Nihonkai Math. J.
Vol.30(2019), 1–17

CLASSIFICATION OF SEMISIMPLE COMMUTATIVE BANACH ALGEBRAS OF TYPE I

JYUNJI INOUE, TAKESHI MIURA, HIROYUKI TAKAGI*, AND SIN-EI TAKAHASI

Abstract. In the first and fourth authors' paper in 2017, it was shown that there exists a BSE-algebra of type I isomorphic to no C*-algebras, which solved negatively a question posed by the fourth author and O. Hatori. However, this result suggests a further investigation of commutative Banach algebra of type I. In the first part of the paper, we classify type I algebras into six families by means of BSE, BED, and Tauberian. It is shown that a Banach algebra of type I is isomorphic to a Segal algebra in some commutative C*-algebra if and only if it is Tauberian. In the second part, we give concrete examples of type I algebras to show that all of six families mentioned above are nonempty.

1. Introduction and overview of main results

Let A be a semisimple commutative Banach algebra with Gelfand space Φ_A, and $C^b(\Phi_A)$ the Banach algebra of all bounded continuous complex-valued functions on Φ_A with supremum norm $\|\cdot\|_\infty$. Put $\widehat{A}=\{\widehat{x}: x\in A\}$, where $\widehat{x}$ is the Gelfand transform of $x\in A$. Let $M(A)$ be the multiplier algebra of A. It is well known that for each $T\in M(A)$ there exists a unique bounded complex-valued continuous function $\widehat{T}$ on Φ_A such that $\widehat{Tx}=\widehat{T}\widehat{x}$ for all $x\in A$ (cf. [5]). Put $\widehat{M}(A)=\{\widehat{T}: T\in M(A)\}$. Then we have $\widehat{A}\subseteq\widehat{M}(A)\subseteq C^b(\Phi_A)$. We say that an algebra A is of *type I* if $\widehat{M}(A)=C^b(\Phi_A)$. Let A_c be the set of all $x\in A$ such that $\widehat{x}$ has compact support. We say that A is *Tauberian* if A_c is norm-dense in A. Any commutative C*-algebra is a typical Tauberian Banach algebra of type I.

2010 *Mathematics Subject Classification.* Primary 46J10; Secondary 46J40, 46H20.

Key words and phrases. Banach algebra of type I, BSE-algebra, BED-algebra, Segal algebra, Tauberian algebra, C*-algebra.

*To our regret, the third author Prof. Hiroyuki Takagi passed away on November 25, 2017. We would like to express our deepest condolences for him.

Research of the second author was supported in part by JSPS KAKENHI Grant Number 15K04921 and 16K05172.

In [3], the first and fourth authors have shown that there exists a BSE-algebra of type I such that it is not isomorphic to any C*-algebra. This solves a question posed by the fourth author and Hatori (see [9, p.153, Problem 1]) negatively. However, this result suggests a further investigation of Banach algebras of type I.

The purpose of this paper is to study type I Banach algebras by classifying them into six families by means of BSE, BED and Tauberian. We also construct a subfamily of each of these six families. Tauberian algebras and Segal algebras will play a crucial role in our arguments because, as stated in the section 3, a Banach algebra of type I is Tauberian if and only if it is isomorphic to a Segal algebra in a certain commutative C*-algebra. Also it will be clear that any unital Banach algebra of type I is just isomorphic to a unital commutaitve C*-algebra, and hence the non-unital case is essential in our arguments.

For the details of Segal algebras, BSE-algebras and BED-algebras, refer to the next section.

Let $\mathcal{B}_{typeI}$ be the collection of all Banach algebras of type I. We define $\mathcal{B}_{typeI}^{1,1}$, $\mathcal{B}_{typeI}^{1,0}$, $\mathcal{B}_{typeI}^{0,1}$ and $\mathcal{B}_{typeI}^{0,0}$ by

$$\begin{aligned} \mathcal{B}_{typeI}^{1,1} &= \{A \in \mathcal{B}_{typeI} : A \text{ is of BSE and of BED}\}, \\ \mathcal{B}_{typeI}^{1,0} &= \{A \in \mathcal{B}_{typeI} : A \text{ is of BSE but not of BED}\}, \\ \mathcal{B}_{typeI}^{0,1} &= \{A \in \mathcal{B}_{typeI} : A \text{ is not of BSE but of BED}\}, \\ \mathcal{B}_{typeI}^{0,0} &= \{A \in \mathcal{B}_{typeI} : A \text{ is not of BSE nor of BED}\}, \end{aligned}$$

respectively. By using the Tauberian property, we divide $\mathcal{B}_{typeI}^{i,j}$ $(i, j = 0, 1)$ into two families:

$$\mathcal{B}_{typeI}^{i,j,1} = \{A \in \mathcal{B}_{typeI}^{i,j} : A \text{ is Tauberian}\}$$

and

$$\mathcal{B}_{typeI}^{i,j,0} = \{A \in \mathcal{B}_{typeI}^{i,j} : A \text{ is not Tauberian}\}.$$

Since any algebra in $\mathcal{B}_{typeI}^{i,j}$ is Tauberian for $j = 1$ (see Theorem 3.2), these divisions are meaningful in the case $j = 0$. Thus we have the following disjoint union representation of $\mathcal{B}_{typeI}$:

$$\mathcal{B}_{typeI} = \mathcal{B}_{typeI}^{1,1} \cup \mathcal{B}_{typeI}^{1,0,1} \cup \mathcal{B}_{typeI}^{1,0,0} \cup \mathcal{B}_{typeI}^{0,1} \cup \mathcal{B}_{typeI}^{0,0,1} \cup \mathcal{B}_{typeI}^{0,0,0}.$$

We prove that $\mathcal{B}_{typeI}^{1,1}$ consists of all commutative C*-algebras up to isomorphism (see Corollary 4.2). We next give a concrete subfamily of $\mathcal{B}_{typeI}^{1,0,1}$ which extends the result obtained in [3]. This will be descreibed in Theorem 5.1. Then we construct a subfamily of $\mathcal{B}_{typeI}^{1,0,0}$ in Theorem 5.2. For $\mathcal{B}_{typeI}^{0,1}$, we give two subfamilies in Theorems 6.1 and 6.2. Finally subfamilies of $\mathcal{B}_{typeI}^{0,0,1}$ and $\mathcal{B}_{typeI}^{0,0,0}$ are given in Theorems 7.1 and 7.3, respectively. We will give such subfamilies by constructing special Banach function algebras on noncompact locally compact Hausdorff spaces.

2. Segal algebras, BSE-algebras and BED-algebras

(I) Segal algebras.

H. Reiter investigated Segal algebras in commutative group algebras. For Reiter's Segal algebras we refer to [6] and [7]. In [2], the first and fourth authors introduced the notion of Segel algebras in semisimple commutative Banach algebras A with the following properties:

(α) A is regular.

