Nihonkai Math. J.
Vol.29(2018), 77–130

LOG DEL PEZZO SURFACES OF RANK ONE CONTAINING THE AFFINE PLANE

HIDEO KOJIMA AND TAKESHI TAKAHASHI

ABSTRACT. Let X be a log del Pezzo surface of rank one. In [8], the first author determined the possible singularity type of X when X contains the affine plane as a Zariski open subset. In this paper, we prove that, if X contains a non-cyclic quotient singular point and its singularity type is one of the list of [8, Appendix C], then it contains the affine plane as a Zariski open subset.

1. Introduction

This paper is a continuation of the paper [8] of the first author. We work over the complex number field $\mathbb{C}$.

Let X be a normal projective surface with only quotient singular points. Then X is called a *log del Pezzo surface* if its anticanonical divisor $-K_X$ is ample. A log del Pezzo surface is said to have rank one if its Picard number equals one. In this paper, we call a log del Pezzo surface of rank one an *LDP1-surface*.

A pair (X, Γ) of a normal compact complex surface X and a subvariety Γ of X is called a *compactification* of the complex affine plane $\mathbb{C}^2$ if $X \setminus \Gamma$ is biholomorphic to $\mathbb{C}^2$. A compactification (X, Γ) of $\mathbb{C}^2$ is said to be *minimal* if Γ is irreducible.

In [8], the first author proved that if (X, Γ) is a minimal compactification of $\mathbb{C}^2$ and X has only quotient singular points, then X is an LDP1-surface and the compactification (X, Γ) is algebraic. Moreover, he determined the possible singularity types of X. See [8, Appendix C]. In this paper, we consider the following problem.

PROBLEM 1. Let X be an LDP1-surface whose the singularity type is one of the list of [8, Appendix C]. Is then X a compactification of $\mathbb{C}^2$, i.e., X has a subvariety Γ such that $X \setminus \Gamma$ is biholomorphic to $\mathbb{C}^2$?

The first author [8] proved that Problem 1 is true provided the index of $X \leq 3$. However, Problem 1 is false in general. See [8, Example 4.2]. In this paper, we prove

2010 *Mathematics Subject Classification.* Primary 14J26; Secondary 14J17.
Key words and phrases. Log del Pezzo surface, compactification of the affine plane, quotient singular point.
This work was supported by JSPS KAKENHI Grant Numbers JP16K05094, JP17K05198.

that Problem 1 is true if X contains a non-cyclic quotient singular point. The main result of this paper is the following theorem.

Theorem 1.1. *Let X be an LDP1-surface and let $\pi : (V, D) \to X$ be the minimal resolution of X, where D is the reduced exceptional divisor. Assume that X contains at least one non-cyclic quotient singular point. Then X is a compactification of $\mathbb{C}^2$ if and only if the weighted dual graph of D is one of the following* (1)−(28), *where we omit the weight of the vertex corresponding to a* (-2)*-curve.*

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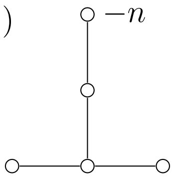

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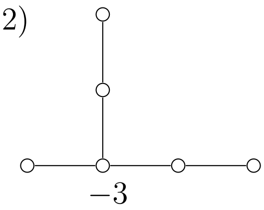

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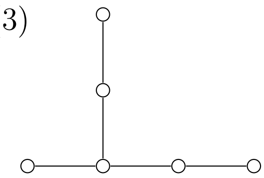

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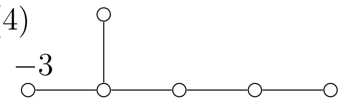

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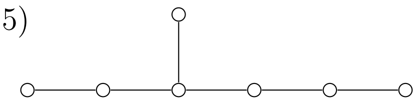

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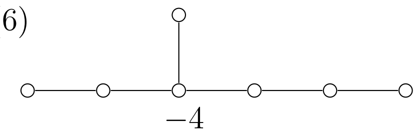

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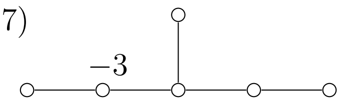

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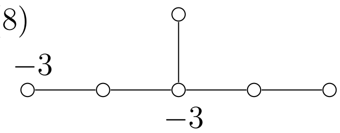

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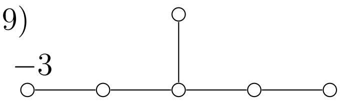

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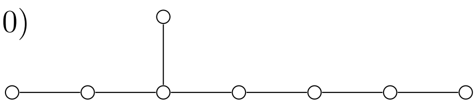

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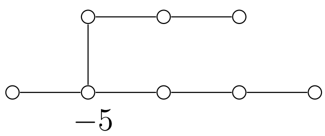

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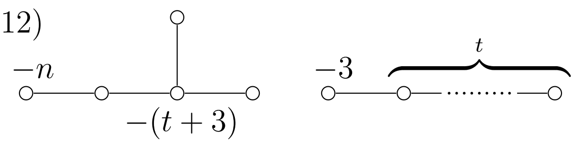

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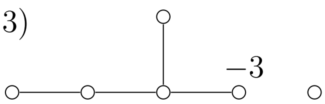

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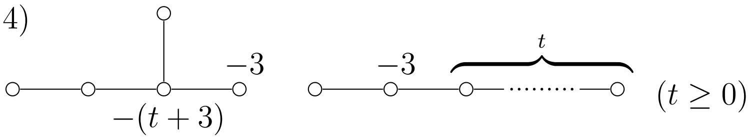

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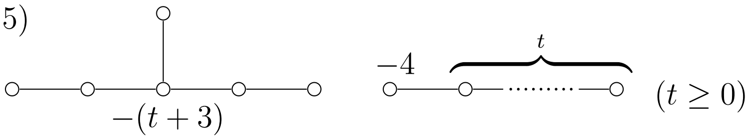

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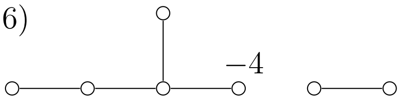

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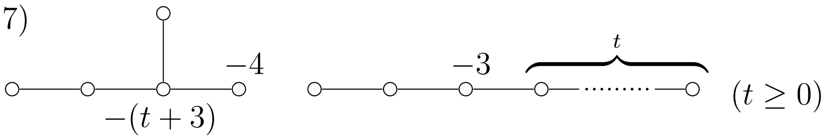

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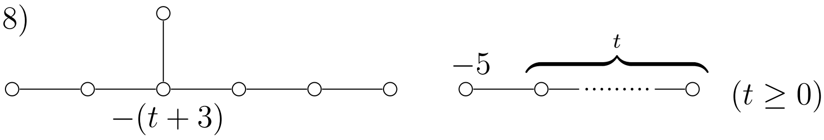

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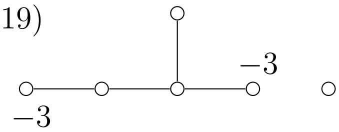

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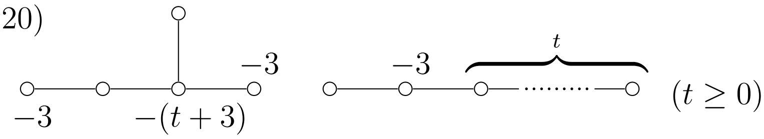

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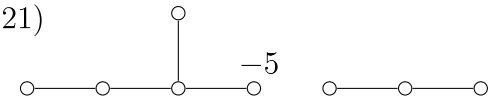

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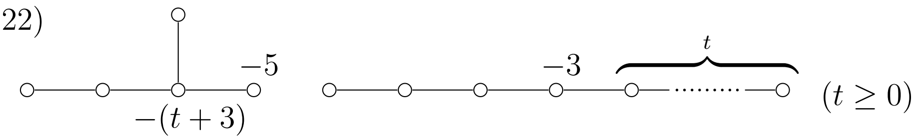

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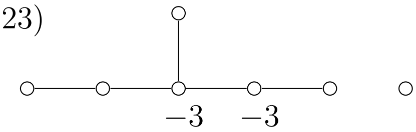

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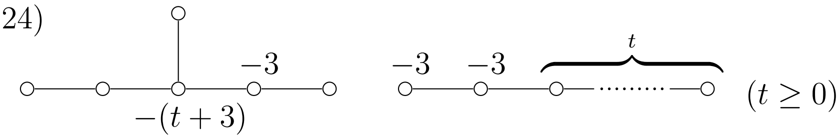

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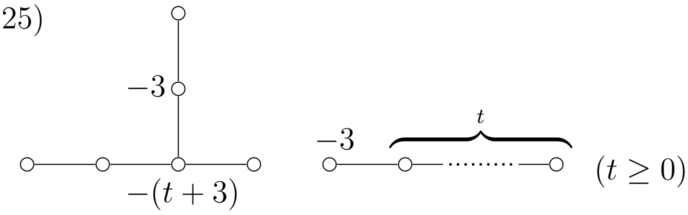

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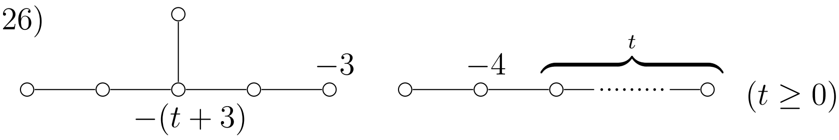

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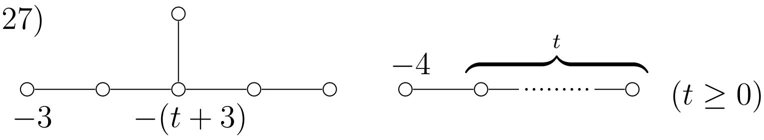

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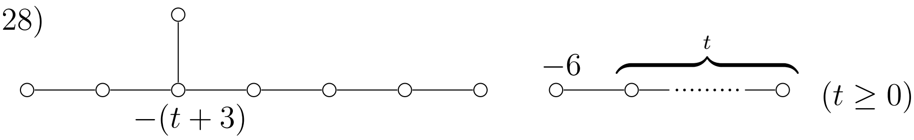


If a normal algebraic surface contains $\mathbb{C}^2$ as a Zariski open subset, then its smooth locus is simply connected. So we obtain the following result as a consequence of Theorem 1.1.

Corollary 1.1. *Let X be an LDP1-surface. If the singularity type of X is one of* (1)–(28) *in Theorem* 1.1*, then $X \setminus \mathrm{Sing}X$ is simply connected.*

It is well-known that the fundamental group of the smooth locus of every log del Pezzo surface is finite. See [5] and [6]. A short proof of the result is given in [4].

TERMINOLOGIES. A $(-n)$-curve is a smooth complete rational curve with self-intersection number $-n$. A reduced effective divisor D is called an SNC divisor if D has only simple normal crossings. We employ the following notations:

K_V: the canonical divisor on V.
$\rho(V)$: the Picard number of V.
$\overline{\kappa}(S)$: the logarithmic Kodaira dimension of S.
$\#D$: the number of all irreducible components in Supp$D$.
Σ_n: the Hirzebruch surface of degree n.

2. Preliminary results on LDP1-surfaces

In this section, we recall some basic results on LDP1-surfaces given in [14] and [15]. The results given in this section are generalized for the normal del Pezzo surfaces of rank one with only rational singularities. See [9] and [10].

Let X be an LDP1-surface and let $\pi : V \to X$ be the minimal resolution of singularities on X.

Lemma 2.1. *With the same notation and assumptions as above, the following assertions hold true.*

(1) *X is a rational surface.*
(2) *X is projective.*
(3) *X is $\mathbb{Q}$-factorial, i.e., for any Weil divisor L on X, there exists an integer $n > 0$ such that nL is a Cartier divisor.*

Proof. Since X has only quotient singular points, it has only rational singular points by [3]. So the assertions follow from results of [1]. □

Let $D = \sum_i D_i$ be the reduced exceptional divisor with respect to π, where the D_i are irreducible components of D. It is well-known that D is an SNC divisor and each connected component of D is a tree of smooth rational curves (cf. [2], [3]). We often denote (V, D) and X interchangeably.

There exists uniquely an effective $\mathbb{Q}$-divisor $D^\# = \sum_i \alpha_i D_i$ such that $D^\# + K_V \equiv \pi^* K_{\overline{V}}$.

Lemma 2.2. *The following assertions hold true.*

(1) *$-(D^\# + K_V)$ is a nef and big $\mathbb{Q}$-Cartier divisor.*
(2) *For any irreducible curve F on V, $-F(D^\# + K_V) = 0$ if and only if F is a component of D.*
(3) *Any $(-n)$-curve with $n \geq 2$ on V is a component of D.*

Proof. See [15, Lemma 1.1]. □

Lemma 2.3. *Let E be a (-1)-curve on V. Then the connected component of* Supp$(E + D)$ *containing E supports a big divisor. In particular, the intersection matrix of $E + D$ is neither negative definite nor negative semi-definite.*

Proof. The assertions follow from $\rho(V) = 1 + \#D$. $\square$

Let p the smallest positive integer such that $pD^{\#}$ is an integral divisor. By Lemmas 2.1 (3) and 2.2 (2), we know that, if C is an irreducible curve not contained in Supp$D$, then $-C(D^{\#} + K_V)$ takes value in $\{n/p| \ n \in \mathbb{Z}_{>0}\}$. So we can find an irreducible curve C such that $-C(D^{\#} + K_V)$ attains the smallest positive value. We denote the set of all such irreducible curves by $\mathrm{MV}(V, D)$.

Definition 2.1. (cf. [15, Definitions 1.2 and 3.2]) Let (V, D) and X be the same as above. (V, D) is said to be of the first kind if there exits a curve $C \in \mathrm{MV}(V, D)$ such that $|C + D + K_V| \neq \emptyset$. (V, D) is said to be of the second kind if (V, D) is not of the first kind, i.e., $|C + D + K_V| = \emptyset$ for any curve $C \in \mathrm{MV}(V, D)$.

Lemma 2.4. *Assume that (V, D) is of the first kind. Then there exists uniquely a decomposition of D as a sum of effective integral divisors $D = D' + D''$ such that the following conditions are satisfied.*

(i) $CD_i = D''D_i = K_V D_i = 0$ *for every component D_i of D'.*
(ii) $C + D'' + K_V \sim 0$.

Proof. See [14, Lemma 2.1]. $\square$

Following lemmas are useful to consider the case where (V, D) is of the second kind.

Lemma 2.5. *If $\rho(V) \geq 3$ and (V, D) is of the second kind, every curve of* $\mathrm{MV}(V, D)$ *is a (-1)-curve.*

Proof. The assertion can be proved by using the proof of [14, Lemma 2.2]. See [9, Lemma 3.6] for a direct proof of the assertion. $\square$

Lemma 2.6. *Let $\Phi : V \to \mathbb{P}^1$ be a $\mathbb{P}^1$-fibration (i.e., Φ is a fibration from V onto $\mathbb{P}^1$ whose general fiber is isomorphic to $\mathbb{P}^1$). Then the following assertions hold true.*

(1) *The number of irreducible components of D not in any fiber of Φ equals $1 + \sum_F (\#\{(-1)\text{-curves in } F\} - 1)$, where F moves over all singular fibers of Φ.*
(2) *If a singular fiber F of Φ consists only of (-1)-curves and (-2)-curves, then its weighted dual graph has one of the configurations* (i)–(iii) *in Figure* 2.1.

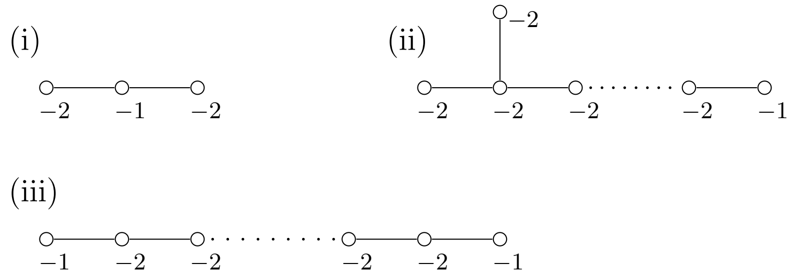


Figure 2.1.

Proof. See [14, Lemma 1.5]. □

Lemma 2.7. *Let $\Phi : V \to \mathbb{P}^1$ be a $\mathbb{P}^1$-fibration. Assume that there exists a singular fiber F of Φ such that it is of type* (i) *or* (ii) *in Figure* 2.1 *and that $C \in \mathrm{MV}(V, D)$, where C is the unique (-1)-curve in* Supp$F$*. Then every singular fiber G consists of (-2)-curves and (-1)-curves, i.e., the weighted dual graph of G is one of* (i), (ii) *and* (iii) *in Figure* 2.1*. Moreover, if E_1 and E_2 (possibly $E_1 = E_2$) are the (-1)-curves $\subset$* Supp$G$*, then $E_i \in \mathrm{MV}(V, D)$ for $i = 1, 2$.*

Proof. See [14, Lemma 1.6]. □

Lemma 2.8. *Let $\Phi : V \to \mathbb{P}^1$ be a $\mathbb{P}^1$-fibration and let C be a (-1)-curve in* $\mathrm{MV}(V, D)$*. Assume that Φ has a singular fiber F such that $F = 3C + \Delta$, where Δ is an effective divisor with* Supp$\Delta \subset$ Supp$D$*. Then every singular fiber of Φ consists of (-1)-curves, (-2)-curves and at most one (-3)-curve.*

Proof. See [9, Lemma 3.8]. The assertion can be proved by using the same argument as in the proof of [14, Lemma 1.6]. □

3. Proof of Theorem 1.1, part I

In Sections 3 and 4, we prove Theorem 1.1. Let X and $\pi : (V, D) \to X$ be the same as in Theorem 1.1. Let $D^{\#}$ be the $\mathbb{Q}$-divisor defined in Section 2 (see before Lemma 2.2). If X contains at least one non-cyclic quotient singular points and is a minimal compactification of $\mathbb{C}^2$, then [8, Theorem 1.1] implies that the weighted dual graph of D is one of (1)–(28) in Theorem 1.1.

