_X^{i,j,k}$. Let $\phi : Y \to (X^1 \times X^2 \times \cdots \times X^n) \setminus Z$ be the blow-up of $(X^1 \times X^2 \times \cdots \times X^n) \setminus Z$ at the subscheme $\bigcup_{i<j} \Delta_X^{i,j} \setminus Z$. We set $E^{i,j} := \phi^{-1}(\Delta_X^{i,j} \setminus Z)$. For $\alpha \in H^*(X, \mathbb{Z})$ and the projection $\varpi_i : X^0 \times Y \times X^a \to X^i = X$, $i = 0, 1, \dots, n, a$, we denote the pull-back of α to $X^0 \times Y \times X^a$ by α^i. Then $H^2(\mathrm{Hilb}_X^n, \mathbb{Z}) \cong H^2(Y, \mathbb{Z})^{\mathfrak{S}_n}$ and $H^2(Y, \mathbb{Z})^{\mathfrak{S}_n}$ is generated by $\sum_{i=1}^n e^i$, $\sum_{i<j} (f^i \cdot g^j - g^i \cdot f^j)$ and $\sum_{i<j} E^{i,j}$ where $e \in H^2(X, \mathbb{Z})$ and $f, g \in H^1(X, \mathbb{Z})$. Let $\mathfrak{a} : X^0 \times \mathrm{Hilb}_X^n \to X^0 \times X$ be the Albanese map such that $\mathfrak{a}((x, I_Z)) = (x, \sum_{i=1}^n x_i)$ for reduced subscheme $Z = \bigcup_i \{x_i\}$.

Lemma 3.2. (1) *Let F be a vector bundle on $C_2^0 \times C_2^a$ such that $F|_{\{t\} \times C_2^a}$, $t \in C_2^0$ is a stable vector bundle of rank r on C_2^a with* $\det F|_{\{t\} \times C_2^a} \cong \mathcal{O}(\Delta_2^{0,a} + (d-1)p_2^a)|_{\{t\} \times C_2^a}$. *Then,*

$$\tag{3.9} \begin{cases} c_1(F) = \Delta_2^{0,a} + (d-1)p_2^a + (r_1 - 1 + kr)p_2^0, & k \in \mathbb{Z}, \\ \mathrm{ch}_2(F) = \dfrac{1}{2r}(c_1(F)^2). \end{cases}$$

If $k = 0$, then $\mathrm{ch}_2(F) = d_1 p_2^0 \cdot p_2^a$.

(2) *Let F_i $(1 \leq i \leq n)$ be a vector bundle on $C_2^i \times C_2^a$ such that $F_i|_{\{t\} \times C_2^a}$, $t \in C_2^i$ is a stable vector bundle of rank r_2 on C_2^a with* $\det F_i|_{\{t\} \times C_2^a} \cong \mathcal{O}(\Delta_2^{i,a} + (d_2 - 1)p_2^a)|_{\{t\} \times C_2^a}$. *Then,*

$$\tag{3.10} \begin{cases} c_1(F_i) = \Delta_2^{i,a} + (d_2-1)p_2^a + (r_1 - 1 + kr_2)p_2^i, & k \in \mathbb{Z}, \\ \mathrm{ch}_2(F_i) = \dfrac{1}{2r_2}(c_1(F_i)^2). \end{cases}$$

If $k = 0$, then $\mathrm{ch}_2(F_i) = d_1 p_2^i \cdot p_2^a$.

Proof. We shall only prove (1). We set $c_1(F) = \Delta_2^{0,a} + (d-1)p_2^a + (r_1 - 1 + x)p_2^0$, $x \in \mathbb{Z}$. Since $F|_{\{t\}\times C_2^a}$, $t \in C_2^0$ is a stable vector bundle, $\Delta(F) = c_2(F) - (c_1(F)^2)(r-1)/2r = 0$. Hence we get that $\mathrm{ch}_2(F) = -(c_2(F) - (c_1(F)^2)/2) = (c_1(F)^2)/2r$. We note that $c_2(F) = (d(r_1+x)-1)(r-1)/r$ is an integer. Hence $d(r_1+x)-1 = rd_1 + dx$ is a multiple of r. Since $(r,d) = 1$, x is a multiple of r. We also see that $(c_1(F)^2)/2r = d_1 p_2^0 \cdot p_2^a$ for the case $x = 0$. ∎

Let F and F_i be vector bundles in Lemma 3.2 and assume that $k = 0$. We also denote the pull-backs of F and F_i to $C_2^0 \times C_2^i \times C_2^a$ by F and F_i respectively. Let $q_{C_2^0\times C_2^i} : C_2^0 \times C_2^i \times C_2^a \to C_2^0 \times C_2^i$ be the projection. We set $\mathcal{L} := \mathrm{Hom}_{q_{C_2^0\times C_2^a}}(F, F_i)$. Then $c_1(\mathcal{L}) = -\Delta_2^{0,i}$.

Proof. By using the Grothendieck-Riemann-Roch theorem and the above lemma, we see that

$$\begin{aligned} c_1(\mathcal{L}) &= \left[q_{C_2^0\times C_2^i *}(\mathrm{ch}(F^\vee)\,\mathrm{ch}(F_i))\right]_2 \\ &= \left[q_{C_2^0\times C_2^i *}(r - c_1(F) + \frac{1}{2r}(c_1(F)^2))(r_2 + c_1(F_i) + \frac{1}{2r_2}(c_1(F_i)^2))\right]_2 \\ &= \left[q_{C_2^0\times C_2^i *}(rr_2 + (rc_1(F_i) - r_2c_1(F)) + \frac{1}{2rr_2}((rc_1(F_i) - r_2c_1(F))^2))\right]_2 \\ &= \frac{1}{2rr_2}\left[q_{C_2^0\times C_2^i *}((rc_1(F_i) - r_2c_1(F))^2)\right]_2 \\ &= -\Delta_2^{0,i}. \end{aligned}$$ ∎

Let Y_0 be the complement of the closed subset $W := \bigcup_{i<j<k}(\widetilde{\Delta}_1^{i,j} \cap \widetilde{\Delta}_1^{j,k}) \cup \bigcup_{i<j}(\widetilde{\Delta}_1^{i,j} \cap E^{i,j})$ of Y, where $\Delta_1^{i,j} = \widetilde{\Delta}_1^{i,j} \cup E^{i,j}$. Since $\operatorname{codim} W = 2$, $H^2(X^0 \times Y_0, \mathbb{Z}) \cong H^2(X^0 \times Y, \mathbb{Z})$.