(β) There is a bounded approximate identity of A composed of elements in A_c.

It is obvious that commutative C*-algebras and group algebras on LCA groups satisfy the conditions (α) and (β).

An ideal S in A is called *a Banach ideal* in A if S itself constitutes a Banach space under a norm $\|\cdot\|_S$ satisfying $\|a\|_A \le \|a\|_S$ $(a \in S)$ and $\|ax\|_S \le \|a\|_S\|x\|_A$ $(a \in S, x \in A)$. A dense Banach ideal in A is called *a Segal algebra* in A if it has approximate units. When A is equal to a group algebra $L^1(G)$ of an LCA group G, Segal algebras in A coincide with Reiter's Segal algebras in $L^1(G)$. We here present the following important lemma which asserts that Segal algebras preserve "type I".

Lemma 2.1. *Let A be a Banach algebra of type I satisfying the conditions (α) and (β), and S be a Segal algebra in A. Then S is also of type I.*

Proof. By [2, Theorem B$'$-(ii)], we identify Φ_S with Φ_A. Take $\sigma \in C^b(\Phi_A)$ arbitrarily. Since A is of type I, we can take $T \in M(A)$ with $\sigma = \widehat{T}$. For any $x \in S$, there are $y \in S$ and $z \in A$ such that $x = yz$ by [2, Theorem A$'$]. Then we have

$$\sigma\widehat{x} = \widehat{T}\widehat{y}\widehat{z} = \widehat{(Tz)y} \in \widehat{S},$$

hence σ yields a linear operator T_σ from S to itself such that $T_\sigma(ab) = (T_\sigma a)b$ for all $a, b \in S$. Note that T_σ is continuous by the closed graph theorem, so T_σ is a multiplier of S with $\widehat{T_\sigma} = \sigma$. This observation implies that $\widehat{M}(S) = C^b(\Phi_S)$, namely, S is of type I. □

(II) BSE-algebras and BED-algebras.

Let A be a semisimple commutative Banach algebra with Gelfand space Φ_A. We denote by $\mathrm{span}(\Phi_\mathrm{A})$ the linear span of Φ_A in the dual space A^* of A. Therefore, an arbitrary element p in $\mathrm{span}(\Phi_\mathrm{A})$ has the unique expression

$$p = \sum_{\varphi \in \Phi_A} \widehat{p}(\varphi)\varphi,$$

where $\widehat{p}$ is a complex-valued function on Φ_A with finite support. A function $\sigma \in C^b(\Phi_A)$ is said to be a *BSE-function* if there is a constant $\beta > 0$ such that

$$\left| \sum_{\varphi \in \Phi_A} \widehat{p}(\varphi)\sigma(\varphi) \right| \leq \beta \|p\|_{A^*}$$

for all $p \in \mathrm{span}(\Phi_\mathrm{A})$. The BSE-norm of σ, denoted by $\|\sigma\|_{BSE(A)}$, is the infimum of all such β. The norm $\|\cdot\|_{BSE(A)}$ is written simply as $\|\cdot\|_{BSE}$ if it will cause no confusion.

Let $C_{BSE}(\Phi_A)$ be the algebra of all BSE-functions, then it is a semisimple commutative Banach algebra under the BSE-norm (see [9, Lemma 1]). An algebra A is said to be a *BSE-algebra* if $\widehat{M(A)} = C_{BSE}(\Phi_A)$ (see [9, p.151, Definition]). If $\{e_\lambda\}$ is a net in A satisfying the condition

$$\lim_\lambda \varphi(e_\lambda) = 1 \quad (\varphi \in \Phi_A),$$

then we call it a Φ-weak approximate identity of A. We note that $\widehat{M(A)} \subseteq C_{BSE}(\Phi_A)$ if and only if A has a bounded Φ-weak approximate identity (see [9, Corollary 5]). Therefore, any BSE-algebra has a bounded Φ-weak approximate identity. For more details on BSE-algebras, we refer the reader to [1, 4].

Let $\mathcal{K}(\Phi_A)$ be the directed set consisting of all compact subsets in Φ_A with respect to the inclusion order. For $\sigma \in C_{BSE}(\Phi_A)$ and $K \in \mathcal{K}(\Phi_A)$, define

$$\|\sigma\|_{BSE,K} = \sup \left\{ \left| \sum_{\varphi \in \Phi_A} \widehat{p}(\varphi)\sigma(\varphi) \right| : p \in \mathrm{span}(\Phi_A),\ \|p\|_{A^*} \leq 1,\ \widehat{p}|_K = 0 \right\},$$

and so we have $\|\sigma\|_{BSE,K} \leq \|\sigma\|_{BSE}$. We set

$$C^0_{BSE}(\Phi_A) = \left\{ \sigma \in C_{BSE}(\Phi_A) : \lim_{K \in \mathcal{K}(\Phi_A)} \|\sigma\|_{BSE,K} = 0 \right\}.$$

Then we see that $C^0_{BSE}(\Phi_A)$ is a closed ideal of $C_{BSE}(\Phi_A)$ (see [1, Corollary 3.9]). An algebra A is said to be a *BED-algebra* if each function in $C^0_{BSE}(\Phi_A)$ is precisely the Gelfand transform of some element of A, that is, $\widehat{A} = C^0_{BSE}(\Phi_A)$ (see [1, Definition 4.13]).

We now give a basic result for Segal algebras in a Banach algebra of type I.

Lemma 2.2. *Let A be a Banach algebra of type I satisfying the conditions (α) and (β), and S be a Segal algebra in A. Then $\widehat{S}$ is a dense subset of $C^0_{BSE}(\Phi_S)$.*

Proof. Let $\{e_\lambda\}_{\lambda \in \Lambda}$ be a bounded approximate identity of A bounded by $\beta > 0$ composed of elements in A_c. Then we see from [2, Theorem A$'$] that $\{e_\lambda\}_{\lambda \in \Lambda}$ is an approximate identity of S and $\|e_\lambda f\|_S \leq \beta \|f\|_S$ holds for all $\lambda \in \Lambda$ and $f \in S$. We see from [2, Lemma 3.4] that S is regular. By Lemma 2.1, S is of type I, and then

$C^0_{BSE}(\Phi_S) \subseteq C^b(\Phi_S) = \widehat{M}(S)$. We observe that $\widehat{S}$ is dense in $C^0_{BSE}(\Phi_S)$ with the help of [1, Proposition 4.4]. $\square$

In the rest of this paper, we will identify Φ_S with Φ_A if S is a Segal algebra in a semisimple commutative Banach algebra A satisfying the conditions (α) and (β). In fact, [2, Theorem B′-(ii)] states that Φ_S is homeomorphic to Φ_A. Especially, in the case where A is a commutative C*-algebra $C_0(X)$, consisting of all continuous complex-valued functions on a locally compact Hausdorff space X vanishing at infinity, we will identify both Φ_S and Φ_A with X. In this case, the Gelfand transforms on S and A are the identity mapping under this identification. Therefore, we may and do write $C_{BSE(S)}(X)$ and $C^0_{BSE(S)}(X)$ instead of $C_{BSE}(\Phi_S)$ and $C^0_{BSE}(\Phi_S)$, respectively. Moreover, we will say that S has a X-weak approximate identity if S has a Φ-weak approximate identity. Also we define $C_c(X) = (C_0(X))_c$, that is, the algebra of all continuous complex-valued functions on X with compact supports.

