Nihonkai Math. J.
Vol.30(2019), 41–76

SURJECTIVE ISOMETRIES ON C^1 SPACES OF UNIFORM ALGEBRA VALUED MAPS

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ABSTRACT. Let $C^1([0,1],A)$ be the Banach algebra of all continuously differentiable maps from the closed unit interval $[0,1]$ to a uniform algebra A with respect to certain norms. We prove that every surjective, not necessarily linear, isometry on $C^1([0,1],A)$ is represented by homeomorphisms on $[0,1]$ and the maximal ideal space of A.

1. Introduction and Preliminaries

The purpose of this paper is to characterize surjective isometries on $C^1([0,1],A)$, the set of all continuously differentiable maps from the closed unit interval $[0,1]$ to a uniform algebra A with respect to certain norms. The main result of this paper generalizes the result of [7] for some of those norms. We will investigate the structure of isometries on $C^1([0,1],A)$ to clarify the difference between the Banach algebra $C^1([0,1])$ and a uniform algebra A. For a strictly convex Banach space E, surjective linear isometries on C^1 spaces of E-valued continuously differentiable maps are characterized in [2, 9, 10]: uniform algebras are not strictly convex.

Let $C(X)$ be the Banach algebra of all continuous complex valued functions on a compact Hausdorff space X with respect to supremum norm $\|u\|_X = \sup_{x\in X}|u(x)|$ for $u \in C(X)$. A uniformly closed subalgebra A of $C(X)$ is said to be a *uniform algebra* on X if A contains the constants and separates the points of X in the following sense: For each distinct points $x, y \in X$ there exists $u \in A$ such that $u(x) \neq u(y)$. We denote by $\mathrm{Ran}(u)$ the range of a function $u \in A$. The *peripheral range* $\mathrm{Ran}_\pi(u)$ of $u \in A$ is defined by $\mathrm{Ran}_\pi(u) = \{z \in \mathrm{Ran}(u) : |z| = \|u\|_X\}$. An element $u \in A$ is said to be a *peaking function* of A if $\mathrm{Ran}_\pi(u) = \{1\}$. A *peak set* E of A is a compact subset of X such that $E = \{x \in X : u(x) = 1\}$ for some peaking function $u \in A$. The strong boundary of A, denoted by $b(A)$, is the set of all $x \in X$ such that $\{x\}$ is the intersection of a family of peak sets of A. It is well-known that

2010 *Mathematics Subject Classification.* 46J10.
Key words and phrases. Banach algebra, extreme point, isometry, maximal ideal space, uniform algebra.

the strong boundary $b(A)$ of A has the following properties (see, for example [12, Propositions 2.2 and 2.3]).

(1) For each $\varepsilon > 0$, $x \in b(A)$ and open neighborhood O of x in X there exists a peaking function $u \in A$ such that $u(x) = 1 = \|u\|_X$ and $|u| < \varepsilon$ on $X \setminus O$.
(2) For each $u \in A$ there exists $x \in b(A)$ such that $|u(x)| = \|u\|_X$.

We denote by ∂A the Shilov boundary of A, i.e., the smallest closed subset of X with the property that $\sup_{x\in\partial A} |u(x)| = \|u\|_X$ for $u \in A$. It is well known that $b(A)$ is contained in ∂A and that $b(A)$ is dense in ∂A (cf. [3, Corollary 2.2.10]).

If A is a uniform algebra on X, then it is a commutative Banach algebra with the supremum norm $\|\cdot\|_X$. We denote by $\mathcal{M}_A$ the maximal ideal space of A, and then $\mathcal{M}_A$ is a compact Hausdorff space with the relative weak *-topology. We may regard X as a subspace of $\mathcal{M}_A$. The Gelfand transform $\widehat{u}$ of $u \in A$ is a continuous function on $\mathcal{M}_A$, defined by $\widehat{u}(\eta) = \eta(u)$ for every $\eta \in \mathcal{M}_A$. Let e_x be the point evaluation functional, defined by $e_x(u) = u(x)$ for $u \in A$ and $x \in X$. Then the map $x \mapsto e_x$ is a homeomorphism from X onto $\{e_x : x \in X\} \subset \mathcal{M}_A$. Identifying X with $\{e_x : x \in X\}$, we may and do assume $X \subset \mathcal{M}_A$. Because $\|\widehat{u}\|_{\mathcal{M}_A} = \|u\|_X = \|u\|_{\partial A}$ for $u \in A$, we observe that ∂A is a boundary for $\widehat{A} = \{\widehat{u} : u \in A\}$.

For a uniform algebra A on X, we denote by $C^1([0,1], A)$ a complex linear space of all A-valued continuously differentiable maps on $[0,1]$ in the following sense: For each $F \in C^1([0,1], A)$ there exists a continuous map $F' \colon [0,1] \to A$ such that, for each $t \in [0,1]$,

$$\lim_{h\to 0} \left\| \frac{F(t+h) - F(t)}{h} - F'(t) \right\|_X = 0;$$

if $t = 0, 1$, then the limit means the right-hand and left-hand one-sided limit, respectively. If X is a singleton, then we may regard A as $\mathbb{C}$, and we write $C^1([0,1])$ instead of $C^1([0,1], \mathbb{C})$. For each $F \in C^1([0,1], A)$ and $x \in X$, the mapping $F_x \colon [0,1] \to \mathbb{C}$, defined by $F_x(t) = F(t)(x)$, belongs to $C^1([0,1])$ with $(F_x)'(t) = F'(t)(x)$; in fact, for each $t \in [0,1]$,

$$\left| \frac{F_x(t+h) - F_x(t)}{h} - F'(t)(x) \right| \le \left\| \frac{F(t+h) - F(t)}{h} - F'(t) \right\|_X \to 0$$

as $h \to 0$.

Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle and D a compact connected subset of $[0,1] \times [0,1]$. We denote by π_j the projection from D to the j-th coordinate of $[0,1] \times [0,1]$ for $j = 1, 2$. For each $F \in C^1([0,1], A)$, we define $\|F\|_{\langle D\rangle}$ by

$$\|F\|_{\langle D\rangle} = \sup_{(t_1,t_2)\in D} (\|F(t_1)\|_X + \|F'(t_2)\|_X).$$

If $\pi_2(D) = [0,1]$, then we see that $\|\cdot\|_{\langle D\rangle}$ is a norm on $C^1([0,1], A)$. For example, if $D_1 = \{(t,t) \in [0,1]\times[0,1] : t \in [0,1]\}$ then $\|F\|_{\langle D_1\rangle} = \sup_{t\in[0,1]}(\|F(t)\|_X + \|F'(t)\|_X)$.

Cambern [4] characterized surjective complex linear isometries on $C^1([0,1])$ with this norm. If $D_2 = [0,1]\times[0,1]$ then $\|F\|_{\langle D_2\rangle} = \sup_{t_1\in[0,1]}\|F(t_1)\|_X + \sup_{t_2\in[0,1]}\|F'(t_2)\|_X$, for which Rao and Roy [14] gave the characterization of surjective complex linear isometries on $C^1([0,1])$. Kawamura and the authors [7] of this paper introduced the norm $\|\cdot\|_{\langle D\rangle}$ for unifying those norms.

The following is the main result of this paper. Theorem 1 says that every surjective isometry on $C^1([0,1],A)$ is represented by homeomorphisms on $[0,1]$ and the maximal ideal space of A. This implies that the Banach algebra $C^1([0,1])$ and a uniform algebra A have different structures. On the other hand, if we consider $C(X, C(Y))$, the Banach space of all $C(Y)$ valued continuous maps on X with the supremum norm, then we may regard $C(X, C(Y))$ as $C(X\times Y)$. By the Banach-Stone theorem, every unital, surjective complex linear isometry from $C(X_1\times Y_1)$ onto $C(X_2\times Y_2)$ is induced by a homeomorphism from $X_2\times Y_2$ onto $X_1\times Y_1$. Generally speaking, neither X_1 and X_2 nor Y_1 and Y_2 are homeomorphic to each other.

Theorem 1. *Let A be a uniform algebra on X, and D a compact connected subset of $[0,1]\times[0,1]$ such that $\pi_1(D) = \pi_2(D) = [0,1]$. If $T\colon C^1([0,1],A)\to C^1([0,1],A)$ is a surjective isometry with respect to*

$$\|F\|_{\langle D\rangle} = \sup_{(t_1,t_2)\in D}(\|F(t_1)\|_X + \|F'(t_2)\|_X)$$

for $F\in C^1([0,1],A)$, then there exist an invertible element $\beta\in A$ with $|\widehat{\beta}| = 1$ on $\mathcal{M}_A$, a homeomorphism $\sigma\colon\mathcal{M}_A\to\mathcal{M}_A$ and closed and open, possibly empty, subsets $M_1^+, M_1^-, M_{-1}^+, M_{-1}^-\subset\mathcal{M}_A$ with $M_1^+\cup M_1^-\cup M_{-1}^+\cup M_{-1}^- = \mathcal{M}_A$, $M_j^+\cap M_j^- = \emptyset$ for $j=\pm1$ and $M_{-1}^+\cup M_{-1}^- = \mathcal{M}_A\setminus(M_1^+\cup M_1^-)$, such that

$$\widehat{T_0(F)(t)}(\rho) = \begin{cases} \widehat{\beta}(\rho)\widehat{F(t)}(\sigma(\rho)) & \rho\in M_1^+ \\ \widehat{\beta}(\rho)\overline{\widehat{F(t)}(\sigma(\rho))} & \rho\in M_1^- \\ \widehat{\beta}(\rho)\widehat{F(1-t)}(\sigma(\rho)) & \rho\in M_{-1}^+ \\ \widehat{\beta}(\rho)\overline{\widehat{F(1-t)}(\sigma(\rho))} & \rho\in M_{-1}^- \end{cases}$$

for all $F\in C^1([0,1],A)$ and $t\in[0,1]$, where $T_0 = T - T(0)$.

Conversely, if T_0 is a map of the above form, then $T = T_0 + F_0$ is a surjective isometry with respect to $\|\cdot\|_{\langle D\rangle}$ for every $F_0\in C^1([0,1],A)$.

2. Characterization of extreme points

Throughout this paper, we denote $D\times K\times K\times\mathbb{T}$ by $\widetilde{D}_K$ for each subset K of $\mathcal{M}_A$. Then $\widetilde{D}_{\partial A}$ is a compact Hausdorff space with respect to the product topology. For each $F \in C^1([0,1],A)$, we define the function $\widetilde{F}$ on $\widetilde{D}_{\partial A}$ by

$$\widetilde{F}(t_1,t_2,x_1,x_2,z) = F(t_1)(x_1) + zF'(t_2)(x_2) \tag{2.1}$$

for $(t_1,t_2,x_1,x_2,z) \in \widetilde{D}_{\partial A}$; for the sake of simplicity, we shall write (t_1,t_2,x_1,x_2,z) instead of $((t_1,t_2),x_1,x_2,z)$. Then $\widetilde{F}$ is a continuous function on $\widetilde{D}_{\partial A}$ with

$$\begin{aligned}\|\widetilde{F}\|_{\widetilde{D}_{\partial A}} &= \sup_{(t_1,t_2,x_1,x_2,z)\in\widetilde{D}_{\partial A}} |\widetilde{F}(t_1,t_2,x_1,x_2,z)| \\ &= \sup_{(t_1,t_2,x_1,x_2,z)\in\widetilde{D}_{\partial A}} |F(t_1)(x_1) + zF'(t_2)(x_2)|.\end{aligned}$$

We may regard $F \in C^1([0,1],A)$ and $F' \colon [0,1] \to A$ as continuous functions on $[0,1]\times X$. Since ∂A is a boundary for A, there exist $(s_1,s_2) \in D$ and $y_1,y_2 \in \partial A$ such that

$$\sup_{(t_1,t_2)\in D} (\|F(t_1)\|_X + \|F'(t_2)\|_X) = |F(s_1)(y_1)| + |F'(s_2)(y_2)|.$$

We can choose $z_0 \in \mathbb{T}$ so that $|F(s_1)(y_1)| + |F'(s_2)(y_2)| = |F(s_1)(y_1) + z_0F'(s_2)(y_2)|$, and thus

$$\begin{aligned}\|F\|_{\langle D\rangle} &= \sup_{(t_1,t_2)\in D} (\|F(t_1)\|_X + \|F'(t_2)\|_X) = |F(s_1)(y_1) + z_0F'(s_2)(y_2)| \\ &\le \sup_{(t_1,t_2,x_1,x_2,z)\in\widetilde{D}_{\partial A}} |F(t_1)(x_1) + zF'(t_2)(x_2)| \\ &\le \sup_{(t_1,t_2)\in D} (\|F(t_1)\|_X + \|F'(t_2)\|_X) = \|F\|_{\langle D\rangle}.\end{aligned}$$

Therefore, $\|F\|_{\langle D\rangle} = \sup_{(t_1,t_2,x_1,x_2,z)\in\widetilde{D}_{\partial A}} |F(t_1)(x_1) + zF'(t_2)(x_2)|$, and hence

$$\|F\|_{\langle D\rangle} = \|\widetilde{F}\|_{\widetilde{D}_{\partial A}} \qquad (F \in C^1([0,1],A)). \tag{2.2}$$

Let $\mathbf{1}_K$ be constant function on a set K such that $\mathbf{1}_K(x) = 1$ for all $x \in K$. Then $\mathbf{1}_{[0,1]} \in C^1([0,1])$ and $\mathbf{1}_X \in A$. In the rest of this paper, we denote $\mathbf{1}_{[0,1]} \otimes \mathbf{1}_X$ by $\mathbf{1}$. We set

$$B = \{\widetilde{F} \in C(\widetilde{D}_{\partial A}) : F \in C^1([0,1],A)\}.$$

Then we see that B is a linear subspace of $C(\widetilde{D}_{\partial A})$ with $\widetilde{\mathbf{1}} \in B$. We define the mapping $U \colon (C^1([0,1],A), \|\cdot\|_{\langle D\rangle}) \to (B, \|\cdot\|_{\widetilde{D}_{\partial A}})$ by

$$U(F) = \widetilde{F} \qquad (F \in C^1([0,1],A)). \tag{2.3}$$

Equalities (2.1) and (2.2) show that U is a surjective complex linear isometry.

For each $f \in C^1([0,1])$ and $u \in A$, we define $f \otimes u \in C^1([0,1],A)$ by

$$(f\otimes u)(t)(x) = f(t)u(x) \qquad (t \in [0,1],\ x \in X).$$

By the definition of the derivative, we see that

$$(f\otimes u)'(t)(x) = f'(t)u(x)$$

for all $f \in C^1([0,1])$, $u \in A$, $t \in [0,1]$ and $x \in X$.

We show that B separates the points of $\widetilde{D}_{\partial A}$. Let $\boldsymbol{p} = (t_1, t_2, x_1, x_2, z) \in \widetilde{D}_{\partial A}$ and $\boldsymbol{q} = (s_1, s_2, y_1, y_2, w) \in \widetilde{D}_{\partial A}$ with $\boldsymbol{p} \neq \boldsymbol{q}$.

If $t_1 \neq s_1$, then choose $f_1 \in C^1([0,1])$ so that $f_1(t_1) \neq f_1(s_1)$ and $f_1'(t_2) = f_1'(s_2) = 0$. Let $F_1 = f_1 \otimes \mathbf{1}_X \in C^1([0,1], A)$, and then $\widetilde{F}_1 \in B$ satisfies $\widetilde{F}_1(\boldsymbol{p}) = f_1(t_1) \neq f_1(s_1) = \widetilde{F}_1(\boldsymbol{q})$ by (2.1).

We now consider the case when $t_1 = s_1$ and $t_2 \neq s_2$. There exists $f_2 \in C^1([0,1])$ such that $f_2(t_1) = 0 = f_2(s_1)$, $f_2'(t_2) = 1$ and $f_2'(s_2) = 0$. For $F_2 = f_2 \otimes \mathbf{1}_X \in C^1([0,1], A)$, we have $\widetilde{F}_2(\boldsymbol{p}) = z \neq 0 = \widetilde{F}_2(\boldsymbol{q})$.

Suppose that $t_j = s_j$ for $j = 1, 2$ and $x_1 \neq y_1$. Since A separates the points of X, there exists $v_1 \in A$ such that $v_1(x_1) = 1$ and $v_1(y_1) = 0$. Then $G_1 = \mathbf{1}_{[0,1]} \otimes v_1 \in C^1([0,1], A)$ satisfies $\widetilde{G}_1(\boldsymbol{p}) = 1 \neq 0 = \widetilde{G}_1(\boldsymbol{q})$.

Now we suppose $x_2 \neq y_2$. We may assume that $t_j = s_j$ for $j = 1, 2$ and $x_1 = y_1$. We can choose $v_2 \in A$ with $v_2(x_2) = 1$ and $v_2(y_2) = 0 = v_2(x_1) = v_2(y_1)$. Let id be the identity function on $[0,1]$. If we define $G_2 = (\mathrm{id} - t_1 \mathbf{1}_{[0,1]}) \otimes v_2 \in C^1([0,1], A)$, then $\widetilde{G}_2(\boldsymbol{p}) = z \neq 0 = \widetilde{G}_2(\boldsymbol{q})$.

Finally, if $z \neq w$, then we may and do assume that $t_j = s_j$ and $x_j = y_j$ for $j = 1, 2$. Then the function $G_3 = (\mathrm{id} - t_1 \mathbf{1}_{[0,1]}) \otimes \mathbf{1}_X \in C^1([0,1], A)$ satisfies $\widetilde{G}_3(\boldsymbol{p}) = z \neq w = \widetilde{G}_3(\boldsymbol{q})$. From the above arguments we have proven that B separates the points of $\widetilde{D}_{\partial A}$, as is claimed.

By (2.1), we see that $\widetilde{\mathbf{1}} \in B$ is the constant function with $\widetilde{\mathbf{1}}(\boldsymbol{p}) = 1$ for all $\boldsymbol{p} \in \widetilde{D}_{\partial A}$. In other words, B is a function space on $\widetilde{D}_{\partial A}$. We denote by B_1^* the closed unit ball of the dual space B^* of $(B, \|\cdot\|_{\widetilde{D}_{\partial A}})$. The set of all extreme points of B_1^* is denoted by $\mathrm{ext}(B_1^*)$. Let $\delta_{\boldsymbol{p}}$ be the point evaluation at $\boldsymbol{p} \in \widetilde{D}_{\partial A}$, that is, $\delta_{\boldsymbol{p}}(\widetilde{F}) = \widetilde{F}(\boldsymbol{p})$ for each $\widetilde{F} \in B$. We define the Choquet boundary for the function space B by the set of all points $\boldsymbol{p} \in \widetilde{D}_{\partial A}$ with the property that $\delta_{\boldsymbol{p}}$ is an extreme point of B_1^*. We may regard uniform algebras as function spaces. By [3, Theorem 2.3.4], the strong boundary $b(A)$ coincides with the Choquet boundary $\mathrm{Ch}(A)$ for a uniform algebra A.

By the Riesz representation theorem, for each $\eta \in B^*$ there exists a regular Borel measure μ on $\widetilde{D}_{\partial A}$ such that $\|\eta\|_{\mathrm{op}} = \|\mu\|$ and $\eta(\widetilde{F}) = \int_{\widetilde{D}_{\partial A}} \widetilde{F}\, d\mu$ for all $\widetilde{F} \in B$, where $\|\cdot\|_{\mathrm{op}}$ and $\|\cdot\|$ are the operator norm and the total variation of a measure, respectively.

Lemma 2.1. *Let $\boldsymbol{p} = (t_1, t_2, x_1, x_2, z_1) \in \widetilde{D}_{b(A)}$ and μ a representing measure for $\delta_{\boldsymbol{p}}$. Then $\mu(\{D \cap ([0,1] \times \{t_2\})\} \times \partial A \times \partial A \times \mathbb{T}) = 1$.*

Proof. Let $\boldsymbol{p} = (t_1, t_2, x_1, x_2, z_1) \in \widetilde{D}_{b(A)} \subset \widetilde{D}_{\partial A}$ be an arbitrary point. There exists a regular Borel measure μ such that $\|\mu\| = \|\delta_{\boldsymbol{p}}\|_{\mathrm{op}}$ and $\delta_{\boldsymbol{p}}(\widetilde{F}) = \int_{\widetilde{D}_{\partial A}} \widetilde{F}\, d\mu$ for every

$\widetilde{F} \in B$. Since $\delta_{\boldsymbol{p}}(\widetilde{\mathbf{1}}) = 1 = \|\delta_{\boldsymbol{p}}\|_{\mathrm{op}}$, any representing measure for $\delta_{\boldsymbol{p}}$ is a probability measure (see, for example, [3, p. 81]). Let $\varepsilon > 0$ be an arbitrary positive real number and $N_2 \subset [0,1]$ an open neighborhood of $t_2 \in [0,1]$. There exists a function $f_2 \in C^1([0,1])$ such that

$$f_2|_{[0,1]\setminus N_2} = 0, \quad \|f_2\|_{[0,1]} < \varepsilon, \quad \text{and} \quad f_2'(t_2) = 1 = \|f_2'\|_{[0,1]}. \tag{2.4}$$

Here we notice that

$$f_2'|_{[0,1]\setminus N_2} = 0. \tag{2.5}$$

Let $F_2 = f_2 \otimes \mathbf{1}_X \in C^1([0,1], A)$, and then $F_2' = f_2' \otimes \mathbf{1}_X$. By the choice of μ,

$$\begin{aligned}
\int_{\widetilde{D}_{\partial A}} \widetilde{F}_2 \, d\mu = \delta_{\boldsymbol{p}}(\widetilde{F}_2) &= \widetilde{F}_2(t_1, t_2, x_1, x_2, z_1) \\
&= F_2(t_1)(x_1) + z_1 F_2'(t_2)(x_2) \\
&= f_2(t_1) + z_1 f_2'(t_2) = f_2(t_1) + z_1.
\end{aligned}$$

Equality (2.4) shows that

$$1 - \varepsilon \le \left| \int_{\widetilde{D}_{\partial A}} \widetilde{F}_2 \, d\mu \right|. \tag{2.6}$$

Recall that $\widetilde{D}_{\partial A} = D \times \partial A \times \partial A \times \mathbb{T}$ with $D \subset [0,1] \times [0,1]$. Let $N_2^c = [0,1] \setminus N_2$ and set, for each $N \subset [0,1]$,

$$O_N = \{D \cap ([0,1] \times N)\} \times \partial A \times \partial A \times \mathbb{T}.$$

Then $\widetilde{D}_{\partial A} = O_{N_2} \cup O_{N_2^c}$ and $O_{N_2} \cap O_{N_2^c} = \emptyset$. By equalities (2.4) and (2.5), we obtain

$$\int_{O_{N_2^c}} \widetilde{F}_2 \, d\mu = \int_{O_{N_2^c}} \{(f_2 \otimes \mathbf{1}_X)(s)(x) + z(f_2' \otimes \mathbf{1}_X)(t)(y)\} \, d\mu = 0.$$