We shall construct a family of stable sheaves on X parametrized by $X^0 \times Y_0$. For simplicity, we denote the pull-backs of F and F_i to $X^0 \times Y_0 \times X^a$ by F and F_i respectively. Then there is a homomorphism:

$$\Lambda : F \otimes \mathcal{O}(\Delta_1^{0,a} - p_1^a) \longrightarrow \bigoplus_{i=1}^{n} \left(F_i|_{\Delta_1^{i,a}} \otimes L^i\right), \tag{3.11}$$

where L^i is a line bundle on $X^0 \times Y_0 \times X^a$ such that $c_1(L^i) = \Delta_1^{0,i} - p_1^i + \Delta_2^{0,i}$. Let $\mathcal{E}$ be the kernel of this homomorphism and $\mathcal{Q}$ the cokernel. Then $\mathcal{Q} \cong \bigoplus_{i<j}\left((F_i/G_j)|_{\Delta_1^{i,a}\cap\widetilde{\Delta}_1^{i,j}} \otimes L^i \oplus (F_i \otimes L^i|_{\Delta_1^{i,a}} \otimes \mathcal{O}_{E^{i,j}})\right)$, where

$G_i := \ker(F \to F_i)$. We first assume that $r_1 \le r_2$. Then $G_j|_{\Delta_1^{i,a}} \to F_i|_{\Delta_1^{i,a}}$ is injective and $(F_i/G_j)|_{\Delta_1^{i,a}}$ is flat over $X^0 \times Y_0$. Hence we see that

$$\operatorname{Tor}_2^{\mathcal{O}_{X^0\times Y_0}}\big((F_i/G_j)|_{\Delta_1^{i,a}\cap\widetilde{\Delta}_1^{i,j}}, k(x)\big) = 0, \quad x \in X^0 \times Y_0. \tag{3.12}$$

Since $F_i \otimes \mathcal{O}(\Delta_2^{0,i})|_{\Delta_1^{i,a}}$ is also flat over $X^0 \times Y_0$, we get that

$$\operatorname{Tor}_2^{\mathcal{O}_{X^0\times Y_0}}\big(F_i \otimes \mathcal{O}(\Delta_2^{0,i})|_{\Delta_1^{i,a}} \otimes \mathcal{O}_{E^{i,j}}, k(x)\big) = 0, \quad x \in X^0 \times Y_0. \tag{3.13}$$

Hence we see that $\operatorname{Tor}_1^{\mathcal{O}_{X^0\times Y_0}}(\operatorname{im}(\Lambda), k(x)) = 0$, which implies that $\mathcal{E}$ is flat over $X^0 \times Y_0$ and $\mathcal{E} \otimes k(x)$ is torsion free. Then $\mathcal{E}$ defines a family of stable sheaves on X parametrized by $X^0 \times Y_0$. It defines a morphism $X^0 \times Y_0 \to M(r, c_1, \Delta)$, which is $\mathfrak{S}_n$-invariant. Hence we get a morphism $\nu : X^0 \times (Y_0/\mathfrak{S}_n) \to M(r, c_1, \Delta)$.

Let $\overline{\kappa_2} : H(r, c_1, \Delta) \to H^2(X^0 \times Y_0, \mathbb{Z})/\mathfrak{a}^* H^2(\operatorname{Alb}(X^0 \times \operatorname{Hilb}_X^n), \mathbb{Z})$ be the homomorphism sending $\alpha \in H(r, c_1, \Delta)$ to $[p_{X^0\times Y_0 *}(\operatorname{ch}(\mathcal{E})\alpha)]_2$ mod $\mathfrak{a}^* H^2(\operatorname{Alb}(X^0 \times \operatorname{Hilb}_X^n), \mathbb{Z})$. Since κ_2 does not depend on the choice of quasi-universal families, we shall compute the image of $\overline{\kappa_2}$.

$$\begin{aligned}
\operatorname{ch}(\mathcal{E}) &= \operatorname{ch}(F \otimes \mathcal{O}(\Delta_1^{0,a} - p_1^a)) - \sum_{i=1}^n \operatorname{ch}\big(F_i \otimes L^i|_{\Delta_1^{i,a}}\big)\\
&\qquad + \sum_{i<j} \operatorname{ch}\big(F_i \otimes L^i|_{\Delta_1^{i,a}} \otimes \mathcal{O}_{E^{i,j}}\big) + \sum_{i<j} \operatorname{ch}\big(F_i/G_j \otimes L^i|_{\Delta_1^{i,a}} \otimes \mathcal{O}_{\widetilde{\Delta}_1^{i,j}}\big)\\
&= (r + c_1(F) + d_1 p_2^0 \cdot p_2^a)(1 + \Delta_1^{0,a} - p_1^a - p_1^0 \cdot p_1^a)\\
&\qquad - \sum_{i=1}^n \Delta_1^{i,a}(r_2 + c_1(F_i) + d_1 p_2^i \cdot p_2^a)(\operatorname{ch} L^i)\\
&\qquad + \sum_{i<j} \operatorname{ch}\big(F_i \otimes L^i|_{\Delta_1^{i,a}} \otimes \mathcal{O}_{E^{i,j}}\big) + \sum_{i<j} \operatorname{ch}\big(F_i/G_j \otimes L^i|_{\Delta_1^{i,a}} \otimes \mathcal{O}_{\widetilde{\Delta}_1^{i,j}}\big).
\end{aligned}$$

Since $[p_{X^0\times Y_0 *}(\operatorname{ch}(F \otimes \mathcal{O}(\Delta_1^{0,a} - p_1^a))\alpha^a)]_2 \equiv 0$, $\sum_{i=1}^n \Delta_1^{0,i} - p_1^i \equiv 0 \bmod \mathfrak{a}^* H^2(\operatorname{Alb}(X^0 \times \operatorname{Hilb}_X^n), \mathbb{Z})$, we get that

$$\begin{aligned}
\overline{\kappa_2}(\alpha) &= -\sum_{i=1}^n [p_{X^0\times Y_0 *}(\Delta_1^{i,a}(r_2 + c_1(F_i) + d_1 p_2^i \cdot p_2^a)(1 + \Delta_2^{0,i})\alpha^a)]_2\\
&\qquad + \sum_{i<j} \big[p_{X^0\times Y_0 *}\big(\operatorname{ch}\big(F_i \otimes \mathcal{O}(\Delta_2^{0,i})|_{\Delta_1^{i,a}} \otimes \mathcal{O}_{E^{i,j}}\big)\alpha^a\big)\big]_2
\end{aligned}$$

$$+\sum_{i<j}\big[p_{X^0\times Y_0*}\big(\operatorname{ch}\big(F_i/G_j\otimes\mathcal{O}(\Delta_2^{0,i})|_{\Delta_1^{i,a}}\otimes\mathcal{O}_{\widetilde{\Delta}_1^{i,j}}\big)\alpha^a\big)\big]_2.$$