3. Tauberian Banach algebras of type I

In this section, we characterize a Tauberian Banach algebra of type I in terms of Segal algebras. Moreover, we show that any BED-algebra of type I is Tauberian.

Theorem 3.1. *Let A be a semisimple commutative Banach algebra. Then A is a Tauberian Banach algebra of type I if and only if it is isomorphic to a Segal algebra in a certain commutative C*-algebra.*

Proof. First, assume that A is isomorphic to a Segal algebra in a commutative C*-algebra. Then A is of type I by Lemma 2.1. Also we see from [2, Theorem A′] that A is Tauberian.

Conversely, assume that A is a Tauberian Banach algebra of type I. Put $S = \widehat{A}$ and $\|\widehat{a}\|_S = \|a\|_A$ for each $a \in A$. Then S becomes a commutative Banach algebra which is isomorphic to A. Put $X = \Phi_A$ and so S becomes a Banach ideal in $C_0(X)$ because A is of type I.

Now we assert that $C_c(X) \subseteq S$. To show this, let $f \in C_c(X)$ and put $K = \mathrm{supp}(f)$. Take $x \in X$ arbitrarily and choose $f_x \in S$ with $f_x(x) \neq 0$. There exist a neighborhood U_x of x and $g_x \in C_0(X)$ such that $g_x(y) = 1/f_x(y)$ for all $y \in U_x$. Put $e_x = f_x g_x$, and then $e_x \in S$ with $e_x = 1$ on U_x. Since K is compact, we can find a finite number of elements $x_1, \cdots, x_n \in K$ such that $\{U_{x_1}, \cdots, U_{x_n}\}$ is a covering of K. We now define

$$u_K = 1 - (1 - e_{x_1}) \cdots (1 - e_{x_n}),$$

and then $u_K \in S$ with $u_K = 1$ on K. Thus $f = f u_K \in S$ as required.

The above assertion implies that S is a Segal algebra in $C_0(X)$. In fact, it is sufficient to show that S has approximate units. To do this, let $f \in S$ and $\varepsilon > 0$ be chosen arbitrarily. Then there is $g \in S$ with compact support such that $\|f-g\|_S < \varepsilon/2$ because S is Tauberian. Put $K' = \mathrm{supp}(g)$ and choose $e_{K'} \in C_c(X)$ such that $0 \le e_{K'} \le 1$ and $e_{K'} = 1$ on K'. Then we have $e_{K'} \in S$ by our assertion. We obtain

$$\|e_{K'}(f-g)\|_S \le \|e_{K'}\|_\infty \|f-g\|_S < \varepsilon/2.$$

The above inequalities show that

$$\|e_{K'}f - f\|_S \le \|e_{K'}f - e_{K'}g\|_S + \|e_{K'}g - f\|_S < \varepsilon/2 + \|g-f\|_S < \varepsilon,$$

and hence S has approximate units. □

The following theorem describes a relationship between "BED" and "Tauberian" in a Banach algebra of type I.

Theorem 3.2. *Any BED-algebra of type I is Tauberian.*

Proof. Let A be a BED-algebra of type I and put $X = \Phi_A$. As observed in the proof of Theorem 3.1, A is isomorphic to a certain Banach ideal S in $C_0(X)$. We will show that S is Tauberian. To do this, let $f \in S$ and $\varepsilon > 0$ be chosen arbitrarily. Since S is of BED, we can find a compact subset K in X with $\|f\|_{BSE,K} < \varepsilon$. Choose $e_K \in C_c(X)$ such that $0 \le e_K \le 1$ and $e_K = 1$ on K. Also, take $p \in \mathrm{span}(X)$ arbitrarily and define

$$q(g) = \sum_{x \in X} (1 - e_K(x))\widehat{p}(x)g(x)$$

for each $g \in S$. Then q is an element of $\mathrm{span}(X)$ such that

$$\widehat{q}(x) = (1 - e_K(x))\widehat{p}(x)$$

for each $x \in X$. Hence we have

$$\begin{aligned}
\|q\|_{S^*} &= \sup\left\{ \left| \sum_{x \in X} (1 - e_K(x))\widehat{p}(x)g(x) \right| : g \in S, \|g\|_S \le 1 \right\} \\
&= \sup\left\{ \left| \sum_{x \in X} \widehat{p}(x)(1 - e_K)g(x) \right| : g \in S, \|g\|_S \le 1 \right\} \\
&\le 2\sup\left\{ \left| \sum_{x \in X} \widehat{p}(x)h(x) \right| : h \in S, \|h\|_S \le 1 \right\} \\
&\le 2\|p\|_{S^*}
\end{aligned}$$

because $\|(1-e_K)g\|_S \leq \|g\|_S + \|e_K\|_\infty \|g\|_S \leq 2\|g\|_S$ for each $g \in S$. Therefore, we obtain

$$\begin{aligned}\left|\sum_{x\in X} \widehat{p}(x)(1-e_K)f(x)\right| &= \left|\sum_{x\in X} \widehat{q}(x)f(x)\right| = \left|\sum_{x\notin K} \widehat{q}(x)f(x)\right| \\ &\leq \|q\|_{S^*}\|f\|_{BSE,K} \\ &\leq 2\|p\|_{S^*} \times \varepsilon,\end{aligned}$$

which implies $\|f - fe_K\|_{BSE(S)} = \|(1-e_K)f\|_{BSE(S)} < 2\varepsilon$. Since S is of BED, it follows that $\|\cdot\|_S$ and $\|\cdot\|_{BSE(S)}$ are equivalent, and hence we conclude that S is Tauberian. □

4. Algebras which belong to $\mathcal{B}^{1,1}_{typeI}$

We have the following result which gives a characterization of commutative C*-algebras in terms of BSE and BED.

Theorem 4.1. *Let A be a Banach algebra of type I. Then the following are equivalent to each other :*

(i) *A has a bounded approximate identity.*
(ii) *A is isomorphic to a C*-algebra.*
(iii) *A is of BSE and of BED.*

Proof. (i)⇔(ii). This is essentially shown in the proof of [9, Theorem 3].

(ii)⇒(iii). If A is isomorphic to a C*-algebra, then it is of BSE and of BED by [9, Theorem 3] and [1, Theorem 5.10], respectively.