Therefore, we have

$$\int_{\widetilde{D}_{\partial A}} \widetilde{F}_2 \, d\mu = \int_{O_{N_2}} \widetilde{F}_2 \, d\mu + \int_{O_{N_2^c}} \widetilde{F}_2 \, d\mu = \int_{O_{N_2}} \{f_2(s) + z f_2'(t)\} \, d\mu.$$

It follows from (2.4) and (2.6) that

$$1 - \varepsilon \le \left| \int_{\widetilde{D}_{\partial A}} \widetilde{F}_2 \, d\mu \right| \le (\varepsilon + 1)\mu(O_{N_2}).$$

By the liberty of the choice of ε, we get $1 \le \mu(O_{N_2})$. Because μ is a probability measure, $\mu(O_{N_2}) \le \mu(\widetilde{D}_{\partial A}) = 1$, and hence $\mu(O_{N_2}) = 1$. Since μ is a regular measure and N_2 is an arbitrary open neighborhood of t_2, we conclude $1 = \mu(O_{\{t_2\}}) = \mu(\{D \cap ([0,1] \times \{t_2\})\} \times \partial A \times \partial A \times \mathbb{T})$. □

Lemma 2.2. *Let $\boldsymbol{p} = (t_1, t_2, x_1, x_2, z_1) \in \widetilde{D}_{b(A)}$ and μ a representing measure for $\delta_{\boldsymbol{p}}$. Then $\mu(\{t_1\} \times \{t_2\} \times \partial A \times \partial A \times \mathbb{T}) = 1$.*

Proof. Let $N_1 \subset [0,1]$ be an open neighborhood of $t_1 \in [0,1]$ and $N_1^c = [0,1] \setminus N_1$. Choose a function $f_1 \in C^1([0,1])$ with

$$f_1(t_1) = 1 = \|f_1\|_{[0,1]}, \quad f_1|_{N_1^c} = a \quad \text{for some } 0 < a < 1, \tag{2.7}$$

and

$$f_1'(t_2) = f_1'|_{N_1^c} = 0. \tag{2.8}$$

Let $F_1 = f_1 \otimes \mathbf{1}_X \in C^1([0,1], A)$. For each $N \subset [0,1]$, we set

$$P_N = [D \cap (N \times \{t_2\})] \times \partial A \times \partial A \times \mathbb{T}.$$

By Lemma 2.1, $\mu(P_{[0,1]}) = \mu(\widetilde{D}_{\partial A}) = 1$. Equalities (2.1), (2.7) and (2.8) yield

$$\begin{aligned}\int_{P_{[0,1]}} f_1(s)\, d\mu + \int_{P_{[0,1]}} z f_1'(t)\, d\mu = \int_{P_{[0,1]}} \widetilde{F}_1\, d\mu &= \int_{\widetilde{D}_{\partial A}} \widetilde{F}_1\, d\mu \\ &= \delta_{\boldsymbol{p}}(\widetilde{F}_1) = f_1(t_1) + z_1 f_1'(t_2) = 1.\end{aligned}$$

As $P_{N_1} \cup P_{N_1^c} = P_{[0,1]}$ and $P_{N_1} \cap P_{N_1^c} = \emptyset$, it follows from (2.7) and (2.8) that

$$\begin{aligned} 1 &\leq \left| \int_{P_{[0,1]}} f_1(s)\, d\mu \right| + \left| \int_{P_{[0,1]}} z f_1'(t)\, d\mu \right| \\ &\leq \left| \int_{P_{N_1}} f_1(s)\, d\mu \right| + \left| \int_{P_{N_1^c}} f_1(s)\, d\mu \right| \\ &\leq \mu(P_{N_1}) + a\mu(P_{N_1^c}). \end{aligned}$$

Since $\mu(P_{N_1}) + \mu(P_{N_1^c}) = \mu(P_{[0,1]}) = 1$, we get $(1-a)\mu(P_{N_1^c}) \leq 0$. Recall that $a < 1$, and thus $(1-a)\mu(P_{N_1^c}) = 0$. Therefore, $\mu(P_{N_1^c}) = 0$, and hence $\mu(P_{N_1}) = 1$. By the regularity of μ, we have $\mu(P_{\{t_1\}}) = 1$, that is, $\mu(\{t_1\} \times \{t_2\} \times \partial A \times \partial A \times \mathbb{T}) = 1$. □

Lemma 2.3. *Let $\boldsymbol{p} = (t_1, t_2, x_1, x_2, z_1) \in \widetilde{D}_{b(A)}$ and μ a representing measure for $\delta_{\boldsymbol{p}}$. Then $\mu(\{t_1\} \times \{t_2\} \times \{x_1\} \times \partial A \times \mathbb{T}) = 1$.*

Proof. Let $W_1 \subset X$ be an open neighborhood of $x_1 \in b(A)$. For each $W \subset X$, we define Q_W by

$$Q_W = \{t_1\} \times \{t_2\} \times (W \cap \partial A) \times \partial A \times \mathbb{T}.$$

Set $W_1^c = X \setminus W_1$, and then $Q_{W_1} \cup Q_{W_1^c} = Q_{\partial A}$ and $Q_{W_1} \cap Q_{W_1^c} = \emptyset$. Since $x_1 \in b(A)$ there exists $v_1 \in A$ such that

$$v_1(x_1) = 1 = \|v_1\|_X \quad \text{and} \quad |v_1| < \varepsilon \quad \text{on } W_1^c. \tag{2.9}$$

We set $G_1 = \mathbf{1}_{[0,1]} \otimes v_1 \in C^1([0,1], A)$. By Lemma 2.2, $\mu(Q_{\partial A}) = 1 = \mu(\widetilde{D}_{\partial A})$, and then

$$\int_{Q_{\partial A}} \widetilde{G}_1\, d\mu = \int_{\widetilde{D}_{\partial A}} \widetilde{G}_1\, d\mu = \delta_{\boldsymbol{p}}(\widetilde{G}_1) = v_1(x_1) = 1$$

by (2.1). According to (2.9), $|\widetilde{G}_1| = |(\mathbf{1}_{[0,1]} \otimes v_1) + z_1(\mathbf{1}'_{[0,1]} \otimes v_1)| \le 1$ on Q_{W_1}, and $|\widetilde{G}_1| < \varepsilon$ on $Q_{W_1^c}$. These imply that

$$1 = \left| \int_{Q_{\partial A}} \widetilde{G}_1 \, d\mu \right| \le \left| \int_{Q_{W_1}} \widetilde{G}_1 \, d\mu \right| + \left| \int_{Q_{W_1^c}} \widetilde{G}_1 \, d\mu \right|$$
$$\le \mu(Q_{W_1}) + \varepsilon \mu(Q_{W_1^c}).$$

Since $\varepsilon > 0$ is arbitrary, we obtain $1 \le \mu(Q_{W_1})$, and then $\mu(Q_{W_1}) = 1$. By the regularity of μ, we get $\mu(Q_{\{x_1\}}) = 1$, that is, $\mu(\{t_1\} \times \{t_2\} \times \{x_1\} \times \partial A \times \mathbb{T}) = 1$. □

Lemma 2.4. *Let $\boldsymbol{p} = (t_1, t_2, x_1, x_2, z_1) \in \widetilde{D}_{b(A)}$. Then the Dirac measure concentrated at $\boldsymbol{p}$ is the unique representing measure for $\delta_{\boldsymbol{p}}$.*

Proof. Let $W_2 \subset X$ be an open neighborhood of $x_2 \in b(A)$, and let μ be a representing measure for $\delta_{\boldsymbol{p}}$. We will prove that μ is the Dirac measure concentrated at $\boldsymbol{p}$. For each $W \subset X$, we set $R_W = \{t_1\} \times \{t_2\} \times \{x_1\} \times (W \cap \partial A) \times \mathbb{T}$ and $W_2^c = X \setminus W_2$. Then $R_{W_2} \cup R_{W_2^c} = R_{\partial A}$ and $R_{W_2} \cap R_{W_2^c} = \emptyset$. For each $\varepsilon > 0$ there exist $g \in C^1([0,1])$ and $v_2 \in A$ such that

$$\|g\|_{[0,1]} < \varepsilon, \quad g'(t_2) = 1 = \|g'\|_{[0,1]}, \quad v_2(x_2) = 1 = \|v_2\|_X \quad \text{and} \quad |v_2| < \varepsilon \quad \text{on } W_2^c.$$

We set $G_2 = g \otimes v_2 \in C^1([0,1], A)$, and then $\left| \int_{R_{W_2^c}} \widetilde{G}_2 \, d\mu \right| \le 2\varepsilon \mu(R_{W_2^c})$. Lemma 2.3 shows $\mu(R_{\partial A}) = 1 = \mu(\widetilde{D}_{\partial A})$, and hence

$$\int_{R_{W_2}} \widetilde{G}_2 \, d\mu + \int_{R_{W_2^c}} \widetilde{G}_2 \, d\mu = \int_{R_{\partial A}} \widetilde{G}_2 \, d\mu = \delta_{\boldsymbol{p}}(\widetilde{G}_2) = g(t_1)v_2(x_1) + z_1.$$

It follows that

$$1 - \varepsilon - 2\varepsilon \mu(R_{W_2^c}) \le \left| \int_{R_{W_2}} \widetilde{G}_2 \, d\mu \right| \le (\varepsilon + 1)\mu(R_{W_2}).$$

Since $\varepsilon > 0$ is arbitrary, $1 \le \mu(R_{W_2})$ and thus $\mu(R_{W_2}) = 1$. By the regularity of μ, we conclude $\mu(\{t_1\} \times \{t_2\} \times \{x_1\} \times \{x_2\} \times \mathbb{T}) = 1$.

Let $J = \{t_1\} \times \{t_2\} \times \{x_1\} \times \{x_2\}$, and then $\mu(J \times \mathbb{T}) = 1$. We finally prove that $\mu(J \times \{z_1\}) = 1$. If we choose $f_3 \in C^1([0,1])$ so that

$$f_3(t_1) = 0, \quad \text{and} \quad f_3'(t_2) = 1,$$

then the function $F_3 = f_3 \otimes \mathbf{1}_X \in C^1([0,1], A)$ satisfies

$$z_1 = \delta_{\boldsymbol{p}}(\widetilde{F}_3) = \int_{\widetilde{D}_{\partial A}} \widetilde{F}_3 \, d\mu$$
$$= \int_{J \times \mathbb{T}} \{(f_3 \otimes \mathbf{1}_X)(t_1)(x_1) + z(f_3 \otimes \mathbf{1}_X)'(t_2)(x_2)\} \, d\mu = \int_{J \times \mathbb{T}} z \, d\mu.$$

Since $\mu(J\times\mathbb{T})=1$, we obtain $\displaystyle\int_{J\times\mathbb{T}}(z-z_1)\,d\mu=0$, and therefore $\displaystyle\int_{J\times\mathbb{T}}(1-\overline{z_1}z)\,d\mu=0$. Because μ is a probability measure,

$$\int_{J\times\mathbb{T}}(1-\mathrm{Re}\,(\overline{z_1}z))\,d\mu=\mathrm{Re}\int_{J\times\mathbb{T}}(1-\overline{z_1}z)\,d\mu=0.$$

Note that $1-\mathrm{Re}\,(\overline{z_1}z)\geq 0$ for all $z\in\mathbb{T}$, and thus there exists $Z\subset J\times\mathbb{T}$ such that

$$\mu(Z)=0 \quad\text{and}\quad 1-\mathrm{Re}\,(\overline{z_1}z)=0 \quad\text{on}\quad (J\times\mathbb{T})\setminus Z.$$

This shows $Z=J\times(\mathbb{T}\setminus\{z_1\})$. Since $\mu(Z)=0$ and $\mu(J\times\mathbb{T})=1$, we obtain $\mu(\boldsymbol{p})=\mu(J\times\{z_1\})=1$. We have proven that μ is a Dirac measure concentrated at $\boldsymbol{p}=(t_1,t_2,x_1,x_2,z_1)$, as is claimed. □

Lemma 2.5. *The Choquet boundary* $\mathrm{Ch}(B)$ *contains* $\widetilde{D}_{b(A)}$.

Proof. Let $\boldsymbol{p}\in\widetilde{D}_{b(A)}$. We will prove $\delta_{\boldsymbol{p}}\in\mathrm{ext}(B_1^*)$. Let $\eta_1,\eta_2\in B_1^*$ be such that $\delta_{\boldsymbol{p}}=(\eta_1+\eta_2)/2$. Recall that $\mathbf{1}=\mathbf{1}_{[0,1]}\otimes\mathbf{1}_X\in C^1([0,1],A)$. Then $\eta_1(\widetilde{\mathbf{1}})+\eta_2(\widetilde{\mathbf{1}})=2\delta_{\boldsymbol{p}}(\widetilde{\mathbf{1}})=2$ by (2.1). Because $\eta_j\in B_1^*$, $|\eta_j(\widetilde{\mathbf{1}})|\leq 1$ and thus $\eta_j(\widetilde{\mathbf{1}})=1=\|\eta_j\|$ for $j=1,2$, where $\|\cdot\|$ is the operator norm on B^*. Let ν_j be a representing measure for η_j, that is, $\eta_j(\widetilde{F})=\displaystyle\int_{\widetilde{D}_{\partial A}}\widetilde{F}\,d\nu_j$ for $\widetilde{F}\in B$. Then ν_j is a probability measure as mentioned in Proof of Lemma 2.1. Because $(\nu_1+\nu_2)/2$ is also a representing measure for $\delta_{\boldsymbol{p}}$, it follows from Lemma 2.4 that $(\nu_1+\nu_2)/2=\tau_{\boldsymbol{p}}$, the Dirac measure concentrated at $\boldsymbol{p}$. Since ν_j is a positive measure, $\nu_j(E)=0$ for each Borel set E with $\boldsymbol{p}\notin E$. Hence $\nu_j=\tau_{\boldsymbol{p}}$ for $j=1,2$, and consequently $\eta_1=\eta_2$. Therefore, $\delta_{\boldsymbol{p}}$ is an extreme point of B_1^*, and thus $\widetilde{D}_{b(A)}\subset\mathrm{Ch}(B)$ as is claimed. □

Lemma 2.6. *The set* $\mathrm{ext}(B_1^*)$ *is* $\{\lambda\delta_{\boldsymbol{p}}:\lambda\in\mathbb{T},\ \boldsymbol{p}\in\widetilde{D}_{b(A)}\}$.

Proof. According to the Arens-Kelley theorem, we see that $\mathrm{ext}(B_1^*)=\{\lambda\delta_{\boldsymbol{p}}:\lambda\in\mathbb{T},\ \boldsymbol{p}\in\mathrm{Ch}(B)\}$ (see [5, Corollary 2.3.6 and Theorem 2.3.8]). By Lemma 2.5, we need to prove that $\mathrm{Ch}(B)\subset\widetilde{D}_{b(A)}$. To this end, let $\boldsymbol{p}\in\mathrm{Ch}(B)$, and then $\delta_{\boldsymbol{p}}$ is an extreme point of B_1^*. There exist $(t_1,t_2)\in D$, $x_1,x_2\in\partial A$ and $z_0\in\mathbb{T}$ such that $\boldsymbol{p}=(t_1,t_2,x_1,x_2,z_0)$. Let e_x be a point evaluational functional on A at $x\in X$, defined by $e_x(u)=u(x)$ for $u\in A$. We denote by A_1^* the closed unit ball of the dual space of A. Let $\zeta_j,\xi_j\in A_1^*$ be such that $e_{x_1}=(\zeta_1+\zeta_2)/2$ and $e_{x_2}=(\xi_1+\xi_2)/2$. We show that $\zeta_1=\zeta_2$ and $\xi_1=\xi_2$. We define $\eta_j\colon B\to\mathbb{C}$ by

$$\eta_j(\widetilde{F})=\zeta_j(F(t_1))+z_0\xi_j(F'(t_2)) \qquad (j=1,2) \tag{2.10}$$

for $F\in C^1([0,1],A)$. Here, we recall that the map $U\colon C^1([0,1],A)\to B$, defined by $U(F)=\widetilde{F}$, is a surjective complex linear isometry (see (2.1), (2.2) and (2.3)). Then

η_j is a well defined, complex linear functional on B. Since $\zeta_j, \xi_j \in A_1^*$, we have, for each $\widetilde{F} \in B$,

$$
\begin{aligned}
|\eta_j(\widetilde{F})| &= |\zeta_j(F(t_1)) + z_0\xi_j(F'(t_2))| \\
&\leq \|\zeta_j\| \, \|F(t_1)\|_X + \|\xi_j\| \, \|F'(t_2)\|_X \\
&\leq \|F(t_1)\|_X + \|F'(t_2)\|_X \leq \|F\|_{\langle D \rangle} = \|\widetilde{F}\|_{\widetilde{D}_{\partial A}},
\end{aligned}
$$

where we have used (2.2). Therefore, $\eta_j \in B_1^*$ for $j = 1, 2$. Since $e_{x_1} = (\zeta_1 + \zeta_2)/2$ and $e_{x_2} = (\xi_1 + \xi_2)/2$,

$$
\begin{aligned}
(\eta_1 + \eta_2)(\widetilde{F}) &= (\zeta_1 + \zeta_2)(F(t_1)) + z_0(\xi_1 + \xi_2)(F'(t_2)) \\
&= 2e_{x_1}(F(t_1)) + 2z_0 e_{x_2}(F'(t_2)) \\
&= 2F(t_1)(x_1) + 2z_0 F'(t_2)(x_2) \\
&= 2\widetilde{F}(\boldsymbol{p}) = 2\delta_{\boldsymbol{p}}(\widetilde{F})
\end{aligned}
$$

for all $\widetilde{F} \in B$, where we have used (2.1). It follows that $\delta_{\boldsymbol{p}} = (\eta_1 + \eta_2)/2$. By the choice of $\boldsymbol{p}$, $\delta_{\boldsymbol{p}}$ is an extreme point of B_1^*, and thus $\eta_1 = \eta_2$. Let $F_u = \mathbf{1}_{[0,1]} \otimes u \in C^1([0,1], A)$ for each $u \in A$. Taking $F = F_u$ in (2.10), we have $\eta_j(\widetilde{F_u}) = \zeta_j(u)$. As $\eta_1 = \eta_2$, $\zeta_1(u) = \zeta_2(u)$ for all $u \in A$, and hence $\zeta_1 = \zeta_2$. This implies that e_{x_1} is an extreme point of A_1^*, i.e. $x_1 \in b(A)$. By the help of (2.10), we now derive $\xi_1(F'(t_2)) = \xi_2(F'(t_2))$ for all $F \in C^1([0,1], A)$. Taking $F = \mathrm{id} \otimes u \in C^1([0,1], A)$ in the last equality, we obtain $\xi_1(u) = \xi_2(u)$ for all $u \in A$. This shows $\xi_1 = \xi_2$, and therefore e_{x_2} is an extreme point of A_1^* as well. Hence $x_2 \in b(A)$, and consequently $\boldsymbol{p} = (t_1, t_2, x_1, x_2, z) \in \widetilde{D}_{b(A)}$. We have shown that $\mathrm{Ch}(B) \subset \widetilde{D}_{b(A)}$, as is claimed. $\square$

3. Auxiliary lemmas

Let T be a surjective isometry on $(C^1([0,1], A), \|\cdot\|_{\langle D \rangle})$. Recall that $B = \{\widetilde{F} \in C(\widetilde{D}_{\partial A}) : F \in C^1([0,1], A)\}$. Define a mapping $T_0 \colon C^1([0,1], A) \to C^1([0,1], A)$ by

$$
T_0 = T - T(0). \tag{3.1}
$$

By the Mazur-Ulam theorem [11, 17], T_0 is a surjective, *real linear* isometry on $(C^1([0,1], A), \|\cdot\|_{\langle D \rangle})$. Recall, by (2.3), that $U \colon (C^1([0,1], A), \|\cdot\|_{\langle D \rangle}) \to (B, \|\cdot\|_{\widetilde{D}_{\partial A}})$ is the surjective complex linear isometry, defined by $U(F) = \widetilde{F}$ for $F \in C^1([0,1], A)$. Denote UT_0U^{-1} by S; the mapping $S \colon B \to B$ is well defined since U is a surjective complex linear isometry.

$$
\begin{array}{ccc}
C^1([0,1], A) & \xrightarrow{T_0} & C^1([0,1], A) \\
{\scriptstyle U}\downarrow & & \downarrow{\scriptstyle U} \\
B & \xrightarrow[S]{} & B
\end{array}
$$

The equality $S = UT_0U^{-1}$ is equivalent to

$$S(\widetilde{F}) = \widetilde{T_0(F)} \qquad (\widetilde{F} \in B). \tag{3.2}$$

By the definition of S, we see that S is a surjective *real linear* isometry on $(B, \|\cdot\|_{\widetilde{D}_{\partial A}})$.

We define $S_*\colon B^* \to B^*$ by

$$S_*(\chi)(\widetilde{F}) = \mathrm{Re}\,[\chi(S(\widetilde{F}))] - i\mathrm{Re}\,[\chi(S(i\widetilde{F}))] \qquad (\chi \in B^*, \widetilde{F} \in B), \tag{3.3}$$

where $\mathrm{Re}\, z$ is the real part of a complex number z. We see that S_* is a surjective real linear isometry with respect to the operator norm (see [15, Proposition 5.17]).

Let $\mathfrak{B} = \{\lambda\delta_{\boldsymbol{p}} \in B_1^* : \lambda \in \mathbb{T}, \boldsymbol{p} \in \widetilde{D}_{\partial A}\}$ be a topological subspace of B_1^* with the relative weak *-topology. We define a map $\mathbf{h}\colon \mathbb{T} \times \widetilde{D}_{\partial A} \to \mathfrak{B}$ by $\mathbf{h}(\lambda, \boldsymbol{p}) = \lambda\delta_{\boldsymbol{p}}$ for $(\lambda, \boldsymbol{p}) \in \mathbb{T} \times \widetilde{D}_{\partial A}$.