From now on, we assume that the weighted dual graph of D is one of (1)–(28) in Theorem 1.1 and prove that X contains $\mathbb{C}^2$ as a Zariski open subset.

3.1. Case where $\#\mathrm{Sing}X = 1$

In this subsection, we consider the case where $\#\mathrm{Sing}X = 1$. Namely, we consider the case where the weighted dual graph of D is one of (1)–(11) of Theorem 1.1. We consider the case (1) only; the other cases can be treated similarly. Let $D = \sum_{i=0}^{4} D_i$ be the decomposition of D into irreducible components such that the weighted dual graph of D is given as in Figure 3.1, where $n \geq 2$ and the weight of the vertex corresponding to a (-2)-curve is omitted. In this case, $\rho(V) = 6$.

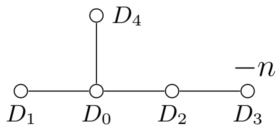


Figure 3.1.

By [7, Main Theorem and Appendix B], there exists a (-1)-curve C such that $CD = CD_i = 1$ for $i = 1$ or 4. We may assume that $CD_1 = 1$. Then the divisor $F = D_2 + D_4 + 2(C + D_1 + D_0)$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F|} : V \to \mathbb{P}^1$ and D_3 becomes a section of Φ. Since $6 = \rho(V) = 2 + (\#F - 1)$, Φ has no singular fibers other than F. Hence $V \setminus \mathrm{Supp}(C + D) \cong \mathbb{C}^2$. Therefore, X contains $\mathbb{C}^2$ as a Zariski open subset.

3.2. Case where $\#\mathrm{Sing}X = 2$, part I

We consider the case where the weighted dual graph of D is one of (13), (16), (19), (21) and (23) in Theorem 1.1. Although the arguments given in this subsection are similar to those given in Section 4, we treat the above cases separately because some of the arguments are different to those given in Section 4. Let $D = D^{(1)} + D^{(2)}$ be the decomposition of D into connected components such that $D^{(2)}$ is a linear chain and consists only of (-2)-curves.

Let C be a curve of $\mathrm{MV}(V, D)$. Then, by Lemma 2.3, $X \setminus \pi_*(C)$ is a normal affine surface with only quotient singular points. So the connected component of $C + D$ containing C supports a big divisor. Since $D^{(1)}$ is not a linear chain and contains a $(-m)$-curve ($m \geq 3$), (V, D) is of the second kind. Lemma 2.5 implies that C is a (-1)-curve and $|C + D + K_V| = \emptyset$. In particular, $CD^{(i)} \leq 1$ for $i = 1, 2$.

3.2.1. Case (13). Let $D = \sum_{i=0}^{5} D_i$ be the decomposition of D into irreducible components such that the weighted dual graph of D is given as in Figure 3.2, where the weight of the vertex corresponding to a (-2)-curve is omitted. In this case, $\rho(V) = 7$.

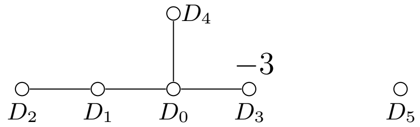


Figure 3.2.

If $CD^{(1)} = 0$, then $CD = CD^{(2)} = 1$. So the divisor $C + D^{(2)}$ is contracted to a smooth point. This contradicts Lemma 2.3. Hence, $CD^{(1)} = 1$.

We assume that $CD^{(2)} = 0$. Then $CD = CD_i = 1$ for some $i \in \{0, 1\}$ by Lemma 2.3. We consider the following subcases separately.

Subcase 1: $i = 1$**.** The divisor $F_1 := D_0 + D_2 + 2(C + D_1)$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F_1|} : V \to \mathbb{P}^1$ and D_3 and D_4 become sections of Φ. Let F_2 be the fiber of Φ containing $D^{(2)} = D_5$. Then $\mathrm{Supp}F_2$ consists only of D_5 and some (-1)-curves. We infer from Lemma 2.6 (2) that $F_2 = E_{2,1} + D_5 + E_{2,2}$, where $E_{2,1}$ and $E_{2,2}$ are (-1)-curves and $E_{2,1}D_5 = E_{2,2}D_5 = 1$. Since D_3 is a section of Φ, we may assume that $E_{2,1}D_3 = 1$. Then $E_{2,2}D_4 = 1$ by Lemma 2.3. We know that $E_{2,1}, E_{2,2} \in \mathrm{MV}(V, D)$. Set $G := D_1 + D_4 + 2(D_0 + D_3 + D_5) + 4E_{2,1}$. Then G defines a $\mathbb{P}^1$-fibraton $\Psi := \Phi_{|G|} : V \to \mathbb{P}^1$ and D_2 becomes a section of Ψ. Since $7 = \rho(V) = 2 + (\#G - 1)$, we see that $V \setminus \mathrm{Supp}(E_1 + D) \cong \mathbb{C}^2$. Therefore, X contains $\mathbb{C}^2$ as a Zariski open subset.

Subcase 2: $i = 0$**.** The divisor $F_1 := D_2 + D_3 + 2D_1 + 3(C + D_0)$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F_1|} : V \to \mathbb{P}^1$, D_4 becomes a 3-section of Φ and $D - D_4$ is contained in fibers of Φ. Let F_2 be the fiber of Φ containing $D^{(2)} = D_5$. By the argument as in Subcase 1, we know that $\#F_2 = 3$. So

$$7 = \rho(V) \geq 2 + (\#F_1 - 1) + (\#F_2 - 1) = 8.$$

This is a contradiction. Therefore, this subcase does not take place.

From now on, we assume that $CD^{(2)} (= CD_5) = 1$. If $CD^{(1)} = CD_3 = 1$, then $V \setminus \mathrm{Supp}(C + D) \cong \mathbb{C}^2$ by the argument as in Subcase 1. Suppose that $CD^{(1)} = CD_i = 1$ for some $i \in \{1, 2, 4\}$. Then the divisor $D_i + D_5 + 2C$ defines a $\mathbb{P}^1$-fibtation $\Phi_{|D_i + D_5 + 2C|} : V \to \mathbb{P}^1$. Then D_3, that is a (-3)-curve, is a fiber component of $\Phi_{|D_i + D_5 + 2C|}$. This contradicts Lemma 2.7.

Suppose that $CD^{(1)} = CD_0 = 1$. Then the divisor $F_1 := D_0 + D_5 + 2C$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F_1|} : V \to \mathbb{P}^1$, D_1, D_3 and D_4 become sections of Φ and $D - (D_0 + D_3 + D_4)$ is contained in fibers of Φ. Let F_2 be the fiber of Φ containing D_2. By using the same argument as in Subcase 1, we know that $F_2 = E_{2,1} + D_2 + E_{2,2}$, where $E_{2,1}$ and $E_{2,2}$ are (-1)-curves and $E_{2,1}D_2 = E_{2,2}D_2 = 1$. Since the intersection matrix of $E_{2,i} + D$ is not negative semi-definite for $i = 1, 2$ and D_3 and D_4 are sections of Φ, we may assume that $E_{2,1}D_3 = E_{2,2}D_4 = 1$. Since

$$7 = \rho(V) > 2 + (\#F_1 - 1) + (\#F_2 - 1) = 6,$$

Φ has a singular fiber $F_3 = E_{3,1} + E_{3,2}$ consisting of two (-1)-curves $E_{3,1}$ and $E_{3.2}$. Since

$$7 = \rho(V) = 2 + (\#F_1 - 1) + (\#F_2 - 1) + (\#F_3 - 1),$$

F_1, F_2 and F_3 exhaust the singular fibers of Φ. By Lemma 2.3, we may assume that $E_{3,1}D_3 = E_{3,2}D_4 = 1$. Let $\nu : V \to \Sigma_3$ be a relatively minimal model of $\Phi : V \to \mathbb{P}^1$ onto the Hirzebruch surface Σ_3 of degree 3 such that $f_*(D_3) = M_3$, the minimal section of Σ_3. By the construction of ν, we know that $\nu_*(D_4)^2 = 1$. However, this is a contradiction because $\nu_*(D_4)$ is a section of the ruling $\Phi \circ \nu^{-1} : \Sigma_3 \to \mathbb{P}^1$.

Therefore, X contains $\mathbb{C}^2$ as a Zariski open subset.

3.2.2. Case (16). Let $D = \sum_{i=0}^{6} D_i$ be the decomposition of D into irreducible components such that the weighted dual graph of D is given as in Figure 3.3, where the weight of the vertex corresponding to a (-2)-curve is omitted. In this case, $\rho(V) = 8$.

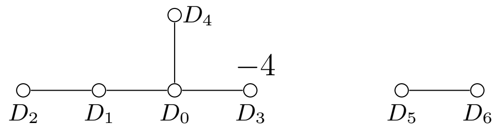


Figure 3.3.

Let α_i ($i = 0, 1, \ldots, 6$) be the coefficient of D_i in $D^{\#}$. Then

$$\alpha_0 = \frac{6}{7}, \quad \alpha_1 = \frac{4}{7}, \quad \alpha_2 = \frac{2}{7}, \quad \alpha_3 = \frac{5}{7}, \quad \alpha_4 = \frac{3}{7}, \quad \alpha_5 = \alpha_6 = 0.$$

If $CD^{(1)} = 0$, then $CD = CD^{(2)} = 1$. So the divisor $C + D^{(2)}$ is contracted to a smooth point. This contradicts Lemma 2.3. Hence, $CD^{(1)} = 1$.

We assume that $CD^{(2)} = 0$. Then $CD = CD_i = 1$ for some $i \in \{0, 1\}$ by Lemma 2.3. We consider the following subcases separately.

Subcase 1: $i = 1$**.** The divisor $F_1 := D_0 + D_2 + 2(C + D_1)$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F_1|} : V \to \mathbb{P}^1$ and D_3 and D_4 become sections of Φ. Let F_2 be the fiber of Φ containing $D^{(2)} = D_5 + D_6$. By Lemmas 2.7 and 2.6 (2), we know that $F_2 = E_{2,1} + D_5 + D_6 + E_{2,2}$, where $E_{2,1}$ and $E_{2,2}$ are (-1)-curves and $E_{2,1}D_5 = E_{2,2}D_6 = 1$. Since

$$8 = \rho(V) = 2 + (\#F_1 - 1) + (\#F_2 - 1),$$

F_1 and F_2 exhaust the singular fibers of Φ. By Lemma 2.3, we know that $E_{2,j}(D_3 + D_4) > 0$ for $j = 1, 2$. Since D_3 and D_4 are sections of Φ, we may assume that $E_{2,1}D_3 = E_{2,2}D_4 = 1$.

Let $\nu : V \to \Sigma_4$ be a relatively minimal model of $\Phi : V \to \mathbb{P}^1$ such that $f_*(D_3) = M_4$, the minimal section of Σ_4. By the construction of ν, we know that $\nu_*(D_4)^2 = 1$. However, this is a contradiction because $\nu_*(D_4)$ is a section of the ruling $\Phi \circ \nu^{-1} : \Sigma_4 \to \mathbb{P}^1$. Therefore, this subcase does not take place.

Subcase 2: $i = 0$**.** The divisor $F_1 := D_1 + D_4 + 2(C + D_0)$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F_1|} : V \to \mathbb{P}^1$, D_3 becomes a 2-section of Φ, D_2 becomes a section of Φ

and $D - (D_2 + D_3)$ is contained in fibers of Φ. Let F_2 be the fiber of Φ containing $D^{(2)} = D_5 + D_6$. By the same argument as in Subcase 1, we know that $F_2 = E_{2,1} + D_5 + D_6 + E_{2,2}$, where $E_{2,1}$ and $E_{2,2}$ are (-1)-curves and $E_{2,1}D_5 = E_{2,2}D_6 = 1$. Since

$$8 = \rho(V) = 2 + (\#F_1 - 1) + (\#F_2 - 1),$$

F_1 and F_2 exhaust the singular fibers of Φ.

Since D_2 is a section of Φ, we may assume that $E_{2,1}D_2 = 1$. Since $E_{2,1}D^{\#} < 1$, $E_{2,1}D_3 = 0$. So $E_{2,2}D_3 > 0$. Since D_3 is a 2-section of Φ and the coefficient of $E_{2,2}$ in F_2 equals one, we know that $E_{2,2}D_3 = 2$. Then $E_{2,2}D^{\#} = 2\alpha_3 = 10/7 > 1$, which is a contradiction. Therefore, this subcase does not take place.

Therefore, we know that $CD^{(2)} = 1$. We may assume that $CD_5 = 1$. Let $i \in \{0, 1, 2, 3, 4\}$ be the integer such that $CD_i = 1$, here we note that $CD^{(1)} = 1$.

If $i = 3$, then the divisor $F := D_1 + D_4 + 2(D_0 + D_3 + D_6) + 4D_5 + 6C$ defines a $\mathbb{P}^1$-fibraton $\Phi_{|F|} : V \to \mathbb{P}^1$ and D_2 becomes a section of $\Phi_{|F|}$. It is then clear that $V \setminus \mathrm{Supp}(C + D) \cong \mathbb{C}^2$.

Suppose that $i \in \{1, 2, 4\}$. Then the divisor $D_i + D_5 + 2C$ defines a $\mathbb{P}^1$-fibration $\Phi_{|D_i+D_5+2C|} : V \to \mathbb{P}^1$ and D_3, that is a (-4)-curve, becomes a fiber component of $\Phi_{|D_i+D_5+2C|}$. This contradicts Lemma 2.7. Suppose that $i = 0$. Then the divisor $F_1 := D_0 + D_5 + 2C$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F_1|} : V \to \mathbb{P}^1$, D_1, D_3, D_4 and D_6 become sections of Φ and $D - (D_1 + D_3 + D_4 + D_6)$ is contained in fibers of Φ. Let F_2 be the fiber of Φ containing D_2. By Lemmas 2.7 and 2.6 (2), we know that $F_2 = E_{2,1} + D_2 + E_{2,2}$, where $E_{2,1}$ and $E_{2,2}$ are (-1)-curves and $E_{2,1}D_2 = E_{2,2}D_2 = 1$. Since D_3 is a section of Φ, we may assume that $E_{2,1}D_3 = 1$. Then

$$E_{2,1}D^{\#} \geq E_{2,1}(\alpha_2 D_2 + \alpha_3 D_3) = 1,$$

which is a contradiction.

Therefore, X contains $\mathbb{C}^2$ as a Zariski open subset.

3.2.3. Case (19). Let $D = \sum_{i=0}^{5} D_i$ be the decomposition of D into irreducible components such that the weighted dual graph of D is given as in Figure 3.4, where the weight of the vertex corresponding to a (-2)-curve is omitted. In this case, $\rho(V) = 7$.

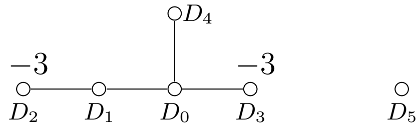


Figure 3.4

Let α_i $(i = 0, 1, \ldots, 5)$ be the coefficient of D_i in $D^{\#}$. Then

$$\alpha_0 = \frac{16}{17}, \quad \alpha_1 = \frac{13}{17}, \quad \alpha_2 = \frac{10}{17}, \quad \alpha_3 = \frac{11}{17}, \quad \alpha_4 = \frac{8}{17}, \quad \alpha_5 = 0.$$

If $CD^{(1)} = 0$, then $CD = CD^{(2)} = 1$. So the divisor $C + D^{(2)}$ is contracted to a smooth point. This contradicts Lemma 2.3. Hence, $CD^{(1)} = 1$.

We assume that $CD^{(2)} = 0$. Then $CD = CD_i = 1$ for some $i \in \{0, 1\}$ by Lemma 2.3. We consider the following subcases separately.

Subcase 1: $i = 0$**.** The divisor $F_1 := D_1 + D_4 + 2(C + D_0)$ defines a $\mathbb{P}^1$-fibration $\Phi_{|F_1|} : V \to \mathbb{P}^1$, D_3 becomes a 2-section of Φ, D_2 becomes a section of Φ and $D - (D_2 + D_3)$ is contained in fibers of Φ. Let F_2 be the fiber of Φ containing $D^{(2)} = D_5$. By Lemmas 2.7 and 2.6 (2), we know that $F_2 = E_{2,1} + D_5 + E_{2,2}$, where $E_{2,1}$ and $E_{2,2}$ are (-1)-curves and $E_{2,1}D_5 = E_{2,2}D_5 = 1$. Since D_2 is a section of Φ, we may assume that $E_{2,1}D_2 = 1$. Since $E_{2,1}D^{\#} < 1$, $E_{2,1}D_3 = 0$. So $E_{2,2}D_3 > 0$. Since D_3 is a 2-section of Φ and the coefficient of $E_{2,2}$ in F_2 equals one, we see that $E_{2,2}D_3 = 2$. Then

$$E_{2,2}D^{\#} = 2\alpha_3 = 22/17 > 1,$$

which is a contradiction. Therefore, this subcase does not take place.