(iii)⇒(ii). Suppose that A is of BSE and of BED. Since A is a BSE-algebra of type I, it follows that $C_{BSE}(\Phi_A)$ is isomorphic to the C*-algebra $C^b(\Phi_A)$. Also, since $C^0_{BSE}(\Phi_A)$ is a closed ideal of $C_{BSE}(\Phi_A)$, we see that $C^0_{BSE}(\Phi_A)$ is isomorphic to a C*-algebra. By the initial assumption, A is of BED, and then we deduce that it must be isomorphic to a C*-algebra. □

Recall that an arbitrary commutative C*-algebra is always of type I, and hence we obtain the following from Theorem 4.1.

Corollary 4.2. *The family $\mathcal{B}^{1,1}_{typeI}$ consists of all commutative C*-algebras up to isomorphism.*

5. Algebras which belong to $\mathcal{B}^{1,0,k}_{typeI}$ $(k = 0, 1)$

The case of $k = 1$.

Let X be a noncompact locally compact Hausdorff space and μ a positive continuous regular Borel measure on X with $\mu(X) = \infty$. Let $L^p(X, \mu)$ be the L^p-space on X, where $1 \le p < \infty$, and define

$$C_{0,p}(X, \mu) = C_0(X) \cap L^p(X, \mu).$$

Then $C_{0,p}(X, \mu)$ becomes a semisimple commutative Banach algebra with l^1-norm:

$$\|f\|_{\infty,p} = \|f\|_\infty + \|f\|_p \ \ (f \in C_{0,p}(X, \mu)).$$

As observed in the proof of [3, Lemma 2.1], $C_{0,p}(X, \mu)$ is a Segal algebra in $C_0(X)$. Theorem 3.1 shows that it is a Tauberian Banach algebra of type I. Moreover, we see that this algebra has a bounded X-weak approximate identity as observed in the proof of [3, Lemma 2.2]. Therefore, we see from [2, Theorem 9.10] that this Segal algebra is of BSE.

Since X is noncompact and $\mu(X) = \infty$, we can find a sequence $\{K_1, K_2, \cdots\}$ of compact subsets of X and a sequence $\{V_1, V_2, \cdots\}$ of open subsets of X with compact closure such that

$$V_i \cap V_j = \varnothing \ (i \ne j), \quad K_n \subseteq V_n \quad \text{and} \quad \mu(K_n) \ge 1 \ (n \in \mathbf{N}).$$

For each $n \in \mathbf{N}$, choose a continuous complex-valued function f_n on X such that

$$f_n(x) = \frac{1}{n^{1/p}} \ (x \in K_n), \quad 0 \le f_n \le \frac{1}{n^{1/p}} \quad \text{and} \quad \operatorname{supp}(f_n) \subseteq V_n.$$

Put

$$f = \sum_{n=1}^{\infty} f_n.$$

Then it is easy to see that $f \in C_0(X)$ and $f \notin L^p(X, \mu)$, hence $C_{0,p}(X, \mu)$ is proper in $C_0(X)$. This yields that $C_{0,p}(X, \mu)$ has no bounded approximate identity by [2, Theorem C$'$-(ii)]. Therefore, we see from Theorem 4.1 that $C_{0,p}(X, \mu)$ is not of BED.

By summarizing the above arguments, the following theorem is obtained.

Theorem 5.1. *Let $C_{0,p}(X, \mu)$ be as in the above. Then it is a Tauberian Banach algebra of type I which is of BSE but is not of BED, that is, this algebra belongs to $\mathcal{B}^{1,0,1}_{typeI}$.*

Remark 1. A family of Segal algebras obtained in [3] is, of course, contained in $\mathcal{B}^{1,0,1}_{typeI}$, but the family obtained in the above theorem is a wider one than this family.

The case of $k = 0$.

Let X be a noncompact σ-compact locally compact Hausdorff space and μ a positive Borel measure on X. Let τ be a continuous complex-valued function on X such that $\tau(x) > 0$ for all $x \in X$ and $1/\tau \in C_0(X)$, where $(1/\tau)(x) = 1/\tau(x)$ $(x \in X)$. Note that there do exist such a function τ because X is σ-compact. Furthermore, let $\{V_x\}_{x\in X}$ be a family of open neighbourhoods V_x of $x \in X$ with compact closure. We define

$$\begin{aligned} S &\equiv S_{\tau,\{V_x\}}(X,\mu) \\ &= \left\{ f \in C_0(X) : \|f\|_\tau := \sup_{x\in X} \int_{V_x} |f(t)|\tau(t)d\mu(t) < \infty \right\} \end{aligned}$$

and

$$\|f\|_S = \|f\|_\infty + \|f\|_\tau$$

for each $f \in S$. Then $(S, \|\cdot\|_S)$ is a natural Banach function algebra on X. Actually, it is apparent that $C_c(X) \subseteq S$, hence S separates strongly the points of X. By a routine argument, we see that S is a Banach module over $C_0(X)$. We now prove that S is natural. Let $\varphi \in \Phi_S$ be chosen arbitrarily and take $e_\varphi \in S$ with $\varphi(e_\varphi) = 1$. For any $f \in C_0(X)$, we can find a sequence $\{f_n\}$ in S which converges uniformly to f because S is uniformly dense in $C_0(X)$. Then we have

$$\lim_{n,m\to\infty} \|f_n e_\varphi - f_m e_\varphi\|_S \le \lim_{n,m\to\infty} \|f_n - f_m\|_\infty \, \|e_\varphi\|_S = 0,$$

and hence $\{\varphi(f_n)\}_{n=1}^\infty$ converges to a complex number. We define

$$\widetilde{\varphi}(f) = \lim_{n\to\infty} \varphi(f_n)$$

for each $f \in C_0(X)$. This is well-defined because we can easily see that $\widetilde{\varphi}(f)$ does not depend on a choice of $\{f_n\}_{n=1}^\infty$. Apparently, $\widetilde{\varphi}$ is an element of $\Phi_{C_0(X)} \cong X$ with $\widetilde{\varphi}|_S = \varphi$, hence S is natural. Therefore, Φ_S can be identified with X by [8, Theorem 3.2.4].

We assume that (X,μ) and $\{V_x\}_{x\in X}$ have the following properties:

- (a) There are two positive constants m_μ and M_μ such that $m_\mu \le \mu(V_x) \le M_\mu$ for all $x \in X$.
- (b) For any compact subset K in X, there is $x \in X$ such that $V_x \subseteq X \setminus K$.
- (c) Given $x \in X$, a neighbourhood V of x and $\varepsilon > 0$, there is a neighbourhood U of x such that $U \subseteq V$ and $\mu(U) < \varepsilon$.

Then we have the following

Theorem 5.2. *Under the assumptions $(a), (b)$ and (c), the Banach algebra $S = S_{\tau,\{V_x\}}(X,\mu)$ has the following properties:*

(i) *S is of type I.*

(ii) *S is not Tauberian.*

(iii) *S is of BSE but is not of BED.*

Namely, S belongs to $\mathcal{B}^{1,0,0}_{typeI}$.