Lemma 3.1. *The map $\mathbf{h}\colon \mathbb{T} \times \widetilde{D}_{\partial A} \to \mathfrak{B}$ is a homeomorphism. In particular, $\mathbf{h}(\mathbb{T} \times \widetilde{D}_{\partial A}) = \mathfrak{B}$.*

Proof. Since B contains the constant function $\widetilde{\mathbf{1}}$ and separates the points of $\widetilde{D}_{\partial A}$, we see that $\mathbf{h}$ is injective. By the definition of the map $\mathbf{h}$, we observe that $\mathbf{h}$ is continuous from the compact space $\mathbb{T} \times \widetilde{D}_{\partial A}$ with the product topology onto the Hausdorff space $\mathfrak{B}$ with the relative weak *-topology. Hence it is a homeomorphism. $\square$

Lemma 3.2. *The map S_* preserves $\mathfrak{B}$, that is, $S_*(\mathfrak{B}) = \mathfrak{B}$.*

Proof. Since S_* is a surjective real linear isometry on B_1^*, we see that $S^*(\mathrm{ext}(B_1^*)) = \mathrm{ext}(B_1^*)$. Let $\mathbf{h}$ be the homeomorphism defined in Lemma 3.1. By Lemma 2.6, $\mathrm{ext}(B_1^*) = \{\lambda\delta_{\boldsymbol{p}} : \lambda \in \mathbb{T}, \boldsymbol{p} \in \widetilde{D}_{b(A)}\} = \mathbf{h}(\mathbb{T} \times \widetilde{D}_{b(A)})$. Hence $S_*(\mathbf{h}(\mathbb{T} \times \widetilde{D}_{b(A)})) = \mathbf{h}(\mathbb{T} \times \widetilde{D}_{b(A)}) \subset \mathbf{h}(\mathbb{T} \times \widetilde{D}_{\partial A}) = \mathfrak{B}$, and therefore, $S_*(\mathbf{h}(\mathbb{T} \times \widetilde{D}_{b(A)})) \subset \mathfrak{B}$. We denote by $\mathrm{cl}(E)$ the closure of a set E. Because $b(A)$ is dense in ∂A, we obtain $\mathfrak{B} = \mathbf{h}(\mathbb{T} \times \widetilde{D}_{\partial A}) = \mathbf{h}(\mathbb{T} \times \mathrm{cl}(\widetilde{D}_{b(A)})) = \mathrm{cl}(\mathbf{h}(\mathbb{T} \times \widetilde{D}_{b(A)}))$, and thus $\mathfrak{B} = \mathrm{cl}(\mathbf{h}(\mathbb{T} \times \widetilde{D}_{b(A)}))$. Since $S_*\colon B_1^* \to B_1^*$ is a surjective isometry with respect to the operator norm, it is a homeomorphism with the relative weak *-topology on B_1^*. It follows that $S_*(\mathfrak{B}) = S_*(\mathrm{cl}(\mathbf{h}(\mathbb{T} \times \widetilde{D}_{b(A)}))) = \mathrm{cl}(S_*(\mathbf{h}(\mathbb{T} \times \widetilde{D}_{b(A)}))) \subset \mathrm{cl}(\mathfrak{B}) = \mathfrak{B}$. Therefore, $S_*(\mathfrak{B}) \subset \mathfrak{B}$. By the same arguments, applied to $(S_*)^{-1}$, we see that $(S_*)^{-1}(\mathfrak{B}) \subset \mathfrak{B}$, and consequently $S_*(\mathfrak{B}) = \mathfrak{B}$. $\square$

Definition 3.1. Suppose that $\mathbf{h}\colon \mathbb{T} \times \widetilde{D}_{\partial A} \to \mathfrak{B}$ is the homeomorphism defined in Lemma 3.1. Let $p_1\colon \mathbb{T} \times \widetilde{D}_{\partial A} \to \mathbb{T}$ and $p_2\colon \mathbb{T} \times \widetilde{D}_{\partial A} \to \widetilde{D}_{\partial A}$ be the natural projections from $\mathbb{T} \times \widetilde{D}_{\partial A}$ to the first and second coordinate, respectively. We define two maps $\alpha\colon \mathbb{T} \times \widetilde{D}_{\partial A} \to \mathbb{T}$ and $\Phi\colon \mathbb{T} \times \widetilde{D}_{\partial A} \to \widetilde{D}_{\partial A}$ by $\alpha = p_1 \circ \mathbf{h}^{-1} \circ S_* \circ \mathbf{h}$ and $\Phi = p_2 \circ \mathbf{h}^{-1} \circ S_* \circ \mathbf{h}$.

$$\begin{array}{ccc} \mathbb{T} \times \widetilde{D}_{\partial A} & \longrightarrow & \mathbb{T} \times \widetilde{D}_{\partial A} \\ {\scriptstyle \mathbf{h}}\big\downarrow & & \big\downarrow{\scriptstyle \mathbf{h}} \\ \mathfrak{B} & \underset{S_*}{\longrightarrow} & \mathfrak{B} \end{array}$$

By the definitions of maps α and Φ, $(\mathbf{h}^{-1} \circ S_* \circ \mathbf{h})(\lambda, \boldsymbol{p}) = (\alpha(\lambda, \boldsymbol{p}), \Phi(\lambda, \boldsymbol{p}))$ for all $(\lambda, \boldsymbol{p}) \in \mathbb{T} \times \widetilde{D}_{\partial A}$. Thus, $(S_* \circ \mathbf{h})(\lambda, \boldsymbol{p}) = \mathbf{h}(\alpha(\lambda, \boldsymbol{p}), \Phi(\lambda, \boldsymbol{p}))$, which is described as $S_*(\lambda\delta_{\boldsymbol{p}}) = \alpha(\lambda, \boldsymbol{p})\delta_{\Phi(\lambda,\boldsymbol{p})}$. For the sake of simplicity of notation, we shall write $\alpha(\lambda, \boldsymbol{p}) = \alpha_\lambda(\boldsymbol{p})$. Then we can write

$$S_*(\lambda\delta_{\boldsymbol{p}}) = \alpha_\lambda(\boldsymbol{p})\delta_{\Phi(\lambda,\boldsymbol{p})} \tag{3.4}$$

for all $(\lambda, \boldsymbol{p}) \in \mathbb{T} \times \widetilde{D}_{\partial A}$. Here, we notice that both α and Φ are surjective continuous maps since $\mathbf{h}$ and S_* are homeomorphisms.

Lemma 3.3. *For each $\boldsymbol{p} \in \widetilde{D}_{\partial A}$, $\alpha_i(\boldsymbol{p}) = i\alpha_1(\boldsymbol{p})$ or $\alpha_i(\boldsymbol{p}) = -i\alpha_1(\boldsymbol{p})$.*

Proof. Let $\boldsymbol{p} \in \widetilde{D}_{\partial A}$, and we set $\lambda_0 = (1+i)/\sqrt{2} \in \mathbb{T}$. By the real linearity of S_*, we obtain

$$\begin{aligned} \sqrt{2}\,\alpha_{\lambda_0}(\boldsymbol{p})\delta_{\Phi(\lambda_0,\boldsymbol{p})} &= S_*(\sqrt{2}\,\lambda_0\delta_{\boldsymbol{p}}) = S_*(\delta_{\boldsymbol{p}}) + S_*(i\delta_{\boldsymbol{p}}) \\ &= \alpha_1(\boldsymbol{p})\delta_{\Phi(1,\boldsymbol{p})} + \alpha_i(\boldsymbol{p})\delta_{\Phi(i,\boldsymbol{p})}. \end{aligned}$$

Hence $\sqrt{2}\,\alpha_{\lambda_0}(\boldsymbol{p})\delta_{\Phi(\lambda_0,\boldsymbol{p})} = \alpha_1(\boldsymbol{p})\delta_{\Phi(1,\boldsymbol{p})} + \alpha_i(\boldsymbol{p})\delta_{\Phi(i,\boldsymbol{p})}$. Evaluating this equality at $\widetilde{\mathbf{1}} \in B$, we get $\sqrt{2}\,\alpha_{\lambda_0}(\boldsymbol{p}) = \alpha_1(\boldsymbol{p}) + \alpha_i(\boldsymbol{p})$. Since $|\alpha_\lambda(\boldsymbol{p})| = 1$ for $\lambda \in \mathbb{T}$, we have $\sqrt{2} = |\alpha_1(\boldsymbol{p}) + \alpha_i(\boldsymbol{p})| = |1 + \alpha_i(\boldsymbol{p})\overline{\alpha_1(\boldsymbol{p})}|$. Then we see that $\alpha_i(\boldsymbol{p})\overline{\alpha_1(\boldsymbol{p})} = i$ or $\alpha_i(\boldsymbol{p})\overline{\alpha_1(\boldsymbol{p})} = -i$, which implies that $\alpha_i(\boldsymbol{p}) = i\alpha_1(\boldsymbol{p})$ or $\alpha_i(\boldsymbol{p}) = -i\alpha_1(\boldsymbol{p})$. □

Lemma 3.4. *There exists a continuous function $\varepsilon_0 \colon \widetilde{D}_{\partial A} \to \{\pm 1\}$ such that $S_*(i\delta_{\boldsymbol{p}}) = i\varepsilon_0(\boldsymbol{p})\alpha_1(\boldsymbol{p})\delta_{\Phi(i,\boldsymbol{p})}$ for every $\boldsymbol{p} \in \widetilde{D}_{\partial A}$.*

Proof. For each $\boldsymbol{p} \in \widetilde{D}_{\partial A}$, $\alpha_i(\boldsymbol{p}) = i\alpha_1(\boldsymbol{p})$ or $\alpha_i(\boldsymbol{p}) = -i\alpha_1(\boldsymbol{p})$ by Lemma 3.3. We define I_+ and I_- by

$$I_+ = \{\boldsymbol{p} \in \widetilde{D}_{\partial A} : \alpha_i(\boldsymbol{p}) = i\alpha_1(\boldsymbol{p})\} \quad \text{and} \quad I_- = \{\boldsymbol{p} \in \widetilde{D}_{\partial A} : \alpha_i(\boldsymbol{p}) = -i\alpha_1(\boldsymbol{p})\}.$$

Then $\widetilde{D}_{\partial A} = I_+ \cup I_-$ and $I_+ \cap I_- = \emptyset$. By the continuity of the functions $\alpha_1 = \alpha(1, \cdot)$ and $\alpha_i = \alpha(i, \cdot)$, we observe that I_+ and I_- are both closed subsets of $\widetilde{D}_{\partial A}$. Hence, the function $\varepsilon_0 \colon \widetilde{D}_{\partial A} \to \{\pm 1\}$, defined by

$$\varepsilon_0(\boldsymbol{p}) = \begin{cases} 1 & \boldsymbol{p} \in I_+ \\ -1 & \boldsymbol{p} \in I_- \end{cases},$$

is continuous on $\widetilde{D}_{\partial A}$. We obtain $\alpha_i(\boldsymbol{p}) = i\varepsilon_0(\boldsymbol{p})\alpha_1(\boldsymbol{p})$ for every $\boldsymbol{p} \in \widetilde{D}_{\partial A}$. This shows $S_*(i\delta_{\boldsymbol{p}}) = i\varepsilon_0(\boldsymbol{p})\alpha_1(\boldsymbol{p})\delta_{\Phi(i,\boldsymbol{p})}$ for all $\boldsymbol{p} \in \widetilde{D}_{\partial A}$. □

Lemma 3.5. *Suppose that ε_0 is the continuous function defined in Lemma 3.4. For each $\lambda = a + ib \in \mathbb{T}$ with $a, b \in \mathbb{R}$ and $\boldsymbol{p} \in \widetilde{D}_{\partial A}$,*

$$\lambda^{\varepsilon_0(\boldsymbol{p})} \widetilde{F}(\Phi(\lambda, \boldsymbol{p})) = a\widetilde{F}(\Phi(1, \boldsymbol{p})) + ib\varepsilon_0(\boldsymbol{p})\widetilde{F}(\Phi(i, \boldsymbol{p})) \tag{3.5}$$

for all $\widetilde{F} \in B$.

Proof. Let $\lambda = a + ib \in \mathbb{T}$ with $a, b \in \mathbb{R}$ and $\boldsymbol{p} \in \widetilde{D}_{\partial A}$. Recall that $S_*(\delta_{\boldsymbol{p}}) = \alpha_1(\boldsymbol{p})\delta_{\Phi(1,\boldsymbol{p})}$, and $S_*(i\delta_{\boldsymbol{p}}) = i\varepsilon_0(\boldsymbol{p})\alpha_1(\boldsymbol{p})\delta_{\Phi(i,\boldsymbol{p})}$ by Lemma 3.4. Because S_* is real linear,

$$\begin{aligned} \alpha_\lambda(\boldsymbol{p})\delta_{\Phi(\lambda,\boldsymbol{p})} &= S_*(\lambda\delta_{\boldsymbol{p}}) = aS_*(\delta_{\boldsymbol{p}}) + bS_*(i\delta_{\boldsymbol{p}}) \\ &= a\alpha_1(\boldsymbol{p})\delta_{\Phi(1,\boldsymbol{p})} + ib\varepsilon_0(\boldsymbol{p})\alpha_1(\boldsymbol{p})\delta_{\Phi(i,\boldsymbol{p})}, \end{aligned}$$

and thus $\alpha_\lambda(\boldsymbol{p})\delta_{\Phi(\lambda,\boldsymbol{p})} = \alpha_1(\boldsymbol{p})\{a\delta_{\Phi(1,\boldsymbol{p})} + ib\varepsilon_0(\boldsymbol{p})\delta_{\Phi(i,\boldsymbol{p})}\}$. The evaluation of this equality at $\widetilde{\mathbf{1}} \in B$ shows that $\alpha_\lambda(\boldsymbol{p}) = \alpha_1(\boldsymbol{p})(a + ib\varepsilon_0(\boldsymbol{p}))$. Because $\lambda = a + ib \in \mathbb{T}$ and $\varepsilon_0(\boldsymbol{p}) \in \{\pm 1\}$, we can write $a + ib\varepsilon_0(\boldsymbol{p}) = \lambda^{\varepsilon_0(\boldsymbol{p})}$. Hence $\alpha_\lambda(\boldsymbol{p}) = \lambda^{\varepsilon_0(\boldsymbol{p})}\alpha_1(\boldsymbol{p})$. We obtain $\lambda^{\varepsilon_0(\boldsymbol{p})}\delta_{\Phi(\lambda,\boldsymbol{p})} = a\delta_{\Phi(1,\boldsymbol{p})} + ib\varepsilon_0(\boldsymbol{p})\delta_{\Phi(i,\boldsymbol{p})}$, which implies $\lambda^{\varepsilon_0(\boldsymbol{p})}\widetilde{F}(\Phi(\lambda, \boldsymbol{p})) = a\widetilde{F}(\Phi(1, \boldsymbol{p})) + ib\varepsilon_0(\boldsymbol{p})\widetilde{F}(\Phi(i, \boldsymbol{p}))$ for all $\widetilde{F} \in B$. □

Definition 3.2. Let q_k be the projection from $\widetilde{D}_{\partial A} = D \times \partial A \times \partial A \times \mathbb{T}$ onto the k-th coordinate of $\widetilde{D}_{\partial A}$ for k with $1 \leq k \leq 4$. For the map $\Phi\colon \mathbb{T} \times \widetilde{D}_{\partial A} \to \widetilde{D}_{\partial A}$, as in Definition 3.1, we define $\phi\colon \mathbb{T} \times \widetilde{D}_{\partial A} \to D$, $\psi\colon \mathbb{T} \times \widetilde{D}_{\partial A} \to \partial A$, $\varphi\colon \mathbb{T} \times \widetilde{D}_{\partial A} \to \partial A$, and $\omega\colon \mathbb{T} \times \widetilde{D}_{\partial A} \to \mathbb{T}$ by $\phi = q_1 \circ \Phi$, $\psi = q_2 \circ \Phi$ $\varphi = q_3 \circ \Phi$ and $\omega = q_4 \circ \Phi$, respectively.

For each $\lambda \in \mathbb{T}$, we also write $\phi(\lambda, \boldsymbol{p}) = \phi_\lambda(\boldsymbol{p})$, $\psi(\lambda, \boldsymbol{p}) = \psi_\lambda(\boldsymbol{p})$, $\varphi(\lambda, \boldsymbol{p}) = \varphi_\lambda(\boldsymbol{p})$ and $\omega(\lambda, \boldsymbol{p}) = \omega_\lambda(\boldsymbol{p})$ for all $\boldsymbol{p} \in \widetilde{D}_{\partial A}$.

Recall that $\pi_j\colon D \to [0, 1]$ is the natural projection of $D \subset [0, 1] \times [0, 1]$ to the j-th coordinate for $j = 1, 2$. By the definition of ϕ, ψ, φ and ω, we have $(\pi_1(\phi_\lambda(\boldsymbol{p})), \pi_2(\phi_\lambda(\boldsymbol{p}))) \in D$ and

$$\Phi(\lambda, \boldsymbol{p}) = (\phi_\lambda(\boldsymbol{p}), \psi_\lambda(\boldsymbol{p}), \varphi_\lambda(\boldsymbol{p}), \omega_\lambda(\boldsymbol{p}))$$

for every $(\lambda, \boldsymbol{p}) \in \mathbb{T} \times \widetilde{D}_{\partial A}$. By (2.1),

$$\widetilde{F}(\Phi(\lambda, \boldsymbol{p})) = F(\pi_1(\phi_\lambda(\boldsymbol{p})))(\psi_\lambda(\boldsymbol{p})) + \omega_\lambda(\boldsymbol{p})F'(\pi_2(\phi_\lambda(\boldsymbol{p})))(\varphi_\lambda(\boldsymbol{p})) \tag{3.6}$$

for all $F \in C^1([0, 1], A)$ and $(\lambda, \boldsymbol{p}) \in \mathbb{T} \times \widetilde{D}_{\partial A}$. Note that ϕ, ψ, φ and ω are surjective and continuous since so is Φ (see Definition 3.1).

Lemma 3.6. *The function $\pi_1 \circ \phi_1\colon \widetilde{D}_{\partial A} \to [0, 1]$ is a surjective continuous function with $\pi_1(\phi_1(\boldsymbol{p})) = \pi_1(\phi_\lambda(\boldsymbol{p}))$ for all $\boldsymbol{p} \in \widetilde{D}_{\partial A}$ and $\lambda \in \mathbb{T}$.*

Proof. Let $\boldsymbol{p} \in \widetilde{D}_{\partial A}$. We will prove $\pi_1(\phi_\lambda(\boldsymbol{p})) \in \{\pi_1(\phi_1(\boldsymbol{p})), \pi_1(\phi_i(\boldsymbol{p}))\}$ for all $\lambda \in \mathbb{T}$. To do this, suppose, on the contrary, that there exists $\lambda_0 \in \mathbb{T} \setminus \{1, i\}$ such that

$\pi_1(\phi_{\lambda_0}(\boldsymbol{p})) \notin \{\pi_1(\phi_1(\boldsymbol{p})), \pi_1(\phi_i(\boldsymbol{p}))\}$. Choose $f_0 \in C^1([0,1])$ so that

$$f_0(\pi_1(\phi_{\lambda_0}(\boldsymbol{p}))) = 1, \quad f_0(\pi_1(\phi_1(\boldsymbol{p}))) = 0 = f_0(\pi_1(\phi_i(\boldsymbol{p}))) \\ \text{and} \quad f_0'(\pi_2(\phi_\mu(\boldsymbol{p}))) = 0 \qquad (\mu = \lambda_0, 1, i).$$

We set $F_0 = f_0 \otimes \mathbf{1}_X \in C^1([0,1], A)$. By (3.6), $\widetilde{F_0}(\Phi(\lambda_0, \boldsymbol{p})) = 1$ and $\widetilde{F_0}(\Phi(1, \boldsymbol{p})) = 0 = \widetilde{F_0}(\Phi(i, \boldsymbol{p}))$. Substituting these equalities into (3.5) to get $\lambda_0^{\varepsilon_0(\boldsymbol{p})} = 0$, which contradicts $\lambda_0 \in \mathbb{T}$. Consequently, we obtain $\pi_1(\phi_\lambda(\boldsymbol{p})) \in \{\pi_1(\phi_1(\boldsymbol{p})), \pi_1(\phi_i(\boldsymbol{p}))\}$ for all $\lambda \in \mathbb{T}$.

We next prove that $\pi_1(\phi_1(\boldsymbol{p})) = \pi_1(\phi_i(\boldsymbol{p}))$. To this end, suppose that $\pi_1(\phi_1(\boldsymbol{p})) \neq \pi_1(\phi_i(\boldsymbol{p}))$. We set $\lambda_1 = (1+i)/\sqrt{2} \in \mathbb{T}$. We obtain $\pi_1(\phi_{\lambda_1}(\boldsymbol{p})) \in \{\pi_1(\phi_1(\boldsymbol{p})), \pi_1(\phi_i(\boldsymbol{p}))\}$ as proved above. We consider the case when $\pi_1(\phi_{\lambda_1}(\boldsymbol{p})) = \pi_1(\phi_1(\boldsymbol{p}))$. Choose $f_1 \in C^1([0,1])$ so that

$$f_1(\pi_1(\phi_i(\boldsymbol{p}))) = 1, \quad f_1(\pi_1(\phi_1(\boldsymbol{p}))) = 0 \\ \text{and} \quad f_1'(\pi_2(\phi_\mu(\boldsymbol{p}))) = 0 \qquad (\mu = \lambda_1, 1, i).$$

Let $F_1 = f_1 \otimes \mathbf{1}_X \in C^1([0,1], A)$. Substituting these equalities into (3.6), we get $\widetilde{F_1}(\Phi(i, \boldsymbol{p})) = 1$ and $\widetilde{F_1}(\Phi(\lambda_1, \boldsymbol{p})) = 0 = \widetilde{F_1}(\Phi(1, \boldsymbol{p}))$. By (3.5), we obtain $0 = i\varepsilon_0(\boldsymbol{p})$, which contradicts $\varepsilon_0(\boldsymbol{p}) \in \{\pm 1\}$. By a similar arguments, we reach a contradiction even if $\pi_1(\phi_{\lambda_1}(\boldsymbol{p})) = \pi_1(\phi_i(\boldsymbol{p}))$. Thus, we get $\pi_1(\phi_1(\boldsymbol{p})) = \pi_1(\phi_i(\boldsymbol{p}))$ for all $\boldsymbol{p} \in \widetilde{D}_{\partial A}$, and consequently $\pi_1(\phi_1(\boldsymbol{p})) = \pi_1(\phi_\lambda(\boldsymbol{p}))$ for all $\lambda \in \mathbb{T}$ and $\boldsymbol{p} \in \widetilde{D}_{\partial A}$.

We show that $\pi_1 \circ \phi_1$ is surjective. Let $t_1 \in \pi_1(D)$, and then $\pi_1(\boldsymbol{t}) = t_1$ for some $\boldsymbol{t} \in D$. Since ϕ is surjective, there exists $(\mu, \boldsymbol{q}) \in \mathbb{T} \times \widetilde{D}_{\partial A}$ such that $\boldsymbol{t} = \phi(\mu, \boldsymbol{q}) = \phi_\mu(\boldsymbol{q})$. By the fact proved in the last paragraph, $\pi_1(\phi_1(\boldsymbol{q})) = \pi_1(\phi_\mu(\boldsymbol{q})) = \pi_1(\boldsymbol{t}) = t_1$. This yields the surjectivity of $\pi_1 \circ \phi_1$. □

By a similar argument to Lemma 3.6, we can prove that $\pi_2(\phi_\lambda(\boldsymbol{p})) = \pi_2(\phi_1(\boldsymbol{p}))$ for all $\lambda \in \mathbb{T}$ and $\boldsymbol{p} \in \widetilde{D}_{\partial A}$. Just for the sake of completeness, here we give its proof.