Subcase 2: $i = 1$**.** The divisor $F_1 := D_2 + D_4 + 2D_0 + 3(C + D_1)$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F_1|} : V \to \mathbb{P}^1$, D_3 becomes a 2-section of Φ and $D - D_3$ is contained in fibers of Φ. Let F_2 be the fiber of Φ containing $D^{(2)} = D_5$. By the same argument as in Subcase 1, we know that $\#F_2 = 3$. Then

$$7 = \rho(V) \geq 2 + (\#F_1 - 1) + (\#F_2 - 1) = 8,$$

which is a contradiction. Therefore, this subcase does not take place.

Therefore, we know that $CD^{(2)} = CD_5 = 1$. Let $i \in \{0, 1, 2, 3, 4\}$ be the integer such that $CD_i = 1$, here we note that $CD^{(1)} = 1$. By Lemma 2.3, $i \neq 2$.

If $i = 3$, then the divisor $F := D_1 + D_4 + 2(D_0 + D_3 + D_5) + 4C$ defines a $\mathbb{P}^1$-fibraton $\Phi_{|F|} : V \to \mathbb{P}^1$ and D_2 becomes a section of $\Phi_{|F|}$. It is then clear that $V \setminus \mathrm{Supp}(C + D) \cong \mathbb{C}^2$.

Suppose that $i \in \{0, 1, 4\}$. Then the divisor $D_i + D_5 + 2C$ defines a $\mathbb{P}^1$-fibration $\Phi_{|D_i+D_5+2C|} : V \to \mathbb{P}^1$. If $i \in \{1, 4\}$ (resp. $i = 0$), then D_3 (resp. D_2), that is a (-3)-curve, becomes a fiber component of $\Phi_{|D_i+D_5+2C|}$. This contradicts Lemma 2.7.

Therefore, X contains $\mathbb{C}^2$ as a Zariski open subset.

3.2.4. Case (23). Let $D = \sum_{i=0}^{6} D_i$ be the decomposition of D into irreducible components such that the weighted dual graph of D is given as in Figure 3.5, where the weight of the vertex corresponding to a (-2)-curve is omitted. In this case, $\rho(V) = 8$.

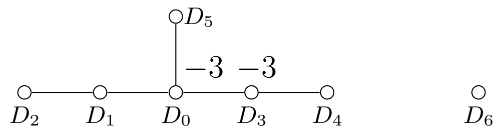


Figure 3.5

Let α_i $(i = 0, 1, \ldots, 6)$ be the coefficient of D_i in $D^{\#}$. Then

$$\alpha_0 = \frac{42}{43}, \quad \alpha_1 = \frac{28}{43}, \quad \alpha_2 = \frac{14}{43}, \quad \alpha_3 = \frac{34}{43}, \quad \alpha_4 = \frac{17}{43}, \quad \alpha_5 = \frac{21}{43}, \quad \alpha_6 = 0.$$

If $CD^{(1)} = 0$, then $CD = CD^{(2)} = 1$. So the divisor $C + D^{(2)}$ is contracted to a smooth point. This contradicts Lemma 2.3. Hence $CD^{(1)} = 1$. If $CD^{(2)} = 0$, then we easily see that the intersection matrix of $C + D$ is negative definite, which contradicts Lemma 2.3. Hence $CD^{(2)} = CD_6 = 1$. Let $i \in \{0, 1, 2, 3, 4, 5\}$ be the integer such that $CD_i = 1$. We consider the following subcases separately

Subcase: $i = 3$**.** The divisor $F := D_1 + D_5 + 2(D_0 + D_4) + 4(D_3 + D_6) + 8C$ defines a $\mathbb{P}^1$-fibration $\Phi_{|F|} : V \to \mathbb{P}^1$ and D_2 becomes a section of $\Phi_{|F|}$. It is then clear that $V \setminus \mathrm{Supp}(C + D) \cong \mathbb{C}^2$.

Subcase: $i \in \{1, 2, 4, 5\}$**.** The divisor $2C + D_i + D_6$ defines a $\mathbb{P}^1$-fibration $\Phi_{|2C+D_i+D_6|} : V \to \mathbb{P}^1$. Then $D^{(1)}$ has a (-3)-curve that is a fiber component of Φ. This contradicts Lemma 2.7. Therefore, this subcase does not take place.

Subcase: $i = 0$**.** The divisor $F_1 := D_1 + D_5 + 2(D_0 + D_6) + 4C$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F_1|} : V \to \mathbb{P}^1$, D_3 becomes a 2-section of Φ, D_2 becomes a section of Φ and $D - (D_2 + D_5)$ is contained in fibers of Φ. Let F_2 be the fiber of Φ containing D_4. Since $\mathrm{Supp}F_2$ consists only of D_4 and some (-1)-curves, $F_2 = E_{2,1} + D_4 + E_{2,2}$, where $E_{2,1}$ and $E_{2,2}$ are (-1)-curves and $E_{2,1}D_4 = E_{2,2}D_4 = 1$. Since D_3 is a 2-section of Φ, $D_2D_3 = 0$ and the coefficient of D_4 in F_2 equals one, we may assume that $E_{2,1}D_3 = 1$. Then

$$E_{2,1}D^{\#} \geq E_{2,1}(\alpha_3 D_3 + \alpha_4 D_4) = \alpha_3 + \alpha_4 = \frac{51}{43} > 1,$$

a contradiction. Therefore, this subcase does not take place.

Therefore, X contains $\mathbb{C}^2$ as a Zariski open subset.

3.2.5. Case (21). Let $D = \sum_{i=0}^{7} D_i$ be the decomposition of D into irreducible components such that the weighted dual graph of D is given as in Figure 3.6, where the weight of the vertex corresponding to a (-2)-curve is omitted. In this case, $\rho(V) = 9$.

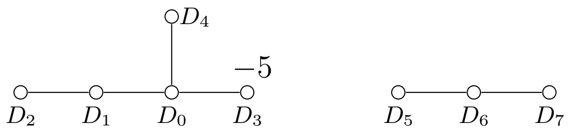


Figure 3.6

Let α_i $(i = 0, 1, \ldots, 7)$ be the coefficient of D_i in $D^{\#}$. Then

$$\alpha_0 = \frac{18}{19}, \quad \alpha_1 = \frac{12}{19}, \quad \alpha_2 = \frac{6}{19}, \quad \alpha_3 = \frac{15}{19}, \quad \alpha_4 = \frac{9}{19}, \quad \alpha_5 = \alpha_6 = \alpha_7 = 0.$$

We consider the following two cases separately.

Case 1: $CD^{(2)} = 0$**.** Then $CD^{(1)} = CD_i = 1$ for some $i \in \{0, 1, 2, 3, 4\}$. By Lemma 2.3, $i = 0$ or 1. We consider the following subcases separately.

Subcase 1-1: $i = 0$**.** The divisor $F_1 := D_1 + D_4 + 2(C + D_0)$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F_1|} : V \to \mathbb{P}^1$, D_3 becomes a 2-section of Φ, D_2 becomes a section of Φ and $D - (D_2 + D_3)$ is contained in fibers of Φ. Let F_2 be the fiber of Φ containing $D^{(2)} = D_5 + D_6 + D_7$. By Lemmas 2.7 and 2.6 (2), SuppF_2 consists either of a (-1)-curve E_2 and Supp$D^{(2)}$ or of two (-1)-curves $E_{2,1}$ and $E_{2,2}$ and Supp$D^{(2)}$. If SuppF_2 consists of a (-1)-curve E_2 and Supp$D^{(2)}$, then E_2 meets both of D_2 and D_3. Then

$$E_2 D^{\#} \geq E_2(\alpha_2 D_2 + \alpha_3 D_4) \geq \frac{21}{19} > 1,$$

which is a contradiction. Suppose SuppF_2 consists of two (-1)-curves $E_{2,1}$ and $E_{2,2}$ and Supp$D^{(2)}$. Then $F_2 = E_{2,1} + D_5 + D_6 + D_7 + E_{2,2}$. We may assume that $E_{2,2}D_2 = 1$ since D_2 is a section of Φ. Then $E_{2,2}D_3 = 0$ since $\alpha_2 + \alpha_3 > 1$. So $E_{2,1}D_3 = 2$ since D_3 is a 2-section of Φ and the coefficient of $E_{2,1}$ in F_2 equals one. Then

$$E_{2,1}D^{\#} = 2\alpha_3 = \frac{30}{19} > 1,$$

a contradiction. Therefore, this subcase does not take place.

Subcase 1-2: $i = 1$**.** The divisor $F_1 := D_0 + D_2 + 2(C + D_1)$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F_1|} : V \to \mathbb{P}^1$, D_3 and D_4 become sections of Φ and $D - (D_3 + D_4)$ is contained in fibers of Φ. Let F_2 be the fiber of Φ containing $D^{(2)} = D_5 + D_6 + D_7$. By the same argument as in Subcase 1-1, we know that SuppF_2 consists only of Supp$D^{(2)}$ and one or two (-1)-curves. The component E' of SuppF_2 meeting D_3 is a (-1)-curve. Then

$$E' D^{\#} \geq \alpha_3 E' D_3 \geq \frac{15}{19} > \frac{12}{19} = CD^{\#}.$$

This is a contradiction. Therefore, this subcase does not take place.

Therefore, Case 1 does not take place.

Case 2: $CD^{(2)} = 1$. If $CD^{(1)} = 0$, then the intersection matrix of $C + D^{(2)}$ is either negative definite or negative semi-definite. This contradicts Lemma 2.3. Hence $CD^{(1)} = 1$. Let $i \in \{0, 1, 2, 3, 4\}$ be the integer such that $CD_i = 1$.

We claim that $CD_6 = 0$. Indeed, if $CD_6 = 1$, then the divisor $G := D_5 + D_7 + 2(C + D_6)$ defines a $\mathbb{P}^1$-fibration $\Phi_{|G|} : V \to \mathbb{P}^1$, D_i becomes a 2-section of $\Phi_{|G|}$ and $D - D_i$ is contained in fibers of $\Phi_{|G|}$. We infer from Lemma 2.7 that $i = 3$. So $D_0 + D_1 + D_2 + D_4$ is contained in a fiber, say G', of $\Phi_{|G|}$. It is clear that $\#G' \geq 6$. Then

$$9 = \rho(V) \geq 2 + (\#G - 1) + (\#G' - 1) \geq 10,$$

a contradiction. Therefore, $CD_6 = 0$. We may assume that $CD_5 = 1$.

We consider the following subcases separately.

Subcase 2-1: $i = 3$. The divisor $F := D_1 + D_4 + 2(D_0 + D_3 + D_7) + 4D_6 + 6D_5 + 8C$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F|} : V \to \mathbb{P}^1$ and D_2 becomes a section of Φ. It is then clear that $V \setminus \mathrm{Supp}(C + D) \cong \mathbb{C}^2$.

Subcase 2-2: $i \in \{1, 2, 4\}$. The divisor $2C + D_i + D_5$ defines a $\mathbb{P}^1$-fibration $\Phi_{|2C+D_i+D_5|} : V \to \mathbb{P}^1$ and D_3, that is a (-5)-curve, becomes a fiber component of $\Phi_{|2C+D_i+D_5|}$. This contradicts Lemma 2.7. Therefore, this subcase does not take place.

Subcase 2-3: $i = 0$. The divisor $F_1 := 2C + D_0 + D_5$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F_1|} : V \to \mathbb{P}^1$, D_1, D_3, D_4 and D_6 become sections of Φ. Let F_2 be the fiber of Φ containing D_2. By Lemma 2.7, $\mathrm{Supp}F_2$ consists only of (-1)-curves and (-2)-curves. Since the component of $\mathrm{Supp}F_2$ meeting D_3, that is a section of Φ, must be a (-1)-curve, $\mathrm{Supp}F_2$ contains at least two (-1)-curves. By Lemma 2.6 (2), $F_2 = E_{2,1} + D_2 + E_{2,2}$, where $E_{2,1}$ and $E_{2,2}$ are (-1)-curves and $E_{2,1}D_2 = E_{2,2}D_2 = 1$. We may assume that $E_{2,1}D_3 = 1$. Then

$$E_{2,1}D^{\#} \geq E_{2,1}(\alpha_2 D_2 + \alpha_3 D_3) = \alpha_2 + \alpha_3 = \frac{21}{19} > 1.$$

This is a contradiction. Therefore, this subcase does not take place.

Therefore, X contains $\mathbb{C}^2$ as a Zariski open subset.

4. Proof of Theorem 1.1, part II

We continue the proof of Theorem 1.1. Let V, D, $D^{\#}$ and $\mathrm{MV}(V, D)$ be the same as in Section 3. In this section, we consider the remaining cases: the dual graph of D is one of (12), (14), (15), (17), (18), (20), (22), (24), (25), (26), (27) and (28) in Theorem 1.1. In these cases, we need more detailed arguments than those given in Section 3. Let $D = D^{(1)} + D^{(2)}$ be the decomposition of D into connected components such that $D^{(2)}$ is a linear chain and let $D = \sum_{i \geq 0} D_i$ be the decomposition of D

into irreducible components. We assume that D_0 is the unique branch component of $D^{(1)}$. Let α_i $(i = 0, 1, \ldots, \#D - 1)$ be the coefficient of D_i in $D^{\#}$. The values α_i are given in the following subsections.

We note that $D^{(i)}$ $(i = 1, 2)$ contains at least one curve of self-intersection number ≤ -3. By Lemma 2.4, the pair (V, D) is of the second kind. By Lemma 2.5, every curve $C \in \mathrm{MV}(V, D)$ is a (-1)-curve.

We prove some general properties for the pairs (V, D), which are used frequently in the cases treated below.

Lemma 4.1. *There exist no* (-1)*-curves meeting* D_0*.*

Proof. Suppose to the contrary that there exists a (-1)-curve E meeting D_0. We note that $\alpha_0 > 1/2$ (see Subsections 4.1∼4.12 below) and that $ED^{\#} < 1$. So $ED_0 = 1$.

Suppose that $E(D - D_0) \geq 1$. Let D_j be the component of $D - D_0$ meeting E. Then

$$ED^{\#} \geq E(\alpha_0 D_0 + \alpha_j D_j) \geq \alpha_0 + \alpha_j > 1,$$

where the last inequality can be proved by calculating α_i's (see Subsections 4.1∼4.12 below). This is a contradiction. So $E(D - D_0) = 0$.

The intersection matrix of $E + D$ is negative definite because $ED = ED_0 = 1$ and $D_0^2 \leq -3$ (see Subsections 4.1∼4.12 below). This contradicts Lemma 2.3. □

Let $C \in \mathrm{MV}(V, D)$ be a curve of $\mathrm{MV}(V, D)$. Then $X \setminus \pi_*(C)$ is a normal affine surface with only quotient singular points. So the connected component of $C + D$ supports a big divisor. Note that $CD^{(i)} \leq 1$ for $i = 1, 2$ because $|C + D + K_V| = \emptyset$.

Lemma 4.2. $CD^{(1)} = 1$.

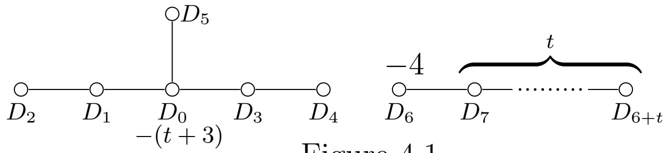
Proof. Suppose to the contrary that $CD^{(1)} = 0$. Since $CD^{(2)} = 1$ and $D^{(2)}$ is a linear chain, we infer from Lemma 2.3 that there exist a positive integer n and an effective divisor Δ such that $\mathrm{Supp}\Delta \subsetneq \mathrm{Supp}D^{(2)}$ and $nC + \Delta$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|nC+\Delta|} : V \to \mathbb{P}^1$. (See the proof of [14, Lemma 6.1].) It then follows from [13, Corollary 2.2.11.1 (p. 82)] (or [12, Corollary I.2.4.3 (p. 16)]) that $V \setminus \mathrm{Supp}(C + D)$ is affine ruled, namely, $V \setminus \mathrm{Supp}(C + D)$ contains a non-empty Zariski open subset isomorphic to $\mathbb{A}^1 \times T$, where T is a smooth curve. Hence $S := X \setminus \pi_*(C)$ is affine ruled. However, this contradicts [11, Theorem 1] because S then contains a non-cyclic quotient singular point that is the image of $D^{(1)}$ by π. □

Let C be the same as above. We will prove that $CD^{(2)} = 1$ by using case by case analysis.

From now on, we consider the remaining cases separately.

4.1. Case (15)

In this subsection, we treat the case where the weighted dual graph of D is (15) in Theorem 1.1. Let $D = \sum_{i=0}^{6+t} D_i$ be the decomposition of D into irreducible components such that the weighted dual graph of D is given as in Figure 4.1, where $D_0^2 = -(t+3)$ and the weight of the vertex corresponding to a (-2)-curve is omitted. In this case, $\rho(V) = 8 + t$.