Let $\alpha = x_1 + x_2p_1 + x_3p_2 + x_4p_1\cdot p_2 + D$ be an element of $H(r, c_1, \Delta)$, $D \in H^1(C_1,\mathbb{Z})\otimes H^1(C_2,\mathbb{Z})$. Then we see that $0 = [p_*((\operatorname{ch} E_0)\alpha)]_0 = [p_*((r+dp_2 - r_2np_1 - d_2np_1\cdot p_2)\alpha)]_0 = -d_2nx_1 - r_2nx_3 + dx_2 + rx_4$. Thus α satisfies

$$dx_2 + rx_4 = d_2nx_1 + r_2nx_3. \tag{3.14}$$

By a simple calculation, we get that

$$\begin{cases} [p_{X^0\times Y_0*}(\operatorname{ch}(F_i\otimes\mathcal{O}(\Delta_2^{0,i})|_{\Delta_1^{i,a}}))]_2 = d_2\Delta_2^{0,i} + d_1p_2^i \\ [p_{X^0\times Y_0*}(\operatorname{ch}(F_i\otimes\mathcal{O}(\Delta_2^{0,i})|_{\Delta_1^{i,a}})p_2^a)]_2 = r_2\Delta_2^{0,i} + r_1p_2^i \\ [p_{X^0\times Y_0*}(\operatorname{ch}(F_i\otimes\mathcal{O}(\Delta_2^{0,i})|_{\Delta_1^{i,a}})p_1^a)]_2 = d_2p_1^i \\ [p_{X^0\times Y_0*}(\operatorname{ch}(F_i\otimes\mathcal{O}(\Delta_2^{0,i})|_{\Delta_1^{i,a}})D^a)]_2 = D^i \\ [p_{X^0\times Y_0*}(\operatorname{ch}(F_i\otimes\mathcal{O}(\Delta_2^{0,i})|_{\Delta_1^{i,a}})(p_1^a\cdot p_2^a))]_2 = r_2p_1^i, \end{cases} \tag{3.15}$$

$$\begin{cases} [p_{X^0\times Y_0*}(\operatorname{ch}(F_i/G_j\otimes\mathcal{O}(\Delta_2^{0,i})|_{\Delta_1^{i,a}}\otimes\mathcal{O}_{\widetilde{\Delta}_1^{i,j}}))]_2 = (2d_2-d)\widetilde{\Delta}_1^{i,j} \\ [p_{X^0\times Y_0*}(\operatorname{ch}(F_i/G_j\otimes\mathcal{O}(\Delta_2^{0,i})|_{\Delta_1^{i,a}}\otimes\mathcal{O}_{\widetilde{\Delta}_1^{i,j}})p_2^a)]_2 = (2r_2-r)\widetilde{\Delta}_1^{i,j} \\ [p_{X^0\times Y_0*}(\operatorname{ch}(F_i/G_j\otimes\mathcal{O}(\Delta_2^{0,i})|_{\Delta_1^{i,a}}\otimes\mathcal{O}_{\widetilde{\Delta}_1^{i,j}})p_1^a)]_2 = 0 \\ [p_{X^0\times Y_0*}(\operatorname{ch}(F_i/G_j\otimes\mathcal{O}(\Delta_2^{0,i})|_{\Delta_1^{i,a}}\otimes\mathcal{O}_{\widetilde{\Delta}_1^{i,j}})D^a)]_2 = 0 \\ [p_{X^0\times Y_0*}(\operatorname{ch}(F_i/G_j\otimes\mathcal{O}(\Delta_2^{0,i})|_{\Delta_1^{i,a}}\otimes\mathcal{O}_{\widetilde{\Delta}_1^{i,j}})(p_1^a\cdot p_2^a))]_2 = 0, \end{cases} \tag{3.16}$$

and

$$\begin{cases} [p_{X^0\times Y_0*}(\operatorname{ch}(F_i\otimes\mathcal{O}(\Delta_2^{0,i})|_{\Delta_1^{i,a}}\otimes\mathcal{O}_{E^{i,j}}))]_2 = d_2E^{i,j} \\ [p_{X^0\times Y_0*}(\operatorname{ch}(F_i\otimes\mathcal{O}(\Delta_2^{0,i})|_{\Delta_1^{i,a}}\otimes\mathcal{O}_{E^{i,j}})p_2^a)]_2 = r_2E^{i,j} \\ [p_{X^0\times Y_0*}(\operatorname{ch}(F_i\otimes\mathcal{O}(\Delta_2^{0,i})|_{\Delta_1^{i,a}}\otimes\mathcal{O}_{E^{i,j}})p_1^a)]_2 = 0 \\ [p_{X^0\times Y_0*}(\operatorname{ch}(F_i\otimes\mathcal{O}(\Delta_2^{0,i})|_{\Delta_1^{i,a}}\otimes\mathcal{O}_{E^{i,j}})D^a)]_2 = 0 \\ [p_{X^0\times Y_0*}(\operatorname{ch}(F_i\otimes\mathcal{O}(\Delta_2^{0,i})|_{\Delta_1^{i,a}}\otimes\mathcal{O}_{E^{i,j}})(p_1^a\cdot p_2^a))]_2 = 0, \end{cases} \tag{3.17}$$

where $D \in H^1(C_1,\mathbb{Z})\otimes H^1(C_2,\mathbb{Z})$. Hence we get that

$$\begin{aligned}\overline{\kappa_2}(\alpha) = &-\sum_{i=1}^n(d_2x_1+r_2x_3)\Delta_2^{0,i} - \sum_{i=1}^n(d_2x_2+r_2x_4)p_1^i \\ &-\sum_{i=1}^n(d_1x_1+r_1x_3)p_2^i - \sum_{i=1}^n D^i \\ &+\sum_{i<j}((2d_2-d)x_1+(2r_2-r)x_3)\widetilde{\Delta}_1^{i,j} + \sum_{i<j}(d_2x_1+r_2x_3)E.\end{aligned}$$

We note that

$$(3.18)\quad \begin{cases} \sum_{i=1}^{n} \Delta_2^{0,i} \equiv \sum_{i=1}^{n} p_2^i \mod \mathfrak{a}^* H^2(\mathrm{Alb}(X^0 \times \mathrm{Hilb}_X^n), \mathbb{Z}) \\ \sum_{i<j} \Delta_1^{i,j} \equiv n \sum_{i=1}^{n} p_1^i \mod \mathfrak{a}^* H^2(\mathrm{Alb}(X^0 \times \mathrm{Hilb}_X^n), \mathbb{Z}) \\ \widetilde{\Delta}_1^{i,j} = \Delta_1^{i,j} - E^{i,j}. \end{cases}$$

Therefore we get that

$$(3.19)\quad \overline{\kappa_2}(\alpha) = y_1\Big(\sum_{i=1}^{n} p_2^i\Big) + y_2\Big(\sum_{i=1}^{n} p_1^i\Big) + y_3\Big(\sum_{i<j} E^{i,j}\Big) - \sum_{i=1}^{n} D^i,$$

where

$$(3.20)\quad \begin{cases} y_1 = -(dx_1 + rx_3) \\ y_2 = -\{(d_2x_2 + r_2x_4) - n((2d_2 - d)x_1 + (2r_2 - r)x_3)\} \\ y_3 = (d_1x_1 + r_1x_3) \\ y_4 = dx_2 + rx_4 - n(d_2x_1 + r_2x_3). \end{cases}$$