Proof. (i) We can easily see that S is a Banach module over $C^b(X)$, and hence $M(S) = C^b(X)$, namely S is of type I.

(ii) We have $\|1/\tau\|_\tau \leq M_\mu < \infty$ by (a), hence $1/\tau \in S$. Let h be an arbitrary function in S with compact support. We can choose $x_0 \in X$ with $V_{x_0} \subseteq X \setminus \mathrm{supp}(h)$ by (b). Then we have from (a) that

$$\left\|\frac{1}{\tau} - h\right\|_S \geq \left\|\frac{1}{\tau} - h\right\|_\tau \geq \int_{V_{x_0}} \frac{1}{\tau(t)}\tau(t)d\mu(t) = \mu(V_{x_0}) \geq m_\mu > 0,$$

which implies that S is not Tauberian.

(iii) To see that S is of BSE, let Λ be the directed set consisting of all finite subsets of X with inclusion order, and take $\lambda = \{x_1, \cdots, x_n\} \in \Lambda$ arbitrarily. By (c), we can find a family $\{U_1, \cdots, U_n\}$ consisting of open subsets in X such that

$$x_i \in U_i \subseteq V_{x_i}, \quad \mu(U_i) < \frac{1}{nM_i} \quad \text{and} \quad U_i \cap U_j = \emptyset \quad (i \neq j)$$

for all $i = 1, \cdots, n$, where $M_i = \sup\{\tau(x) : x \in V_{x_i}\}$. Next, we choose a finite set $\{e_{x_1}, \cdots, e_{x_n}\}$ in $C_c(X)$ such that

$$e_{x_i}(x_i) = 1, \quad 0 \leq e_{x_i} \leq 1 \quad \text{and} \quad \mathrm{supp}(e_{x_i}) \subseteq U_i$$

for all $i = 1, \cdots, n$. We define e_λ by

$$e_\lambda = \sum_{i=1}^{n} e_{x_i}.$$

Then we have that $e_\lambda(x_i) = 1$ $(1 \leq i \leq n)$ and

$$\begin{aligned}\|e_\lambda\|_S = \|e_\lambda\|_\infty + \|e_\lambda\|_\tau &= 1 + \sup_{x \in X} \sum_{i=1}^{n} \int_{V_x} e_{x_i}(t)\tau(t)d\mu(t) \\ &\leq 1 + \sum_{i=1}^{n} \int_{U_i} e_{x_i}(t)\tau(t)d\mu(t) \leq 1 + \sum_{i=1}^{n} \int_{U_i} M_i\mu(t) < 2.\end{aligned}$$

Thus, $\{e_\lambda\}_{\lambda \in \Lambda}$ is a bounded X-weak approximate identity of S. Therefore, we have that $M(S) \subseteq C_{BSE(S)}(X)$ by [9, Corollary 5]. Hence S must be of BSE because S is of type I by (i). S is not Tauberian by (ii), and then Theorem 3.2 shows that S is not of BED. □

Remark 2. If X is a locally compact σ-compact Hausdorff space with a positive continuous regular Borel measure μ, then (c) is automatically satisfied.

Remark 3. Let G be a locally compact σ-compact noncompact group with continuous left Harr measure μ. Then the conditions $(a), (b)$ and (c) are automatically satisfied. In fact, take an open neighbourhood V_e of the identity element e with compact support and put $V_x = xV_e$ for each $x \in G$. Then $m_\mu = M_\mu = \mu(V_e) = \mu(V_x) > 0$ and hence (a) is satisfied. Let K be an arbitrary compact set in G. If $K \cap V_x \neq \varnothing$ for all $x \in G \setminus K$, then $G \setminus K \subseteq KV_e^{-1}$, hence G must be compact because KV_e^{-1} has compact closure. This contradicts that G is noncompact. Thus, we see that (b) is satisfied. Also, since μ is continuous and regular, it follows from Remark 2 that (c) is satisfied.

6. Algebras which belong to $\mathcal{B}^{0,1}_{typeI}$

In this section, we will give two subfamilies of $\mathcal{B}^{0,1}_{typeI}$. To do this, let X be a noncompact locally compact Hausdorff space.

(I)

Assume that X is σ-compact. Then we can choose a sequence $\{U_1, U_2, \cdots\}$ of open subsets of X with compact closure such that $\overline{U_1} \subsetneqq U_2 \subseteq \overline{U_2} \subsetneqq U_3 \subseteq \cdots$ and $\cup_{n=1}^{\infty} U_n = X$. For each $n \in \mathbf{N}$, choose $x_n \in U_n \setminus \overline{U_{n-1}}$, where $U_0 = \emptyset$, and define

$$C_{0,\,p,\{x_i\}}(X) = \left\{ f \in C_0(X) : \sum_{i=1}^{\infty} |f(x_i)|^p < \infty \right\},$$

for $1 \leq p < \infty$. Then it becomes a semisimple commutative Banach algebra under l^1-norm:

$$\|f\|_{\infty,p,\{x_i\}} = \|f\|_\infty + \left(\sum_{i=1}^{\infty} |f(x_i)|^p \right)^{1/p} \qquad (f \in C_{0,p,\{x_i\}}(X)).$$

In this case, we have the following

Theorem 6.1. *The Banach algebra $C_{0,p,\{x_i\}}(X)$ is a BED-algebra of type I but is not a BSE-algebra, that is, this algebra belongs to $\mathcal{B}^{0,1}_{typeI}$.*

Proof. We first show that $C_{0,p,\{x_i\}}(X)$ is a Segal algebra in $C_0(X)$. Since each compact subset of X is contained in some U_n, it follows that $C_c(X) \subseteq C_{0,\,p,\{x_i\}}(X)$, hence $C_{0,\,p,\{x_i\}}(X)$ is a dense Banach ideal in $C_0(X)$. Then it is sufficient to show that $C_{0,\,p,\{x_i\}}(X)$ has approximate units. To see this, take $f \in C_{0,\,p,\{x_i\}}(X)$ and $\varepsilon > 0$ arbitrarily. Then there is $N_\varepsilon \in \mathbf{N}$ such that

$$\sum_{i=N_\varepsilon+1}^{\infty} |f(x_i)|^p < \varepsilon^p/2^p.$$

Put

$$K = \{x \in X : |f(x)| \geq \varepsilon/2\} \cup \overline{U_{N_\varepsilon}},$$

and let U be an open neighbourhood of K with compact closure. Choose a continuous function e on X such that $e|_K = 1, e|_{X \setminus U} = 0$ and $0 \leq e \leq 1$. Then e is in $C_c(X)$ and

$$\begin{aligned} \|f - fe\|_{\infty,p,\{x_i\}} &= \sup_{x \notin K} |f(x)(1 - e(x))| + \left(\sum_{i=N_\varepsilon+1}^{\infty} |f(x_i)(1 - e(x_i))|^p \right)^{1/p} \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon \end{aligned}$$

holds. Then $C_{0,p,\{x_i\}}(X)$ has approximate units as required.