Lemma 3.7. *The function $\pi_2 \circ \phi_1 \colon \widetilde{D}_{\partial A} \to [0,1]$ is a surjective continuous function with $\pi_2(\phi_1(\boldsymbol{p})) = \pi_2(\phi_\lambda(\boldsymbol{p}))$ for all $\boldsymbol{p} \in \widetilde{D}_{\partial A}$ and $\lambda \in \mathbb{T}$.*

Proof. Let $\boldsymbol{p} \in \widetilde{D}_{\partial A}$. By Lemma 3.6, $\phi_\lambda(\boldsymbol{p}) = (\pi_1(\phi_1(\boldsymbol{p})), \pi_2(\phi_\lambda(\boldsymbol{p})))$ and $\Phi(\lambda, \boldsymbol{p}) = (\phi_\lambda(\boldsymbol{p}), \psi_\lambda(\boldsymbol{p}), \varphi_\lambda(\boldsymbol{p}), \omega_\lambda(\boldsymbol{p}))$ for $\lambda \in \mathbb{T}$. Equality (3.6) is reduced to

$$\widetilde{F}(\Phi(\lambda, \boldsymbol{p})) = F(\pi_1(\phi_1(\boldsymbol{p})))(\psi_\lambda(\boldsymbol{p})) + \omega_\lambda(\boldsymbol{p}) F'(\pi_2(\phi_\lambda(\boldsymbol{p})))(\varphi_\lambda(\boldsymbol{p})) \qquad (3.7)$$

for all $F \in C^1([0,1], A)$ and $\lambda \in \mathbb{T}$.

First, we show that $\pi_2(\phi_\lambda(\boldsymbol{p})) \in \{\pi_2(\phi_1(\boldsymbol{p})), \pi_2(\phi_i(\boldsymbol{p}))\}$ for all $\lambda \in \mathbb{T}$. Suppose, on the contrary, that $\pi_2(\phi_{\lambda_0}(\boldsymbol{p})) \notin \{\pi_2(\phi_1(\boldsymbol{p})), \pi_2(\phi_i(\boldsymbol{p}))\}$ for some $\lambda_0 \in \mathbb{T} \setminus \{1, i\}$.

Then there exists $f_0 \in C^1([0,1])$ such that

$$f_0(\pi_1(\phi_1(\boldsymbol{p}))) = 0, \quad f_0'(\pi_2(\phi_{\lambda_0}(\boldsymbol{p}))) = 1$$
$$\text{and} \quad f_0'(\pi_2(\phi_1(\boldsymbol{p}))) = 0 = f_0'(\pi_2(\phi_i(\boldsymbol{p}))).$$

For $F_0 = f_0 \otimes \mathbf{1}_X \in C^1([0,1], A)$, $\widetilde{F_0}(\Phi(\lambda_0, \boldsymbol{p})) = \omega_{\lambda_0}(\boldsymbol{p})$ and $\widetilde{F_0}(\Phi(1, \boldsymbol{p})) = 0 = \widetilde{F_0}(\Phi(i, \boldsymbol{p}))$ by (3.7). If we substitute these equalities into (3.5), we have $\lambda_0^{\varepsilon_0(\boldsymbol{p})}\omega_{\lambda_0}(\boldsymbol{p}) = 0$, which contradicts $\lambda_0, \omega_{\lambda_0}(\boldsymbol{p}) \in \mathbb{T}$. Consequently, $\pi_2(\phi_\lambda(\boldsymbol{p})) \in \{\pi_2(\phi_1(\boldsymbol{p})), \pi_2(\phi_i(\boldsymbol{p}))\}$ for all $\lambda \in \mathbb{T}$.

We next prove $\pi_2(\phi_1(\boldsymbol{p})) = \pi_2(\phi_i(\boldsymbol{p}))$. Suppose that $\pi_2(\phi_1(\boldsymbol{p})) \neq \pi_2(\phi_i(\boldsymbol{p}))$. For $\lambda_1 = (1+i)/\sqrt{2} \in \mathbb{T}$, $\pi_2(\phi_{\lambda_1}(\boldsymbol{p})) \in \{\pi_2(\phi_1(\boldsymbol{p}), \pi_2(\phi_i(\boldsymbol{p}))\}$ by the last paragraph. If we assume $\pi_2(\phi_{\lambda_1}(\boldsymbol{p})) = \pi_2(\phi_1(\boldsymbol{p}))$, then we can choose $f_1 \in C^1([0,1])$ so that

$$f_1(\pi_1(\phi_1(\boldsymbol{p}))) = 0 = f_1'(\pi_2(\phi_1(\boldsymbol{p}))) \quad \text{and} \quad f_1'(\pi_2(\phi_i(\boldsymbol{p}))) = 1.$$

Applying these equalities to (3.7), we obtain $\widetilde{F_1}(\Phi(i, \boldsymbol{p})) = \omega_i(\boldsymbol{p})$ and $\widetilde{F_1}(\Phi(1, \boldsymbol{p})) = 0 = \widetilde{F_1}(\Phi(\lambda_1, \boldsymbol{p}))$ for $F_1 = f_1 \otimes \mathbf{1}_X \in C^1([0,1], A)$, where we have used $\pi_2(\phi_{\lambda_1}(\boldsymbol{p})) = \pi_2(\phi_1(\boldsymbol{p}))$. By (3.5), we have $0 = i\varepsilon_0(\boldsymbol{p})\omega_i(\boldsymbol{p})$, which is impossible. We reach a similar contradiction even if $\pi_2(\phi_{\lambda_1}(\boldsymbol{p})) = \pi_2(\phi_i(\boldsymbol{p}))$. Therefore, we conclude $\pi_2(\phi_1(\boldsymbol{p})) = \pi_2(\phi_i(\boldsymbol{p}))$. Consequently $\pi_2(\phi_1(\boldsymbol{p})) = \pi_2(\phi_\lambda(\boldsymbol{p}))$ for all $\lambda \in \mathbb{T}$.

Finally, since ϕ is surjective, for each $t_2 \in [0,1] = \pi_2(D)$ there exists $(\mu, \boldsymbol{q}) \in \mathbb{T} \times \widetilde{D}_{\partial A}$ such that $\pi_2(\phi(\mu, \boldsymbol{q})) = t_2$. By the last paragraph, we see that $t_2 = \pi_2(\phi_\mu(\boldsymbol{q})) = \pi_2(\phi_1(\boldsymbol{q}))$, which shows the surjectivity of $\pi_2 \circ \phi_1$. □

Notation. For the sake of simplicity of notation, we will write $\pi_1(\phi_1(\boldsymbol{p})) = d_1(\boldsymbol{p})$ and $\pi_2(\phi_1(\boldsymbol{p})) = d_2(\boldsymbol{p})$ for $\boldsymbol{p} \in \widetilde{D}_{\partial A}$. Then $\phi_1(\boldsymbol{p})$ is written as $(d_1(\boldsymbol{p}), d_2(\boldsymbol{p}))$.

Lemma 3.8. *The function $\psi_1 \colon \widetilde{D}_{\partial A} \to \partial A$ is a surjective continuous function with $\psi_1(\boldsymbol{p}) = \psi_\lambda(\boldsymbol{p})$ for all $\boldsymbol{p} \in \widetilde{D}_{\partial A}$ and $\lambda \in \mathbb{T}$.*

Proof. Let $\boldsymbol{p} \in \widetilde{D}_{\partial A}$. By Lemma 3.7, equality (3.7) is reduced to

$$\widetilde{F}(\Phi(\lambda, \boldsymbol{p})) = F(d_1(\boldsymbol{p}))(\psi_\lambda(\boldsymbol{p})) + \omega_\lambda(\boldsymbol{p})F'(d_2(\boldsymbol{p}))(\varphi_\lambda(\boldsymbol{p})) \tag{3.8}$$

for all $F \in C^1([0,1], A)$ and $\lambda \in \mathbb{T}$.

First, we show that $\psi_\lambda(\boldsymbol{p}) \in \{\psi_1(\boldsymbol{p}), \psi_i(\boldsymbol{p})\}$ for all $\lambda \in \mathbb{T}$. Suppose, on the contrary, that there exists $\lambda_0 \in \mathbb{T} \setminus \{1, i\}$ such that $\psi_{\lambda_0}(\boldsymbol{p}) \notin \{\psi_1(\boldsymbol{p}), \psi_i(\boldsymbol{p})\}$. Then there exists $u_0 \in A$ such that

$$u_0(\psi_{\lambda_0}(\boldsymbol{p}))) = 1 \quad \text{and} \quad u_0(\psi_1(\boldsymbol{p})) = 0 = u_0(\psi_i(\boldsymbol{p})).$$

For $G_0 = \mathbf{1}_{[0,1]} \otimes u_0 \in C^1([0,1], A)$, we obtain $\widetilde{G_0}(\Phi(\lambda_0, \boldsymbol{p})) = 1$ and $\widetilde{G_0}(\Phi(1, \boldsymbol{p})) = 0 = \widetilde{G_0}(\Phi(i, \boldsymbol{p}))$ by (3.8). If we substitute these equalities into (3.5), we get $\lambda_0^{\varepsilon_0(\boldsymbol{p})} = 0$, which contradicts $\lambda_0 \in \mathbb{T}$. Consequently, $\psi_\lambda(\boldsymbol{p}) \in \{\psi_1(\boldsymbol{p}), \psi_i(\boldsymbol{p})\}$ for all $\lambda \in \mathbb{T}$.

We next prove that $\psi_1(\boldsymbol{p}) = \psi_i(\boldsymbol{p})$. Suppose that $\psi_1(\boldsymbol{p}) \neq \psi_i(\boldsymbol{p})$. We set $\lambda_1 = (1+i)/\sqrt{2} \in \mathbb{T}$, and then $\psi_{\lambda_1}(\boldsymbol{p}) \in \{\psi_1(\boldsymbol{p}), \psi_i(\boldsymbol{p})\}$ by the fact obtained in the last paragraph. If $\psi_{\lambda_1}(\boldsymbol{p}) = \psi_1(\boldsymbol{p})$, then we can choose $u_1 \in A$ so that

$$u_1(\psi_{\lambda_1}(\boldsymbol{p})) = 0 = u_1(\psi_1(\boldsymbol{p})) \quad \text{and} \quad u_1(\psi_i(\boldsymbol{p})) = 1.$$

Equality (3.8), applied to $F = \mathbf{1}_{[0,1]} \otimes u_1 \in C^1([0,1], A)$, shows that $\widetilde{F}(\Phi(\lambda_1, \boldsymbol{p})) = 0 = \widetilde{F}(\Phi(1, \boldsymbol{p}))$ and $\widetilde{F}(\Phi(i, \boldsymbol{p})) = 1$. By (3.5), we have $0 = i\varepsilon_0(\boldsymbol{p})$, which is impossible. We can reach a similar contradiction even if $\psi_{\lambda_1}(\boldsymbol{p}) = \psi_i(\boldsymbol{p})$. Therefore, we conclude $\psi_1(\boldsymbol{p}) = \psi_i(\boldsymbol{p})$. Consequently $\psi_1(\boldsymbol{p}) = \psi_\lambda(\boldsymbol{p})$ for all $\lambda \in \mathbb{T}$.

Finally, we show that $\psi_1 \colon \widetilde{D}_{\partial A} \to \partial A$ is surjective. Since $\psi \colon \mathbb{T} \times \widetilde{D}_{\partial A} \to \partial A$ is surjective, for each $x \in \partial A$ there exists $(\mu, \boldsymbol{q}) \in \mathbb{T} \times \widetilde{D}_{\partial A}$ such that $\psi(\mu, \boldsymbol{q}) = x$. By the last paragraph, we see $x = \psi_\mu(\boldsymbol{q}) = \psi_1(\boldsymbol{q})$, which shows that ψ_1 is surjective. $\square$

Lemma 3.9. *The function $\varphi_1 \colon \widetilde{D}_{\partial A} \to \partial A$ is a surjective continuous function with $\varphi_1(\boldsymbol{p}) = \varphi_\lambda(\boldsymbol{p})$ for all $\boldsymbol{p} \in \widetilde{D}_{\partial A}$ and $\lambda \in \mathbb{T}$.*

Proof. Let $\boldsymbol{p} \in \widetilde{D}_{\partial A}$. By Lemma 3.8, equality (3.8) is reduced to

$$\widetilde{F}(\Phi(\lambda, \boldsymbol{p})) = F(d_1(\boldsymbol{p}))(\psi_1(\boldsymbol{p})) + \omega_\lambda(\boldsymbol{p}) F'(d_2(\boldsymbol{p}))(\varphi_\lambda(\boldsymbol{p})) \tag{3.9}$$

for all $F \in C^1([0,1], A)$ and $\lambda \in \mathbb{T}$.

First, we show that $\varphi_\lambda(\boldsymbol{p}) \in \{\varphi_1(\boldsymbol{p}), \varphi_i(\boldsymbol{p})\}$ for all $\lambda \in \mathbb{T}$. Suppose, on the contrary, that there exists $\lambda_0 \in \mathbb{T} \setminus \{1, i\}$ such that $\varphi_{\lambda_0}(\boldsymbol{p}) \notin \{\varphi_1(\boldsymbol{p}), \varphi_i(\boldsymbol{p})\}$. Then there exists $u_0 \in A$ such that

$$u_0(\varphi_{\lambda_0}(\boldsymbol{p}))) = 1 \quad \text{and} \quad u_0(\varphi_1(\boldsymbol{p})) = 0 = u_0(\varphi_i(\boldsymbol{p})).$$

For $G_0 = (\mathrm{id} - d_1(\boldsymbol{p})\mathbf{1}_{[0,1]}) \otimes u_0 \in C^1([0,1], A)$, we obtain $\widetilde{G_0}(\Phi(\lambda_0, \boldsymbol{p})) = \omega_{\lambda_0}(\boldsymbol{p})$ and $\widetilde{G_0}(\Phi(1, \boldsymbol{p})) = 0 = \widetilde{G_0}(\Phi(i, \boldsymbol{p}))$ by (3.9). If we substitute these equalities into (3.5), we get $\lambda_0^{\varepsilon_0(\boldsymbol{p})} \omega_{\lambda_0}(\boldsymbol{p}) = 0$, which contradicts $\lambda_0, \omega_{\lambda_0}(\boldsymbol{p}) \in \mathbb{T}$. Consequently, $\varphi_\lambda(\boldsymbol{p}) \in \{\varphi_1(\boldsymbol{p}), \varphi_i(\boldsymbol{p})\}$ for all $\lambda \in \mathbb{T}$.

We next prove that $\varphi_1(\boldsymbol{p}) = \varphi_i(\boldsymbol{p})$. Suppose that $\varphi_1(\boldsymbol{p}) \neq \varphi_i(\boldsymbol{p})$. Set $\lambda_1 = (1+i)/\sqrt{2} \in \mathbb{T}$, and then $\varphi_{\lambda_1}(\boldsymbol{p}_0) \in \{\varphi_1(\boldsymbol{p}), \varphi_i(\boldsymbol{p})\}$ by the previous paragraph. If $\varphi_{\lambda_1}(\boldsymbol{p}) = \varphi_1(\boldsymbol{p})$, then we can choose $u_1 \in A$ so that

$$u_1(\varphi_{\lambda_1}(\boldsymbol{p})) = 0 = u_1(\varphi_1(\boldsymbol{p})) \quad \text{and} \quad u_1(\varphi_i(\boldsymbol{p})) = 1.$$

Equality (3.9), applied to $F = (\mathrm{id} - d_1(\boldsymbol{p})\mathbf{1}_{[0,1]}) \otimes u_1 \in C^1([0,1], A)$, shows that $\widetilde{F}(\Phi(\lambda_1, \boldsymbol{p})) = 0 = \widetilde{F}(\Phi(1, \boldsymbol{p}))$ and $\widetilde{F}(\Phi(i, \boldsymbol{p})) = \omega_i(\boldsymbol{p})$. According to (3.5), we get $0 = i\varepsilon_0(\boldsymbol{p})\omega_i(\boldsymbol{p})$, which is impossible. We can reach a similar contradiction even if $\varphi_{\lambda_1}(\boldsymbol{p}) = \varphi_i(\boldsymbol{p})$. Therefore, we conclude $\varphi_1(\boldsymbol{p}) = \varphi_i(\boldsymbol{p})$. Consequently $\varphi_\lambda(\boldsymbol{p}) = \varphi_1(\boldsymbol{p})$ for all $\lambda \in \mathbb{T}$.

Finally, we show that $\varphi_1 \colon \widetilde{D}_{\partial A} \to \partial A$ is surjective. Since $\varphi \colon \mathbb{T} \times \widetilde{D}_{\partial A} \to \partial A$ is surjective, for each $x \in \partial A$ there exists $(\mu, \boldsymbol{q}) \in \mathbb{T} \times \widetilde{D}_{\partial A}$ such that $\varphi(\mu, \boldsymbol{q}) = x$.

By the last paragraph, we see that $x = \varphi_\mu(\boldsymbol{q}) = \varphi_1(\boldsymbol{q})$, which shows that φ_1 is surjective. □

Lemma 3.10. *There exists a continuous function $\varepsilon_1\colon \widetilde{D}_{\partial A} \to \{\pm 1\}$ such that $\omega_i(\boldsymbol{p}) = \varepsilon_1(\boldsymbol{p})\omega_1(\boldsymbol{p})$ for all $\boldsymbol{p} \in \widetilde{D}_{\partial A}$.*

Proof. Let $\boldsymbol{p} \in \widetilde{D}_{\partial A}$. According to Lemmas from 3.6 to 3.9, we can write $\Phi(\lambda, \boldsymbol{p}) = ((d_1(\boldsymbol{p}), d_2(\boldsymbol{p})), \psi_1(\boldsymbol{p}), \varphi_1(\boldsymbol{p}), \omega_\lambda(\boldsymbol{p}))$ for all $\lambda \in \mathbb{T}$. We set $\lambda_0 = (1+i)/\sqrt{2} \in \mathbb{T}$ and $f_0 = \mathrm{id} - d_1(\boldsymbol{p})\mathbf{1}_{[0,1]} \in C^1([0,1])$. Then $f_0(d_1(\boldsymbol{p})) = 0$ and $f_0' = 1$ on $[0,1]$. By (3.9), $\widetilde{F_0}(\Phi(\mu, \boldsymbol{p})) = \omega_\mu(\boldsymbol{p})$ for $F_0 = f_0 \otimes \mathbf{1}_X \in C^1([0,1], A)$ and $\mu = \lambda_0, 1, i$. If we apply these equalities to (3.5), then we obtain $\sqrt{2}\,\lambda_0^{\varepsilon_0(\boldsymbol{p})}\omega_{\lambda_0}(\boldsymbol{p}) = \omega_1(\boldsymbol{p}) + i\varepsilon_0(\boldsymbol{p})\omega_i(\boldsymbol{p})$. As $\omega_\lambda(\boldsymbol{p}) \in \mathbb{T}$ for all $\lambda \in \mathbb{T}$,

$$\sqrt{2} = |\omega_1(\boldsymbol{p}) + i\varepsilon_0(\boldsymbol{p})\omega_i(\boldsymbol{p})| = |1 + i\varepsilon_0(\boldsymbol{p})\omega_i(\boldsymbol{p})\overline{\omega_1(\boldsymbol{p})}|.$$

Then we get $i\varepsilon_0(\boldsymbol{p})\omega_i(\boldsymbol{p})\overline{\omega_1(\boldsymbol{p})} = i$ or $i\varepsilon_0(\boldsymbol{p})\omega_i(\boldsymbol{p})\overline{\omega_1(\boldsymbol{p})} = -i$. Thus, for each $\boldsymbol{p} \in \widetilde{D}_{\partial A}$, we derive $\omega_i(\boldsymbol{p}) = \varepsilon_0(\boldsymbol{p})\omega_1(\boldsymbol{p})$ or $\omega_i(\boldsymbol{p}) = -\varepsilon_0(\boldsymbol{p})\omega_1(\boldsymbol{p})$. By the continuity of ω_1 and ω_i, there exists a continuous function $\varepsilon_1\colon \widetilde{D}_{\partial A} \to \{\pm 1\}$ such that $\omega_i(\boldsymbol{p}) = \varepsilon_1(\boldsymbol{p})\omega_1(\boldsymbol{p})$ for all $\boldsymbol{p} \in \widetilde{D}_{\partial A}$. □

Notation. In the rest of this paper, we denote $a + ib\varepsilon$ by $[a+ib]^\varepsilon$ for $a, b \in \mathbb{R}$ and $\varepsilon \in \{\pm 1\}$. Therefore, $[\lambda\mu]^\varepsilon = [\lambda]^\varepsilon[\mu]^\varepsilon$ for all $\lambda, \mu \in \mathbb{C}$. If, in addition, $\lambda \in \mathbb{T}$, then $[\lambda]^\varepsilon = \lambda^\varepsilon$.