Since $dr_1 - rd_1 = d_2r - dr_2 = 1$, the homomorphism $\psi : \mathbb{Z}^{\oplus 4} \to \mathbb{Z}^{\oplus 4}$ sending (x_1, x_2, x_3, x_4) to (y_1, y_2, y_3, y_4) is an isomorphism. The condition (3.14) implies that $y_4 = 0$. Therefore,

$$(3.21)\quad \overline{\kappa_2} : H(r, c_1, \Delta) \longrightarrow H^2(X^0 \times Y_0, \mathbb{Z})^{\mathfrak{S}_n}/\mathfrak{a}^* H^2(\mathrm{Alb}(X^0 \times \mathrm{Hilb}_X^n), \mathbb{Z})$$

is an isomorphism. Since $H^2(X^0 \times Y_0, \mathbb{Z})^{\mathfrak{S}_n} \cong H^2(X^0 \times \mathrm{Hilb}_X^n, \mathbb{Z})$, we get that

$$(3.22)\quad H(r, c_1, \Delta) \longrightarrow H^2(X^0 \times \mathrm{Hilb}_X^n, \mathbb{Z})/\mathfrak{a}^* H^2(\mathrm{Alb}(X^0 \times \mathrm{Hilb}_X^n), \mathbb{Z})$$

is an isomorphism.

We next treat the case $r_1 > r_2$. Since $G_j \to F_i$ is surjective, we get that

$$\begin{aligned} \overline{\kappa_2}(\alpha) = &- \sum_{i=1}^{n} (d_2x_1 + r_2x_3)\Delta_2^{0,i} - \sum_{i=1}^{n} (d_2x_2 + r_2x_4)p_1^i \\ &- \sum_{i=1}^{n} (d_1x_1 + r_1x_3)p_2^i - \sum_{i=1}^{n} D^i + \sum_{i<j} (d_2x_1 + r_2x_3)E. \end{aligned}$$

In the same way as in the case $r_1 \le r_2$, we see that

$$(3.23) \quad H(r, c_1, \Delta) \longrightarrow H^2(X^0 \times \mathrm{Hilb}^n_X, \mathbb{Z})/\mathfrak{a}^* H^2(\mathrm{Alb}(X^0 \times \mathrm{Hilb}^n_X), \mathbb{Z})$$

is an isomorphism.

Therefore κ_2 is injective and $H^2(M_H(r, c_1, \Delta), \mathbb{Z})$ is generated by $\mathrm{im}(\kappa_2)$ and $\mathrm{im}(\mathfrak{a})$. By using similar computations, we see that κ_1 is an isomorphism. Hence Theorem 3.1 (1), (2) hold for this case.

3.3.

We next treat general cases. Replacing c_1 by $c_1 + rc_1(H)$, we may assume that c_1 belongs to the ample cone.

PROPOSITION 3.3. *Let (X, L) be a pair consisting of Abelian surface X and an ample divisor L of type (d_1, d_2), where d_1 and d_2 are positive integers of $d_1 | d_2$ and $(r, d_1) = 1$. Then Theorem* 3.1 (1), (2) *hold for $M_H(r, c_1(L), \Delta)$, where H is a general polarization.*

Proof. Let (X, L) be a pair consisting of Abelian surface X and an ample divisor L of type (d_1, d_2), where d_1 and d_2 are positive integers of $d_1 | d_2$ and $(r, d_1) = 1$. We shall choose an ample line bundle H on X which is not lie on walls. Let T be a connected smooth curve and $(\mathcal{X}, \mathcal{L})$ a pair of a smooth family of Abelian surface $p_T : \mathcal{X} \to T$ and a relatively ample line bundle $\mathcal{L}$ of type (d_1, d_2). For points $t_0, t_1 \in T$, we assume that $(\mathcal{X}_{t_0}, \mathcal{L}_{t_0}) = (X, L)$ and $\mathcal{X}_{t_1}$ is an Abelian surface of $\mathrm{NS}(\mathcal{X}_{t_1}) \cong \mathbb{Z}$. Let $g : \mathrm{Pic}_{\mathcal{X}/T} \to T$ be the relative Picard scheme. We denote the connected component of $\mathrm{Pic}_{\mathcal{X}/T}$ containing the section of g which corresponds to the family $\mathcal{L}$ by $\mathrm{Pic}^{\xi}_{\mathcal{X}/T}$. Since $\mathrm{Pic}^0_{\mathcal{X}/T} \cong \mathrm{Pic}^{\xi}_{\mathcal{X}/T}$, $\mathrm{Pic}^{\xi}_{\mathcal{X}/T} \to T$ is a smooth morphism. Let $h : \overline{\mathcal{M}}_{\mathcal{X}/T}(r, \xi, \Delta) \to T$ be the moduli scheme parametrizing S-equivalence classes of $\mathcal{L}_t$-semi-stable sheaves E on $\mathcal{X}_t$ with $(\mathrm{rk}(E), c_1(E), \Delta(E)) = (r, c_1(\mathcal{L}_t), \Delta)$ [Ma1]. Let D be the closed subset of $\overline{\mathcal{M}}_{\mathcal{X}/T}(r, \xi, \Delta)$ consisting of properly $\mathcal{L}_t$-semi-stable sheaves on $\mathcal{X}_t$. Since h is a proper morphism, $h(D)$ is a closed subset of T. Since $h(D)$ does not contain t_1 and T is an irreducible curve, $h(D)$ is a finite point set. Replacing T by the open subscheme $T \setminus (h(D) \setminus \{t_0\})$, we may assume that $\mathcal{L}_t$-semi-stable sheaves are $\mathcal{L}_t$-stable for $t \ne t_0$. Let $s : \mathrm{Spl}_{\mathcal{X}/T}(r, \xi, \Delta) \to T$ be the moduli of simple sheaves E on $\mathcal{X}_t, t \in T$ with $(\mathrm{rk}(E), c_1(E), \Delta(E)) = (r, c_1(\mathcal{L}_t), \Delta)$ [A-K, Thm. 7.4]. Let U_1 be the closed subset of $s^{-1}(T \setminus \{t_0\})$ consisting of simple sheaves on $\mathcal{X}_t$, $t \in T \setminus \{t_0\}$ which are not stable with respect to $\mathcal{L}_t$ and