By the above argument and Theorem 3.1, we see that $C_{0,p,\{x_i\}}(X)$ is a Tauberian Banach algebra of type I. Furthermore, we see that $C_{0,p,\{x_i\}}(X)$ has no bounded X-weak approximate identity. In fact, suppose on the contrary that it has a bounded X-weak approximate identity $\{e_\lambda\}_{\lambda \in \Lambda}$ bounded by β. Then for each $n \in \mathbf{N}$, we can find $\lambda_n \in \Lambda$ such that

$$n^{\frac{1}{p}}/2 \leq \|e_{\lambda_n}\|_{\infty,p,\{x_i\}} \leq \beta.$$

This is a contradiction because n is arbitrary. Thus we see that $C_{0,p,\{x_i\}}(X)$ is not of BSE by [9, Corollary 5].

Finally, we show that $C_{0,p,\{x_i\}}(X)$ is of BED. In fact, since $C_{0,p,\{x_i\}}(X)$ is Tauberian, it follows from [1, Proposition 4.1] that

$$C_{0,p,\{x_i\}}(X) \subseteq C^0_{BSE(C_{0,p,\{x_i\}}(X))}(X).$$

To show the reverse inclusion, let $f \in C^0_{BSE(C_{0,p,\{x_i\}}(X))}(X)$. Take $\varepsilon > 0$ arbitrarily, and hence there is $K_0 \in \mathcal{K}(X)$ with $\|f\|_{BSE,K_0} < \varepsilon$. Let $p_x(g) = g(x)$ for $x \in X$ and $g \in C_{0,p,\{x_i\}}(X)$. Since $\|p_x\|_{C_{0,p,\{x_i\}}(X)^*} \leq 1$ and $\widehat{p_x}|_{K_0} = 0$ for all $x \notin K_0$, it follows that

$$|f(x)| \leq \|f\|_{BSE,K_0} < \varepsilon$$

for all $x \notin K_0$, and hence $f \in C_0(X)$. Next, we need to show $\sum_{i=1}^{\infty} |f(x_i)|^p < \infty$. To do this, take $n \in \mathbf{N}$ arbitrarily and consider the space l^q_n, where $1/p + 1/q = 1$. Then we can choose $a = (a_1, \cdots, a_n) \in l^q_n$ such that

$$\|a\|_{l^q_n} = 1 \quad \text{and} \quad \left| \sum_{i=1}^{n} a_i f(x_i) \right| = \left(\sum_{i=1}^{n} |f(x_i)|^p \right)^{1/p}. \tag{6.1}$$

Moreover, let $p_a \in$ span(X) be a functional defined by $\widehat{p_a}(x_i) = a_i$ for $i = 1, 2, \cdots, n$ and $\widehat{p_a}(x) = 0$ otherwise. Then we have from Hörder-Rogers' inequality and the first equation of (6.1) that

$$\begin{aligned}\|p_a\|_{C_{0,\,p,\{x_i\}}(X)^*} &= \sup_{\substack{g\in C_{0,p,\{x_i\}}(X)\\ \|g\|_{\infty,p,\{x_i\}}\le 1}} \left|\sum_{i=1}^n a_i g(x_i)\right| \\ &\le \sup_{\substack{g\in C_{0,p,\{x_i\}}(X)\\ \|g\|_{\infty,p,\{x_i\}}\le 1}} \left(\sum_{i=1}^n |a_i|^q\right)^{1/q} \left(\sum_{i=1}^n |g(x_i)|^p\right)^{1/p} \\ &\le 1.\end{aligned}$$

Hence,

$$\begin{aligned}\sum_{i=1}^n |f(x_i)|^p = \left|\sum_{i=1}^n a_i f(x_i)\right|^p &\le \sup_{\substack{p\in \operatorname{span}(X)\\ \|p\|_{C_{0,p,\{x_i\}}(X)^*}\le 1}} \left|\sum_{i=1}^n \widehat{p}(x_i) f(x_i)\right|^p \\ &= \|f\|^p_{BSE(C_{0,p,\{x_i\}}(X))} < \infty\end{aligned}$$

holds by the second equation of (6.1). Since n is arbitrary, it follows that

$$\sum_{i=1}^\infty |f(x_i)|^p \le \|f\|^p_{BSE(C_{0,p,\{x_i\}}(X))} < \infty$$

as required, that is, $f \in C_{0,\,p,\{x_i\}}(X)$. Consequently, we have

$$C_{0,\,p,\{x_i\}}(X) = C^0_{BSE(C_{0,\,p,\{x_i\}}(X))}(X),$$

namely, $C_{0,\,p,\{x_i\}}(X)$ is of BED. □

(II)

Let $A = C_0(X)$ and τ an unbounded complex-valued continuous function on X. For $n \in \mathbf{N}$, define

$$A_{\tau(n)} = \{f \in A : f\tau^k \in A \ (0 \le k \le n)\}$$

and

$$\|f\|_{\tau(n)} = \sum_{k=0}^n \|f\tau^k\|_\infty \qquad (f \in A_{\tau(n)}).$$

Note that τ is a local A-function, that is, $f\tau \in A$ holds for all $f \in A_c$. Therefore, it follows from [2, Theorem 5.4 (ii)] that $A_{\tau(n)}$ is a Segal algebra in A, hence it is of type I from Lemma 2.1. Moreover, we see from [2, Remark 9.11 (b)] that $A_{\tau(n)}$ is of BED but is not of BSE.

By summarizing the above arguments, the following theorem is obtained.

Theorem 6.2. *Let $A_{\tau(n)}$ be as in the above. Then $A_{\tau(n)}$ is a BED-algebra of type I but is not a BSE-algebra, that is, this algebra belongs to $\mathcal{B}^{0,1}_{typeI}$.*

In addition, we have from [2, Proposition 8.2 (ii)] that

$$A \supsetneqq A_{\tau(1)} \supsetneqq A_{\tau(2)} \supsetneqq \cdots \supsetneqq A_{\tau(n)} \supsetneqq \cdots .$$

7. Algberas which belong to $\mathcal{B}^{0,0,k}_{typeI}$ $(k = 0, 1)$

The case of $k = 1$.