$(t \geq 0)$

Figure 4.1.

Let α_i $(i = 0, 1, \dots, 6+t)$ be the coefficient of D_i in $D^{\#}$. Then

$$\alpha_0 = \frac{6(t+1)}{6t+7}, \quad \alpha_1 = \alpha_3 = \frac{4(t+1)}{6t+7}, \quad \alpha_2 = \alpha_4 = \frac{2(t+1)}{6t+7},$$

$$\alpha_5 = \frac{3(t+1)}{6t+7}, \quad \alpha_{6+i} = \frac{2(t+1-i)}{3t+4} \quad (i = 0, 1, \dots, t).$$

Let $C \in \mathrm{MV}(V, D)$. By Lemmas 4.1 and 4.2, $CD_0 = 0$ and $CD^{(1)} = 1$.

Claim 4.1.1. $CD^{(2)} = 1$.

Proof. Suppose to the contrary that $CD^{(2)} = 0$. Then $CD^{(1)} = CD_i = 1$ for some $i \in \{1, 2, 3, 4, 5\}$. By Lemma 2.3, we know that $i = 1$ or 3 and $t = 0$. We may assume that $i = 1$. Then the divisor $F_1 := D_3 + D_5 + 2(D_0 + D_2) + 4(C + D_1)$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F_1|} : V \to \mathbb{P}^1$, D_4 becomes a section of Φ and $D - D_4$ is contained in fibers of Φ. Let F_2 be the fiber of Φ containing $D^{(2)} = D_6$. Then $\#F_2 \geq 5$ because $D_6^2 = -4$. Then we have

$$8 = \rho(V) \geq 2 + (\#F_1 - 1) + (\#F_2 - 1) \geq 10,$$

a contradiction. □

We take $i \in \{1, 2, 3, 4, 5\}$ and $j \in \{6, 7, \dots, 6+t\}$ such that $CD_i = CD_j = 1$. By the shape of the dual graph of $D^{(1)}$, we may assume that $i \neq 3, 4$.

Claim 4.1.2. If $j = 6$, then $i = 2$ and $V \setminus \mathrm{Supp}(C + D) \cong \mathbb{C}^2$.

Proof. Assume that $i = 2$. Then there exists a positive integer n and an effective divisor Δ such that $\mathrm{Supp}\Delta = \mathrm{Supp}(D - D_4)$, $nC + \Delta$ defines a $\mathbb{P}^1$-fibration $\Phi_{|nC+\Delta|} : V \to \mathbb{P}^1$ and D_4 becomes a section of Φ. (We can wright down the divisor $nC + \Delta$ explicitly; we omit the description.) It is then clear that $V \setminus \mathrm{Supp}(C + D) \cong \mathbb{C}^2$. (See the arguments in Section 3.)

Suppose that $i \neq 2$. Then $i = 1$ or 5. If $i = 1$, then

$$CD^{\#} = \alpha_1 + \alpha_6 > \frac{4t+4}{3t+4} \geq 1,$$

which is a contradiction. If $i = 5$, then $t = 0$ because $CD^{\#} = \alpha_5 + \alpha_6 < 1$. However, this is a contradiction because the intersection matrix of $C + D$ is then negative definite. □

Claim 4.1.3. The case $j \geq 7$ does not take place.

Proof. Suppose to the contrary that $j \geq 7$. Then $t \geq 1$ and the both D_i and D_j are (-2)-curves. So the divisor $F_1 := D_i + D_j + 2C$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F_1|} : V \to \mathbb{P}^1$. By Lemma 2.7, D_0 and D_6 are horizontal components of Φ. Hence, $j = 7$ and $i \in \{1, 5\}$. We consider the following subcases separately.

Subcase 1: $i = 5$**.** Let F_2 (resp. F_3) be the fiber of Φ containing $D_1 + D_2$ (resp. $D_3 + D_4$). Then F_1, F_2 and F_3 exhaust the singular fibers of Φ. Indeed, if G is a singular fiber of Φ other than F_1, F_2 and F_3, then the component of G meeting D_0 is a (-1)-curve. This contradicts Lemma 4.1. By Lemmas 2.7 and 2.6 (2), we know that $F_2 = E_{2,1} + D_1 + D_2 + E_{2,2}$ and $F_3 = E_{3,1} + D_3 + D_4 + E_{3,2}$, where $E_{2,1}$, $E_{2,2}$, $E_{3,1}$ and $E_{3,2}$ are (-1)-curves and $E_{2,1}D_1 = E_{2,2}D_2 = E_{3,1}D_3 = E_{3,2}D_4 = 1$. Since

$$8 + t = \rho(V) = 2 + (\#F_1 - 1) + (\#F_2 - 1) + (\#F_3 - 1) = 10,$$

$t = 2$. So D_8 becomes a section of Φ.

Since the divisor $E_{2,1} + D$ supports a big divisor by Lemma 2.3, $E_{2,1}$ meets at least one of D_6 and D_8. Then we have

$$-E_{2,1}(D^{\#} + K_V) \leq 1 - (\alpha_1 + \alpha_8) = 1 - \left(\frac{12}{19} + \frac{1}{5}\right) = \frac{16}{95}$$

and

$$-C(D^{\#} + K_V) = 1 - (\alpha_5 + \alpha_7) = 1 - \left(\frac{9}{19} + \frac{2}{5}\right) = \frac{17}{95}.$$

This contradicts $C \in \mathrm{MV}(V, D)$. Therefore, this subcase does not take place.

Subcase 2: $i = 1$**.** Then D_0, D_2 and D_6 become sections of Φ. Let F_2 (resp. F_3) be the fiber of Φ containing $D_3 + D_4$ (resp. D_5). By using the argument as in Subcase 1, we know that F_1, F_2 and F_3 exhaust the singular fibers of Φ, $\#F_2 = 4$, $\#F_3 = 3$ and $t = 1$. The fiber F_2 is expressed as $F_2 = E_{2,1} + D_3 + D_4 + E_{2,2}$, where $E_{2,1}$ and $E_{2,2}$ are (-1)-curves and $E_{2,1}D_3 = E_{2,2}D_4 = 1$.

Since the divisor $E_{2,1} + D$ supports a big divisor by Lemma 2.3, $E_{2,1}$ meets at least one of D_2 and D_6. If $E_{2,1}D_6 = 1$, then $-E_{2,1}(D^{\#} + K_V) < -C(D^{\#} + K_V)$, which contradicts $C \in \mathrm{MV}(V, D)$. If $E_{2,1}D_6 = 0$ and $E_{2,1}D_2 = 1$, then

$$-E_{2,1}(D^{\#} + K_V) = 1 - (\alpha_2 + \alpha_3) = \frac{1}{13}$$

since $t = 1$. On the other hand,

$$-C(D^{\#} + K_V) = 1 - (\alpha_1 + \alpha_7) = \frac{9}{91} > -E_{2,1}(D^{\#} + K_V),$$

which is a contradiction. Therefore, this subcase does not take place.

The proof of Claim 4.1.3 is thus completed. □

Therefore, X contains $\mathbb{C}^2$ as a Zariski open subset.

4.2. Case (18)

In this subsection, we treat the case where the weighted dual graph of D is (18) in Theorem 1.1. Let $D = \sum_{i=0}^{7+t} D_i$ be the decomposition of D into irreducible components such that the weighted dual graph of D is given as in Figure 4.2, where $D_0^2 = -(t+3)$ and the weight of the vertex corresponding to a (-2)-curve is omitted. In this case, $\rho(V) = 9 + t$.

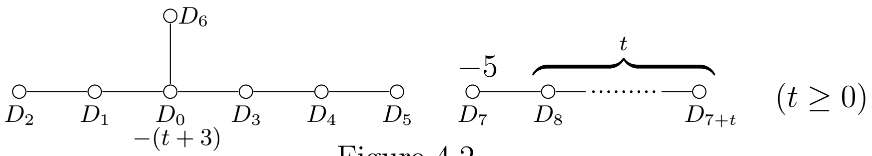


Figure 4.2.

Let α_i $(i = 0, 1, \ldots, 7+t)$ be the coefficient of D_i in $D^{\#}$. Then

$$\alpha_0 = \frac{12(t+1)}{12t+13}, \quad \alpha_1 = \frac{8(t+1)}{12t+13}, \quad \alpha_2 = \frac{4(t+1)}{12t+13}, \quad \alpha_3 = \frac{9(t+1)}{12t+13},$$

$$\alpha_4 = \alpha_6 = \frac{6(t+1)}{12t+13}, \quad \alpha_5 = \frac{3(t+1)}{12t+13}, \quad \alpha_{7+i} = \frac{3(t+1-i)}{4t+5} \quad (i = 0, 1, \ldots, t).$$

Let $C \in \mathrm{MV}(V, D)$. By Lemmas 4.1 and 4.2, $CD_0 = 0$ and $CD^{(1)} = 1$.

Claim 4.2.1. $CD^{(2)} = 1$.

Proof. Suppose to the contrary that $CD^{(2)} = 0$. Then $CD^{(1)} = CD_i = 1$ for some $i \in \{1, 2, 3, 4, 5, 6\}$. By Lemma 2.3, we know that $i \in \{1, 3, 4\}$. We consider the following subcases separately.

Subcase 1: $i = 4$. The divisor $D_3 + D_5 + 2(C + D_4)$ defines a $\mathbb{P}^1$-fibration $\Phi_{|D_3+D_5+2(C+D_4)|} : V \to \mathbb{P}^1$. Then D_7, that is a (-5)-curve, is a fiber component of $\Phi_{|D_3+D_5+2(C+D_4)|}$. This contradicts Lemma 2.7.

Subcase 2: $i = 1$. By Lemma 2.3, we know that $t = 0$. So the divisor $F_1 := D_3 + D_6 + 2(D_0 + D_2) + 4(C + D_1)$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F_1|} : V \to \mathbb{P}^1$. Let F_2 be the fiber of Φ containing $D^{(2)} = D_7$. Then $\#F_2 \geq 6$ because $D_7^2 = -5$. Then we have

$$9 = \rho(V) \geq 2 + (\#F_1 - 1) + (\#F_2 - 1) = 6 + \#F_2 \geq 12,$$

which is a contradiction.

Subcase 3: $i = 3$. By Lemma 2.3, we know that $t \le 1$. If $t = 1$ (resp. $t = 0$), then the divisor $F := D_1 + D_6 + 2(D_0 + D_5) + 4D_4 + 6(C + D_3)$ (resp. $F = D_0 + D_5 + 2D_4 + 3(C + D_1)$) defines a $\mathbb{P}^1$-fibration $\Phi_{|F|} : V \to \mathbb{P}^1$ and $D^{(2)}$ is contained in a fiber of $\Phi_{|F|}$. By using the same argument as in Subcase 2, we derive a contradiction.

The proof of Claim 4.2.1 is thus completed. □

We take $i \in \{1, 2, \ldots, 6\}$ and $j \in \{7, 8, \ldots, 7 + t\}$ such that $CD_i = CD_j = 1$.

Claim 4.2.2. If $j = 7$, then $i = 5$ and $V \setminus \mathrm{Supp}(C + D) \cong \mathbb{C}^2$.

Proof. Assume that $i = 5$. Then there exists a positive integer n and an effective divisor Δ such that $\mathrm{Supp}\Delta = \mathrm{Supp}(D - D_2)$, $nC + \Delta$ defines a $\mathbb{P}^1$-fibration $\Phi_{|nC+\Delta|} : V \to \mathbb{P}^1$ and D_2 becomes a section of $\Phi_{|nC+\Delta|}$. (We can wright down the divisor $nC + \Delta$ explicitly; we omit the description.) It is then clear that $V \setminus \mathrm{Supp}(C + D) \cong \mathbb{C}^2$. (See the arguments as in Section 3.)

Suppose that $i \neq 5$. Since

$$\alpha_4 + \alpha_7 = \alpha_6 + \alpha_7 > \frac{2(t+1)}{4t+5} + \frac{3(t+1)}{4t+5} \ge 1,$$

we have $i = 2$. Further, since $CD^{\#} = \alpha_2 + \alpha_7 < 1$, we have $t = 0$. Then the intersection matrix of $C + D$ is negative definite, which contradicts Lemma 2.3. □

Claim 4.2.3. The case $j \ge 8$ does not take place.

Proof. Suppose to the contrary that $j \ge 8$. Then $t \ge 1$ and the both D_i and D_j are (-2)-curves. So the divisor $F_1 := D_i + D_j + 2C$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F_1|} : V \to \mathbb{P}^1$. By Lemma 2.7, D_0 and D_7 are horizontal components of Φ. Hence, $j = 8$ and $i \in \{1, 3, 6\}$. We consider the following subcases separately.

Subcase 1: $i = 6$. Let F_2 (resp. F_3) be the fiber of Φ containing $D_1 + D_2$ (resp. $D_3 + D_4 + D_5$). Then F_1, F_2 and F_3 exhaust the singular fibers of Φ. Indeed, if G is a singular fiber of Φ other than F_1, F_2 and F_3, then the component of G meeting D_0 is a (-1)-curve. This contradicts Lemma 4.1. By Lemmas 2.7 and 2.6 (2), we know that $\#F_2 = 4$ and $\#F_3 = 4$ or 5. So we have

$$9 + t = \rho(V) = 2 + (\#F_1 - 1) + (\#F_2 - 1) + (\#F_3 - 1) = 10 \quad \text{or} \quad 11.$$

If $\#F_3 = 4$, then we infer from Lemma 2.6 (2) that $F_3 = D_3 + D_5 + 2(E_3 + D_4)$, where E_3 is a (-1)-curve and $E_3 D_4 = 1$. Since D_7 is a section of Φ and $D_7(D_3 + D_4 + D_5) = 0$, E_3 meets D_7. This is a contradiction. Hence $\#F_3 = 5$ and $\rho(V) = 11$. In particular, $t = 2$ and D_9 becomes a section of Φ.

Since $\#F_3 = 5$ and by Lemma 2.6 (2), we know that $F_3 = E_{3,1}+D_3+D_4+D_5+E_{3,2}$, where $E_{3,1}$ and $E_{3,2}$ are (-1)-curves and $E_{3,1}D_3 = E_{3,2}D_5 = 1$. Then $E_{3,1}D^{(1)} = E_{3,2}D^{(1)} = 1$. By Lemma 2.3, we know that $E_{3,1}$ meets at least one of D_7 and D_9. So we have

$$-E_{3,1}(D^{\#} + K_V) = 1 - E_{3,1}D^{\#} \leq 1 - (\alpha_3 + \alpha_9) = 1 - \left(\frac{27}{37} + \frac{3}{13}\right) = \frac{19}{481}.$$

On the other hand,

$$-C(D^{\#} + K_V) = 1 - (\alpha_6 + \alpha_8) = 1 - \left(\frac{12}{37} + \frac{6}{13}\right) = \frac{103}{481} > -E_{3,1}(D^{\#} + K_V).$$

This is a contradiction. Therefore, this subcase does not take place.

Subcase 2: $i = 1$**.** Then D_0, D_2 and D_7 become sections of Φ. Let F_2 (resp. F_3) be the fiber of Φ containing $D_3 + D_4 + D_5$ (resp. D_6). By using the argument as in Subcase 1, we know that F_1, F_2 and F_3 exhaust the singular fibers of Φ, $\#F_2 = 4$ or 5 and $\#F_3 = 3$. Moreover, we know that $\#F_2 = 5$ because D_7 is a section of Φ and the component of SuppF_2 meeting D_7 is a (-1)-curve. So

$$9 + t = \rho(V) = 2 + (\#F_1 - 1) + (\#F_2 - 1) + (\#F_3 - 1) = 10$$

and hence $t = 1$. We have $F_2 = E_{2,1} + D_3 + D_4 + D_5 + E_{2,2}$, where $E_{2,1}$ and $E_{2,2}$ are (-1)-curves and $E_{2,1}D_3 = E_{2,2}D_5 = 1$. By Lemma 2.3, $E_{2,1}$ meets at least one of D_2 and D_7. However, this is a contradiction because $\alpha_2 + \alpha_3 \geq 1$ and $\alpha_3 + \alpha_7 > 1$. Therefore, this subcase does not take place.

Subcase 3: $i = 3$**.** Then D_0, D_4 and D_7 become sections of Φ. Let F_2 (resp. F_3) be the fiber of Φ containing $D_1 + D_2$ (resp. D_6). By using the argument as in Subcase 1, we know that F_1, F_2 and F_3 exhaust the singular fibers of Φ, $\#F_2 = 4$ and $\#F_3 = 3$. So

$$9 + t = \rho(V) = 2 + (\#F_1 - 1) + (\#F_2 - 1) + (\#F_3 - 1) = 9.$$

This contradicts $t \geq 1$. Therefore, this subcase does not take place.

The proof of Claim 4.2.3 is thus completed. □

Theorefore, X contains $\mathbb{C}^2$ as a Zariski open subset.

4.3. Case (28)

In this subsection, we treat the case where the weighted dual graph of D is (28) in Theorem 1.1. Let $D = \sum_{i=0}^{8+t} D_i$ be the decomposition of D into irreducible components such that the weighted dual graph of D is given as in Figure 4.3, where $D_0^2 = -(t+3)$ and the weight of the vertex corresponding to a (-2)-curve is omitted. In this case, $\rho(V) = 10 + t$.