$\overline{U_1}$ the closure of U_1 in $\mathrm{Spl}_{\mathcal{X}/T}(r,\xi,\Delta)$. Let U_2 be the closed subset of $s^{-1}(t_0)$ consisting of simple sheaves which are not semi-stable with respect to H. Then we can show that $\overline{U_1} \cap s^{-1}(t_0)$ is a subset of U_2 (see the second paragraph of the proof of Lemma 3.4). We set $\mathcal{M} := \mathrm{Spl}_{\mathcal{X}/T}(r,\xi,\Delta) \setminus (\overline{U_1} \cup U_2)$. Then $\mathcal{M}$ is an open subspace of $\mathrm{Spl}_{\mathcal{X}/T}(r,\xi,\Delta)$ which is of finite type and contains all H-stable sheaves on $\mathcal{X}_{t_0}$. By using valuative criterion of separatedness and properness, we get that $s : \mathcal{M} \to T$ is a proper morphism. In fact, since $\mathcal{M} \times_T (T \setminus \{t_0\}) \to T \setminus \{t_0\}$ is proper, it is sufficient to check these properties near the fibre $\mathcal{X}_{t_0}$. The separatedness follows from base change theorem and stability with respect to H (cf. [A-K, Lem. 7.8]), and the properness follows from the following lemma (Lemma 3.4) and the projectivity of $\mathcal{M}_{t_0}$. Since $Pic^{\xi}_{\mathcal{X}/T} \to T$ is a smooth morphism, [Mu2, Thm. 1.17] implies that $s : \mathcal{M} \to T$ is a smooth morphism. Let $\mathfrak{a}_T : \mathcal{M} \to \mathrm{Alb}_{\mathcal{M}/T}$ be the family of Albanese map. Let $\mathcal{F}_T$ be a quasi-universal family of similitude ρ on $\mathcal{M} \times_T \mathcal{X}$ and we shall consider the homomorphism over T.

$$\begin{cases} \kappa_{1,t} : H^{odd}(\mathcal{X}_t,\mathbb{Z}) \to H^1(\mathcal{M}_t,\mathbb{Z}) \\ \kappa_{2,t} : H(r,c_1(\mathcal{L}_t),\Delta) \to H^2(\mathcal{M}_t,\mathbb{Z}) \end{cases} \tag{3.24}$$

such that $\kappa_{i,t}(\alpha_{i,t}) = \frac{1}{\rho}[p_{\mathcal{M}_t *}((\mathrm{ch}\,\mathcal{F}_t)\alpha_t)]_i$, where $\alpha_{1,t} \in H^{odd}(\mathcal{X}_t,\mathbb{Z})$, $\alpha_{2,t} \in H(r,c_1(\mathcal{L}_t),\Delta)$. We assume that $\mathcal{X}_{t_0}$ is a product of elliptic curves. Since p_T and s are smooth, Theorem 3.1 (1), (2) for the pair $(\mathcal{X}_{t_0},\mathcal{L}_{t_0})$ imply that Theorem 3.1 (1), (2) also hold for all pairs $(\mathcal{X}_t,\mathcal{L}_t)$, $t \in T$. By the connectedness of the moduli of (d_1,d_2)-polarized Abelian surfaces (cf. [L-B, 8]), (3.7) holds for all pairs (X,L) of (d_1,d_2)-polarized Abelian surfaces. ∎

LEMMA 3.4. *Let R be a discrete valuation ring, K the quotient field of R, and k the residue field of R. Let* $\mathrm{Spec}(R) \to T$ *be a dominant morphism such that* $\mathrm{Spec}(k) \to T$ *defines the point t_0. For a stable sheaf E_K on X_K, there is a R-flat coherent sheaf E on X_R such that $E \otimes_R K = E_K$ and $E \otimes_R k$ is a H-stable sheaf.*

If $H = \mathcal{L}_{t_0}$, then Langton [L] proved our claim. If $H \neq \mathcal{L}_{t_0}$, then we need some modifications, which will be done in the second paragraph of our proof.

Proof. Let E^0 be an R-flat coherent sheaf on X_R such that $E^0 \otimes_R K = E_K$ and $E^0_k := E^0 \otimes_R k$ is torsion free. If E^0_k is H-stable, then we put $E = E^0$. We assume that E^0_k is not H-stable. Let $F^0_k (\subset E^0_k)$ be the first filter of the Harder-Narasimhan filtration of E^0_k with respect to H. We set $E^1 := \ker(E^0 \to E^0_k/F^0_k)$. Then E^1 is an R-flat coherent sheaf on X_R with $E^1_K = E_K$. If E^1_k is not H-stable, then we shall consider the first filter F^1_k of the Harder-Narasimhan filtration of E^1_k and set $E^2 := \ker(E^1 \to E^1_k/F^1_k)$. Continuing this procedure successively, we obtain a decreasing sequence of R-flat coherent sheaves on X_R: $E^0 \supset E^1 \supset E^2 \supset \cdots$. We assume that this sequence is infinite. Then in the same way as in [L, Lem. 2], we see that there is an integer i such that $E^i \otimes_R \widehat{R}$ has a subsheaf F of rank r' with $F \otimes_R k = F^i_k$, where $\widehat{R}$ is the completion of R.

We set $\widehat{K} := K \otimes_R \widehat{R}$ and $D := \det(E^i \otimes_R \widehat{R})^{\otimes r'} \otimes \det(F)^{\otimes(-r)}$. Let $P(x)$ be the Hilbert polynomial of D with respect to $\mathcal{L}_{\widehat{R}}$. Let V be a locally free sheaf on $\mathcal{X}$ such that there is a surjective homomorphism $V \otimes_{\mathcal{O}_T} \widehat{R} \to D$, and we shall consider the quot scheme $\mathcal{Q} := \mathrm{Quot}^{P(x)}_{V/\mathcal{X}/T}$. Then D defines a morphism $\tau : \mathrm{Spec}(\widehat{R}) \to \mathcal{Q}$ such that $D = (\tau \times_T 1_{\mathcal{X}})^* \mathcal{D}$, where $\mathcal{D}$ is the universal quotient. Let $\mathcal{Q}_0$ be the connected component of $\mathcal{Q}$ which contains the image of $\mathrm{Spec}(\widehat{R})$. Since $\mathrm{Spec}(\widehat{R}) \to T$ is dominant, $\mathfrak{q} : \mathcal{Q}_0 \to T$ is dominant, and hence surjective. Since $E^i_{\widehat{K}} \cong E_K \otimes_K \widehat{K}$ is a stable sheaf on $X_{\widehat{K}}$, $(\mathcal{D}_{q_1}, \mathcal{L}_{q_1}) = (\mathcal{D}_{\widehat{K}}, \mathcal{L}_{\widehat{K}}) > 0$, where q_1 is a point of $\mathfrak{q}^{-1}(t_1)$. Since $\mathrm{NS}(\mathcal{X}_{t_1}) \cong \mathbb{Z}$, we get that $c_1(\mathcal{D}_{q_1}) = l c_1(\mathcal{L}_{q_1})$, $l > 0$. Hence we obtain that $(\mathcal{D}^2_{\tau(t_0)}) > 0$ and $(\mathcal{D}_{\tau(t_0)}, \mathcal{L}_{\tau(t_0)}) > 0$. By the Riemann-Roch theorem and the Serre duality, we see that $\mathcal{D}_{\tau(t_0)}$ is an effective divisor. Therefore $(\mathcal{D}_{\tau(t_0)}, H) > 0$, which is a contradiction. Hence there is an integer n such that $E^n \otimes_R k$ is H-stable. ∎