Let X be a locally compact Hausdorff space. Let S_1 and S_2 be two Segal algebras in $C_0(X)$. Then $S_1 \cap S_2$ is a Segal algebra in $C_0(X)$ with norm $\|f\|_{S_1} + \|f\|_{S_2}$ for $f \in S_1 \cap S_2$ (see [2, Theorem D′]). We denote by $S_1 \wedge S_2$ such a Segal algebra in $C_0(X)$. Under this notation, we have the following

Theorem 7.1. *Assume that S_1 is not of BSE, S_2 is of BSE and $S_1 \nsubseteq S_2$. Then*

(i) *$S_1 \wedge S_2$ is a Tauberian Banach algebra of type I.*

(ii) *$S_1 \wedge S_2$ is neither of BSE nor of BED.*

Namely, $S_1 \wedge S_2$ belongs to $\mathcal{B}^{0,0,1}_{typeI}$.

Proof. (i) This follows directly from Theorem 3.1.

(ii) We first show that $S_1 \wedge S_2$ is not of BSE. In fact, suppose on the contrary that $S_1 \wedge S_2$ is of BSE, so it has a bounded X-weak approximate identity, say $\{e_\lambda\}_{\lambda\in\Lambda}$. Since $\|e_\lambda\|_{S_1} \le \|e_\lambda\|_{S_1\wedge S_2}$ for all $\lambda \in \Lambda$, it follows that $\{e_\lambda\}_{\lambda\in\Lambda}$ is also a bounded X-weak approximate identity of S_1. Therefore, S_1 must be of BSE with the help of [2, Theorem 9.10]. This contradicts that S_1 is not of BSE.

We next show that $S_1 \wedge S_2$ is not of BED. We first assert that $C_{BSE(S_1\wedge S_2)}(X) = C_{BSE(S_1)}(X)$ holds. Clearly $C_{BSE(S_1\wedge S_2)}(X) \subseteq C_{BSE(S_1)}(X)$ because $\|p\|_{(S_1\wedge S_2)^*} \le \|p\|_{S_1^*}$ for all $p \in \mathrm{span}(X)$. To show the reverse inclusion, take $\sigma \in C_{BSE(S_1)}(X)$ arbitrarily. Then by [9, Theorem 4], we can find a bounded net $\{\sigma_\lambda\}_{\lambda\in\Lambda}$ in S_1 such that $\lim_\lambda \sigma_\lambda(x) = \sigma(x)$ for all $x \in X$. Also, since S_2 is of BSE, it has a bounded X-weak approximate identity, say $\{u_i\}_{i\in I}$. Put $\sigma_{\lambda,i} = \sigma_\lambda u_i$ for each $\lambda \in \Lambda$ and $i \in I$. Since both S_1 and S_2 are ideals of $C_0(X)$, it follows that $\{\sigma_{\lambda,i}\}_{(\lambda,i)\in\Lambda\times I}$ is a net in $S_1 \wedge S_2$ such that $\lim_{\lambda,i} \sigma_{\lambda,i}(x) = \sigma(x)$ for all $x \in X$. Moreover,

$$\begin{aligned}
\|\sigma_{\lambda,i}\|_{S_1\wedge S_2} &= \|\sigma_\lambda u_i\|_{S_1} + \|\sigma_\lambda u_i\|_{S_2} \le \|\sigma_\lambda\|_{S_1}\|u_i\|_\infty + \|\sigma_\lambda\|_\infty\|u_i\|_{S_2} \\
&\le \|\sigma_\lambda\|_{S_1}\|u_i\|_{S_2} + \|\sigma_\lambda\|_{S_1}\|u_i\|_{S_2} \\
&\le 2\sup_{\lambda\in\Lambda}\|\sigma_\lambda\|_{S_1} \times \sup_{i\in I}\|u_i\|_{S_2} < \infty
\end{aligned}$$

for all $\lambda \in \Lambda$ and $i \in I$. Thus, we see that $\{\sigma_{\lambda,i}\}_{(\lambda,i)\in\Lambda\times I}$ is a bounded net in $S_1 \wedge S_2$. Then it follows that $\sigma \in C_{BSE(S_1\wedge S_2)}(X)$ with the help of [9, Theorem 4] again. Thus, we see that the reverse inclusion holds as required. Our assertion implies that two BSE-norms $\|\cdot\|_{BSE(S_1\wedge S_2)}$ and $\|\cdot\|_{BSE(S_1)}$ on $C_{BSE(S_1\wedge S_2)}(X) = C_{BSE(S_1)}(X)$ are equivalent. Note also that S_1 is the $\|\cdot\|_{S_1}$-norm closure of $C_c(X)$ by [2, Theorem A′]

and that the $\|\cdot\|_{S_1}$-norm closure of $C_c(X)$ is contained in the $\|\cdot\|_{BSE(S_1)}$-norm closure of $S_1 \wedge S_2$ because $C_c(X) \subseteq S_1 \wedge S_2$ and $\|f\|_{BSE(S_1)} \leq \|f\|_{S_1}$ for all $f \in S_1$. Moreover, note that $\overline{S_1 \wedge S_2}^{\|\cdot\|_{BSE(S_1 \wedge S_2)}} = C^0_{BSE(S_1 \wedge S_2)}(X)$ holds by Lemma 2.2. Therefore, we have

$$\begin{aligned} S_1 \wedge S_2 \subsetneqq S_1 = \overline{C_c(X)}^{\|\cdot\|_{S_1}} &\subseteq \overline{S_1 \wedge S_2}^{\|\cdot\|_{BSE(S_1)}} = \overline{S_1 \wedge S_2}^{\|\cdot\|_{BSE(S_1 \wedge S_2)}} \\ &= C^0_{BSE(S_1 \wedge S_2)}(X), \end{aligned}$$

which implies that $S_1 \wedge S_2$ is not of BED. □

The above theorem gives many Segal algebras belonging to $\mathcal{B}^{0,0,1}_{typeI}$. For example, let $S_1 = C_0(\mathbf{R}^n)_{\tau(1)}$ and $S_2 = C_{0,p}(\mathbf{R}^n, dx)$, where $n \in \mathbf{N}, 1 \leq p < 1/\alpha$ and dx is the Lebesgue measure on $\mathbf{R}^n$ with $\tau(x) = |x|^\alpha + 1$ $(x \in \mathbf{R}^n)$. Then S_1 is a Segal algebra of type I in $C_0(\mathbf{R}^n)$ which is not of BSE by Theorem 6.2. Also S_2 is a BSE Segal algebra of type I in $C_0(\mathbf{R}^n)$ by Theorem 5.1. Define

$$f(x) = \begin{cases} 1/|x|^{1/p} & (|x| > 1) \\ 1 & (|x| \leq 1). \end{cases}$$

Then we can easily see that $f \in S_1 \setminus S_2$, and hence $S_1 \not\subseteq S_2$. Thus, by Theorem 7.1, we obtain the following

Corollary 7.2. *If $\tau(x) = |x|^\alpha + 1$ $(x \in \mathbf{R}^n)$, $1 \leq p < n/\alpha$ and dx is the Lebesgue measure on $\mathbf{R}^n$, then $C_0(\mathbf{R}^n)_{\tau(1)} \wedge C_{0,p}(\mathbf{R}^n, dx)$ belongs to $\mathcal{B}^{0,0,1}_{typeI}$.*

The case of $k = 0$.