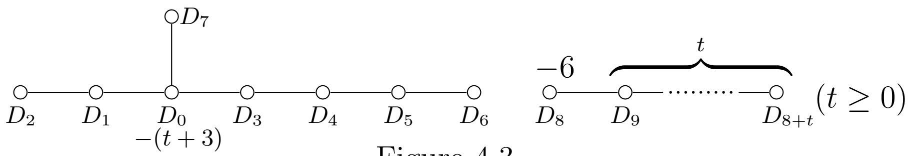


Figure 4.3.

Let α_i $(i = 0, 1, \ldots, 8+t)$ be the coefficient of D_i in $D^{\#}$. Then

$$\alpha_0 = \frac{30(t+1)}{30t+31}, \quad \alpha_1 = \frac{20(t+1)}{30t+31}, \quad \alpha_2 = \frac{10(t+1)}{30t+31}, \quad \alpha_3 = \frac{24(t+1)}{30t+31},$$

$$\alpha_4 = \frac{18(t+1)}{30t+31}, \quad \alpha_5 = \frac{12(t+1)}{30t+31}, \quad \alpha_6 = \frac{6(t+1)}{30t+31}, \quad \alpha_7 = \frac{15(t+1)}{30t+31},$$

$$\alpha_{8+i} = \frac{4(t+1-i)}{5t+6} \quad (i = 0, 1, \ldots, t).$$

Let $C \in \mathrm{MV}(V, D)$. By Lemmas 4.1 and 4.2, $CD_0 = 0$ and $CD^{(1)} = 1$.

Claim 4.3.1. $CD^{(2)} = 1$.

Proof. Suppose to the contrary that $CD^{(2)} = 0$. Then $CD^{(1)} = CD_i = 1$ for some $i \in \{1, 2, 3, 4, 5, 6\}$. By Lemma 2.3, we know that $i \in \{1, 3, 4, 5\}$. We consider the following subcases separately.

Subcase 1: $i = 4$ **or** 5. The divisor $D_{i-1} + D_{i+1} + 2(C + D_i)$ defines a $\mathbb{P}^1$-fibration $\Phi_{|D_{i-1}+D_{i+1}+2(C+D_i)|} : V \to \mathbb{P}^1$. Then D_8, that is a (-6)-curve, is a fiber component of $\Phi_{|D_{i-1}+D_{i+1}+2(C+D_i)|}$. This contradicts Lemma 2.7.

Subcase 2: $i = 1$. By Lemma 2.3, we know that $t = 0$. So the divisor $F_1 := D_3 + D_7 + 2(D_0 + D_2) + 4(C + D_1)$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F_1|} : V \to \mathbb{P}^1$. Let F_2 be the fiber of Φ containing $D^{(2)} = D_8$. Then $\#F_2 \geq 7$ because $D_8^2 = -6$. So we have

$$10 = \rho(V) \geq 2 + (\#F_1 - 1) + (\#F_2 - 1) = 6 + \#F_2 \geq 13,$$

which is a contradiction.

Subcase 3: $i = 3$. By Lemma 2.3, we know that $t \leq 2$. If $t = 0$ (resp. $t = 1$, $t = 2$), then the divisor $F_1 := D_1 + D_5 + 2D_4 + 3(C + D_3)$ (resp. $F_1 = D_0 + D_6 + 2D_5 + 3D_4 + 4(C + D_3)$, $F_1 = D_1 + D_7 + 2(D_0 + D_6) + 4D_5 + 6D_4 + 8(C + D_3)$) defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F_1|} : V \to \mathbb{P}^1$ and $D^{(2)}$ is contained in a fiber of Φ. By using the same argument as in Subcase 2, we derive a contradiction.

The proof of Claim 4.3.1 is thus completed. $\square$

We take $i \in \{1, 2, \ldots, 7\}$ and $j \in \{8, 9, \ldots, 8+t\}$ such that $CD_i = CD_j = 1$.

Claim 4.3.2. If $j = 8$, then $i = 6$ and $V \setminus \mathrm{Supp}(C + D) \cong \mathbb{C}^2$.

Proof. Assume that $i = 6$. Then there exists a positive integer n and an effective divisor Δ such that $\mathrm{Supp}\Delta = \mathrm{Supp}(D - D_2)$, $nC + \Delta$ defines a $\mathbb{P}^1$-fibration $\Phi_{|nC+\Delta|} : V \to \mathbb{P}^1$ and D_2 becomes a section of $\Phi_{|nC+\Delta|}$. (We can wright down the divisor $nC + \Delta$ explicitly; we omit the description.) It is then clear that $V \setminus \mathrm{Supp}(C + D) \cong \mathbb{C}^2$. (See the arguments as in Section 3.)

Suppose that $i \neq 6$. Since

$$\alpha_5 + \alpha_8 > \frac{2(t+1)}{5t+6} + \frac{4(t+1)}{5t+6} \geq 1,$$

we have $i = 2$. Further, since $CD^{\#} = \alpha_2 + \alpha_8 < 1$, we have $t = 0$. Then the intersection matrix of $C + D$ is negative definite, which contradicts Lemma 2.3. □

Claim 4.3.3. The case $j \geq 9$ does not take place.

Proof. Suppose to the contrary that $j \geq 9$. Then $t \geq 1$ and the both D_i and D_j are (-2)-curves. So the divisor $F_1 := D_i + D_j + 2C$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F_1|} : V \to \mathbb{P}^1$. By Lemma 2.7, D_0 and D_8 are horizontal components of Φ. Hence, $j = 9$ and $i \in \{1, 3, 7\}$. In particular, D_0 and D_8 are sections of Φ. We consider the following subcases separately.

Subcase 1: $i = 7$**.** Let F_2 (resp. F_3) be the fiber of Φ containing $D_1 + D_2$ (resp. $D_3 + D_4 + D_5 + D_6$). Then F_1, F_2 and F_3 exhaust the singular fibers of Φ. Indeed, if G is a singular fiber of Φ other than F_1, F_2 and F_3, then the component of G meeting D_0 is a (-1)-curve. This contradicts Lemma 4.1. By Lemmas 2.7 and 2.6 (2), we know that $\#F_2 = 4$ and $\#F_3 = 6$. Since

$$10 + t = \rho(V) = 2 + (\#F_1 - 1) + (\#F_2 - 1) + (\#F_3 - 1) = 12,$$

we have $t = 2$. In particular, D_{10} becomes a section of Φ.

Since $\#F_3 = 6$ and by Lemma 2.6 (2), we know that $F_3 = E_{3,1} + D_3 + D_4 + D_5 + D_6 + E_{3,2}$, where $E_{3,1}$ and $E_{3,2}$ are (-1)-curves and $E_{3,1}D_3 = E_{3,2}D_6 = 1$. Then $E_{3,1}D^{(1)} = E_{3,2}D^{(1)} = 1$. Since $E_{3,1} \in \mathrm{MV}(V, D)$ by Lemma 2.7, we know that $E_{3,1}$ meets at least one of D_8 and D_{10}. So we have

$$E_{3,1}D^{\#} \geq \alpha_3 + \alpha_{10} = \frac{72}{91} + \frac{1}{4} > 1,$$

which is a contradiction. Therefore, this subcase does not take place.

Subcase 2: $i = 1$**.** Then D_2 becomes a section of Φ. Let F_2 (resp. F_3) be the fiber of Φ containing $D_3 + D_4 + D_5 + D_6$ (resp. D_7). By using the argument as in Subcase 1, we know that F_1, F_2 and F_3 exhaust the singular fibers of Φ, $\#F_2 = 6$ and $\#F_3 = 3$. Then

$$10 + t = \rho(V) = 2 + (\#F_1 - 1) + (\#F_2 - 1) + (\#F_3 - 1) = 11$$

and hence $t = 1$. We know that $F_2 = E_{2,1} + D_3 + D_4 + D_5 + D_6 + E_{2,2}$, where $E_{2,1}$ and $E_{2,2}$ are (-1)-curves and $E_{2,1}D_3 = E_{2,2}D_6 = 1$. Since $\alpha_2 + \alpha_3$, $\alpha_3 + \alpha_8 > 1$, $E_{2,1}$ meets none of D_2 and D_8. Then $E_{2,2}$ meets both of D_2 and D_8 and so

$$E_{2,2}D^{\#} \geq \alpha_2 + \alpha_6 + \alpha_8 = \frac{32}{61} + \frac{8}{11} > 1,$$

which is a contradiction. Therefore, this subcase does not take place.

Subcase 3: $i = 3$. Then D_4 becomes a section of Φ. Let F_2 (resp. F_3) be the fiber of Φ containing $D_1 + D_2$ (resp. D_7). By using the argument as in Subcase 1, we know that F_1, F_2 and F_3 exhaust the singular fibers of Φ, $\#F_2 = 4$ and $\#F_3 = 3$. Then

$$10 + t = \rho(V) = 2 + (\#F_1 - 1) + (\#F_2 - 1) + (\#F_3 - 1) = 9,$$

which is a contradiction. Therefore, this subcase does not take place.

The proof of Claim 4.3.3 is thus completed. □

Theorefore, X contains $\mathbb{C}^2$ as a Zariski open subset.

4.4. Case (27)

In this subsection, we treat the case where the weighted dual graph of D is (27) in Theorem 1.1. Let $D = \sum_{i=0}^{6+t} D_i$ be the decomposition of D into irreducible components such that the weighted dual graph of D is given as in Figure 4.4, where $D_0^2 = -(t+3)$ and the weight of the vertex corresponding to a (-2)-curve is omitted. In this case, $\rho(V) = 8 + t$.

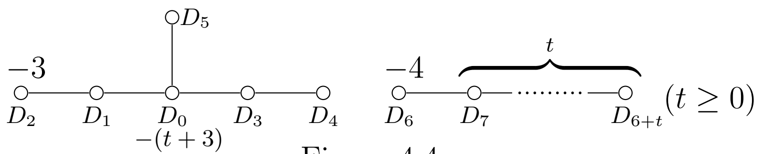


Figure 4.4.

Let α_i $(i = 0, 1, \ldots, 6+t)$ be the coefficient of D_i in $D^{\#}$. Then

$$\alpha_0 = \frac{30t+36}{30t+37}, \quad \alpha_1 = \frac{24t+29}{30t+37}, \quad \alpha_2 = \frac{18t+22}{30t+37}, \quad \alpha_3 = \frac{20t+24}{30t+37},$$

$$\alpha_4 = \frac{10t+12}{30t+37}, \quad \alpha_5 = \frac{15t+18}{30t+37}, \quad \alpha_{6+i} = \frac{2(t+1-i)}{3t+4} \quad (i = 0, 1, \ldots, t).$$

Let $C \in \mathrm{MV}(V, D)$. By Lemmas 4.1 and 4.2, $CD_0 = 0$ and $CD^{(1)} = 1$.

Claim 4.4.1. $CD^{(2)} = 1$.

Proof. Suppose to the contrary that $CD^{(2)} = 0$. Then $CD^{(1)} = CD_i = 1$ for some $i \in \{1, 2, 3, 4, 5\}$. By Lemma 2.3, we know that $i = 3$ and $t = 0$. So the divisor $F_1 := D_1 + D_5 + 2(D_0 + D_4) + 4(C + D_3)$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F_1|} : V \to \mathbb{P}^1$, D_2 becomes a section of Φ and $D - D_2$ is contained in fibers of Φ. Let F_2 be the fiber of Φ containing $D^{(2)} = D_6$. Then $\#F_2 \geq 5$ because $D_6^2 = -4$. So we have

$$8 = \rho(V) \geq 2 + (\#F_1 - 1) + (\#F_2 - 1) = 6 + \#F_2 \geq 11,$$

which is a contradiction. □

We take $i \in \{1, 2, \ldots, 5\}$ and $j \in \{6, 7, \ldots, 6 + t\}$ such that $CD_i = CD_j = 1$.

Claim 4.4.2. If $j = 6$, then $i = 4$ and $V \setminus \mathrm{Supp}(C + D) \cong \mathbb{C}^2$.

Proof. Assume that $i = 4$. Then there exists a positive integer n and an effective divisor Δ such that $\mathrm{Supp}\Delta = \mathrm{Supp}(D - D_2)$, $nC + \Delta$ defines a $\mathbb{P}^1$-fibration $\Phi_{|nC+\Delta|} : V \to \mathbb{P}^1$ and D_2 becomes a section of $\Phi_{|nC+\Delta|}$. (We can wright down the divisor $nC + \Delta$ explicitly; we omit the description.) It is then clear that $V \setminus \mathrm{Supp}(C + D) \cong \mathbb{C}^2$. (See the arguments as in Section 3.)

Suppose that $i \neq 6$. Since $\alpha_2 + \alpha_6 > 1$, we have $i = 5$. Further, since $CD^{\#} = \alpha_5 + \alpha_6 < 1$, we have $t = 0$. Then the intersection matrix of $C + D$ is negative definite, which contradicts Lemma 2.3. □

Claim 4.4.3. The case $j \geq 7$ does not take place.

Proof. Suppose that $j \geq 7$. Then D_j is a (-2)-curve and $t \geq 1$. We consider the following subcases separately.

Subcase 1: $i \in \{1, 3, 4, 5\}$. Then D_i is a (-2)-curve and so the divisor $D_i + D_j + 2C$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|D_i+D_j+2C|} : V \to \mathbb{P}^1$. Since D_0, D_2 and D_6 become horizontal components of Φ by Lemma 2.7, we know that $i = 1$ and $j = 7$. Then

$$CD^{\#} = \alpha_1 + \alpha_7 > \frac{24t + 29}{30t + 40} + \frac{2t}{3t + 4} > 1,$$

which is a contradiction. Therefore, this subcase does not take place.

Subcase 2: $i = 2$ and $t = 1$. Then $j = 7$ and so the intersection matrix of $C + D$ is negative definite. This contradicts Lemma 2.3. Therefore, this subcase does not take place.

Subcase 3: $i = 2$, $t \geq 3$ and $8 \leq j \leq 5 + t$. The divisor $D_{j-1} + D_{j+1} + 2(C + D_j)$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|D_{j-1}+D_{j+1}+2(C+D_j)|} : V \to \mathbb{P}^1$. Then D_0, that is a $(-t-3)$-curve, is a fiber component of Φ. This contradicts Lemma 2.7 since $t+3 \geq 6$. Therefore, this subcase does not take place.

Subcase 4: $i = 2$, $t \geq 2$ and $j \in \{7, 6 + t\}$. If $j = 7$ (resp. $j = 6 + t$), then the divisor $F_1 := D_2 + D_8 + 2D_7 + 3C$ (resp. $F_1 := D_2 + D_{5+t} + 2D_{6+t} + 3C$)

defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F_1|} : V \to \mathbb{P}^1$. Then D_0, that is a $(-t-3)$-curve, is a fiber component of Φ. This contradicts Lemma 2.8 because $D_0^2 = -(t+3) \le -5$. Therefore, this subcase does not take place.

The proof of Claim 4.4.3 is thus completed. □

Theorefore, X contains $\mathbb{C}^2$ as a Zariski open subset.

4.5. Case (25)

In this subsection, we treat the case where the weighted dual graph of D is (25) in Theorem 1.1. Let $D = \sum_{i=0}^{6+t} D_i$ be the decomposition of D into irreducible components such that the weighted dual graph of D is given as in Figure 4.5, where $D_0^2 = -(t+3)$ and the weight of the vertex corresponding to a (-2)-curve is omitted. In this case, $\rho(V) = 8 + t$.

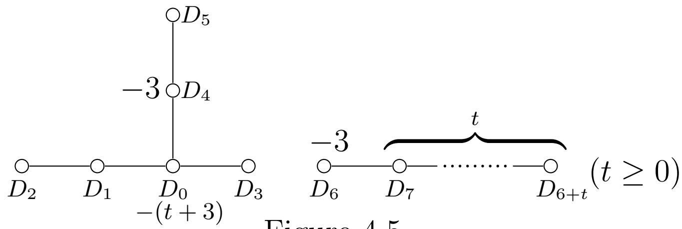


Figure 4.5.

Let α_i $(i = 0, 1, \ldots, 6+t)$ be the coefficient of D_i in $D^{\#}$. Then

$$\alpha_0 = \frac{30t+42}{30t+43}, \quad \alpha_1 = \frac{20t+28}{30t+43}, \quad \alpha_2 = \frac{10t+14}{30t+43}, \quad \alpha_3 = \frac{15t+21}{30t+43},$$

$$\alpha_4 = \frac{24t+34}{30t+43}, \quad \alpha_5 = \frac{12t+17}{30t+43}, \quad \alpha_{6+i} = \frac{t+1-i}{2t+3} \quad (i = 0, 1, \ldots, t).$$

Let $C \in \mathrm{MV}(V, D)$. By Lemmas 4.1 and 4.2, $CD_0 = 0$ and $CD^{(1)} = 1$.

Claim 4.5.1. $CD^{(2)} = 1$.