Proof of Theorem 3.1 (3). Let $\kappa'_2 : H(r, c_1, \Delta) \otimes \mathbb{C} \to H^2(M(r, c_1, \Delta), \mathbb{C})$ be the homomorphism induced by κ_2. We note that $H^{2,0}(X)$ and $H^{0,2}(X)$ are subsets of $H(r, c_1, \Delta) \otimes \mathbb{C}$. Since $\mathrm{ch}_i(\mathcal{F})$ is of type (i, i), we see that

$$(3.25) \qquad \begin{cases} \kappa'_2(H^{2,0}(X)) \subset H^{2,0}(M_H(r, c_1, \Delta)) \\ \kappa'_2(\bigoplus_{p=0}^2 H^{p,p}(X)) \subset H^{1,1}(M_H(r, c_1, \Delta)) \\ \kappa'_2(H^{0,2}(X)) \subset H^{0,2}(M_H(r, c_1, \Delta)). \end{cases}$$

Since $H(r, c_1, \Delta) \otimes \mathbb{C} = H^{2,0}(X) \oplus (\bigoplus_{p=0}^2 H^{p,p}(X)) \cap H(r, c_1, \Delta) \otimes \mathbb{C} \oplus H^{0,2}(X)$ and $\mathfrak{a}^*$ preserves the type, we obtain that

$$\begin{aligned} H^{1,1}(M_H(r, c_1, \Delta)) = \kappa'_2((&\textstyle\bigoplus_{p=0}^2 H^{p,p}(X)) \cap H(r, c_1, \Delta) \otimes \mathbb{C}) \\ &\oplus \mathfrak{a}^*(H^{1,1}(\mathrm{Alb}(M_H(r, c_1, \Delta)))). \end{aligned}$$

Hence we get Theorem 3.1 (3). ∎

Combining [Y4, Thm. 2.1] with the proof of Proposition 3.3, we get the following theorem.

THEOREM 3.5. *Let X be an Abelian surface defined over $\mathbb{C}$ and $c_1 \in \mathrm{NS}(X)$ a primitive element. Then*

$$P(M_H(2,c_1,\Delta),z) = P(M_H(1,0,2\Delta),z)$$

for a general polarization H, where $P(\cdot\,,z)$ is the Poincaré polynomial.

3.4.

We shall next consider the Albanese variety of $M_H(r,c_1,\Delta)$, if dim $M_H(r,c_1,\Delta) \geq 4$. Let $\mathcal{P}$ be the Poincaré line bundle on $\widehat{X} \times X$, where $\widehat{X}$ is the dual of X. For an element $E_0 \in M_H(r,c_1,\Delta)$, let $\alpha_{E_0} : M_H(r,c_1,\Delta) \to X$ be the morphism sending $E \in M_H(r,c_1,\Delta)$ to $\det p_{\widehat{X}!}((E-E_0)\otimes(\mathcal{P}-\mathcal{O}_{\widehat{X}\times X})) \in \mathrm{Pic}^0(\widehat{X}) = X$, and $\det_{E_0} : M_H(r,c_1,\Delta) \to \widehat{X}$ the morphism sending E to $\det E \otimes \det E_0^{\vee} \in \widehat{X}$ (cf. [Y3, Sect. 5]). We shall show that $\mathfrak{a}_{E_0} := \det_{E_0} \times \alpha_{E_0}$ is the Albanese map of $M_H(r,c_1,\Delta)$. Let B be an effective divisor on X. Then we see that

$$\begin{aligned}
&\det p_{\widehat{X}!}((E-E_0)\otimes\mathcal{O}_B\otimes(\mathcal{P}-\mathcal{O}_{\widehat{X}\times X}))\\
&\quad = \det p_{\widehat{X}!}((\det E|_B - \det E_0|_B)\otimes(\mathcal{P}-\mathcal{O}_{\widehat{X}\times X}))\\
&\quad = \zeta(\det_{E_0}(E)),
\end{aligned}$$

where $\zeta : \widehat{X} \to X$ is the morphism sending $L \in \widehat{X}$ to $\bigotimes_i \mathcal{P}_{\widehat{X}\times\{x_i\}} \in \mathrm{Pic}^0(\widehat{X}) = X$, $L \cdot B = \sum_i x_i$. Therefore if $\mathfrak{a}_{E_0}$ is the Albanese map for $M_H(r,c_1,\Delta)$, then $\mathfrak{a}_{E_0(B)}$ is the Albanese map for $M_H(r,c_1+rc_1(\mathcal{O}_X(B)),\Delta)$. Hence we may assume that c_1 belongs to the ample cone. In the notation of Proposition 3.3, we assume that there is a section $\sigma : T \to \mathcal{M}$ of s. Then we can also construct a morphism $\mathfrak{a}_\sigma : \mathcal{M} \to \mathrm{Pic}^0_{\mathcal{X}/T} \times_T \mathcal{X}$. In fact, it is sufficient to construct the morphism on small neighbourhoods U (in the sense of classical topology) of each point. By using a universal family on $U \times_T \mathcal{X}$, we get the morphism. Since $s : \mathcal{M} \to T$ and $\mathrm{Pic}^0_{\mathcal{X}/T} \times_T \mathcal{X} \to T$ are smooth over T, it is sufficient to prove that

$$\mathfrak{a}_{E_0}^* : H^1(\widehat{X}\times X,\mathbb{Z}) \longrightarrow H^1(M_H(r,c_1,\Delta),\mathbb{Z}) \tag{3.26}$$

is an isomorphism, if X is a product of elliptic curves. In order to prove this assertion, we shall show that

$$\mathfrak{a}_{E_0}^* : \operatorname{Pic}^0(\widehat{X} \times X) \longrightarrow \operatorname{Pic}^0(M(r, c_1, \Delta)) \tag{3.27}$$

is an isomorphism. Let $\mathcal{E}$ be a universal family on $M(r, c_1, \Delta)$. For simplicity, we set $M := M(r, c_1, \Delta)$. Let $\widehat{X} \times X \to \operatorname{Pic}^0(X \times \widehat{X})$ be the isomorphism sending $(\hat{x}, x) \in \widehat{X} \times X$ to $\mathcal{P}|_{\{\hat{x}\}\times X} \otimes \mathcal{P}|_{\widehat{X}\times\{x\}}$. We set $\mathcal{R} := \det p_{\widehat{X}\times M!}((\mathcal{E} - E_0 \otimes \mathcal{O}_M) \otimes (\mathcal{P} - \mathcal{O}_{\widehat{X}\times X}))$. By the construction of α_{E_0}, we get that $\mathcal{R} \cong (1_{\widehat{X}} \times \alpha_{E_0})^*\mathcal{P} \otimes L$, where L is the pull-back of a line bundle on M. Since $\mathcal{R}|_{\{0\}\times M} \cong \mathcal{O}_M$, we get that $L \cong \mathcal{O}_{\widehat{X}\times M}$. Hence we see that

$$\begin{aligned} \alpha_{E_0}^*(\mathcal{P}|_{\{\hat{x}\}\times X}) &= \det p_{M!}((\mathcal{E} - E_0 \otimes \mathcal{O}_M) \otimes (\mathcal{P}|_{\{\hat{x}\}\times X} - \mathcal{O}_X)) \\ &= \det p_{M!}(\mathcal{E} \otimes (\mathcal{P}|_{\{\hat{x}\}\times X} - \mathcal{O}_X)). \end{aligned} \tag{3.28}$$