Let X be a noncompact locally compact Hausdorff space and τ a continuous complex-valued function on X such that $\inf_{x \in X} \tau(x) \geq 1$ and $1/\tau \in C_0(X)$. Define

$$C_0(X; \tau) = \{f \in C^b(X) : \sup_{x \in X} |f(x)|\tau(x) < \infty\}$$

and

$$\|f\|_{\infty,\tau} = \sup_{x \in X} |f(x)|\tau(x)$$

for each $f \in C_0(X; \tau)$. By a routine argument, we see that $C_0(X; \tau)$ is a commutative Banach algebra with norm $\|\cdot\|_{\infty,\tau}$ such that

$$C_c(X) \subseteq C_0(X; \tau) \subseteq C_0(X).$$

Therefore, $C_0(X; \tau)$ becomes a dense Banach ideal in $C_0(X)$. Also $C_0(X; \tau)$ is natural. In fact, let φ be an arbitrary element of $\Phi_{C_0(X;\tau)}$. Choose $h \in C_0(X; \tau)$ with $\varphi(h) \neq 0$ and define

$$\widetilde{\varphi}(f) = \varphi(hf)/\varphi(h)$$

for each $f \in C_0(X)$. This is well-defined because the right hand side of the above equation is independent of a choice of h. By an easy calculation, we see that $\widetilde{\varphi}$ is a nonzero complex homomorphism on $C_0(X)$, and hence we can find $x \in X$ such that $\widetilde{\varphi}(f) = f(x)$ holds for all $f \in C_0(X)$. This implies that $\varphi(f) = f(x)$ holds for all $f \in C_0(X;\tau)$, namely, $C_0(X;\tau)$ is natural as required. Therefore, $\Phi_{C_0(X;\tau)}$ can be identified with X by [8, Theorem 3.2.4]. Then we have the following

Theorem 7.3. *Let $C_0(X;\tau)$ be as in the above. Then*

(i) *$C_0(X;\tau)$ is not Tauberian.*

(ii) *$C_0(X;\tau)$ is a Banach algebra of type I but is neither of BSE nor of BED. Namely, $C_0(X;\tau)$ belongs to $\mathcal{B}^{0,0,0}_{typeI}$.*

Proof. (i) Suppose that $C_0(X;\tau)$ is Tauberian. Put $h = 1/\tau$, and then it must be in $C_0(X;\tau)$. Therefore, we can find $f \in C_0(X;\tau)$ with compact support such that $\|h - f\|_{\infty,\tau} < 1$ by hypothesis. On the other hand, we have

$$\|h - f\|_{\infty,\tau} = \sup_{x\in X} \left| \frac{1}{\tau(x)} - f(x) \right| \tau(x) = \sup_{x\in X} |1 - f(x)\tau(x)| \geq 1.$$

Thus, we reach to a contradiction.

(ii) Note that $C_0(X;\tau)$ is an ideal of $C^b(X)$, and hence $C^b(X) \subseteq M(C_0(X;\tau))$ holds. Also, since $\Phi_{C_0(X;\tau)}$ can be identified with X, it follows that $M(C_0(X;\tau)) \subseteq C^b(X)$. Thus we obtain $M(C_0(X;\tau)) = C^b(X)$, that is, $C_0(X;\tau)$ is of type I. Also, since $C_0(X;\tau)$ is a Banach algebra of type I but is not Tauberian, it follows from Theorem 3.2 that $C_0(X;\tau)$ is not of BED. Finally, we show that $C_0(X;\tau)$ is not of BSE. Suppose on the contrary that $C_0(X;\tau)$ is of BSE, hence it has a bounded X-weak approximate identity, say, $\{e_\lambda\}_{\lambda\in\Lambda}$ bounded by β. Then we can choose $x_0 \in X$ and $\lambda_0 \in \Lambda$ such that $\tau(x_0) \geq 2\beta + 1$ and $|e_{\lambda_0}(x_0) - 1| \leq 1/2$ because $\sup_{x\in X} \tau(x) = \infty$ by the assumption on τ. Then we have

$$\beta \geq \|e_{\lambda_0}\|_{\infty,\tau} = \sup_{x\in X} |e_{\lambda_0}(x)|\tau(x) \geq |e_{\lambda_0}(x_0)|\tau(x_0) \geq \frac{2\beta+1}{2} = \beta + \frac{1}{2},$$

which is a contradiction. □

References

1. J. Inoue and S.-E. Takahasi, *On characterizations of the image of the Gelfand transform of commutative Banach algebras*, Math. Nachr., **280** (2007), 105–126.
2. J. Inoue and S.-E. Takahasi, *Segal algebras in commutative Banach algebras*, Rocky Mountain J. Math., **44-2** (2014), 539–589.
3. J. Inoue and S.-E. Takahasi, *A construction of a BSE-algebra of type I which is isomorphic to no C*-algebras*, Rocky Mountain J. Math., **47-8** (2017), 2693–2697.

4. E. Kaniuth and A. Ülger, *The Bochner-Schoenberg-Eberlein property for commutative Banach algebras, especially Fourier and Fourier Stieltjes algebras*, Trans. Amer. Math. Soc. **362** (2010), 4331–4356.
5. R. Larsen, An itroduction to the theory of multipliers, Springer-Verlag, New York, 1971.
6. H. Reiter, L^1-algebras and Segal algebras, Lect. Notes Math. 231, Springer-Verlag, Berlin, 1971.
7. H. Reiter and J.D. Stegeman, Classical Harmonic Analsis and Locally compact groups, Oxford Science Publications, Oxford, 2000.
8. C. E. Rickart, General Theory of Banach Algebras, D. Van Nostrand Company, Inc. Princeton, New Jersey, Toronto, London, New York, 1960.
9. S.-E. Takahasi and O. Hatori, *Commutative Banach algebras which satisfy a Bochner-Schoenberg-Eberlein type-theorem*, Proc. Amer. Math. Soc., **110-1** (1990), 149–158.

(Jyunji Inoue) Hokkaido University, Sapporo 060-0808, Japan
Email address: `rqstw4a1@xk9.so-net.ne.jp`

(Takeshi Miura) Department of Mathematics, Faculty of Science, Niigata University, Niigata 950-2181, Japan
Email address: `miura@math.sc.niigata-u.ac.jp`

(Hiroyuki Takagi) Department of Mathematics, Faculty of Science, Shinshu University, Matsumoto 390-8621, Japan

(Sin-Ei Takahasi) Yamagata University, Yonezawa 992-8510, Japan, Laboratory of Mathematics and Games, Chiba 273-0032, Japan
Email address: `sin_ei1@yahoo.co.jp`

Received 2018.10.29
Revised 2019.4.5