Proof. Suppose to the contrary that $CD^{(2)} = 0$. Then $CD^{(1)} = 1$. Since $D_0^2 = -(t+3) \le -3$, we easily see that the intersection matrix of $C + D^{(1)}$ is negative definite. This contradicts Lemma 2.3. □

We take $i \in \{1, 2, \ldots, 5\}$ and $j \in \{6, 7, \ldots, 6+t\}$ such that $CD_i = CD_j = 1$.

Claim 4.5.2. If $j = 6$, then $i = 3$ and $V \setminus \mathrm{Supp}(C+D) \cong \mathbb{C}^2$.

Proof. Assume that $i = 3$. Then there exists a positive integer n and an effective divisor Δ such that $\mathrm{Supp}\Delta = \mathrm{Supp}(D - D_5)$, $nC + \Delta$ defines a $\mathbb{P}^1$-fibration $\Phi_{|nC+\Delta|} : V \to \mathbb{P}^1$ and D_5 becomes a section of $\Phi_{|nC+\Delta|}$. (We can wright down the divisor $nC+\Delta$ explicitly; we omit the description.) It is then clear that $V \setminus \mathrm{Supp}(C+D) \cong \mathbb{C}^2$. (See the arguments as in Section 3.)

Suppose that $i \neq 3$. Since $CD^{\#} = \alpha_i + \alpha_6 < 1$, $i \neq 4$. We consider the following subcases separately.

Subcase 1: $i = 1$. Since $CD^{\#} = \alpha_1 + \alpha_6 < 1$, $t = 0$. So the divisor $F_1 := D_2 + D_6 + 2D_1 + 3C$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F_1|} : V \to \mathbb{P}^1$, D_0 becomes a 2-section of Φ and $D - D_0$ is contained in fibers of Φ. Let F_2 be the fiber of Φ containing $D_4 + D_5$. Since $\mathrm{Supp}(D_4 + D_5) \subset \mathrm{Supp}F_2$, we see that $\#F_2 \geq 5$. Then we have

$$8 = \rho(V) \geq 2 + (\#F_1 - 1) + (\#F_2 - 1) \geq 8.$$

Hence, F_1 and F_2 exhaust the singular fibers of Φ and $\#F_2 = 5$. Since D_3 is a fiber component of Φ, it is a component of $\mathrm{Supp}F_2$. So there exists a (-1)-curve, say $E_{2,1}$, of $\mathrm{Supp}F_2$ such that $E_{2,1}D_5 = E_{2,1}(D_3 + D_4) = 1$. In particular, $E_{2,1}D_4 = 1$. Let $E_{2,2}$ be another (-1)-curve of $\mathrm{Supp}F_2$, here we note that $\#F_2 = 5$ and $\mathrm{Supp}F_2$ consists of D_3,D_4, D_5, $E_{2,1}$ and $E_{2,2}$. Then $E_{2,2}D = E_{2,2}(D_4 + D_5) = 1$ and so the intersection matrix of $E_{2,2} + D$ is negative definite. This contradicts Lemma 2.3. Therefore, this subcase does not take place.

Subcase 2: $i = 5$. By Lemma 2.3, we know that $t \geq 1$. So the divisor $F_1 := D_4+D_7+2D_6+3D_5+5C$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F_1|} : V \to \mathbb{P}^1$ and D_0 becomes a section of Φ. Further, if $t \geq 2$, then D_8 is a section of Φ and $D - (D_0 + D_8)$ is contained in fibers of Φ. Let F_2 (resp. F_3) be the fiber of Φ containing $D_1 + D_2$ (resp. D_3). Then F_1, F_2 and F_3 exhaust the singular fibers of Φ. Indeed, if G is a singular fiber of Φ other than F_1, F_2 and F_3, then the component of G meeting D_0 is a (-1)-curve. This contradicts Lemma 4.1. Since $D - (D_0 + D_4 + D_6)$ consists only of (-1)-curves and (-2)-curves, we infer from Lemma 2.6 (2) that $\#F_2 = 4$ and $\#F_3 = 3$. Then

$$8 + t = \rho(V) = 2 + (\#F_1 - 1) + (\#F_2 - 1) + (\#F_3 - 1) = 11$$

and so $t = 3$. Furthermore, $F_2 = E_{2,1} + D_1 + D_2 + D_{2,2}$, where $E_{2,1}$ and $E_{2,2}$ are (-1)-curves and $E_{2,1}D_1 = E_{2,2}D_2 = 1$. Then either $E_{2,1}$ or $E_{2,2}$ does not meet D_8, a section of Φ. So $E_{2,k}D = E_{2,k}D^{(1)} = 1$ for $k = 1$ or 2. This contradicts Lemma 2.3 because the divisor $E_{2,k} + D$ has negative definite intersection matrix for $k = 1$ or 2. Therefore, this subcase does not take place.

Subcase 3: $i = 2$. The divisor $F_1 := D_1 + D_6 + 2D_2 + 3C$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F_1|} : V \to \mathbb{P}^1$ and D_0 becomes sections of Φ. Further, if $t \geq 1$, then D_7 is a section of Φ and $D - (D_0 + D_7)$ is contained in fibers of Φ. Let F_2 (resp. F_3) be the fiber of Φ containing D_3 (resp. $D_4 + D_5$). By using the same argument as in Subcase 2, we know that F_1, F_2 and F_3 exhaust the singular fibers of Φ. Since $\mathrm{Supp}F_2$ consists only of (-1)-curves and (-2)-curves, we infer form Lemma 2.6 (2) that $\#F_2 = 3$. Further, since $\mathrm{Supp}F_3$ contains D_4 and D_5, we know that $\#F_3 \geq 5$.

Then

$$8 + t = \rho(V) = 2 + (\#F_1 - 1) + (\#F_2 - 1) + (\#F_3 - 1) = 6 + \#F_3 \geq 11$$

and so $t \geq 3$. We note that $D_8 + \cdots + D_{6+t}$ is contained in a fiber of Φ. Since $\#(D_8 + \cdots + D_{6+t}) \geq 2$, $D_8 + \cdots + D_{6+t}$ is contained in $\mathrm{Supp}F_3$. Since D_3 is then a unique (-2)-curve in $\mathrm{Supp}F_2$, we know that $F_2 = E_{2,1} + D_3 + E_{2,2}$, where $E_{2,1}$ and $E_{2,2}$ are (-1)-curves and $E_{2,1}D_3 = E_{2,2}D_3 = 1$. We may assume that $E_{2,2}D_7 = 1$ because D_7 is a section of Φ. Then $E_{2,1}D = E_{2,1}D_3 = 1$. This contradicts Lemma 2.3. Therefore, this subcase does not take place.

The proof of Claim 4.5.2 is thus completed. □

Claim 4.5.3. The case $j \geq 7$ does not take place.

Proof. Suppose to the contrary that $j \geq 7$. Then $t \geq 1$ and D_j is a (-2)-curve. We consider the following subcases separaely.

Subcase 1: $i \in \{1, 2, 3, 5\}$. Then D_i is a (-2)-curve and so the divisor $D_i + D_j + 2C$ defines a $\mathbb{P}^1$-fibration $\Phi_{|D_i+D_j+2C|} : V \to \mathbb{P}^1$. Then D_0 or D_4 is a fiber component of $\Phi_{|D_i+D_j+2C|}$. This contradicts Lemma 2.7. Therefore, this subcase does not take place.

Subcase 2: $i = 4$, $t \geq 3$ and $8 \leq j \leq 5 + t$. The divisor $D_{j-1} + D_{j+1} + 2(C + D_j)$ defines a $\mathbb{P}^1$-fibration $\Phi_{|D_{j-1}+D_{j+1}+2(C+D_j)|} : V \to \mathbb{P}^1$. Then D_0, that is a $(-t-3)$-curve, is a fiber component of $\Phi_{|D_{j-1}+D_{j+1}+2(C+D_j)|}$. This contradicts Lemma 2.7 since $t + 3 \geq 3$. Therefore, this subcase does not take place.

Subcase 3: $i = 4$, $t = 1$ and $j = 7$. The divisor $F_1 := D_5 + D_6 + 2D_4 + 3D_6 + 5C$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F_1|} : V \to \mathbb{P}^1$, D_0 becomes a 2-section of Φ and $D - D_0$ is contained in fibers of Φ. Let F_2 (resp. F_3) be the fiber of Φ containing $D_1 + D_2$ (resp. D_3). Then $\mathrm{Supp}F_2$ and $\mathrm{Supp}F_3$ consist only of (-1)-curves and (-2)-curves. We infer from Lemma 2.6 (2) that $F_2 \neq F_3$, $\#F_2 = 4$ and $\#F_3 = 3$. Then

$$9 = \rho(V) \geq 2 + (\#F_1 - 1) + (\#F_2 - 1) + (\#F_3 - 1) = 11,$$

which is a contradiction. Therefore, this subcase does not take place.

Subcase 4: $i = 4$, $t \geq 2$ and $j = 7$. The divisor $F_1 := D_4 + D_8 + 2D_7 + 3C$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F_1|} : V \to \mathbb{P}^1$, D_0 and D_5 become sections of Φ and D_6 becomes a 2-section of Φ. Further, if $t \geq 3$, then D_9 becomes a section of Φ and $D - (D_0 + D_5 + D_6 + D_9)$ is contained in fibers of Φ. Let F_2 (resp. F_3) be the fiber of Φ containing $D_1 + D_2$ (resp. D_3). By using the same argument as in Subcase 2 in the proof of Claim 4.5.2, we know that F_1, F_2 and F_3 exhaust the singular fibers of Φ. Furthermore, by the argument as in Subcase 3, we know that $\#F_2 = 4$

and $\#F_3 = 3$. In particular, $F_2 = E_{2,1} + D_1 + D_2 + E_{2,2}$, where $E_{2,1}$ and $E_{2,2}$ are (-1)-curves and $E_{2,1}D_1 = E_{2,2}D_2 = 1$. Since

$$8 + t = \rho(V) = 2 + (\#F_1 - 1) + (\#F_2 - 1) + (\#F_3 - 1) = 10,$$

$t = 2$. Since D_5 is a section of Φ and $\alpha_1 + \alpha_5 > 1$, $E_{2,2}D_5 = 1$ and $E_{2,1}D^{(1)} = E_{2,1}D_1 = 1$. By Lemma 2.3, we know that $E_{2,1}$ meets D_6. Since $\alpha_2 + \alpha_5 + \alpha_6 = 864/791 > 1$, $E_{2,2}D_6 = 0$. Since D_6 is a 2-section of Φ and the coefficient of $E_{2,1}$ in F_2 equals one, $E_{2,1}D_6 = 2$. So we have

$$E_{2,1}D^{\#} = \alpha_1 + 2\alpha_6 > 1,$$

which is a contradiction. Therefore, this subcase does not take place.

Subcase 5: $i = 4$, $t \geq 2$ and $j = 6 + t$. The divisor $F_1 := D_4 + D_{5+t} + 2D_{6+t} + 3C$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F_1|} : V \to \mathbb{P}^1$, D_0, D_5 and D_{4+t} become sections of Φ and $D - (D_0 + D_5 + D_{4+t})$ is contained in fibers of Φ. Let F_2 (resp. F_3) be the fiber of Φ containing $D_1 + D_2$ (resp. D_3). By using the same argument as in Subcase 2 in the proof of Claim 4.5.2, we know that F_1, F_2 and F_3 exhaust the singular fibers of Φ. At least one of $\mathrm{Supp}F_2$ and $\mathrm{Supp}F_3$ contains no components of $D^{(2)}$.

(5-1) Assume that $\mathrm{Supp}F_2$ contains no components of $D^{(2)}$. Then F_2 consists only of (-1)-curves and (-2)-curves. By Lemma 2.6 (2), we have $F_2 = E_{2,1}+D_1+D_2+E_{2,2}$, where $E_{2,1}$ and $E_{2,2}$ are (-1)-curves and $E_{2,1}D_1 = E_{2,2}D_2 = 1$. Since $E_{2,k}D^{\#} < 1$ for $k = 1, 2$, we know that $E_{2,1}D_5 = 0$ and $E_{2,2}D_5 = 1$ (cf. Subcase 4). Since $E_{2,1}D^{(1)} = 1$ and by Lemma 2.3, we know that $E_{2,1}D_{4+t} = 1$. Let $E_{3,1}$ be the component of $\mathrm{Supp}F_3$ meeting D_5. Then $E_{3,1}$ is a (-1)-curve and so $\mathrm{Supp}F_3$ has another (-1)-curve, say $E_{3,2}$. We infer from Lemma 2.6 (1) that $E_{3,1}$ and $E_{3,2}$ exhaust the (-1)-curves in $\mathrm{Supp}F_3$. Hence, if $t \geq 3$ (resp. $t = 2$), then $\mathrm{Supp}F_3$ consists only of $E_{3,1}$, $E_{3,2}$, D_3, $D_6, \ldots, D_{4+t}$ (resp. $E_{3,1}$, $E_{3,2}$ and D_3). Suppose that $t = 2$. Then $F_2 = E_{3,1} + D_3 + E_{3,2}$ and $E_{3,1}D_3 = E_{3,2}D_3 = 1$. Let $\mu : V \to \Sigma_3$ be a relatively minimal model of $\Phi : V \to \mathbb{P}^1$ such that $\mu(D_6) = M_3$, the minimal section of Σ_3. Then we know that $\mu(D_0)^2 = D_0^2 + 3 = -t < 0$, here we note that $E_{3,1}D_6 = 1$. This is a contradiction. Hence, $t \geq 3$ and $\mathrm{Supp}F_3$ contains $D_6, \ldots, D_{3+t}$.

Since $E_{3,1}$ meets D_5, the coefficient of $E_{3,1}$ in F_3 equals one. So $E_{3,2}$ connects D_3 and $D_6 + \cdots + D_{3+t}$, namely, $E_{3,2}D_3 = E_{3,2}(D_6 + \cdots + D_{3+t}) = 1$. Since the intersection matrix of $E_{3,2} + D_3 + D_6 + \cdots + D_{3+t}$ is negative definite, $(E_{3,2}D_3 =)$ $E_{3,2}D_6 = 1$. Since $\mathrm{Supp}F_3 = E_{3,1} \cup E_{3,2} \cup D_3 \cup D_6 \cup \cdots \cup D_{3+t}$, we know that $E_{3,1}D_{3+t} = 1$. Here, note that $E_{3,1}D_5 = 1$. Let $\nu : V \to \Sigma_2$ be a relatively minimal model of $\Phi : V \to \mathbb{P}^1$ such that $\nu(D_5) = M_2$, the minimal section of Σ_2. Then $\nu_*(F_1) = \nu(D_4)$, $\nu_*(F_2) = \nu(E_{2,2})$ and $\nu_*(F_3) = \nu(E_{3,1})$. So $\nu(D_{4+t})^2 = D_{4+t}^2 + 5$, $\nu(D_{4+t})$ is a section of the ruling $\Phi \circ \nu^{-1}$ on Σ_2 and $\nu(D_{4+t})\nu(D_5) = 0$. Then $\nu(D_{4+t})^2 = D_{4+t}^2 + 5 = 2$ and so $D_{4+t}^2 = -3$. This contradicts $t \geq 3$.

(5-2) Assume that SuppF_2 contains some components of $D^{(2)}$. Then $t \geq 3$ and SuppF_3 contains no components of $D^{(2)}$. Since SuppF_3 consists only of (-1)-curves and (-2)-curves, we infer from Lemma 2.6 (2) that $F_3 = E_{3,1} + D_3 + E_{3,2}$, where $E_{3,1}$ and $E_{3,2}$ are (-1)-curves and $E_{3,1}D_3 = E_{3,2}D_3 = 1$. By Lemma 2.6 (1) and $t \geq 3$, SuppF_2 has just two (-1)-curves, say $E_{2,1}$ and $E_{2,2}$. We may assume that $E_{2,1}$ meets both of $D_1 + D_2$ and $D_6 + \cdots + D_{3+t}$. Since $E_{2,1}D^{\#} < 1$ and the intersection matrix of $E_{2,1} + D_1 + D_2 + D_6 + \cdots + D_{3+t}$ is negative definite, we know that $E_{2,1}(D_1 + D_2) = E_{2,1}D_2 = 1$ and $E_{2,1}(D_6 + \cdots + D_{3+t}) = E_{2,1}D_6 = 1$. Then, $(3E_{2,1} + 2D_2 + D_1 + D_6)^2 = 0$, which is a contradiction.

Thus, we know that this subcase does not take place.

The proof of Claim 4.5.3 is thus completed. □

Therefore, X contains $\mathbb{C}^2$ as a Zariski open subset.

4.6. Case (24)

In this subsection, we treat the case where the weighted dual graph of D is (24) in Theorem 1.1. Let $D = \sum_{i=0}^{7+t} D_i$ be the decomposition of D into irreducible components such that the weighted dual graph of D is given as in Figure 4.6, where $D_0^2 = -(t+3)$ and the weight of the vertex corresponding to a (-2)-curve is omitted. In this case, $\rho(V) = 9 + t$.

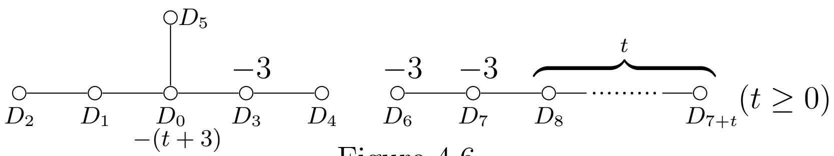


Figure 4.6.