In the same way, we see that

$$\begin{aligned} \det{}_{E_0}^*(\mathcal{P}|_{\widehat{X}\times\{x\}}) &= (\det \mathcal{E} \otimes \det E_0^{\vee} \otimes \det \mathcal{E}^{\vee}|_{M\times\{0\}})|_{M\times\{x\}} \\ &= \det p_{M!}(\mathcal{E} \otimes (k_x - k_0)), \end{aligned} \tag{3.29}$$

where $0 \in X$ is the zero of the group low. In order to prove (3.27), we shall consider the pull-backs of $\alpha_{E_0}^*(\mathcal{P}|_{\{\hat{x}\}\times X})$ and $\det_{E_0}^*(\mathcal{P}|_{\widehat{X}\times\{x\}})$ to $X^0 \times Y_0$.

We denote the zero of C_1 and C_2 by 0_1 and 0_2 respectively. For a point q_k of C_k, $k = 1, 2$, we set $l_k := q_k - 0_k$. We also denote the pull-back of l_k to $X = C_1 \times C_2$ by l_k. In the same way as in **3.2**, we denote $\varpi_i^!(G)$, $i = 0, 1, \dots, n, a$ by G^i, $G \in K(X)$. We also denote $\mathcal{O}_X(D)^i$ by $\mathcal{O}_{X^0\times Y_0}(D^i)$. By simple calculations, we see that

$$\begin{cases} \det p_{X^0\times Y_0!}(\mathcal{E} \otimes (\mathcal{O}_X(l_1) - \mathcal{O}_X)^a) = \mathcal{O}_{X^0\times Y_0}\left(dl_1^0 - d_2 \sum_{i=1}^n l_1^i\right) \\ \det p_{X^0\times Y_0!}(\mathcal{E} \otimes (\mathcal{O}_X(l_2) - \mathcal{O}_X)^a) = \mathcal{O}_{X^0\times Y_0}\left(\sum_{i=1}^n l_2^i\right) \\ \det p_{X^0\times Y_0!}(\mathcal{E} \otimes (k_{(q_1,0_2)} - k_{(0_1,0_2)})^a) = \mathcal{O}_{X^0\times Y_0}\left(rl_1^0 - r_2 \sum_{i=1}^n l_1^i\right) \\ \det p_{X^0\times Y_0!}(\mathcal{E} \otimes (k_{(0_1,q_2)} - k_{(0_1,0_2)})^a) = \mathcal{O}_{X^0\times Y_0}(l_2^0). \end{cases} \tag{3.30}$$

Since $d_2 r - dr_2 = 1$ and $\operatorname{Pic}^0(X^0 \times \operatorname{Hilb}_X^n) \cong \operatorname{Pic}^0(X^0 \times Y_0)^{\mathfrak{S}_n}$, (3.28), (3.29) and (3.30) implies that (3.27) holds.

We set

$$K(r, c_1, \Delta) := \{\alpha \in K(X) \mid \chi(\alpha \otimes E_0) = 0, E_0 \in M_H(r, c_1, \Delta)\}. \tag{3.31}$$

Let $\{U_i\}$ be an open covering of $M_H(r, c_1, \Delta)$ such that there are universal family $\mathcal{F}_i$ on each $U_i \times X$ and $\mathcal{F}_i|_{(U_i \cap U_j)\times X} \cong \mathcal{F}_j|_{(U_i \cap U_j)\times X}$. Since the action of $\mathcal{O}_{U_i}^{\times}$ to $\det p_{U_i!}(\mathcal{F}_i \otimes \alpha)$ is trivial, we get a line bundle $\tilde{\kappa}(\alpha)$ on $M_H(r, c_1, \Delta)$. Thus we obtain a homomorphism

$$\tilde{\kappa} : K(r, c_1, \Delta) \longrightarrow \operatorname{Pic}(M_H(r, c_1, \Delta)). \tag{3.32}$$

We note that there is a commutative diagram:

$$\begin{array}{ccc} K(r, c_1, \Delta) & \xrightarrow{\tilde{\kappa}} & \operatorname{Pic}(M_H(r, c_1, \Delta)) \\ {\scriptstyle \mathrm{ch}}\downarrow & & \downarrow{\scriptstyle c_1} \\ H(r, c_1, \Delta) & \xrightarrow{\kappa_2} & H^2(M_H(r, c_1, \Delta), \mathbb{Z}) \end{array} \tag{3.33}$$

Let K^2 be the subgroup of $K(r, c_1, \Delta)$ generated by $k_P - k_0$, $P \in X$ and N the kernel of the Albanese map $K^2 \to X$. Since $\ker(\mathrm{ch})$ is generated by $\mathcal{O}_X(D) - \mathcal{O}_X$, $\mathcal{O}_X(D) \in \operatorname{Pic}^0(X)$ and $k_P - k_0$, $P \in X$, (3.28) and (3.29) implies that $\tilde{\kappa}$ induces an isomorphism $\ker(\mathrm{ch})/N \to \operatorname{Pic}^0(M_H(r, c_1, \Delta))$. By using Theorem 3.1 (3), we get the following theorem, which is similar to [Y2, Thm. 0.1].

THEOREM 3.6. *Under the same assumption as in Theorem* 3.1*, the following holds.*
(1) $\mathfrak{a}_{E_0} : M_H(r, c_1, \Delta) \to \widehat{X} \times X$ *is an Albanese map.*
(2) $\tilde{\kappa} : K(r, c_1, \Delta)/N \to \operatorname{Pic}(M_H(r, c_1, \Delta))$ *is injective.*
(3) $\operatorname{Pic}(M_H(r, c_1, \Delta))/\mathfrak{a}_{E_0}^*(\operatorname{Pic}(\widehat{X} \times X))$ *is generated by* $\tilde{\kappa}(K(r, c_1, \Delta))$*.*
(4) $\mathfrak{a}_{E_0}^*(\operatorname{Pic}(\widehat{X} \times X)) \cap \tilde{\kappa}(K(r, c_1, \Delta)) \cong X \times \widehat{X}$*.*

§4. Appendix

In this appendix, we shall show the following. We shall also show that $M_H(r, c_1, \Delta) \cong \widehat{X} \times X$, if $\dim M_H(r, c_1, \Delta) = 4$.