Lemma 3.11. *For each $F \in C^1([0,1], A)$ and $\boldsymbol{p} \in \widetilde{D}_{\partial A}$,*

$$\begin{aligned} S(\widetilde{F})(\boldsymbol{p}) = {} & [\alpha_1(\boldsymbol{p})F(d_1(\boldsymbol{p}))(\psi_1(\boldsymbol{p}))]^{\varepsilon_0(\boldsymbol{p})} \\ & + [\alpha_1(\boldsymbol{p})\omega_1(\boldsymbol{p})F'(d_2(\boldsymbol{p}))(\varphi_1(\boldsymbol{p}))]^{\varepsilon_0(\boldsymbol{p})\varepsilon_1(\boldsymbol{p})}. \end{aligned} \tag{3.10}$$

Proof. Let $F \in C^1([0,1], A)$ and $\boldsymbol{p} \in \widetilde{D}_{\partial A}$. By the definition (3.3) of S_*, we have $\mathrm{Re}\,[S_*(\chi)(\widetilde{F})] = \mathrm{Re}\,[\chi(S(\widetilde{F}))]$ for every $\chi \in B^*$. Taking $\chi = \delta_{\boldsymbol{p}}$ and $\chi = i\delta_{\boldsymbol{p}}$ into the last equality, we get

$$\mathrm{Re}\,[S_*(\delta_{\boldsymbol{p}})(\widetilde{F})] = \mathrm{Re}\,[S(\widetilde{F})(\boldsymbol{p})] \quad \text{and} \quad \mathrm{Re}\,[S_*(i\delta_{\boldsymbol{p}})(\widetilde{F})] = -\mathrm{Im}\,[S(\widetilde{F})(\boldsymbol{p})],$$

respectively, where $\mathrm{Im}\, z$ is the imaginary part of a complex number z. Therefore,

$$S(\widetilde{F})(\boldsymbol{p}) = \mathrm{Re}\,[S_*(\delta_{\boldsymbol{p}})(\widetilde{F})] - i\,\mathrm{Re}\,[S_*(i\delta_{\boldsymbol{p}})(\widetilde{F})]. \tag{3.11}$$

By definition, $S_*(\delta_{\boldsymbol{p}}) = \alpha_1(\boldsymbol{p})\delta_{\Phi(1,\boldsymbol{p})}$, and $S_*(i\delta_{\boldsymbol{p}}) = i\varepsilon_0(\boldsymbol{p})\alpha_1(\boldsymbol{p})\delta_{\Phi(i,\boldsymbol{p})}$ by Lemma 3.4. Substitute these two equalities into (3.11) to obtain

$$S(\widetilde{F})(\boldsymbol{p}) = \mathrm{Re}\,[\alpha_1(\boldsymbol{p})\widetilde{F}(\Phi(1,\boldsymbol{p}))] + i\,\mathrm{Im}\,[\varepsilon_0(\boldsymbol{p})\alpha_1(\boldsymbol{p})\widetilde{F}(\Phi(i,\boldsymbol{p}))]. \tag{3.12}$$

Lemmas from 3.6 to 3.10 imply that $\Phi(1,\boldsymbol{p}) = (\phi_1(\boldsymbol{p}), \psi_1(\boldsymbol{p}), \varphi_1(\boldsymbol{p}), \omega_1(\boldsymbol{p}))$ and $\Phi(i,\boldsymbol{p}) = (\phi_1(\boldsymbol{p}), \psi_1(\boldsymbol{p}), \varphi_1(\boldsymbol{p}), \varepsilon_1(\boldsymbol{p})\omega_1(\boldsymbol{p}))$. It follows from (2.1) that

$$\begin{aligned}\widetilde{F}(\Phi(1,\boldsymbol{p})) &= F(d_1(\boldsymbol{p})(\psi_1(\boldsymbol{p})) + \omega_1(\boldsymbol{p})F'(d_2(\boldsymbol{p}))(\varphi_1(\boldsymbol{p})),\\ \widetilde{F}(\Phi(i,\boldsymbol{p})) &= F(d_1(\boldsymbol{p}))(\psi_1(\boldsymbol{p})) + \varepsilon_1(\boldsymbol{p})\omega_1(\boldsymbol{p})F'(d_2(\boldsymbol{p}))(\varphi_1(\boldsymbol{p})).\end{aligned}$$

Applying these two equalities to (3.12), we derive

$$S(\widetilde{F})(\boldsymbol{p}) = [\alpha_1(\boldsymbol{p})F(d_1(\boldsymbol{p}))(\psi_1(\boldsymbol{p}))]^{\varepsilon_0(\boldsymbol{p})} + [\alpha_1(\boldsymbol{p})\omega_1(\boldsymbol{p})F'(d_2(\boldsymbol{p}))(\varphi_1(\boldsymbol{p}))]^{\varepsilon_0(\boldsymbol{p})\varepsilon_1(\boldsymbol{p})}.$$

This completes the proof. □

4. Properties of induced maps

In this section, we shall simplify equality (3.10). By (3.2), $S(\widetilde{F}) = \widetilde{T_0(F)}$ for $F \in C^1([0,1],A)$. Applying (2.1), we can rewrite (3.10) as

$$\begin{aligned}&T_0(F)(t_1)(x_1) + zT_0(F)'(t_2)(x_2)\\ &\quad= [\alpha_1(\boldsymbol{p})F(d_1(\boldsymbol{p}))(\psi_1(\boldsymbol{p}))]^{\varepsilon_0(\boldsymbol{p})} + [\alpha_1(\boldsymbol{p})\omega_1(\boldsymbol{p})F'(d_2(\boldsymbol{p}))(\varphi_1(\boldsymbol{p}))]^{\varepsilon_0(\boldsymbol{p})\varepsilon_1(\boldsymbol{p})} \qquad (4.1)\end{aligned}$$

for all $F \in C^1([0,1],A)$ and $\boldsymbol{p} = (t_1,t_2,x_1,x_2,z) \in \widetilde{D}_{\partial A}$.

Lemma 4.1. *Let $(t_1,t_2) \in D$, $x_1,x_2 \in \partial A$ and $\boldsymbol{p}_z = (t_1,t_2,x_1,x_2,z) \in \widetilde{D}_{\partial A}$ for each $z \in \mathbb{T}$. Then $\phi_1(\boldsymbol{p}_{z_1}) = \phi_1(\boldsymbol{p}_{z_2})$ for each $z_1,z_2 \in \mathbb{T}$.*

Proof. Let $z_1,z_2,z_3 \in \mathbb{T}$. We first show that $d_1(\boldsymbol{p}_{z_k}) = d_1(\boldsymbol{p}_{z_l})$ for some $k,l \in \{1,2,3\}$ with $k \neq l$. To this end, it is enough to consider the case when z_1,z_2,z_3 are mutually distinct. Suppose that $d_1(\boldsymbol{p}_{z_1}), d_1(\boldsymbol{p}_{z_2}), d_1(\boldsymbol{p}_{z_3})$ are mutually distinct. There exists $f_0 \in C^1([0,1])$ such that

$$\begin{aligned}&f_0(d_1(\boldsymbol{p}_{z_1})) = 1, \quad f_0(d_1(\boldsymbol{p}_{z_2})) = 0 = f_0(d_1(\boldsymbol{p}_{z_3}))\\ &\qquad\qquad\text{and}\quad f_0'(d_2(\boldsymbol{p}_{z_j})) = 0 \qquad (j=1,2,3).\end{aligned}$$

Equality (4.1), applied to $F_0 = f_0 \otimes \mathbf{1}_X \in C^1([0,1],A)$, implies that

$$\begin{aligned}T_0(F_0)(t_1)(x_1) + z_1T_0(F_0)'(t_2)(x_2) &= [\alpha_1(\boldsymbol{p}_{z_1})]^{\varepsilon_0(\boldsymbol{p}_{z_1})},\\ T_0(F_0)(t_1)(x_1) + z_jT_0(F_0)'(t_2)(x_2) &= 0 \qquad (j=2,3).\end{aligned}$$

Since $z_2 \neq z_3$, we have $T_0(F_0)'(t_2)(x_2) = 0 = T_0(F_0)(t_1)(x_1)$. This is impossible since $|\alpha_1(\boldsymbol{p}_{z_1})| = 1$, which shows that $d_1(\boldsymbol{p}_{z_k}) = d_1(\boldsymbol{p}_{z_l})$ for some $k,l \in \{1,2,3\}$ with $k \neq l$.

Next, we prove that $d_2(\boldsymbol{p}_{z_m}) = d_2(\boldsymbol{p}_{z_n})$ for some $m,n \in \{1,2,3\}$ with $m \neq n$. Suppose not, that is, $d_2(\boldsymbol{p}_{z_1}), d_2(\boldsymbol{p}_{z_2})$ and $d_2(\boldsymbol{p}_{z_3})$ are all distinct. There exists $f_1 \in$

$C^1([0,1])$ such that

$$f_1'(d_2(\boldsymbol{p}_{z_1})) = 1, \quad f_1'(d_2(\boldsymbol{p}_{z_2})) = 0 = f_1'(d_2(\boldsymbol{p}_{z_3}))$$
$$\text{and} \quad f_1(d_1(\boldsymbol{p}_{z_j})) = 0 \qquad (j = 1, 2, 3).$$

Let $F_1 = f_1 \otimes \mathbf{1}_X \in C^1([0,1], A)$. According to (4.1), we obtain

$$T_0(F_1)(t_1)(x_1) + z_1 T_0(F_1)'(t_2)(x_2) = [\alpha_1(\boldsymbol{p}_{z_1})\omega_1(\boldsymbol{p}_{z_1})]^{\varepsilon_0(\boldsymbol{p}_{z_1})\varepsilon_1(\boldsymbol{p}_{z_1})},$$
$$T_0(F_1)(t_1)(x_1) + z_j T_0(F_1)'(t_2)(x_2) = 0 \qquad (j = 2, 3).$$

Since $z_2 \neq z_3$, we have $T_0(F_1)'(t_2)(x_2) = 0 = T_0(F_1)(t_1)(x_1)$. This contradicts $\alpha_1(\boldsymbol{p}_{z_1}), \omega_1(\boldsymbol{p}_{z_1}) \in \mathbb{T}$, and consequently $d_2(\boldsymbol{p}_{z_m}) = d_2(\boldsymbol{p}_{z_n})$ for some $m, n \in \{1, 2, 3\}$ with $m \neq n$.

Let $z_1, z_2 \in \mathbb{T}$. Now we prove $\phi_1(\boldsymbol{p}_{z_1}) = \phi_1(\boldsymbol{p}_{z_2})$. Suppose, on the contrary, $d_j(\boldsymbol{p}_{z_1}) \neq d_j(\boldsymbol{p}_{z_2})$ for some $j \in \{1, 2\}$. By the fact prove in the last paragraph, we get $d_j(\boldsymbol{p}_z) \in \{d_j(\boldsymbol{p}_{z_1}), d_j(\boldsymbol{p}_{z_2})\}$ for all $z \in \mathbb{T}$. Note that ϕ_1 is continuous by the definition (see Definition 3.2). Since $\mathbb{T}$ is connected and the map $z \mapsto d_j(\boldsymbol{p}_z) = \pi_j(\phi_1(\boldsymbol{p}_z))$ is continuous on $\mathbb{T}$, the image of $\mathbb{T}$ under this map is connected as well. This contradicts $d_j(\boldsymbol{p}_{z_1}) \neq d_j(\boldsymbol{p}_{z_2})$. We thus conclude that $d_j(\boldsymbol{p}_{z_1}) = d_j(\boldsymbol{p}_{z_2})$ for $j = 1, 2$. Hence $\phi_1(\boldsymbol{p}_{z_1}) = (d_1(\boldsymbol{p}_{z_1}), d_2(\boldsymbol{p}_{z_1})) = \phi_1(\boldsymbol{p}_{z_2})$. □

Lemma 4.2. *Let* $(t_1, t_2) \in D$*,* $x_1, x_2 \in \partial A$ *and* $\boldsymbol{p}_z = (t_1, t_2, x_1, x_2, z) \in \widetilde{D}_{\partial A}$ *for each* $z \in \mathbb{T}$*. Then* $\psi_1(\boldsymbol{p}_{z_1}) = \psi_1(\boldsymbol{p}_{z_2})$ *and* $\varphi_1(\boldsymbol{p}_{z_1}) = \varphi_1(\boldsymbol{p}_{z_2})$ *for each* $z_1, z_2 \in \mathbb{T}$*.*

Proof. Let $z_1, z_2, z_3 \in \mathbb{T}$. We first show that $\psi_1(\boldsymbol{p}_{z_k}) = \psi_1(\boldsymbol{p}_{z_l})$ for some $k, l \in \{1, 2, 3\}$ with $k \neq l$. To do this, we need to consider the case when z_1, z_2, z_3 are mutually distinct. Suppose that $\psi_1(\boldsymbol{p}_{z_1})$, $\psi_1(\boldsymbol{p}_{z_2})$, $\psi_1(\boldsymbol{p}_{z_3})$ are mutually distinct. There exists a function $u_0 \in A$ such that

$$u_0(\psi_1(\boldsymbol{p}_{z_1})) = 1 \quad \text{and} \quad u_0(\psi_1(\boldsymbol{p}_{z_2})) = 0 = u_0(\psi_1(\boldsymbol{p}_{z_3})).$$

Let $F_0 = \mathbf{1}_{[0,1]} \otimes u_0 \in C^1([0,1], A)$. As an application of (4.1) to $F = F_0$ shows

$$T_0(F_0)(t_1)(x_1) + z_1 T_0(F_0)'(t_2)(x_2) = [\alpha_1(\boldsymbol{p}_{z_1})]^{\varepsilon_0(\boldsymbol{p}_{z_1})},$$
$$T_0(F_0)(t_1)(x_1) + z_j T_0(F_0)'(t_2)(x_2) = 0 \qquad (j = 2, 3).$$

Since $z_2 \neq z_3$, we obtain $T_0(F_0)'(t_2)(x_2) = 0 = T_0(F_0)(t_1)(x_1)$. This is impossible since $|\alpha_1(\boldsymbol{p}_{z_1})| = 1$. This yields $\psi_1(\boldsymbol{p}_{z_k}) = \psi_1(\boldsymbol{p}_{z_l})$ for some $k, l \in \{1, 2, 3\}$ with $k \neq l$.

Next, we prove that $\varphi_1(\boldsymbol{p}_{z_m}) = \varphi_1(\boldsymbol{p}_{z_n})$ for some $m, n \in \{1, 2, 3\}$ with $m \neq n$. Suppose not, and then, $\varphi_1(\boldsymbol{p}_{z_1}), \varphi_1(\boldsymbol{p}_{z_2})$ and $\varphi_1(\boldsymbol{p}_{z_3})$ are mutually distinct. Choose $u_1 \in A$ so that

$$u_1(\varphi_1(\boldsymbol{p}_{z_1})) = 1 \quad \text{and} \quad u_1(\varphi_1(\boldsymbol{p}_{z_2})) = 0 = u_1(\varphi_1(\boldsymbol{p}_{z_3})).$$

Notice, by Lemma 4.1, that $d_1(\boldsymbol{p}_{z_j})$ and $d_2(\boldsymbol{p}_{z_j})$ are independent of j. There exists $f_1 \in C^1([0,1])$ such that

$$f_1(d_1(\boldsymbol{p}_{z_j})) = 0 \quad \text{and} \quad f_1'(d_2(\boldsymbol{p}_{z_j})) = 1.$$

Let $F_1 = f_1 \otimes u_1 \in C^1([0,1], A)$. According to (4.1), we obtain

$$\begin{aligned} &T_0(F_1)(t_1)(x_1) + z_1 T_0(F_1)'(t_2)(x_2) = [\alpha_1(\boldsymbol{p}_{z_1})\omega_1(\boldsymbol{p}_{z_1})]^{\varepsilon_0(\boldsymbol{p}_{z_1})\varepsilon_1(\boldsymbol{p}_{z_1})}, \\ &T_0(F_1)(t_1)(x_1) + z_j T_0(F_1)'(t_2)(x_2) = 0 \qquad (j = 2, 3). \end{aligned}$$

Since $z_2 \neq z_3$, we get $T_0(F_1)'(t_2)(x_2) = 0 = T_0(F_1)(t_1)(x_1)$. This contradicts $\alpha_1(\boldsymbol{p}_{z_1}), \omega_1(\boldsymbol{p}_{z_1}) \in \mathbb{T}$, and consequently $\varphi_1(\boldsymbol{p}_{z_m}) = \varphi_1(\boldsymbol{p}_{z_n})$ for some $m, n \in \{1,2,3\}$ with $m \neq n$, as is claimed.

We show that $\psi_1(\boldsymbol{p}_{z_1}) = \psi_1(\boldsymbol{p}_{z_2})$ and $\varphi_1(\boldsymbol{p}_{z_1}) = \varphi_1(\boldsymbol{p}_{z_2})$ for each $z_1, z_2 \in \mathbb{T}$. Let $z_1, z_2 \in \mathbb{T}$. Note that ψ_1 and φ_1 are continuous by the definition (see Definition 3.2). Since the maps $z \mapsto \psi_1(\boldsymbol{p}_z)$ and $z \mapsto \varphi_1(\boldsymbol{p}_z)$ are continuous on the connected set $\mathbb{T}$, the ranges $\{\psi_1(\boldsymbol{p}_z) : z \in \mathbb{T}\}$ and $\{\varphi_1(\boldsymbol{p}_z) : z \in \mathbb{T}\}$ are both connected sets. If $\psi_1(\boldsymbol{p}_{z_1}) \neq \psi_1(\boldsymbol{p}_{z_2})$, then $\psi_1(\boldsymbol{p}_z) \in \{\psi_1(\boldsymbol{p}_{z_1}), \psi_1(\boldsymbol{p}_{z_2})\}$ for all $z \in \mathbb{T}$ by the fact proved above. This contradicts the connectedness of the set $\{\psi_1(\boldsymbol{p}_z) : z \in \mathbb{T}\}$. We thus conclude $\psi_1(\boldsymbol{p}_{z_1}) = \psi_1(\boldsymbol{p}_{z_2})$. By a similar reasoning, we get $\varphi_1(\boldsymbol{p}_{z_1}) = \varphi_1(\boldsymbol{p}_{z_2})$. ☐

Proposition 4.3. *Let $\lambda, \mu \in \mathbb{C}$. If $|\lambda + z\mu| = 1$ for all $z \in \mathbb{T}$, then $\lambda\mu = 0$ and $|\lambda| + |\mu| = 1$.*

Proof. Suppose, on the contrary, that $\lambda\mu \neq 0$. Choose $z_1 \in \mathbb{T}$ so that $\mu z_1 = \lambda|\mu||\lambda|^{-1}$, and set $z_2 = -z_1$. By hypothesis, $|\lambda + z_1\mu| = 1 = |\lambda + z_2\mu|$, that is,

$$\left|\lambda + \frac{\lambda|\mu|}{|\lambda|}\right| = 1 = \left|\lambda - \frac{\lambda|\mu|}{|\lambda|}\right|.$$

These equalities yield $|\lambda| + |\mu| = \big||\lambda| - |\mu|\big|$. This implies that $\lambda = 0$ or $\mu = 0$, which contradicts the hypothesis that $\lambda\mu \neq 0$. Thus $\lambda\mu = 0$, and then $|\lambda| + |\mu| = 1$. ☐

Recall that $\mathbf{1}_K$ denotes the constant function on a set K with $\mathbf{1}_K(x) = 1$ for $x \in K$. We also note that $\mathbf{1} = \mathbf{1}_{[0,1]} \otimes \mathbf{1}_X \in C^1([0,1], A)$.

Lemma 4.4. *Let $\lambda \in \{1, i\}$. Then $T_0(\lambda\mathbf{1})'(t)(x) = 0$ for all $t \in [0,1]$ and $x \in \partial A$.*

Proof. Let $\lambda \in \{1, i\}$. For each $(t_1, t_2) \in D$, $x \in \partial A$ and $z \in \mathbb{T}$, we set $\boldsymbol{p} = (t_1, t_2, x, x, z) \in \widetilde{D}_{\partial A}$. By the equality (4.1), $T_0(\lambda\mathbf{1})(t_1)(x) + zT_0(\lambda\mathbf{1})'(t_2)(x) = [\lambda\alpha_1(\boldsymbol{p})]^{\varepsilon_0(\boldsymbol{p})}$, and thus

$$|T_0(\lambda\mathbf{1})(t_1)(x) + zT_0(\lambda\mathbf{1})'(t_2)(x)| = 1$$

for all $z \in \mathbb{T}$. Proposition 4.3 shows that

$$T_0(\lambda\mathbf{1})(t_1)(x) \cdot T_0(\lambda\mathbf{1})'(t_2)(x) = 0, \tag{4.2}$$

$$|T_0(\lambda\mathbf{1})(t_1)(x)| + |T_0(\lambda\mathbf{1})'(t_2)(x)| = 1 \tag{4.3}$$

for each $(t_1, t_2) \in D$ and $x \in \partial A$. Let $x \in \partial A$, and we set

$$D_1(x) = \{(t_1, t_2) \in D : T_0(\lambda\mathbf{1})(t_1)(x) = 0 \quad \text{and} \quad |T_0(\lambda\mathbf{1})'(t_2)(x)| = 1\},$$
$$D_2(x) = \{(t_1, t_2) \in D : T_0(\lambda\mathbf{1})'(t_2)(x) = 0 \quad \text{and} \quad |T_0(\lambda\mathbf{1})(t_1)(x)| = 1\}.$$

Equalities (4.2) and (4.3) show that $D_1(x) \cup D_2(x) = D$ and $D_1(x) \cap D_2(x) = \emptyset$. Since the functions $t \mapsto T_0(\lambda\mathbf{1})(t)(x)$ and $t \mapsto T_0(\lambda\mathbf{1})'(t)(x)$ are continuous on $[0, 1]$, $D_1(x)$ and $D_2(x)$ are both closed subsets of D. By the connectedness of D, we derive $D_1(x) = D$ or $D_2(x) = D$. Suppose that $D_1(x) = D$, and hence $(t_1, t_2) \in D$ implies $T_0(\lambda\mathbf{1})(t_1)(x) = 0$ and $|T_0(\lambda\mathbf{1})'(t_2)(x)| = 1$. Therefore,

$$T_0(\lambda\mathbf{1})(t)(x) = 0 \qquad (\forall t \in \pi_1(D)).$$

Since $\pi_1(D) = [0, 1]$, we obtain

$$T_0(\lambda\mathbf{1})(t)(x) = 0 = T_0(\lambda\mathbf{1})'(t)(x) \qquad (\forall t \in [0, 1]).$$

This contradicts (4.3), and hence $D_2(x) = D$. By the liberty of the choice of $x \in \partial A$, we get $D_2(x) = D$ for all $x \in \partial A$. Since $\pi_2(D) = [0, 1]$, we conclude $T_0(\lambda\mathbf{1})'(t)(x) = 0$ for all $t \in [0, 1]$ and $x \in \partial A$. □

Lemma 4.5. *The values $\varepsilon_0(t_1, t_2, x_1, x_2, z)$ and $\varepsilon_1(t_1, t_2, x_1, x_2, z)$, as in Lemmas 3.4 and 3.10, respectively, are independent of variables $(t_1, t_2) \in D$ and $z \in \mathbb{T}$; we shall write $\varepsilon_k(t_1, t_2, x_1, x_2, z) = \varepsilon_k(x_1, x_2)$ for $k = 0, 1$.*

Proof. Let $k = 0, 1$ and $x_1, x_2 \in \partial A$. The function $\varepsilon_k(\cdot, \cdot, x_1, x_2, \cdot)$, which sends (t_1, t_2, z) to $\varepsilon_k(t_1, t_2, x_1, x_2, z)$, is continuous on the connected set $D \times \mathbb{T}$. Hence, the image of $D \times \mathbb{T}$ under the function is a connected subset of $\{\pm 1\}$. Then we deduce that the value $\varepsilon_k(t_1, t_2, x_1, x_2, z)$ is independent of $(t_1, t_2) \in D$ and $z \in \mathbb{T}$. □

Lemma 4.6. (1) *The value $\varepsilon_0(x_1, x_2)$ is independent of $x_2 \in \partial A$; we shall write $\varepsilon_0(x_1, x_2) = \varepsilon_0(x_1)$.*

(2) *There exists $\beta \in A$ with $|\beta| = 1$ on ∂A such that*
   - (a) *$T_0(\mathbf{1})(t)(x) = \beta(x)$ for all $t \in [0, 1]$ and $x \in \partial A$,*
   - (b) *$T_0(i\mathbf{1})(t)(x) = i\varepsilon_0(x)T_0(\mathbf{1})(t)(x) = i\varepsilon_0(x)\beta(x)$ for all $t \in [0, 1]$ and $x \in \partial A$,*
   - (c) *$[\alpha_1(\boldsymbol{p})]^{\varepsilon_0(x_1)} = \beta(x_1)$ for all $\boldsymbol{p} = (t_1, t_2, x_1, x_2, z) \in \widetilde{D}_{\partial A}$.*

Proof. Let $\lambda \in \{1, i\}$. For each $x \in \partial A$, the function $T_0(\lambda\mathbf{1})_x \colon [0, 1] \to \mathbb{C}$, defined by $T_0(\lambda\mathbf{1})_x(t) = T_0(\lambda\mathbf{1})(t)(x)$ for $t \in [0, 1]$, is continuously differentiable with $(T_0(\lambda\mathbf{1})_x)'(t) = T_0(\lambda\mathbf{1})'(t)(x)$ for all $t \in [0, 1]$. Thus, Lemma 4.4 shows that

$(T_0(\lambda\mathbf{1})_x)'(t) = 0$ for all $t \in [0,1]$. Hence, $T_0(\lambda\mathbf{1})_x$ is constant on $[0,1]$. There exists $\beta_\lambda(x) \in \mathbb{C}$ such that $T_0(\lambda\mathbf{1})_x = \beta_\lambda(x)$. We may regard β_λ as a function on X. Since $T_0(\lambda\mathbf{1}) \in C^1([0,1], A)$, we obtain $\beta_\lambda \in A$.