Let α_i $(i = 0, 1, \ldots, 7+t)$ be the coefficient of D_i in $D^{\#}$. Then

$$\alpha_0 = \frac{30t + 42}{30t + 43}, \quad \alpha_1 = \frac{20t + 28}{30t + 43}, \quad \alpha_2 = \frac{10t + 14}{30t + 43}, \quad \alpha_3 = \frac{24t + 34}{30t + 43},$$

$$\alpha_4 = \frac{12t + 17}{30t + 43}, \quad \alpha_5 = \frac{15t + 21}{30t + 43}, \quad \alpha_6 = \frac{3t + 4}{5t + 8},$$

$$\alpha_{7+i} = \frac{4(t + 1 - i)}{5t + 8} \quad (i = 0, 1, \ldots, t).$$

Let $C \in \mathrm{MV}(V, D)$. By Lemmas 4.1 and 4.2, $CD_0 = 0$ and $CD^{(1)} = 1$.

Claim 4.6.1. $CD^{(2)} = 1$.

Proof. The assertion can be proved by using the same argument as in the proof of Claim 4.5.1 □

We take $i \in \{1, 2, \ldots, 5\}$ and $j \in \{6, 7, \ldots, 7+t\}$ such that $CD_i = CD_j = 1$.

Claim 4.6.2. If $j = 6$, then $i = 4$ and $V \setminus \mathrm{Supp}(C + D) \cong \mathbb{C}^2$.

Proof. Assume that $i = 4$. Then there exists a positive integer n and an effective divisor Δ such that $\mathrm{Supp}\Delta = \mathrm{Supp}(D - D_2)$, $nC + \Delta$ defines a $\mathbb{P}^1$-fibration $\Phi_{|nC+\Delta|} : V \to \mathbb{P}^1$ and D_2 becomes a section of $\Phi_{|nC+\Delta|}$. (We can wright down the divisor $nC + \Delta$ explicitly; we omit the description.) It is then clear that $V \setminus \mathrm{Supp}(C + D) \cong \mathbb{C}^2$. (See the arguments as in Section 3.)

Suppose that $i \neq 6$. Since $CD^{\#} = \alpha_i + \alpha_6 < 1$, $i = 2$ or 5. We consider the following subcases separately.

Subcase 1: $i = 5$**.** Since $CD^{\#} = \alpha_5 + \alpha_6 < 1$, we have $t = 0$. The divisor $F_1 := D_2 + D_3 + 2D_1 + 3(D_0 + D_6) + 6D_5 + 9C$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F_1|} : V \to \mathbb{P}^1$, D_4 becomes a section of Φ and D_7 becomes a 3-section of Φ. Since $9 = \rho(V) > 2 + (\#F_1 - 1)$, there exists a singular fiber F_2 of Φ other than F_1. Since $\mathrm{Supp}F_2$ contains no components of D, $F_2 = E_{2,1} + E_{2,2}$, where $E_{2,1}$ and $E_{2,2}$ are (-1)-curves and $E_{2,1}E_{2,2} = 1$. Since $F_2D_7 = (E_{2,1} + E_{2,2})D_7 = 3$, we may assume that $E_{2,1}D_7 \geq 2$. Then

$$E_{2,1}D^{\#} \geq \alpha_7 E_{2,1}D_7 = \frac{1}{2}E_{2,1}D_7 \geq 1,$$

which is a contradiction. Therefore, this subcase does not take place.

Subcase 2: $i = 2$**.** The divisor $F_1 := D_1 + D_6 + 2D_2 + 3C$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F_1|} : V \to \mathbb{P}^1$, D_0 and D_7 become sections of Φ and $D - (D_0 + D_7)$ is contained in fibers of Φ. Let F_2 (resp. F_3) be the fiber of Φ containing $D_3 + D_4$ (resp. D_5). Then F_1, F_2 and F_3 exhaust the singular fibers of Φ. Indeed, if G is a singular fiber of Φ other than F_1, F_2 and F_3, then the component of G meeting D_0 is a (-1)-curve. This contradicts Lemma 4.1. Moreover, since $\mathrm{Supp}F_3$ consists only of (-1)-curves and (-2)-curves, we infer from Lemma 2.6 (3) that $\#F_3 = 3$.

Suppose that $\mathrm{Supp}F_3$ contains no components of $D^{(2)}$. Then $F_3 = E_{3,1} + D_5 + E_{3,2}$, where $E_{3,1}$ and $E_{3,2}$ are (-1)-curves and $E_{3,1}D_5 = E_{3,2}D_5 = 1$. We may assume that $E_{3,2}D_7 = 1$ since D_7 is a section of Φ. Then $E_{3,1}D = E_{3,1}D_5 = 1$ and so the intersection matrix of $E_{3,1} + D$ is negative definite. This contradicts Lemma 2.3. Hence, $\mathrm{Supp}F_3$ contains at least one component of $D^{(2)}$. Since $D^{(2)} - (D_6 + D_7)$ is contained in $\mathrm{Supp}F_3$, we know that $F_3 = D_5 + D_8 + 2E_3$, where E_3 is a (-1)-curve and $E_3D_5 = E_3D_8 = 1$. In particular, $t = 1$ and $\mathrm{Supp}F_2$ contains no components of $D^{(2)}$. Since $\mathrm{Supp}F_2$ contains D_3 and D_4 and $D_3D_4 = 1$, we know that $\#F_2 \geq 5$. Then we have

$$10 = \rho(V) = 2 + (\#F_1 - 1) + (\#F_2 - 1) + (\#F_3 - 1) = 6 + \#F_2 \geq 11,$$

which is a contradiction. Therefore, this subcase does not take place.

The proof of Claim 4.6.2 is thus completed. □

Claim 4.6.3. If $j = 7$, then $t = 0$ and $V \setminus \mathrm{Supp}(C + D) \cong \mathbb{C}^2$.

Proof. If $t = 0$, then Claim 4.6.2 implies that $i = 4$ and $V \setminus \mathrm{Supp}(C + D) \cong \mathbb{C}^2$. So we assume that $t \geq 1$. Since $CD^{\#} = \alpha_i + \alpha_7 < 1$, we know that $i = 2$ or 5. As seen from the argument as in Subcase 1 in the proof of Claim 4.6.2, we know that $i = 2$ because $t \geq 1$. Further, $t = 1$ or 2 because $\alpha_2 + \alpha_7 < 1$.

The divisor $F_1 := D_1 + D_7 + 2D_2 + 3C$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F_1|} : V \to \mathbb{P}^1$, D_0, D_6 and D_8 become sections of Φ and $D - (D_0 + D_6 + D_8)$ is contained in fibers of Φ. Let F_2 (resp. F_3) be the fiber of Φ containing $D_3 + D_4$ (resp. D_5). By the argument as in Subcase 2 in the proof of Claim 4.6.2, we know that F_1, F_2 and F_3 exhaust the singular fibers of Φ. Moreover, $\#F_2 \geq 5$. Since $\mathrm{Supp}F_3$ consists only of (-1)-curves and (-2)-curves, we infer from Lemma 2.6 (2) that $\#F_3 = 3$.

If $\mathrm{Supp}F_3$ contains no components of $D^{(2)}$, then $F_3 = E_{3,1} + D_5 + E_{3,2}$, where $E_{3,1}$ and $E_{3,2}$ are (-1)-curves and $E_{3,1}D_5 = E_{3,2}D_5 = 1$. We may assume that $E_{3,1}D_6 = 1$ since D_6 is a section of Φ. Then

$$E_{3,1}D^{\#} \geq \alpha_5 + \alpha_6 > 1,$$

where the last inequality follows from $t \geq 1$. This is a contradiction. Hence, $\mathrm{Supp}F_3$ contains at least one component of $D^{(2)}$. Since $D^{(2)} - (D_6 + D_7)$ is contained in $\mathrm{Supp}F_3$, we know that $t = 2$ and $F_3 = D_5 + D_9 + 2E_3$, where E_3 is a (-1)-curve and $E_3D_5 = E_3D_9 = 1$, and that $\mathrm{Supp}F_2$ consists only of D_3, D_4 and three (-1)-curves $E_{2,1}$, $E_{2,2}$ and $E_{2,3}$. Here we note that $\mathrm{Supp}F_2$ contains just three (-1)-curve because $\mathrm{Supp}F_2$ contains no components of $D^{(2)}$ and

$$11 = \rho(V) = 2 + (\#F_1 - 1) + (\#F_2 - 1) + (\#F_3 - 1) = \#F_2 + 6.$$

Then at least one of $E_{2,1}$, $E_{2,2}$ and $E_{2,3}$ does not meet $D^{(2)} = D_6 + D_7 + D_8 + D_9$ because D_6 and D_8 are sections of Φ and D_7 and D_9 are fiber components of Φ. We may assume that $E_{2,1}D^{(2)} = 0$. Then $E_{2,1}D = E_{2,1}(D_3 + D_4) = 1$, which contradicts Lemma 2.3. This proves the claim. □

Claim 4.6.4. The case $j \geq 8$ does not take place.

Proof. Suppose to the contrary that $j \geq 8$. Then $t \geq 1$ and D_j is a (-2)-curve. We consider the following subcases separately.

Subcase 1: $i \neq 3$. Then D_i is a (-2)-curve and so the divisor $D_i + D_j + 2C$ defines a $\mathbb{P}^1$-fibration $\Phi_{|D_i+D_j+2C|} : V \to \mathbb{P}^1$. Then D_6, that is a (-3)-curve, becomes a fiber component of $\Phi_{|D_i+D_j+2C|}$. This contradicts Lemma 2.7. Therefore, this subcase does not take place.

Subcase 2: $i = 3$, $t \geq 3$ and $9 \leq j \leq 6+t$. The divisor $D_{j-1} + D_{j+1} + 2(C + D_j)$ defines a $\mathbb{P}^1$-fibration $\Phi_{|D_{j-1}+D_{j+1}+2(C+D_j)|} : V \to \mathbb{P}^1$. Then D_0, that is a $(-t-3)$-curve, becomes a fiber component of $\Phi_{|D_{j-1}+D_{j+1}+2(C+D_j)|}$. This contradicts Lemma 2.7. Therefore, this subcase does not take place.

Subcase 3: $i = 3$ and $j = 7+t$. Since $\alpha_3+\alpha_8 > 1$ by $t \geq 1$, we know that $t \geq 2$. So the divisor $F_1 := D_3 + D_{6+t} + 2D_{7+t} + 3C$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F_1|} : V \to \mathbb{P}^1$, D_0, D_4 and D_{5+t} become sections of Φ and $D - (D_0 + D_4 + D_{5+t})$ is contained in fibers of Φ. Let F_2 (resp. F_3) be the fiber of Φ containing $D_1 + D_2$ (resp. D_5). By the same argument as in Subcase 2 in the proof of Claim 4.6.2, we know that F_1, F_2 and F_3 exhaust the singular fibers of Φ. At least one of SuppF_2 and SuppF_3 contains no components of $D^{(2)}$.

Suppose that SuppF_2 contains no components of $D^{(2)}$. Then SuppF_3 consists only of $D_5, D_6, D_7, \ldots, D_{4+t}$ and some (-1)-curves. So SuppF_3 contains a (-1)-curve E_3 such that $E_3D_5 = E_3(D_6 + D_7) = 1$, here we note that E_3 does not meet $D^{(2)} - (D_6 + D_7)$. We have

$$E_3D^{\#} \geq \alpha_5E_3D_5 + \alpha_6E_3(D_6 + D_7) > 1,$$

which is a contradiction.

Suppose next that SuppF_3 contains no components of $D^{(2)}$. Then SuppF_2 consists only of $D_1, D_2, D_6, D_7, \ldots, D_{4+t}$ and some (-1)-curves. So SuppF_2 contains a (-1)-curve E_2 such that $E_2(D_1 + D_2) = E_2(D_6 + D_7) = 1$, here we note that E_2 does not meet $D^{(2)} - (D_6 + D_7)$. Since $E_2D^{\#} < 1$, we know that $E_2D_2 = E_2D_6 = 1$. So $F_2 = D_1 + D_6 + 2D_2 + 3E_2$. This is a contradiction because D_4 is a section of Φ and $D_4(D_1 + D_2 + D_6) = 0$.

Therefore, this subcase does not take place.

Subcase 4: $i = 3$ and $j = 8$. This subcase does not take place because $\alpha_3 + \alpha_8 > 1$ since $t \geq 1$.

The proof of Claim 4.6.4 is thus completed. □

Theorefore, X contains $\mathbb{C}^2$ as a Zariski open subset.

4.7. Case (14)

In this subsection, we treat the case where the weighted dual graph of D is (14) in Theorem 1.1. Let $D = \sum_{i=0}^{6+t} D_i$ be the decomposition of D into irreducible components such that the weighted dual graph of D is given as in Figure 4.7, where $D_0^2 = -(t+3)$ and the weight of the vertex corresponding to a (-2)-curve is omitted. In this case, $\rho(V) = 8 + t$.

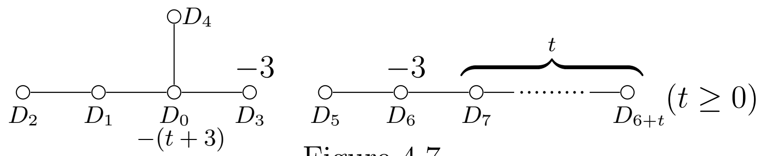


Figure 4.7.

Let α_i $(i = 0, 1, \ldots, 6+t)$ be the coefficient of D_i in $D^{\#}$. Then

$$\alpha_0 = \frac{6t+8}{6t+9}, \quad \alpha_1 = \frac{12t+16}{18t+27}, \quad \alpha_2 = \frac{6t+8}{18t+27}, \quad \alpha_3 = \frac{12t+17}{18t+27},$$
$$\alpha_4 = \frac{3t+4}{6t+9}, \quad \alpha_5 = \frac{t+1}{3t+5}, \quad \alpha_{6+i} = \frac{2(t+1-i)}{3t+5} \quad (i = 0, 1, \ldots, t).$$

Let $C \in \mathrm{MV}(V, D)$. By Lemmas 4.1 and 4.2, $CD_0 = 0$ and $CD^{(1)} = 1$.

Claim 4.7.1. $CD^{(2)} = 1$.

Proof. The assertion can be proved by using the same argument as in the proof of Claim 4.5.1 □

We take $i \in \{1, 2, 3, 4\}$ and $j \in \{5, 6, \ldots, 6+t\}$ such that $CD_i = CD_j = 1$.

Claim 4.7.2. If $j = 5$, then $i = 3$ and $V \setminus \mathrm{Supp}(C + D) \cong \mathbb{C}^2$.

Proof. Assume that $i = 3$. Then there exists a positive integer n and an effective divisor Δ such that $\mathrm{Supp}\Delta = \mathrm{Supp}(D - D_2)$, $nC + \Delta$ defines a $\mathbb{P}^1$-fibration $\Phi_{|nC+\Delta|} : V \to \mathbb{P}^1$ and D_2 becomes a section of $\Phi_{|nC+\Delta|}$. (We can wright down the divisor $nC + \Delta$ explicitly; we omit the description.) It is then clear that $V \setminus \mathrm{Supp}(C + D) \cong \mathbb{C}^2$. (See the arguments as in Section 3.)

Suppose that $i \neq 3$. Then D_i is a (-2)-curve and so the divisor $D_i + D_5 + 2C$ defines a $\mathbb{P}^1$-fibration $\Phi_{|D_i+D_5+2C|} : V \to \mathbb{P}^1$. Then D_3, that is a (-3)-curve, becomes a fiber component of $\Phi_{|D_i+D_5+2C|}$. This contradicts Lemma 2.7. □

Claim 4.7.3. The case $j = 6$ does not take place.

Proof. Suppose to the contrary that $j = 6$. We consider the following subcases separately.

Subcase 1: $i = 3$**.** Since $CD^{\#} = \alpha_3 + \alpha_6 < 1$, $t = 0$. Then the intersection matrix of $C + D$ is negative definite. This contradicts Lemma 2.3. Therefore, this subcase does not take place.

Subcase 2: $i = 1$**.** Since $CD^{\#} = \alpha_1 + \alpha_6 < 1$, $t = 0$. The divisor $F_1 := D_2 + D_6 + 2D_1 + 3C$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F_1|} : V \to \mathbb{P}^1$, D_5 becomes a section of Φ, D_0 becomes a 2-section of Φ and $D - (D_0 + D_5)$ is contained in fibers of Φ. Let F_2 be the fiber of Φ containing D_3. Then $\#F_2 \geq 4$ since $D_3^2 = -3$. Since

$$8 = \rho(V) \geq 2 + (\#F_1 - 1) + (\#F_2 - 1) \geq 8,$$

$\#F_2 = 4$ and F_1 and F_2 exhaust the singular fibers of Φ. Then SuppF_2 consists of D_3, D_4 and two (-1)-curves $E_{2,1}$ and $E_{2,2}$. We may assume that $E_{2,1}D_3 = E_{2,1}D_4 = 1$. Then

$$E_{2,1}D^{\#} \geq \alpha_3 + \alpha_4 > 1,$$

which is a contradiction. Therefore, this subcase does not take place.