PROPOSITION 4.1. *Let L be an ample line bundle on an Abelian surface X. We assume that $\chi(L) = (c_1(L)^2)/2$ and r are relatively prime. Then $M_H(r, c_1(L), \Delta) \cong M_H(r, L, \Delta) \times \widehat{X}$, where $M_H(r, L, \Delta)$ is the moduli space of sheaves of determinant L. In particular, $P(M_H(2, L, \Delta), z) = P(\operatorname{Hilb}_X^{2\Delta}, z)$ for a general polarization H.*

Proof. For a stable sheaf $E \in M_H(r, c_1(L), \Delta)$, $\lambda(E)$ denotes the point of $\widehat{X}$ which correspond to the line bundle $\det(E) \otimes L^{-1}$. Let $\phi_L : X \to \widehat{X}$ be the morphism sending $x \in X$ to $T_x^* L \otimes L^{-1}$, and $\varphi : \widehat{X} \to X$ the morphism such that $\phi_L \circ \varphi = n^2_{\widehat{X}}$, where $T_x : X \to X$ is the translation defined by x and $n^2 = \chi(L)^2 = \deg \phi_L$. Since $(r, n^2) = 1$, there are integers k and k' such that $rk + n^2 k' = 1$. We denote the Poincaré line bundle on $X \times \widehat{X}$ by $\mathcal{P}$. Let $A : M_H(r, c_1(L), \Delta) \to M_H(r, L, \Delta) \times \widehat{X}$ be the morphism sending $F \in M_H(r, c_1(L), \Delta)$ to $(T^*_{-k'\varphi\circ\lambda(F)}(F \otimes \mathcal{P}_{-k\lambda(F)}), \lambda(F))$ and $B : M_H(r, L, \Delta) \times \widehat{X} \to M_H(r, c_1(L), \Delta)$ the morphism sending $(E, x) \in M_H(r, L, \Delta) \times \widehat{X}$ to $T^*_{k'\varphi(X)} E \otimes \mathcal{P}_{kx}$. For an element (E, x) of $M_H(r, L, \Delta) \times \widehat{X}$, $\det(T^*_{k'\varphi(x)} E \otimes \mathcal{P}_{kx}) \cong T^*_{k'\varphi(x)} L \otimes \mathcal{P}_{rkx} \cong L \otimes \mathcal{P}_{k'\phi_L \circ \varphi(x)} \otimes \mathcal{P}_{rkx} \cong L \otimes \mathcal{P}_{(n^2k'+rk)x} = L \otimes \mathcal{P}_x$. Hence $\lambda \circ B((E, x)) = x$. Then it is easy to see that $A \circ B$ and $B \circ A$ are identity morphisms. Hence $A : M_H(r, c_1(L), \Delta) \to M_H(r, L, \Delta) \times \widehat{X}$ is an isomorphism. ∎

Let $\mathbb{D}(X)$ and $\mathbb{D}(\widehat{X})$ be the derived categories of X and $\widehat{X}$ respectively. Let $\mathcal{S} : \mathbb{D}(X) \to \mathbb{D}(\widehat{X})$ be the Fourier-Mukai transform [Mu4]. Then the morphism $\alpha := \alpha_{E_0}$ defined in **3.4** satisfies that $\alpha(E) = \det \mathcal{S}(E) \otimes (\det \mathcal{S}(E_0))^{-1}$. Thus α_{E_0} is also defined by Fourier-Mukai transform. By using [Mu4], we shall treat the case $2r\Delta = 2$ (at least, Mukai treated the case where X is a principally polarized Abelian surface).

Proposition 4.2. *Let L be an ample divisor. If $2r\Delta = 2$, then for a general polarization H, the Albanese map $\alpha : M_H(r, L, \Delta) \to X$ is an isomorphism.*

Proof. Since $rc_2 - (r-1)(L^2)/2 = 1$ and $\chi(L) = (L^2)/2$, r and $\chi(L)$ are relatively prime. We shall choose an element E of $M_H(r, L, \Delta)$ and let $\xi : X \times \widehat{X} \to M(r, c_1(L), \Delta)$ be the morphism sending $(x, y) \in X \times \widehat{X}$ to $T_x^* E \otimes \mathcal{P}_y$. Then $\lambda \circ \xi(x, y) = \phi_L(x) + ry$. Let $f : X \to X \times \widehat{X}$ be the morphism such that $f(x) = (rx, -\phi_L(x))$. Since $\# \ker \phi_L = \chi(L)^2$ and r are relatively prime, f is injective. Let $g : \widehat{X} \to X \times \widehat{X}$ be the morphism such that $g(y) = (k'\varphi(y), ky)$. Then $f \times g : X \times \widehat{X} \to X \times \widehat{X}$ is an isomorphism. In fact, if $(rx + k'\varphi(y), -\phi_L(x) + ky) = (0, 0)$, then $\phi_L(rx + k'\varphi(y)) = r\phi_L(x) + n^2k'y = 0$. Hence $y = (n^2k' + rk)y = 0$. Since f is injective, $x = 0$, which implies that $f \times g$ is injective. Therefore $f \times g$ is an isomorphism. Then we get a morphism $\xi \circ f : X \to M(r, L, \Delta)$. Replacing E by $E \otimes L^{\otimes m}$, we may assume that there is an exact sequence $0 \to \mathcal{O}_X^{\oplus(r-1)} \to E \to I_Z \otimes L \to 0$, where I_Z

is the ideal sheaf of a codimension 2 subscheme Z of X. By our assumption on Chern classes, $1/r = \Delta(E) = \deg Z - (r-1)/r\chi(L)$. For simplicity, we denote $\det \mathcal{S}(\cdot)$ by $\delta(\cdot)$. Then we see that $\delta(T_x^* E \otimes \mathcal{P}_y) = \delta(I_{T_{-x}(Z)} \otimes T_x^* L \otimes \mathcal{P}_y) = \delta(I_{Z-(\deg Z)x} \otimes L \otimes \mathcal{P}_{\phi_L(x)+y}) = \delta(L \otimes \mathcal{P}_{\phi_L(x)+y}) \otimes \mathcal{P}_{-Z+(\deg Z)x} = \det T^*_{\phi_L(x)+y}(\mathcal{S}(L)) \otimes \mathcal{P}_{-Z+(\deg Z)x} = \delta(L) \otimes \mathcal{P}_{\phi_{\delta(L)}(\phi_L(x)+y)+(\deg Z)x-Z}$. Hence $\alpha \circ \xi \circ f(x) = \alpha \circ \xi \circ f(0) + (r-1)\phi_{\delta(L)} \circ \phi_L(x) + r(\deg Z)x$. By the proof of [Mu4, Prop. 1.23], $\phi_{\delta(L)}(\phi_L(x)) = -\chi(L)x$. Since $r \deg Z = 1 + (r-1)\chi(L)$, we get that $\alpha \circ \xi \circ f(x) = \alpha \circ \xi \circ f(0) + x$. Thus $\alpha \circ \xi \circ f(x)$ is an isomorphism. Therefore we get that $\alpha : M_H(r, L, \Delta) \to X$ is an isomorphism. ∎

Corollary 4.3. *$M_H(r, c_1, \Delta) \cong \widehat{X} \times X$, if $\dim M_H(r, c_1, \Delta) = 4$.*

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