Let $(t_1, t_2) \in D$, $x_1, x_2 \in \partial A$ and $z \in \mathbb{T}$. Then $T_0(\lambda\mathbf{1})'(t_2)(x_2) = 0$ by Lemma 4.4. According to (4.1), we get $\beta_\lambda(x_1) = T_0(\lambda\mathbf{1})(t_1)(x_1) = [\lambda\alpha_1(t_1, t_2, x_1, x_2, z)]^{\varepsilon_0(x_1,x_2)}$. This implies $|\beta_\lambda| = 1$ on ∂A with

$$\begin{aligned}\beta_i(x_1) &= [i\alpha_1(t_1, t_2, x_1, x_2, z)]^{\varepsilon_0(x_1,x_2)} \\ &= [i]^{\varepsilon_0(x_1,x_2)}[\alpha_1(t_1, t_2, x_1, x_2, z)]^{\varepsilon_0(x_1,x_2)} = i\varepsilon_0(x_1, x_2)\beta_1(x_1),\end{aligned}$$

that is, $\beta_i(x_1) = i\varepsilon_0(x_1, x_2)\beta_1(x_1)$ for all $x_1, x_2 \in \partial A$. This shows that the value $\varepsilon_0(x_1, x_2)$ is independent of the variable $x_2 \in \partial A$. If we write $\varepsilon_0(x_1)$ instead of $\varepsilon_0(x_1, x_2)$, then $[\alpha_1(t_1, t_2, x_1, x_2, z)]^{\varepsilon_0(x_1)} = \beta_1(x_1)$ for all $(t_1, t_2, x_1, x_2, z) \in \widetilde{D}_{\partial A}$. By the choice of $\beta_\lambda \in A$, we have $T_0(\mathbf{1})(t)(x) = \beta_1(x)$ and $T_0(i\mathbf{1})(t)(x) = \beta_i(x) = i\varepsilon_0(x)\beta_1(x)$ for all $t \in [0,1]$ and $x \in \partial A$. □

For the function $\beta \in A$ as in Lemma 4.6, we set

$$\beta_0(x) = [\beta(x)]^{\varepsilon_0(x)} \qquad (x \in \partial A). \tag{4.4}$$

Then $\alpha_1(\boldsymbol{p}) = [\beta(x_1)]^{\varepsilon_0(x_1)} = \beta_0(x_1)$ for $\boldsymbol{p} = (t_1, t_2, x_1, x_2, z) \in \widetilde{D}_{\partial A}$. By the help of Lemmas 4.1, 4.2, 4.5 and 4.6, we can rewrite (4.1) as

$$\begin{aligned}T_0(F)(t_1)(x_1) + zT_0(F)'(t_2)(x_2) &= [\beta_0(x_1)F(d_1(\boldsymbol{p}_1))(\psi_1(\boldsymbol{p}_1))]^{\varepsilon_0(x_1)} \\ &\quad + [\beta_0(x_1)\omega_1(\boldsymbol{p}_z)F'(d_2(\boldsymbol{p}_1))(\varphi_1(\boldsymbol{p}_1))]^{\varepsilon_0(x_1)\varepsilon_1(x_1,x_2)}\end{aligned} \tag{4.5}$$

for $F \in C^1([0,1], A)$ and $\boldsymbol{p}_z = (t_1, t_2, x_1, x_2, z) \in \widetilde{D}_{\partial A}$.

Lemma 4.7. *Let* $\mathrm{id} \in C^1([0,1])$ *be the identity function. Then*

$$\pi_1(\phi_1(t_1, t_2, x_1, x_2, z)) = [\beta_0(x_1)]^{-\varepsilon_0(x_1)}\, T_0(\mathrm{id} \otimes \mathbf{1}_X)(t_1)(x_1) \tag{4.6}$$

for all $(t_1, t_2) \in D$, $x_1, x_2 \in \partial A$ *and* $z \in \mathbb{T}$.

Proof. Let $(t_1, t_2) \in D$, $x_1, x_2 \in \partial A$ and $\boldsymbol{p}_z = (t_1, t_2, x_1, x_2, z) \in \widetilde{D}_{\partial A}$ for each $z \in \mathbb{T}$. Set $G = (\mathrm{id} - d_1(\boldsymbol{p}_1)\mathbf{1}_{[0,1]}) \otimes \mathbf{1}_X \in C^1([0,1], A)$. Then, we see that $G(d_1(\boldsymbol{p}_1)) = 0$ on X and $G'(t) = \mathbf{1}_X$ for all $t \in [0,1]$. According to (4.5),

$$T_0(G)(t_1)(x_1) + zT_0(G)'(t_2)(x_2) = [\beta_0(x_1)\omega_1(\boldsymbol{p}_z)]^{\varepsilon_0(x_1)\varepsilon_1(x_1,x_2)} \tag{4.7}$$

for all $z \in \mathbb{T}$. Since $|\beta_0(x_1)| = |\omega_1(\boldsymbol{p}_z)| = 1$, we obtain

$$|T_0(G)(t_1)(x_1) + zT_0(G)'(t_2)(x_2)| = 1$$

for all $z \in \mathbb{T}$. By Proposition 4.3, $T_0(G)(t_1)(x_1) \cdot T_0(G)'(t_2)(x_2) = 0$ and $|T_0(G)(t_1)(x_1)| + |T_0(G)'(t_2)(x_2)| = 1$.

We prove that $T_0(G)(t_1)(x_1) = 0$. Suppose not, and then we have $T_0(G)'(t_2)(x_2) = 0$. Thus, $T_0(G)(t_1)(x_1) = [\beta_0(x_1)\omega_1(\boldsymbol{p}_z)]^{\varepsilon_0(x_1)\varepsilon_1(x_1,x_2)}$ for all $z \in \mathbb{T}$ by (4.7). It follows that the function ω_1 is independent of $z \in \mathbb{T}$. Hence we may write $\omega_1(\boldsymbol{p}_z) = w_0$ for all $z \in \mathbb{T}$. Let $h \in C^1([0,1])$ be such that $h(t_1) = 0$ and $h'(t_2) = 1$. Since T_0 is surjective, there exists $H \in C^1([0,1], A)$ such that $T_0(H) = h \otimes \mathbf{1}_X$. Then $T_0(H)(t_1)(x_1) = 0$ and $T_0(H)'(t_2)(x_2) = 1$. Equality (4.5), applied to $F = H$, shows that

$$\begin{aligned} z &= T_0(H)(t_1)(x_1) + zT_0(H)'(t_2)(x_2) \\ &= [\beta_0(x_1)H(d_1(\boldsymbol{p}_1))(\psi_1(\boldsymbol{p}_1))]^{\varepsilon_0(x_1)} + [\beta_0(x_1)w_0H'(d_2(\boldsymbol{p}_1))(\varphi_1(\boldsymbol{p}_1))]^{\varepsilon_0(x_1)\varepsilon_1(x_1,x_2)} \end{aligned}$$

for all $z \in \mathbb{T}$. This is impossible, since the rightmost hand side of the above equalities does not depend on $z \in \mathbb{T}$. Consequently $T_0(G)(t_1)(x_1) = 0$ as is claimed.

By the choice of G, $T_0(G) = T_0(\mathrm{id} \otimes \mathbf{1}_X) - T_0(d_1(\boldsymbol{p}_1)\mathbf{1}_{[0,1]} \otimes \mathbf{1}_X)$. Recall that $d_1(\boldsymbol{p}) \in [0,1]$ for all $\boldsymbol{p} \in \widetilde{D}_{\partial A}$ by Definition 3.2. Since $T_0(G)(t_1)(x_1) = 0$, the real linearity of T_0 implies that

$$T_0(\mathrm{id} \otimes \mathbf{1}_X)(t_1)(x_1) = d_1(\boldsymbol{p}_1)T_0(\mathbf{1}_{[0,1]} \otimes \mathbf{1}_X)(t_1)(x_1) = d_1(\boldsymbol{p}_1)T_0(\mathbf{1})(t_1)(x_1).$$

Notice that $T_0(\mathbf{1})(t_1)(x_1) = [\beta_0(x_1)]^{\varepsilon_0(x_1)}$ by Lemma 4.6 with (4.4). This implies $T_0(\mathrm{id} \otimes \mathbf{1}_X)(t_1)(x_1) = d_1(\boldsymbol{p}_1)[\beta_0(x_1)]^{\varepsilon_0(x_1)}$. By the liberty of the choice of $(t_1, t_2) \in D$ and $x_1, x_2 \in \partial A$, we conclude that

$$\pi_1(\phi_1(t_1, t_2, x_1, x_2, z)) = [\beta_0(x_1)]^{-\varepsilon_0(x_1)}\, T_0(\mathrm{id} \otimes \mathbf{1}_X)(t_1)(x_1)$$

for all $(t_1, t_2) \in D$, $x_1, x_2 \in \partial A$ and $z \in \mathbb{T}$. □

By Lemma 4.7, $\pi_1(\phi_1(t_1, t_2, x_1, x_2, z))$ is independent of variables $t_2 \in [0,1]$, $x_2 \in \partial A$ and $z \in \mathbb{T}$. We will write

$$d_1(t_1, t_2, x_1, x_2, z) = d_1(t_1)(x_1) \qquad (t_1, t_2, x_1, x_2, z) \in \widetilde{D}_{\partial A}. \tag{4.8}$$

By the definition of d_1,

$$d_1(t)(x) \in [0,1] \qquad (t \in [0,1],\ x \in \partial A) \tag{4.9}$$

(see Definition 3.2), where we have used our hypothesis $\pi_1(D) = [0,1]$. We may regard d_1 as a map from $[0,1]$ to $C(\partial A)$. Recall, by Lemma 4.6, that $\beta \in A$ satisfies $|\beta| = 1$ on ∂A. By equalities (4.4) and (4.6), $d_1\colon [0,1] \to C(\partial A)$ is a continuously differentiable map with

$$T_0(\mathrm{id} \otimes \mathbf{1}_X)(t)(x) = \beta(x)d_1(t)(x) \tag{4.10}$$

for all $t \in [0,1]$ and $x \in \partial A$.

Lemma 4.8. *Let $F_1 = \mathrm{id} \otimes \mathbf{1}_X \in C^1([0,1], A)$ and $\kappa(t)(x) = T_0(F_1)'(t)(x)$ for $t \in [0,1]$ and $x \in \partial A$. Then*

$$\begin{aligned} \omega_1(\boldsymbol{p}_z) &= \overline{\beta_0(x_1)}\, [z\kappa(t_2)(x_2)]^{\varepsilon_0(x_1)\varepsilon_1(x_1,x_2)} \\ &= [z]^{\varepsilon_0(x_1)\varepsilon_1(x_1,x_2)}\, \omega_1(\boldsymbol{p}_1) \end{aligned} \tag{4.11}$$

for each $\boldsymbol{p}_z = (t_1, t_2, x_1, x_2, z) \in \widetilde{D}_{\partial A}$. In particular, $|\kappa(t)(x)| = 1$ for all $t \in [0,1]$ and $x \in \partial A$.

Proof. Let $F_1 = \mathrm{id} \otimes \mathbf{1}_X$, $(t_1, t_2) \in D$ and $x_1, x_2 \in \partial A$. Set $\boldsymbol{p}_z = (t_1, t_2, x_1, x_2, z) \in \widetilde{D}_{\partial A}$ for each $z \in \mathbb{T}$. Equalities (4.5) and (4.8), applied to $F = F_1$, yield

$$\begin{aligned} &T_0(F_1)(t_1)(x_1) + zT_0(F_1)'(t_2)(x_2) \\ &\qquad = [\beta_0(x_1)d_1(t_1)(x_1)]^{\varepsilon_0(x_1)} + [\beta_0(x_1)\omega_1(\boldsymbol{p}_z)]^{\varepsilon_0(x_1)\varepsilon_1(x_1,x_2)}. \end{aligned}$$

By (4.4), (4.9) and (4.10), we derive

$$T_0(F_1)(t_1)(x_1) = \beta(x_1)d_1(t_1)(x_1) = [\beta_0(x_1)d_1(t_1)(x_1)]^{\varepsilon_0(x_1)}.$$

Hence

$$zT_0(F_1)'(t_2)(x_2) = [\beta_0(x_1)\omega_1(\boldsymbol{p}_z)]^{\varepsilon_0(x_1)\varepsilon_1(x_1,x_2)}.$$

We thus obtain

$$\begin{aligned} \omega_1(\boldsymbol{p}_z) &= \overline{\beta_0(x_1)}\, [zT_0(F_1)'(t_2)(x_2)]^{\varepsilon_0(x_1)\varepsilon_1(x_1,x_2)} \\ &= \overline{\beta_0(x_1)}\, [z\kappa(t_2)(x_2)]^{\varepsilon_0(x_1)\varepsilon_1(x_1,x_2)} = [z]^{\varepsilon_0(x_1)\varepsilon_1(x_1,x_2)}\, \omega_1(\boldsymbol{p}_1). \end{aligned}$$

In particular, $|\kappa(t_2)(x_2)| = |\omega_1(\boldsymbol{p}_z)| = 1$ for all $t_2 \in [0,1]$ and $x_2 \in \partial A$. □

By (4.11), equality (4.5) is reduced to

$$\begin{aligned} &T_0(F)(t_1)(x_1) + zT_0(F)'(t_2)(x_2) \\ &= [\beta_0(x_1)F(d_1(t_1)(x_1))(\psi_1(\boldsymbol{p}_1))]^{\varepsilon_0(x_1)} + z\kappa(t_2)(x_2)[F'(d_2(\boldsymbol{p}_1))(\varphi_1(\boldsymbol{p}_1))]^{\varepsilon_0(x_1)\varepsilon_1(x_1,x_2)} \end{aligned}$$

for all $F \in C^1([0,1], A)$, $\boldsymbol{p}_1 = (t_1, t_2, x_1, x_2, 1) \in \widetilde{D}_{\partial A}$ and $z \in \mathbb{T}$. Comparing z-term and constant term in the last equality, we get

$$\begin{aligned} T_0(F)(t_1)(x_1) &= [\beta_0(x_1)F(d_1(t_1)(x_1))(\psi_1(\boldsymbol{p}_1))]^{\varepsilon_0(x_1)} \\ T_0(F)'(t_2)(x_2) &= \kappa(t_2)(x_2)[F'(d_2(\boldsymbol{p}_1))(\varphi_1(\boldsymbol{p}_1))]^{\varepsilon_0(x_1)\varepsilon_1(x_1,x_2)} \end{aligned} \tag{4.12}$$

for all $F \in C^1([0,1], A)$ and $\boldsymbol{p}_1 = (t_1, t_2, x_1, x_2, 1) \in \widetilde{D}_{\partial A}$.

Lemma 4.9. *If $x \in \partial A$, then either $d_1'(t)(x) = 1$ for all $t \in [0,1]$, or $d_1'(t)(x) = -1$ for all $t \in [0,1]$.*

Proof. Fix $x_0 \in \partial A$ arbitrarily, and let $F_1 = \mathrm{id} \otimes \mathbf{1}_X \in C^1([0,1], A)$. Lemma 4.8 shows

$$|T_0(F_1)'(t)(x_0)| = |\kappa(t)(x_0)| = 1 \tag{4.13}$$

for all $t \in [0,1]$. According to (4.10), we derive

$$T_0(F_1)'(t) = (\beta d_1(t))' = \beta d_1'(t)$$

for all $t \in [0,1]$, and then the map which sends t to $d_1'(t)(x_0)$ is continuous on $[0,1]$, since $d_1'(t)(x_0) = \overline{\beta(x_0)} T_0(F_1)'(t)(x_0)$ for $t \in [0,1]$. Equality (4.13) yields

$$|d_1'(t)(x_0)| = |T_0(F_1)'(t)(x_0)| = 1$$

for $t \in [0,1]$. For each $t \in [0,1]$, $|d_1'(t)(x_0)| = 1$ implies $d_1'(t)(x_0) \in \{\pm 1\}$. The map $t \mapsto d_1'(t)(x_0)$ is continuous on the connected set $[0,1]$, and then $d_1'(t)(x_0) = 1$ for all $t \in [0,1]$, or $d_1'(t)(x_0) = -1$ for all $t \in [0,1]$. □

Lemma 4.10. *The value $\psi_1(t_1, t_2, x_1, x_2, 1)$ is independent of variables $t_2 \in [0,1]$ and $x_2 \in \partial A$; we will write $\psi_1(t_1, t_2, x_1, x_2, 1) = \psi_1(t_1, x_1)$. Then (4.12) is reduced to*

$$T_0(F)(t_1)(x) = [\beta_0(x) F(d_1(t_1)(x))(\psi_1(t_1, x))]^{\varepsilon_0(x)} \tag{4.14}$$

for all $F \in C^1([0,1], A)$, $t_1 \in [0,1]$ and $x \in \partial A$.

Proof. Let $(t_1, t_2), (t_1, s_2) \in D$ and $x_1, x_2, y_2 \in \partial A$. We set $\boldsymbol{p}_1 = (t_1, t_2, x_1, x_2, 1)$ and $\boldsymbol{q}_1 = (t_1, s_2, x_1, y_2, 1)$. We will prove $\psi_1(\boldsymbol{p}_1) = \psi_1(\boldsymbol{q}_1)$. Let $F_u = \mathbf{1}_{[0,1]} \otimes u \in C^1([0,1], A)$ for each $u \in A$. Equality (4.12) yields

$$[\beta_0(x_1) u(\psi_1(\boldsymbol{p}_1))]^{\varepsilon_0(x_1)} = T(F_u)(t_1)(x_1) = [\beta_0(x_1) u(\psi_1(\boldsymbol{q}_1))]^{\varepsilon_0(x_1)},$$

and therefore $u(\psi_1(\boldsymbol{p}_1)) = u(\psi_1(\boldsymbol{q}_1))$. Since A separates the points of ∂A, we obtain $\psi_1(\boldsymbol{p}_1) = \psi_1(\boldsymbol{q}_1)$. Hence, ψ_1 is independent of variables $t_2 \in [0,1]$ and $x_2 \in \partial A$. □

Let $C^1([0,1])$ be the normed linear space with

$$\|f\|_{\langle D \rangle} = \sup_{(t_1, t_2) \in D} (|f(t_1)| + |f'(t_2)|) \qquad (f \in C^1([0,1])).$$

For each $x \in \partial A$, we define a linear map $V_x \colon A \to C^1([0,1])$ by

$$V_x(u)(t) = T_0(\mathbf{1}_{[0,1]} \otimes u)(t)(x) \qquad (u \in A,\ t \in [0,1]).$$

If we identify $f \in C^1([0,1])$ with $f \otimes \mathbf{1}_X \in C^1([0,1], A)$, we may regard $C^1([0,1])$ as a normed linear subspace of $C^1([0,1], A)$. We note, by (4.14), that

$$V_x(u)(t) = [\beta_0(x) u(\psi_1(t, x))]^{\varepsilon_0(x)} \qquad (u \in A,\ t \in [0,1]) \tag{4.15}$$

for each $x \in \partial A$.

Lemma 4.11. *For each $x \in \partial A$, the map $V_x \colon A \to C^1([0,1])$ is a bounded linear operator with $\|V_x\|_{\mathrm{op}} \leq 1$.*

Proof. For each $x \in \partial A$ and $u \in A$,

$$\begin{aligned}
\|V_x(u)\|_{\langle D\rangle} &= \sup_{(t_1,t_2)\in D} (|T_0(\mathbf{1}_{[0,1]} \otimes u)(t_1)(x)| + |T_0(\mathbf{1}_{[0,1]} \otimes u)'(t_2)(x)|) \\
&\leq \sup_{(t_1,t_2)\in D} (\|T_0(\mathbf{1}_{[0,1]} \otimes u)(t_1)\|_X + \|T_0(\mathbf{1}_{[0,1]} \otimes u)'(t_2)\|_X) \\
&= \|T_0(\mathbf{1}_{[0,1]} \otimes u)\|_{\langle D\rangle} = \|\mathbf{1}_{[0,1]} \otimes u\|_{\langle D\rangle} \\
&= \sup_{(t_1,t_2)\in D} (\|(\mathbf{1}_{[0,1]} \otimes u)(t_1)\|_X + \|(\mathbf{1}_{[0,1]} \otimes u)'(t_2)\|_X) = \|u\|_X,
\end{aligned}$$

where we have used that T_0 is a real linear isometry on $C^1([0,1], A)$ with respect to $\|\cdot\|_{\langle D\rangle}$. Thus, V_x is a bounded linear map with the operator norm $\|V_x\|_{\mathrm{op}} \leq 1$. □

Recall that e_y denotes the point evaluation at $y \in \partial A$, defined by $e_y(u) = u(y)$ for $u \in A$. For each $t \in [0,1]$, we define a map $\Delta_t \colon C^1([0,1]) \to \mathbb{C}$ by $\Delta_t(f) = f(t)$ for $f \in C^1([0,1])$. Then we observe that Δ_t is a bounded linear functional on $(C^1([0,1]), \|\cdot\|_{\langle D\rangle})$. In fact, for each $t \in [0,1]$ there exists $s \in [0,1]$ such that $(t,s) \in D$, since $\pi_1(D) = [0,1]$. By the definition of $\|f\|_{\langle D\rangle}$,

$$|\Delta_t(f)| \leq |f(t)| + |f'(s)| \leq \|f\|_{\langle D\rangle}$$

for all $f \in C^1([0,1])$. Hence, $\|\Delta_t\|_{\mathrm{op}} = 1$ for all $t \in [0,1]$.

Lemma 4.12. *For each $x \in \partial A$ and $s_1, s_2 \in [0,1]$, let $y_j = \psi_1(s_j, x)$ for $j = 1,2$. Then $\|e_{y_1} - e_{y_2}\|_{\mathrm{op}} \leq 2|s_1 - s_2|$.*

Proof. Let $x \in \partial A$ and $s_1, s_2 \in [0,1]$. We need to consider the case when $s_1 \neq s_2$. Then

$$\begin{aligned}
\|e_{y_1} - e_{y_2}\|_{\mathrm{op}} &= \sup_{\|u\|_X \leq 1} |u(y_1) - u(y_2)| \\
&= \sup_{\|u\|_X \leq 1} |u(\psi_1(s_1, x)) - u(\psi_1(s_2, x))| \\
&= \sup_{\|u\|_X \leq 1} |V_x(u)(s_1) - V_x(u)(s_2)| \\
&= \sup_{\|u\|_X \leq 1} |\Delta_{s_1}(V_x(u)) - \Delta_{s_2}(V_x(u))|,
\end{aligned}$$

where we have used equality (4.15) with $|\beta_0(x)| = 1$. By Lemma 4.11, the adjoint operator $V_x^* \colon C^1([0,1])^* \to A^*$ of V_x between the dual spaces of $C^1([0,1])$ and A is well defined with $\|V_x^*\|_{\mathrm{op}} = \|V_x\|_{\mathrm{op}} \le 1$. It follows that

$$\begin{aligned}\|e_{y_1} - e_{y_2}\|_{\mathrm{op}} &= \sup_{\|u\|_X \le 1} |V_x^*(\Delta_{s_1})(u) - V_x^*(\Delta_{s_2})(u)| \\ &= \|V_x^*(\Delta_{s_1} - \Delta_{s_2})\|_{\mathrm{op}} \le \|V_x^*\|_{\mathrm{op}} \|\Delta_{s_1} - \Delta_{s_2}\|_{\mathrm{op}} \\ &\le \|\Delta_{s_1} - \Delta_{s_2}\|_{\mathrm{op}} = \sup_{\|f\|_{\langle D\rangle} \le 1} |\Delta_{s_1}(f) - \Delta_{s_2}(f)| \\ &= \sup_{\|f\|_{\langle D\rangle} \le 1} |f(s_1) - f(s_2)|,\end{aligned}$$

and consequently, we obtain

$$\|e_{y_1} - e_{y_2}\|_{\mathrm{op}} \le \sup_{\|f\|_{\langle D\rangle} \le 1} |f(s_1) - f(s_2)|. \tag{4.16}$$

Let $f \in C^1([0,1])$ be such that $\|f\|_{\langle D\rangle} \le 1$. By the definition of $\|f\|_{\langle D\rangle}$ with $\pi_2(D) = [0,1]$, we see that $\|f'\|_{[0,1]} \le \|f\|_{\langle D\rangle}$, and hence $\|f'\|_{[0,1]} \le 1$. Since $s_1, s_2 \in [0,1]$ with $s_1 \ne s_2$, the mean value theorem shows that

$$\begin{aligned}\frac{|f(s_1) - f(s_2)|}{|s_1 - s_2|} &\le \frac{|\mathrm{Re}\, f(s_1) - \mathrm{Re}\, f(s_2)|}{|s_1 - s_2|} + \frac{|\mathrm{Im}\, f(s_1) - \mathrm{Im}\, f(s_2)|}{|s_1 - s_2|} \\ &\le \|\mathrm{Re}\, f'\|_{[0,1]} + \|\mathrm{Im}\, f'\|_{[0,1]} \\ &\le 2\|f'\|_{[0,1]} \le 2.\end{aligned}$$

It follows that $|f(s_1) - f(s_2)| \le 2|s_1 - s_2|$ for all $f \in C^1([0,1])$ with $\|f\|_{\langle D\rangle} \le 1$. Therefore, by equality (4.16), $\|e_{y_1} - e_{y_2}\|_{\mathrm{op}} \le 2|s_1 - s_2|$. □

Lemma 4.13. *The function $\psi_1(t_1, x_1)$ appeared in Lemma 4.10 is independent of the variable $t_1 \in [0,1]$; we will write $\psi_1(t_1, x_1) = \psi_1(x_1)$.*

Proof. Let $x \in \partial A$. We set $I = \{t_1 \in [0,1] : \psi_1(t_1, x) = \psi_1(0, x)\}$. Then $0 \in I$ and thus $I \ne \emptyset$. Since ψ_1 is continuous, I is a closed subset of $[0,1]$. Put $I^c = [0,1] \setminus I$. We will prove that I^c is a closed set as well. Let $\{s_n\}$ be a sequence in I^c converging to $s_0 \in [0,1]$. We need to show that $s_0 \in I^c$, that is, $\psi_1(s_0, x) \ne \psi_1(0, x)$. Set $y_n = \psi_1(s_n, x)$ for $n \in \mathbb{N} \cup \{0\}$. By the choice of $\{s_n\}$, $y_n \ne \psi_1(0, x)$ for all $n \in \mathbb{N}$. Lemma 4.12 shows that $\|e_{y_n} - e_{y_0}\|_{\mathrm{op}} \le 2|s_n - s_0|$ for all $n \in \mathbb{N}$. Because $\{s_n\}$ converges to s_0, there exists $m \in \mathbb{N}$ such that $\|e_{y_m} - e_{y_0}\|_{\mathrm{op}} \le 1$. By [3, Lemma 2.6.1], we obtain $e_{y_m} = e_{y_0}$ (see also [13, Lemma 6]). That is, $u(y_m) = e_{y_m}(u) = e_{y_0}(u) = u(y_0)$ for all $u \in A$. We derive $y_m = y_0$ since A separates the points of X. By the choice of $\{s_n\}$, $\psi_1(0, x) \ne y_m = y_0 = \psi_1(s_0, x)$, and consequently, $\psi_1(0, x) \ne \psi_1(s_0, x)$ as is claimed.

Because I and $I^c = [0,1] \setminus I$ are both disjoint closed subsets of the connected set $[0,1]$ with $I \ne \emptyset$, we have $I = [0,1]$. Therefore $\psi_1(t_1, x) = \psi_1(0, x)$ for all $t_1 \in [0,1]$, and hence ψ_1 does not depend on the variable $t_1 \in [0,1]$. □

5. Proof of the main theorem

By Lemmas 3.8 and 4.13 with (4.4) and (4.14), there exists a surjective continuous map $\psi_1 \colon \partial A \to \partial A$ such that

$$T_0(F)(t)(x) = \beta(x)[F(d_1(t)(x))(\psi_1(x))]^{\varepsilon_0(x)} \tag{5.1}$$

for all $F \in C^1([0,1], A)$, $t \in [0,1]$ and $x \in \partial A$.

Recall, by Lemma 4.9, that for each $x \in \partial A$, either $d_1'(t)(x) = 1$ for all $t \in [0,1]$, or $d_1'(t)(x) = -1$ for all $t \in [0,1]$. For $j \in \{\pm 1\}$, we define

$$K_j = \{x \in \partial A : d_1'(t)(x) = j \quad (\forall t \in [0,1])\}.$$

Let $j \in \{\pm 1\}$ and $x_0 \in K_j$. By the definition of K_j, $d_1'(t)(x_0) = j$ for all $t \in [0,1]$. There exists $k \in \mathbb{R}$ such that $d_1(t)(x_0) = jt + k$ for all $t \in [0,1]$. Recall, by the definition of d_1, that $d_1(t)(x_0) \in [0,1]$ for all $t \in [0,1]$. We have $k, j+k \in [0,1]$, which implies that $k = 0$ if $j = 1$, and $k = 1$ if $j = -1$. Consequently,

$$d_1(t)(x) = \begin{cases} t & x \in K_1 \\ 1 - t & x \in K_{-1} \end{cases} \tag{5.2}$$

for all $t \in [0,1]$.

Lemma 5.1. *The function $\beta \in A$ is invertible.*

Proof. We set $Y = [0,1] \times \partial A$. We may and do assume that $C^1([0,1], A|_{\partial A}) \subset C(Y)$. Under this identification, let

$$\mathcal{A} = \{F|_Y \in C(Y) : F \in C^1([0,1], A)\};$$

we will write $F(t,x)$ instead of $F(t)(x)$ for $F \in C^1([0,1], A)$, $t \in [0,1]$ and $x \in \partial A$. We define a map $\mathcal{U} \colon \mathcal{A} \to \mathcal{A}$ by

$$\mathcal{U}(F|_Y) = T_0(F)|_Y \qquad (F \in C^1([0,1], A)).$$

Since ∂A is a boundary for A, we observe that $\mathcal{U}$ is a well defined, surjective real linear isometry on $(\mathcal{A}, \|\cdot\|_Y)$. Equality (5.1) shows

$$\mathcal{U}(F|_Y)(t,x) = \beta(x)[F(d_1(t)(x), \psi_1(x))]^{\varepsilon_0(x)} \qquad (F|_Y \in \mathcal{A},\ (t,x) \in Y). \tag{5.3}$$

Let $\mathrm{cl}(\mathcal{A})$ be the uniform closure of $\mathcal{A}$ in $C(Y)$. We see that $\mathrm{cl}(\mathcal{A})$ is a uniform algebra on Y. Let $\widetilde{\mathcal{U}}$ be the unique extension of $\mathcal{U}$ to $\mathrm{cl}(\mathcal{A})$. Then $\widetilde{\mathcal{U}}$ is a surjective real linear isometry on $(\mathrm{cl}(\mathcal{A}), \|\cdot\|_Y)$. Let $\partial(\mathrm{cl}(\mathcal{A}))$ be the Shilov boundary for $\mathrm{cl}(\mathcal{A})$. By [6, Theorem 3.3], there exist a continuous function $\mathcal{K} \colon \partial(\mathrm{cl}(\mathcal{A})) \to \mathbb{T}$, a homeomorphism $\varrho \colon \partial(\mathrm{cl}(\mathcal{A})) \to \partial(\mathrm{cl}(\mathcal{A}))$ and a closed and open set N of $\partial(\mathrm{cl}(\mathcal{A}))$ such that

$$\widetilde{\mathcal{U}}(\mathcal{F})(y) = \begin{cases} \mathcal{K}(y)\mathcal{F}(\varrho(y)) & y \in N \\ \mathcal{K}(y)\overline{\mathcal{F}(\varrho(y))} & y \in \partial(\mathrm{cl}(\mathcal{A})) \setminus N \end{cases} \tag{5.4}$$

for all $\mathcal{F} \in \mathrm{cl}(\mathcal{A})$. Then $\mathcal{K} = \widetilde{\mathcal{U}}(\mathbf{1}|_Y) = \mathbf{1}_{[0,1]} \otimes (\beta|_{\partial A})$ on $\partial(\mathrm{cl}(\mathcal{A}))$ by (5.3). Without loss of generality, we may assume $\mathcal{K} = \mathbf{1}_{[0,1]} \otimes (\beta|_{\partial A})$ on Y, and then $\mathcal{K} \in \mathrm{cl}(\mathcal{A})$. Since $\widetilde{\mathcal{U}}$ is surjective, there exists $\mathcal{G} \in \mathrm{cl}(\mathcal{A})$ such that $\widetilde{\mathcal{U}}(\mathcal{G}) = \mathbf{1}|_Y$. By (5.4), $\mathcal{K} \cdot \widetilde{\mathcal{U}}(\mathcal{G}^2) = \{\widetilde{\mathcal{U}}(\mathcal{G})\}^2 = \mathbf{1}|_Y$ on $\partial(\mathrm{cl}(\mathcal{A}))$. Since $\partial(\mathrm{cl}(\mathcal{A}))$ is a boundary for $\mathrm{cl}(\mathcal{A})$, we have that $\mathcal{K} = \mathbf{1}_{[0,1]} \otimes (\beta|_{\partial A})$ is invertible in $\mathrm{cl}(\mathcal{A})$. For $\mathcal{K}^{-1} \in \mathrm{cl}(\mathcal{A})$, there exists $G \in C^1([0,1], A)$ such that $\|G|_Y - \mathcal{K}^{-1}\|_Y < 1$. Note that $\|F(0)\|_{\partial A} = \sup_{x\in\partial A} |F(0)(x)| \le \sup_{(t,x)\in Y} |F(t)(x)| = \|F|_Y\|_Y$ for all $F|_Y \in \mathcal{A}$. We set $g = G(0)$, and then $g \in A$. It follows that

$$\begin{aligned}\|\beta g - \mathbf{1}_X\|_X &= \|\beta g - \mathbf{1}_X\|_{\partial A} = \|\mathcal{K}(0)(G(0) - \mathcal{K}^{-1}(0))\|_{\partial A} \\ &\le \|\mathcal{K}(G|_Y - \mathcal{K}^{-1})\|_Y = \|G|_Y - \mathcal{K}^{-1}\|_Y < 1,\end{aligned}$$

where we have used $|\beta| = 1$ on ∂A. Hence, $\beta g \in A^{-1}$, and there exists $h \in A$ such that $\beta g h = \mathbf{1}_X$ on X. Consequently $\beta \in A^{-1}$, as is claimed. □

Lemma 5.2. *The Gelfand transform $\widehat{\beta}$ of β is of modulus one on the maximal ideal space $\mathcal{M}_A$ of A.*

Proof. Note that $\|\widehat{\beta}\|_{\mathcal{M}_A} = \|\beta\|_X = \|\beta\|_{\partial A} = 1$, and therefore $|\widehat{\beta}| \le 1$ on $\mathcal{M}_A$. Because β is invertible, $\widehat{\beta}\,\widehat{\beta^{-1}} = 1$ on $\mathcal{M}_A$. In particular, $|\beta^{-1}| = 1$ on ∂A because $|\beta| = 1$ on ∂A. Thus,

$$\left\| \frac{1}{\widehat{\beta}} \right\|_{\mathcal{M}_A} = \|\widehat{\beta^{-1}}\|_{\mathcal{M}_A} = \|\beta^{-1}\|_{\partial A} = 1,$$

and hence $|1/\widehat{\beta}| \le 1$ on $\mathcal{M}_A$. Consequently, $|\widehat{\beta}| = 1$ on $\mathcal{M}_A$. □

Lemma 5.3. *Let $F_1 = \mathrm{id} \otimes \mathbf{1}_X \in C^1([0,1], A)$, $v_1 = \beta^{-1} T_0(F_1)(1) \in A$ and $v_{-1} = \mathbf{1}_X - v_1 \in A$. We set $\gamma_1(t) = t$ and $\gamma_{-1}(t) = 1 - t$ for $t \in [0,1]$. For each $j \in \{\pm 1\}$*

$$T_0(F)(t)(x) v_j(x) = \beta(x)[F(\gamma_j(t))(\psi_1(x))]^{\varepsilon_0(x)} v_j(x) \tag{5.5}$$

for all $F \in C^1([0,1], A)$, $t \in [0,1]$ and $x \in \partial A$.

Proof. By (5.1), we obtain $v_1 = [d_1(1)]^{\varepsilon_0}$ on ∂A. Equality (5.2) implies that $v_1 = 1$ on K_1 and $v_1 = 0$ on K_{-1}. Hence $v_j^2 = v_j$ on ∂A for $j = \pm 1$. Since ∂A is a boundary for $\widehat{A}$, we see that $\widehat{v_j}^2 = \widehat{v_j}$ on $\mathcal{M}_A$, that is, both $\widehat{v_1}$ and $\widehat{v_{-1}}$ are idempotents for $\widehat{A}$. We define

$$M_j = \{\rho \in \mathcal{M}_A : \widehat{v_j}(\rho) = 1\} \qquad (j = \pm 1). \tag{5.6}$$

We observe that both M_1 and M_{-1} are, possibly empty, closed and open subsets of $\mathcal{M}_A$ such that $M_{-1} \cup M_1 = \mathcal{M}_A$ and $M_{-1} \cap M_1 = \emptyset$. By (5.2), $K_j \subset M_j$ for $j = \pm 1$. Since $v_j = 1$ on $\partial A \cap M_j$ and $v_j = 0$ on $\partial A \cap M_{-j}$, we obtain

$$T_0(F)(t)(x) v_j(x) = \beta(x)[F(\gamma_j(t))(\psi_1(x))]^{\varepsilon_0(x)} v_j(x) \qquad (j = \pm 1)$$

for all $F \in C^1([0,1], A)$, $t \in [0,1]$ and $x \in \partial A$. □

Lemma 5.4. *The map ψ_1 is injective.*

Proof. Since T_0^{-1} has the same properties as T_0, there exist $\beta_{-1} \in A^{-1}$, $\rho_{-1}\colon [0,1] \times \partial A \to [0,1]$, $\psi_{-1}\colon \partial A \to \partial A$ and $\varepsilon_{-1}\colon \partial A \to \{\pm 1\}$ such that

$$T_0^{-1}(F)(t)(x) = \beta_{-1}(x)[F(\rho_{-1}(t)(x))(\psi_{-1}(x))]^{\varepsilon_{-1}(x)}$$

for all $F \in C^1([0,1],A)$, $t \in [0,1]$ and $x \in \partial A$ (see (5.1)). Let $F_u = \mathbf{1}_{[0,1]} \otimes u \in C^1([0,1],A)$ for each $u \in A$. If we set $s = d_1(t)(x)$ and $y = \psi_1(x)$, then

$$\begin{aligned} u(x) &= F_u(t)(x) = T_0(T_0^{-1}(F_u))(t)(x) \\ &= \beta(x)[T_0^{-1}(F_u)(d_1(t)(x))(\psi_1(x))]^{\varepsilon_0(x)} = \beta(x)[T_0^{-1}(F_u)(s)(y)]^{\varepsilon_0(x)} \\ &= \beta(x)\left[\beta_{-1}(y)[F_u(\rho_{-1}(s)(y))(\psi_{-1}(y))]^{\varepsilon_{-1}(y)}\right]^{\varepsilon_0(x)} \\ &= \beta(x)\left[\beta_{-1}(y)[u(\psi_{-1}(y))]^{\varepsilon_{-1}(y)}\right]^{\varepsilon_0(x)}, \end{aligned}$$

and thus $[\beta^{-1}(x)u(x)]^{\varepsilon_0(x)} = \beta_{-1}(\psi_1(x))[u(\psi_{-1}(\psi_1(x)))]^{\varepsilon_{-1}(\psi_1(x))}$. If $\psi_1(x_1) = \psi_1(x_2)$, then the last equality shows that

$$[\beta^{-1}(x_1)u(x_1)]^{\varepsilon_0(x_1)} = [\beta^{-1}(x_2)u(x_2)]^{\varepsilon_0(x_2)}$$

for all $u \in A$. If $x_1 \neq x_2$, then we could choose $u \in A$ so that $u(x_1) = \beta(x_1)$ and $u(x_2) = 0$, which contradicts the above equality. Hence, we have $x_1 = x_2$, and consequently ψ_1 is injective. □

Lemma 5.5. *We define*

$$A_{\varepsilon_0} = \{u \circ \psi_1 : u \in A\} \subset C(\partial A).$$

Then the map $\Psi\colon A \to A_{\varepsilon_0}$, defined by

$$\Psi(u) = u \circ \psi_1 \qquad (u \in A), \tag{5.7}$$

is a complex algebra isomorphism.

Proof. Equality (5.1) impiles that $\beta^{-1} \cdot T_0(\mathbf{1}_{[0,1]} \otimes u)(0) = [u \circ \psi_1]^{\varepsilon_0}$ on ∂A for all $u \in A$. Since A separates the points of ∂A and since ψ_1 is injective, we see that A_{ε_0} separates the points of ∂A, as well. We observe that A_{ε_0} is a uniform algebra on ∂A. The mapping $\Psi\colon A \to A_{\varepsilon_0}$, defined by (5.7) is a complex algebra homomorphism on A. Since ∂A is a boundary for A and since ψ_1 is surjective on ∂A (see Lemmas 3.8, 4.10 and 4.13), we see that Ψ is injective. □

Lemma 5.6. *Let $\Psi^*\colon (A_{\varepsilon_0})^* \to A^*$ be the adjoint of Ψ and let $\mathcal{M}_{A_{\varepsilon_0}}$ be the maximal ideal space of A_{ε_0}. We define $\varepsilon_A = -i\beta^{-1}T_0(\mathbf{1}_{[0,1]} \otimes (i\mathbf{1}_X))(0) \in A$. Then*

$\Psi^*|_{\mathcal{M}_{A_{\varepsilon_0}}} : \mathcal{M}_{A_{\varepsilon_0}} \to \mathcal{M}_A$ *is a homeomorphism with* $\Psi^* = \psi_1$ *on* ∂A *and*

$$\widehat{T_0(F)(t)} \cdot \widehat{v_j} = \widehat{\beta} \cdot [\widehat{F(\gamma_j(t))} \circ \Psi^*]^{\widehat{\varepsilon_A}} \cdot \widehat{v_j} \tag{5.8}$$

on ∂A *for all* $F \in C^1([0,1], A)$ *and* $t \in [0,1]$.

Proof. By definition, Ψ^* is continuous with respect to the weak *-topology. Since Ψ is a homomorphism, $\Psi^*(\eta)$ is multiplicative, that is, $\Psi^*(\eta)(uv) = \Psi^*(\eta)(u)\cdot\Psi^*(\eta)(v)$ for all $\eta \in \mathcal{M}_{A_{\varepsilon_0}}$, the maximal ideal space of A_{ε_0}, and $u, v \in A$. Then we see that $\Psi^*(\mathcal{M}_{A_{\varepsilon_0}}) \subset \mathcal{M}_A$. By the surjectivity of Ψ, we observe that $(\Psi^{-1})^* \colon A^* \to (A_{\varepsilon_0})^*$ is well defined with $(\Psi^{-1})^*(\mathcal{M}_A) \subset \mathcal{M}_{A_{\varepsilon_0}}$. Note that $(\Psi^{-1})^* = (\Psi^*)^{-1}$, and hence $\Psi^*|_{\mathcal{M}_{A_{\varepsilon_0}}} : \mathcal{M}_{A_{\varepsilon_0}} \to \mathcal{M}_A$ is a homeomorphism with the relative weak *-topology. We have

$$\widehat{u}(\Psi^*(\zeta)) = \Psi^*(\zeta)(u) = \zeta(\Psi(u)) = \zeta(u \circ \psi_1)$$

for all $u \in A$ and $\zeta \in \mathcal{M}_{A_{\varepsilon_0}}$. Under the identification of ∂A with $\{e_x \in \mathcal{M}_{A_{\varepsilon_0}} : x \in \partial A\}$, we obtain $\widehat{u} \circ \Psi^* = u \circ \psi_1$ on ∂A for all $u \in A$. Since $\widehat{A}$ separates the points of $\mathcal{M}_A$, we see that $\Psi^* = \psi_1$ on ∂A. By (5.1), we see that $\varepsilon_A = \varepsilon_0$ on ∂A. Equality (5.5) is rewritten as

$$\widehat{T_0(F)(t)} \cdot \widehat{v_j} = \widehat{\beta} \cdot [\widehat{F(\gamma_j(t))} \circ \Psi^*]^{\widehat{\varepsilon_A}} \cdot \widehat{v_j}$$

on ∂A for all $F \in C^1([0,1], A)$ and $t \in [0,1]$. □

Lemma 5.7. *Let* $A|_{\partial A} = \{u|_{\partial A} : u \in A\}$. *Then* $A|_{\partial A} = \{[u \circ \psi_1]^{\varepsilon_0} : u \in A\}$.

Proof. For each $u \in A$, we have $T_0(\mathbf{1}_{[0,1]} \otimes u) = \mathbf{1}_{[0,1]} \otimes (\beta \cdot [u \circ \psi_1]^{\varepsilon_0})$ on ∂A by (5.1). Because $T_0(\mathbf{1}_{[0,1]} \otimes u) \in C^1([0,1], A)$, we see that $[u \circ \psi_1]^{\varepsilon_0} \in A|_{\partial A}$ for all $u \in A$. Hence $\{[u \circ \psi_1]^{\varepsilon_0} : u \in A\} \subset A|_{\partial A}$. Conversely, for each $u \in A$ there exists $G_u \in C^1([0,1], A)$ such that $T_0(G_u) = \mathbf{1}_{[0,1]} \otimes \beta u$, since T_0 is surjective. Equality (5.5) shows $T_0(G_u)(t)\cdot v_j = \beta\cdot[G_u(\gamma_j(t))\circ\psi_1]^{\varepsilon_0}\cdot v_j$ on ∂A for $j = \pm 1$ and $t \in [0,1]$. By the choice of G_u, we have $u\cdot v_j = [G_u(\gamma_j(t))\circ\psi_1]^{\varepsilon_0}\cdot v_j$ on ∂A for $j = \pm 1$ and $t \in [0,1]$. This implies that $[G_u(t) \circ \psi_1]^{\varepsilon_0} = u$, and therefore, $G_u = \mathbf{1}_{[0,1]} \otimes [u \circ \psi_1^{-1}]^{\varepsilon_0 \circ \psi_1^{-1}}$ on $[0,1] \times \partial A$. It follows that

$$[u \circ \psi_1^{-1}]^{\varepsilon_0 \circ \psi_1^{-1}} \in A|_{\partial A} \qquad (u \in A). \tag{5.9}$$

Now choose $v \in A$ arbitrarily, and then $[v \circ \psi_1^{-1}]^{\varepsilon_0 \circ \psi_1^{-1}} \in A|_{\partial A}$ by (5.9). There exists $v_{\varepsilon_0} \in A$ such that $[v \circ \psi_1^{-1}]^{\varepsilon_0 \circ \psi_1^{-1}} = v_{\varepsilon_0}|_{\partial A}$. By the choice of v_{ε_0}, we obtain $[v_{\varepsilon_0} \circ \psi_1]^{\varepsilon_0} = \big[[v \circ \psi_1^{-1}]^{\varepsilon_0 \circ \psi_1^{-1}} \circ \psi_1\big]^{\varepsilon_0} = v$ on ∂A, which shows that $v|_{\partial A} \in \{[u \circ \psi_1]^{\varepsilon_0} : u \in A\}$ for all $v \in A|_{\partial A}$. We thus conclude that $A|_{\partial A} = \{[u \circ \psi_1]^{\varepsilon_0} : u \in A\}$. □

Lemma 5.8. *Let* ε_A *be the element of* A *defined in Lemma 5.6. For each* $\xi \in \mathcal{M}_{A|_{\partial A}}$, *we define a map* $\xi_{\varepsilon_0} \colon A_{\varepsilon_0} \to \mathbb{C}$ *by*

$$\xi_{\varepsilon_0}(u \circ \psi_1) = \big[\xi([u \circ \psi_1]^{\varepsilon_0})\big]^{\xi(\varepsilon_A|_{\partial A})} \tag{5.10}$$

for $u \circ \psi_1 \in A_{\varepsilon_0}$. *Then* $\xi_{\varepsilon_0} \in \mathcal{M}_{A_{\varepsilon_0}}$.

Proof. Recall that $\varepsilon_A = \varepsilon_0$ on ∂A by (5.1). Because $\varepsilon_0(x) \in \{\pm 1\}$ for $x \in \partial A$, we get $\{(\varepsilon_A + \mathbf{1}_X)/2\}^2 = (\varepsilon_A + \mathbf{1}_X)/2$ on ∂A. As ∂A is a boundary for $\widehat{A}$, we obtain $\widehat{\varepsilon_A}(\rho) \in \{\pm 1\}$ for $\rho \in \mathcal{M}_A$. Therefore, $(\varepsilon_A|_{\partial A})^2 = \mathbf{1}_X|_{\partial A}$, the unit element of $A|_{\partial A}$. We obtain $\{\xi(\varepsilon_A|_{\partial A})\}^2 = \xi(\mathbf{1}_X|_{\partial A}) = 1$ for all $\xi \in \mathcal{M}_{A|_{\partial A}}$. For each $\xi \in \mathcal{M}_{A|_{\partial A}}$, let $\xi_{\varepsilon_0} \colon A_{\varepsilon_0} \to \mathbb{C}$ be the map described in (5.10). Here we notice that $A|_{\partial A} = \{[u \circ \psi_1]^{\varepsilon_0} : u \in A\}$, and hence ξ_{ε_0} is well defined. By definition, ξ_{ε_0} is a non-zero, real linear and multiplicative functional on A_{ε_0}. We will prove that ξ_{ε_0} is complex linear. Since $\varepsilon_0 = \varepsilon_A|_{\partial A}$, we see that $[i\mathbf{1}_X \circ \psi_1]^{\varepsilon_0} = i\varepsilon_A|_{\partial A}$, and hence $\xi([i\mathbf{1}_X \circ \psi_1]^{\varepsilon_0}) = \xi(i\varepsilon_A|_{\partial A}) = i\xi(\varepsilon_A|_{\partial A})$ for $\zeta \in \mathcal{M}_{A|_{\partial A}}$. By the definition of ξ_{ε_0}, we derive $\xi_{\varepsilon_0}(i\mathbf{1}_X \circ \psi_1) = [i\xi(\varepsilon_A|_{\partial A})]^{\xi(\varepsilon_A|_{\partial A})} = i$. Since ξ_{ε_0} is real linear, we now obtain

$$\xi_{\varepsilon_0}(\lambda \mathbf{1}_X \circ \psi_1) = \lambda \xi_{\varepsilon_0}(\mathbf{1}_X \circ \psi_1) = \lambda[\xi(\mathbf{1}_X|_{\partial A})]^{\xi(\varepsilon_A|_{\partial A})} = \lambda$$

for $\lambda \in \mathbb{C}$. By the multiplicativity of ξ_{ε_0}, we get

$$\xi_{\varepsilon_0}(\lambda(u \circ \psi_1)) = \xi_{\varepsilon_0}(\lambda \mathbf{1}_X \circ \psi_1)\, \xi_{\varepsilon_0}(u \circ \psi_1) = \lambda \xi_{\varepsilon_0}(u \circ \psi_1)$$

for $u \in A$ and $\lambda \in \mathbb{C}$. This shows that ξ_{ε_0} is complex linear, and thus $\xi_{\varepsilon_0} \in \mathcal{M}_{A_{\varepsilon_0}}$. □

Lemma 5.9. *Define* $\Gamma \colon \mathcal{M}_{A|_{\partial A}} \to \mathcal{M}_{A_{\varepsilon_0}}$ *by*

$$\Gamma(\xi) = \xi_{\varepsilon_0} \qquad (\xi \in \mathcal{M}_{A|_{\partial A}}).$$

Then Γ *is an injective and continuous map with the relative weak *-topology.*

Proof. Suppose that $\xi_1 \neq \xi_2$ for $\xi_1, \xi_2 \in \mathcal{M}_{A|_{\partial A}}$. Then there exists $u_0 \in A$ such that $\xi_1([u_0 \circ \psi_1]^{\varepsilon_0}) = 1$ and $\xi_2([u_0 \circ \psi_1]^{\varepsilon_0}) = 0$; this is possible since $A|_{\partial A} = \{[u \circ \psi_1]^{\varepsilon_0} : u \in A\}$. By the definition of Γ with (5.10), we have $\Gamma(\xi_1)(u_0 \circ \psi_1) = 1 \neq 0 = \Gamma(\xi_2)(u_0 \circ \psi_1)$, which shows the injectivity of the map Γ. Now let $\{\xi_\vartheta\}$ be a net in $\mathcal{M}_{A|_{\partial A}}$ converging to $\xi_0 \in \mathcal{M}_{A|_{\partial A}}$. Because $(\varepsilon_A|_{\partial A})^2 = \mathbf{1}_X|_{\partial A}$, $(\xi_\vartheta(\varepsilon_A|_{\partial A}))^2 = 1 = (\xi_0(\varepsilon_A|_{\partial A}))^2$, and thus for each ϑ, $\xi_\vartheta(\varepsilon_A|_{\partial A}) = \xi_0(\varepsilon_A|_{\partial A})$ or $\xi_\vartheta(\varepsilon_A|_{\partial A}) = -\xi_0(\varepsilon_A|_{\partial A})$. Since $\{\xi_\vartheta(\varepsilon_A|_{\partial A})\}$ converges to $\xi_0(\varepsilon_A|_{\partial A})$, we may assume that $\xi_\vartheta(\varepsilon_A|_{\partial A}) = \xi_0(\varepsilon_A|_{\partial A})$ for all ϑ. By the definition of Γ with (5.10),

$$\begin{aligned}\Gamma(\xi_\vartheta)(u \circ \psi_1) &= \left[\xi_\vartheta([u \circ \psi_1]^{\varepsilon_0})\right]^{\xi_0(\varepsilon_A|_{\partial A})} \to \left[\xi_0([u \circ \psi_1]^{\varepsilon_0})\right]^{\xi_0(\varepsilon_A|_{\partial A})} \\ &= \Gamma(\xi_0)(u \circ \psi_1)\end{aligned}$$

for each $u \in A$. This shows that the net $\{\Gamma(\xi_\vartheta)\}$ converges to $\Gamma(\xi_0)$ with respect to the relative weak *-topology, and hence $\Gamma \colon \mathcal{M}_{A|_{\partial A}} \to \mathcal{M}_{A_{\varepsilon_0}}$ is continuous. □

Lemma 5.10. *The map* Γ *as in Lemma 5.9 is a homeomorphism with* $\Gamma(x) = x$ *for* $x \in \partial A$.

Proof. We need to prove that Γ is surjective. By (5.9), $\varepsilon_A \circ \psi_1^{-1} = [\varepsilon_A \circ \psi_1^{-1}]^{\varepsilon_0} \in A|_{\partial A}$. Then there exists $u_{\varepsilon_A} \in A$ such that $u_{\varepsilon_A}|_{\partial A} = \varepsilon_A \circ \psi_1^{-1}$; such a function u_{ε_A} is uniquely determined since ∂A is a boundary for A. Take $\zeta \in \mathcal{M}_{A_{\varepsilon_0}}$ arbitrarily. Since $u_{\varepsilon_A} \circ \psi_1 = (\varepsilon_A \circ \psi_1^{-1}) \circ \psi_1 = \varepsilon_A|_{\partial A}$, we get

$$\varepsilon_A|_{\partial A} = u_{\varepsilon_A} \circ \psi_1 \in A_{\varepsilon_0},$$

and thus $\zeta(\varepsilon_A|_{\partial A}) = \zeta(u_{\varepsilon_A} \circ \psi_1)$. By the choice of ε_A, we obtain $(\varepsilon_A|_{\partial A})^2 = \mathbf{1}_X|_{\partial A}$, and then $\zeta(\varepsilon_A|_{\partial A}) \in \{\pm 1\}$. Now we define a map $\xi_\zeta \colon A|_{\partial A} \to \mathbb{C}$ by

$$\xi_\zeta([u \circ \psi_1]^{\varepsilon_0}) = [\zeta(u \circ \psi_1)]^{\zeta(\varepsilon_A|_{\partial A})} \qquad (u \in A);$$

the map ξ_ζ is well defined, since $A|_{\partial A} = \{[u \circ \psi_1]^{\varepsilon_0} : u \in A\}$. Then ξ_ζ is non-zero, since $\zeta(u_1 \circ \psi_1) \neq 0$ for some $u_1 \circ \psi_1 \in A_{\varepsilon_0}$. We observe that ξ_ζ is a real linear and multiplicative functional on $A|_{\partial A}$. Recall $\varepsilon_A|_{\partial A} = \varepsilon_0$, and then $i = [i\varepsilon_A|_{\partial A}]^{\varepsilon_A|_{\partial A}} = [iu_{\varepsilon_A} \circ \psi_1]^{\varepsilon_0} \in A|_{\partial A}$. Because $\zeta \in \mathcal{M}_{A_{\varepsilon_0}}$, we have

$$\begin{aligned}\xi_\zeta(i) &= \xi_\zeta([iu_{\varepsilon_A} \circ \psi_1]^{\varepsilon_0}) = [\zeta(iu_{\varepsilon_A} \circ \psi_1)]^{\zeta(\varepsilon_A|_{\partial A})} \\ &= [i\zeta(u_{\varepsilon_A} \circ \psi_1)]^{\zeta(\varepsilon_A|_{\partial A})} = [i\zeta(\varepsilon_A|_{\partial A})]^{\zeta(\varepsilon_A|_{\partial A})} = i.\end{aligned}$$

For each $u \in A$, the multiplicativity of ξ_ζ shows that

$$\xi_\zeta(i[u \circ \psi_1]^{\varepsilon_0}) = \xi_\zeta(i)\,\xi_\zeta([u \circ \psi_1]^{\varepsilon_0}) = i\,\xi_\zeta([u \circ \psi_1]^{\varepsilon_0}).$$

Hence $\xi_\zeta(i[u \circ \psi_1]^{\varepsilon_0}) = i\,\xi_\zeta([u \circ \psi_1]^{\varepsilon_0})$ for all $[u \circ \psi_1]^{\varepsilon_0} \in A|_{\partial A}$. By the real linearity of ξ_ζ, we infer that ξ_ζ is complex linear, and thus $\xi_\zeta \in \mathcal{M}_{A|_{\partial A}}$. Since $\varepsilon_A|_{\partial A} = u_{\varepsilon_A} \circ \psi_1$, we get $\zeta(u_{\varepsilon_A} \circ \psi_1) \in \{\pm 1\}$. This shows that

$$\begin{aligned}\zeta(\varepsilon_A|_{\partial A}) &= \zeta(u_{\varepsilon_A} \circ \psi_1) = [\zeta(u_{\varepsilon_A} \circ \psi_1)]^{\zeta(\varepsilon_A|_{\partial A})} \\ &= \xi_\zeta([u_{\varepsilon_A} \circ \psi_1]^{\varepsilon_0}) = \xi_\zeta(\varepsilon_A|_{\partial A})\end{aligned}$$

by the definition of ξ_ζ, that is, $\zeta(\varepsilon_A|_{\partial A}) = \xi_\zeta(\varepsilon_A|_{\partial A})$. We derive

$$\begin{aligned}\Gamma(\xi_\zeta)(u \circ \psi_1) &= \left[\xi_\zeta([u \circ \psi_1]^{\varepsilon_0})\right]^{\xi_\zeta(\varepsilon_A|_{\partial A})} = \left[[\zeta(u \circ \psi_1)]^{\zeta(\varepsilon_A|_{\partial A})}\right]^{\zeta(\varepsilon_A|_{\partial A})} \\ &= \zeta(u \circ \psi_1)\end{aligned}$$

for all $u \in A$. We thus conclude that Γ is surjective. Therefore, $\Gamma \colon \mathcal{M}_{A|_{\partial A}} \to \mathcal{M}_{A_{\varepsilon_0}}$ is a homeomorphism. In particular, if we identify $x \in \partial A$ with the evaluation functional e_x, then for each $u \in A$,

$$\Gamma(x)(u \circ \psi_1) = \left[[u(\psi_1(x))]^{\varepsilon_0(x)}\right]^{\varepsilon_A(x)} = u(\psi_1(x))$$

by (5.10), where we have used $\varepsilon_0 = \varepsilon_A|_{\partial A}$. Namely, $\widehat{u \circ \psi_1}(\Gamma(x)) = (u \circ \psi_1)(x)$ for all $u \in A$, and hence $\Gamma(x) = x$ for $x \in \partial A$. □

Proof of Theorem. Let $\mathcal{R}\colon A \to A|_{\partial A}$ be the restriction, which maps $u \in A$ to $u|_{\partial A}$. Since ∂A is a boundary for A, $\mathcal{R}$ is a complex algebra isomorphism. For the adjoint $\mathcal{R}^*$ of $\mathcal{R}$, we see that $\mathcal{R}^*|_{\mathcal{M}_{A|_{\partial A}}}$ is a homeomorphism from $\mathcal{M}_{A|_{\partial A}}$ onto $\mathcal{M}_A$ with the relative weak *-topology. For each $u \in A$ and $\xi \in \mathcal{M}_{A|_{\partial A}}$,

$$\widehat{u|_{\partial A}}(\xi) = \xi(\mathcal{R}(u)) = \mathcal{R}^*(\xi)(u) = \widehat{u}(\mathcal{R}^*(\xi)).$$

If $x \in \partial A$, then $u(x) = \widehat{u|_{\partial A}}(x) = \widehat{u}(\mathcal{R}^*(x))$ for all $u \in A$. Thus we see that $\mathcal{R}^*(x) = x$ for $x \in \partial A$. Recall that the maps $\Psi^*|_{\mathcal{M}_{A_{\varepsilon_0}}}\colon \mathcal{M}_{A_{\varepsilon_0}} \to \mathcal{M}_A$, $\Gamma\colon \mathcal{M}_{A|_{\partial A}} \to \mathcal{M}_{A_{\varepsilon_0}}$ and $\mathcal{R}^*|_{\mathcal{M}_{A|_{\partial A}}}\colon \mathcal{M}_{A|_{\partial A}} \to \mathcal{M}_A$ are all homeomorphisms. We infer $(\mathcal{R}^*|_{\mathcal{M}_{A|_{\partial A}}})^{-1} = (\mathcal{R}^{-1})^*|_{\mathcal{M}_A}$. Thus, the map $\sigma\colon \mathcal{M}_A \to \mathcal{M}_A$, defined by $\sigma = (\Psi^*|_{\mathcal{M}_{A_{\varepsilon_0}}}) \circ \Gamma \circ (\mathcal{R}^{-1})^*|_{\mathcal{M}_A}$, is a well defined homeomorphism on $\mathcal{M}_A$. For each $x \in \partial A$, $\Gamma(x) = x = \mathcal{R}^*(x)$, and thus $\sigma = \Psi^*$ on ∂A. Therefore, (5.8) is rewritten as

$$\widehat{T_0(F)(t)} \cdot \widehat{v_j} = \widehat{\beta} \cdot [\widehat{F(\gamma_j(t))} \circ \sigma]^{\widehat{\varepsilon_A}} \cdot \widehat{v_j} \tag{5.11}$$

on ∂A for all $F \in C^1([0,1], A)$ and $t \in [0,1]$.

Let $F \in C^1([0,1], A)$, $t \in [0,1]$ and $\rho \in \mathcal{M}_A$. We set $v = F(\gamma_j(t)) \in A$. By the definition of σ, $\sigma(\rho) = \Psi^*(\Gamma((\mathcal{R}^{-1})^*(\rho)))$. Hence

$$\widehat{v}(\sigma(\rho)) = \sigma(\rho)(v) = \Gamma((\mathcal{R}^{-1})^*(\rho))(\Psi(v)).$$

According to (5.7), $\Psi(v) = v \circ \psi_1$. Thus $\sigma(\rho)(v) = \Gamma((\mathcal{R}^{-1})^*(\rho))(v \circ \psi_1)$. By the definition of the map Γ with (5.10),

$$\begin{aligned}\Gamma((\mathcal{R}^{-1})^*(\rho))(v \circ \psi_1) &= [(\mathcal{R}^{-1})^*(\rho)([v \circ \psi_1]^{\varepsilon_0})]^{(\mathcal{R}^{-1})^*(\rho)(\varepsilon_A|_{\partial A})} \\ &= [\rho(\mathcal{R}^{-1}([v \circ \psi_1]^{\varepsilon_0}))]^{\rho(\mathcal{R}^{-1}(\varepsilon_A|_{\partial A}))}.\end{aligned}$$

Here, we notice $\mathcal{R}^{-1}(\varepsilon_A|_{\partial A}) = \varepsilon_A$ by the definition of the map $\mathcal{R}$. Therefore,

$$\begin{aligned}[\widehat{v}(\sigma(\rho))]^{\widehat{\varepsilon_A}(\rho)} &= \left[[\rho(\mathcal{R}^{-1}([v \circ \psi_1]^{\varepsilon_0}))]^{\rho(\varepsilon_A)}\right]^{\rho(\varepsilon_A)} = \rho(\mathcal{R}^{-1}([v \circ \psi_1]^{\varepsilon_0})) \\ &= \widehat{\mathcal{R}^{-1}([v \circ \psi_1]^{\varepsilon_0})}(\rho).\end{aligned}$$

It follows that $[\widehat{F(\gamma_j(t))} \circ \sigma]^{\widehat{\varepsilon_A}} = [\widehat{v} \circ \sigma]^{\widehat{\varepsilon_A}} = \widehat{\mathcal{R}^{-1}([v \circ \psi_1]^{\varepsilon_0})} \in \widehat{A}$. Equality (5.11) is valid on the boundary ∂A for $\widehat{A}$, we observe that (5.11) holds on $\mathcal{M}_A$. We set

$$L_+ = \{\rho \in \mathcal{M}_A : \widehat{\varepsilon_A}(\rho) = 1\}, \quad \text{and} \quad L_- = \{\rho \in \mathcal{M}_A : \widehat{\varepsilon_A}(\rho) = -1\}.$$

By the continuity of $\widehat{\varepsilon_A}$, both L_+ and L_- are closed and open sets satisfying $L_+ \cup L_- = \mathcal{M}_A$ and $L_+ \cap L_- = \emptyset$. We define $M_j^+ = M_j \cap L_+$ and $M_j^- = M_j \cap L_-$ for

$j = \pm 1$ (see (5.6)). Then we obtain

$$\widehat{T_0(F)(t)}(\rho) = \begin{cases} \widehat{\beta}(\rho)\widehat{F(t)}(\sigma(\rho)) & \rho \in M_1^+ \\ \widehat{\beta}(\rho)\overline{\widehat{F(t)}(\sigma(\rho))} & \rho \in M_1^- \\ \widehat{\beta}(\rho)\widehat{F(1-t)}(\sigma(\rho)) & \rho \in M_{-1}^+ \\ \widehat{\beta}(\rho)\overline{\widehat{F(1-t)}(\sigma(\rho))} & \rho \in M_{-1}^- \end{cases}$$

for all $F \in C^1([0,1], A)$ and $t \in [0,1]$. □

Acknowledgement. The authors are grateful to the referee for his/her valuable comments and suggestions which improved the manuscript.

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Received 2019.10.31
Revised 2020.1.10