Subcase 3: $i = 4$**.** Since $CD^{\#} = \alpha_4 + \alpha_6 < 1$, $t = 0$ or 1. Suppose that $t = 0$. Then the divisor $F := D_0 + D_5 + 2D_6 + 3D_4 + 5C$ defines a $\mathbb{P}^1$-fibration $\Psi := \Phi_{|F|} : V \to \mathbb{P}^1$, D_1 and D_3 become sections of Ψ and $D - (D_0 + D_3)$ is contained in sections of Φ. Let F' be the fiber of Ψ containing D_2. Since SuppF' consists only of D_2 and some (-1)-curves, we infer from Lemma 2.6 (2) that $F' = E + D_2 + E'$, where E and E' are (-1)-curves and $ED_2 = E'D_2 = 1$. Since D_3 is a section of Ψ, we may assume that $E'D_3 = 1$. Then $ED = ED_2 = 1$ and so the intersection matrix of $E + D$ is negative definite. This contradicts Lemma 2.3.

Suppose that $t = 1$. Then $\rho(V) = 9$ and the divisor $F_1 := D_5 + D_7 + 2(D_4 + D_6) + 4C$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F_1|} : V \to \mathbb{P}^1$, D_0 becomes a 2-section of Φ and $D - D_0$ is contained in fibers of Φ. Let F_2 be the fiber of Φ containing D_3. Since $D_3^2 = -3$, $\#F_2 \geq 4$. Then

$$9 = \rho(V) \geq 2 + (\#F_1 - 1) + (\#F_2 - 1) = 5 + \#F_2 \geq 9$$

and so $\#F_2 = 4$ and F_1 and F_2 exhaust the singular fibers of Φ. In particular, SuppF_2 consists only of D_1, D_2, D_3 and a (-1)-curve E_2. Since $E_2(D_1 + D_2) = E_2D_3 = 1$ and $E_2D^{\#} < 1$, $E_2D_2 = 1$.

Let $\mu : V \to W$ be the contraction of C, D_4, D_6, D_7, E_2, D_2 and D_1. Then W is a Hirzebruch surface of degree n ($n = 0$ or 1) and $\mu(D_0)$ is a 2-section of the ruling $\Phi \circ \mu^{-1}$ on W. We know that $\mu(D_0)^2 = -4 + 4 = 0$ and $\mu(D_0)$ is a smooth rational curve. On the other hand, $\mu(D_0) \sim 2M_n + \alpha\ell$, where M_n is a minimal section of W, ℓ is a fiber of the ruling $\Phi \circ \mu^{-1}$ on W and $\alpha \in \mathbb{Z}$. Then $\alpha = n$ since $0 = \mu(D_0)^2 = -4n + 4\alpha$, and so

$$\mu(D_0)K_W = (2M_n + n\ell)(-2M_n - (n+2)\ell) = -4.$$

This is a contradiction. Therefore, this subcase does not take place.

Subcase 4: $i = 2$**.** The divisor $F_1 := D_1 + D_6 + 2D_2 + 3C$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F_1|} : V \to \mathbb{P}^1$, D_0 and D_5 become sections of Φ. Further, if $t \geq 1$ then D_7 becomes a section of Φ and $D - (D_0 + D_5 + D_7)$ is contained in fibers of Φ. Let F_2 (resp. F_3) be the fiber of Φ containing D_3 (resp. D_4). Then F_1, F_2 and F_3 exhaust the singular fibers of Φ. Indeed, if G is a singular fiber of Φ other than F_1, F_2 and F_3, then the component of G meeting D_0 is a (-1)-curve. This contradicts Lemma 4.1. Since SuppF_2 contains a (-3)-curve, $\#F_2 \geq 4$. Since SuppF_3 consists only of

(−1)-curves and (−2)-curves, we infer from Lemma 2.6 (2) that $\#F_3 = 3$. We have

$$8 + t = \rho(V) = 2 + (\#F_1 - 1) + (\#F_2 - 1) + (\#F_3 - 1) = 6 + \#F_2.$$

Suppose that $\mathrm{Supp}F_2$ contains no components of $D^{(2)}$. Then $\mathrm{Supp}F_2$ consists only of D_3 and some (−1)-curves. So $\#F_2 = 4$ and $F_2 = D_3 + E_{2,1} + E_{2,2} + E_{2,3}$, where $E_{2,1}$, $E_{2,2}$ and $E_{2,3}$ are (−1)-curves and $E_{2,1}D_3 = E_{2,2}D_3 = E_{2,3}D_3 = 1$. In particular, $t = 2$. Since D_0, D_5 and D_7 are sections of Φ, we may assume that $E_{2,1}D_5 = E_{2,1}D_7 = 0$. Then $E_{2,1}D = E_{2,1}D_3 = 1$ and so the intersection matrix of $E_{2,1} + D$ is negative definite. This contradicts Lemma 2.3. Therefore, $\mathrm{Supp}F_2$ contains some components of $D^{(2)}$. Then $t \geq 2$ and $\mathrm{Supp}F_3$ contains no components of of $D^{(2)}$. So $F_3 = E_{3,1} + D_4 + E_{3,2}$, where $E_{3,1}$ and $E_{3,2}$ are (−1)-curves and $E_{3,1}D_4 = E_{3,2}D_4 = 1$. $\mathrm{Supp}F_2$ consists only of $D_3, D_8, \ldots, D_{6+t}$ and some (−1)-curves. We infer from Lemma 2.6 (1) that

$$\begin{aligned} 3 &= 1 + \sum_{\ell=1}^{3} (\#\{(-1)\text{−curves} \quad \text{in} \quad F_\ell\} - 1) \\ &= 2 + \#\{(-1) - \text{curves} \quad \text{in} \quad F_2\}, \end{aligned}$$

which implies that $\mathrm{Supp}F_2$ has a unique (−1)-curve, say E_2. Then $E_2D_3 = E_2(D_8 + \cdots + D_{6+t}) = 1$. However, this is a contradiction because D_5 is a section of Φ and $D_5(E_2 + D_3 + D_8 + \cdots + D_{6+t}) = 0$. Therefore, this subcase does not take place.

The proof of Claim 4.7.3 is thus completed. □

Claim 4.7.4. If $j \geq 7$, then $t = 1$ and $V \setminus \mathrm{Supp}(C + D) \cong \mathbb{C}^2$.

Proof. Suppose that $j \geq 7$. Then D_j is a (−2)-curve and $t \geq 1$. We consider the following subcases separately.

Subase 1: $i \neq 3$. By using the same argument as in the second paragraph of the proof of Claim 4.7.2, we know that this subcase does not take place.

Subcase 2: $i = 3$, $t \geq 3$ and $8 \leq j \leq 5 + t$. The divisor $D_{j-1} + D_{j+1} + 2(C + D_j)$ defines a $\mathbb{P}^1$-fibration $\Phi_{|D_{j-1}+D_{j+1}+2(C+D_j)|} : V \to \mathbb{P}^1$. Then D_0 is a fiber component of $\Phi_{|D_{j-1}+D_{j+1}+2(C+D_j)|}$. This contradicts Lemma 2.7. Therefore, this subcase does not take place.

Subcase 3: $i = 3$ and $j = 6 + t$. If $t = 1$, then $j = 7$ and Claim 4.7.2 implies that $V \setminus \mathrm{Supp}(C + D) \cong \mathbb{C}^2$. We assume that $t \geq 2$ and derive a contradiction. The divisor $F_1 := D_3 + D_{5+t} + 2D_{6+t} + 3C$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F_1|} : V \to \mathbb{P}^1$, D_0 and D_{4+t} become sections and $D - (D_0 + D_{4+t})$ is contained in fibers of Φ. Let F_2 (resp. F_3) be the fiber of Φ containing $D_1 + D_2$ (resp. D_4). By using the same argument as in the first paragraph of Subcase 4 in the proof of Claim 4.7.3, we know that F_1, F_2 and F_3 exhaust the singular fibers of Φ. At least one of $\mathrm{Supp}F_2$ and $\mathrm{Supp}F_3$ contains no components of $D^{(2)}$.

Suppose that $\mathrm{Supp}F_2$ contains no components of $D^{(2)}$. Then $\mathrm{Supp}F_2$ consists only of D_1, D_2 and some (-1)-curves. We infer from Lemma 2.6 (2) that $F_2 = E_{2,1}+D_1+D_2+E_{2,2}$, where $E_{2,1}$ and $E_{2,2}$ are (-1)-curves and $E_{2,1}D_1 = E_{2,2}D_2 = 1$. Since D_{4+t} is a section of Φ and $D^{(2)} - D_{4+t}$ is contained in fibers of Φ, we know that either $E_{2,1}$ or $E_{2,2}$ does not meet $D^{(2)}$. So $E_{2,k}D = E_{2,k}D^{(1)} = 1$ for $k = 1$ or 2 and hence the intersection matrix of $E_{2,k} + D$ is negative definite for $k = 1$ or 2. This contradicts Lemma 2.3.

Suppose next that $\mathrm{Supp}F_3$ contains no components of $D^{(2)}$. By using the same argument as in the previous paragraph, we know that $F_3 = E_{3,1} + D_3 + E_{3,2}$, where $E_{3,1}$ and $E_{3,2}$ are (-1)-curves and $E_{3,1}D_3 = E_{3,2}D_3 = 1$. Then the intersection matrix of $E_{3,k} + D$ is negative definite for $k = 1$ or 2, which contradicts Lemma 2.3. Therefore, this subcase does not take place.

Subcase 4: $i = 3$, $t \geq 2$ and $j = 7$. The divisor $F_1 := D_3 + D_8 + 2D_7 + 3C$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F_1|} : V \to \mathbb{P}^1$, D_0 becomes a section of Φ and D_6 becomes a 2-section of Φ. Moreover, if $t \geq 3$, then D_9 becomes a section of Φ and $D - (D_0 + D_6 + D_9)$ is contained in fibers of Φ. Let F_2 (resp. F_3) be the fiber of Φ containing $D_1 + D_2$ (resp. D_4). As seen from the argument as in Subcase 3, we know that F_1, F_2 and F_3 exhaust the singular fibers of Φ. Since $D-(D_0+D_3+D_6)$ consists only of (-2)-curves, $\mathrm{Supp}F_2$ and $\mathrm{Supp}F_3$ consist only of (-1)-curves and (-2)-curves. We infer from Lemma 2.6 (2) that $\#F_2 = 4$ and $\#F_3 = 3$. Then

$$8 + t = \rho(V) = 2 + (\#F_1 - 1) + (\#F_2 - 1) + (\#F_3 - 1) = 10,$$

and so $t = 2$. Furthermore, we know that $F_2 = E_{2,1} + D_1 + D_2 + E_{2,2}$, where $E_{2,1}$ and $E_{2,2}$ are (-1)-curves and $E_{2,1}D_1 = E_{2,2}D_2 = 1$, and that $D_5 \subset \mathrm{Supp}F_3$. Since D_6 is a 2-section of Φ, $D^{(2)} - D_6$ is contained in fibers of Φ and the intersection matrix of $E_{2,k} + D$ is not negative definite for $k = 1, 2$, we know that $E_{2,k}D_6 = 1$ for $k = 1, 2$. Then

$$E_{2,1}D^{\#} \geq \alpha_1 + \alpha_6 > 1,$$

a contradiction. Therefore, this subcase does not take place.

The proof of Claim 4.7.4 is thus completed. □

Theorefore, X contains $\mathbb{C}^2$ as a Zariski open subset.

4.8. Case (26)

In this subsection, we treat the case where the weighted dual graph of D is (26) in Theorem 1.1. Let $D = \sum_{i=0}^{7+t} D_i$ be the decomposition of D into irreducible components such that the weighted dual graph of D is given as in Figure 4.8, where $D_0^2 = -(t+3)$ and the weight of the vertex corresponding to a (-2)-curve is omitted. In this case, $\rho(V) = 9 + t$.

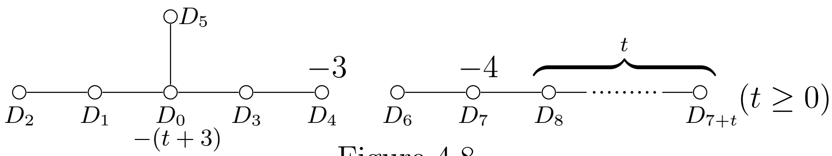


Figure 4.8.

Let α_i $(i = 0, 1, \ldots, 7+t)$ be the coefficient of D_i in $D^{\#}$. Then

$$\alpha_0 = \frac{30t+36}{30t+37}, \quad \alpha_1 = \frac{20t+24}{30t+37}, \quad \alpha_2 = \frac{10t+12}{30t+37}, \quad \alpha_3 = \frac{24t+29}{30t+37},$$

$$\alpha_4 = \frac{18t+22}{30t+37}, \quad \alpha_5 = \frac{15t+18}{30t+37}, \quad \alpha_6 = \frac{2(t+1)}{5t+7},$$

$$\alpha_{7+i} = \frac{4(t+1-i)}{5t+7} \quad (i = 0, 1, \ldots, t).$$

Let $C \in \mathrm{MV}(V, D)$. By Lemmas 4.1 and 4.2, $CD_0 = 0$ and $CD^{(1)} = 1$.

Claim 4.8.1. $CD^{(2)} = 1$.

Proof. The assertion can be proved by using the same argument as in the proof of Claim 4.4.1 □

We take $i \in \{1, 2, \ldots, 5\}$ and $j \in \{6, 7, \ldots, 7+t\}$ such that $CD_i = CD_j = 1$.

Claim 4.8.2. If $j = 6$, then $i = 4$ and $V \setminus \mathrm{Supp}(C + D) \cong \mathbb{C}^2$.

Proof. Assume that $i = 4$. Then there exists a positive integer n and an effective divisor Δ such that $\mathrm{Supp}\Delta = \mathrm{Supp}(D - D_2)$, $nC + \Delta$ defines a $\mathbb{P}^1$-fibration $\Phi_{|nC+\Delta|} : V \to \mathbb{P}^1$ and D_2 becomes a section of $\Phi_{|nC+\Delta|}$. (We can wright down the divisor $nC + \Delta$ explicitly; we omit the description.) It is then clear that $V \setminus \mathrm{Supp}(C+D) \cong \mathbb{C}^2$. (See the arguments as in Section 3.)

Suppose that $i \neq 4$. Since $CD^{\#} = \alpha_i + \alpha_6 < 1$, $i \neq 3$ and so D_i is a (-2)-curve. The divisor $D_i + D_6 + 2C$ defines a $\mathbb{P}^1$-fibration $\Phi_{|D_i+D_6+2C|} : V \to \mathbb{P}^1$. Then D_4, that is a (-3)-curve, is a fiber component of $\Phi_{|D_i+D_6+2C|}$. This contradicts Lemma 2.7. This proves Claim 4.8.2. □

Claim 4.8.3. The case $j = 7$ does not take place.

Proof. Suppose to the contrary that $j = 7$. Since $CD^{\#} = \alpha_i + \alpha_7 < 1$, we know that $i = 2$ and $t = 0$. Then the divisor $F_1 := D_0 + D_6 + 2D_7 + 3D_1 + 5D_2 + 7C$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F_1|} : V \to \mathbb{P}^1$, D_3 and D_5 become sections of Φ and $D - (D_3 + D_5)$ is contained in fibers of Φ. Let F_2 be the fiber of Φ containing D_4. Since $\mathrm{Supp}F_2$ consists only of D_4 and some (-1)-curves, $\#F_2 = 4$. Then we have

$$9 = \rho(V) \geq 2 + (\#F_1 - 1) + (\#F_2 - 1) = 10,$$

a contradiction. □

Claim 4.8.4. If $j \geq 8$, then $t = 1$ and $V \setminus \mathrm{Supp}(C + D) \cong \mathbb{C}^2$.

Proof. Suppose that $j \geq 8$. Then D_j is a (-2)-curve and $t \geq 1$. We consider the following subcases separately.

Subase 1: $i \in \{1, 2, 5\}$**.** By using the same argument as in the second paragraph of the proof of Claim 4.8.2, we know that this subcase does not take place.

Subcase 2: $i = 3$**.** The divisor $D_3 + D_j + 2C$ defines a $\mathbb{P}^1$-fibration $\Phi_{|D_3+D_j+2C|} : V \to \mathbb{P}^1$. Since D_7, that is a (-4)-curve, is not a fiber component of $\Phi_{|D_3+D_j+2C|}$ by Lemma 2.7, we know that $j = 8$. Then $CD^{\#} = \alpha_3 + \alpha_8 > 1$ since $t \geq 1$. This is a contradiction. Therefore, this subcase does not tae place.

Subcase 3: $i = 4$**,** $t \geq 3$ **and** $9 \leq j \leq 6 + t$**.** The divisor $D_{j-1} + D_{j+1} + 2(C + D_j)$ defines a $\mathbb{P}^1$-fibration $\Phi_{|D_{j-1}+D_{j+1}+2(C+D_j)|} : V \to \mathbb{P}^1$. Then D_0 is a fiber component of $\Phi_{|D_{j-1}+D_{j+1}+2(C+D_j)|}$. This contradicts Lemma 2.7. Therefore, this subcase does not take place.

Subcase 4: $i = 4$ **and** $j = 7 + t$**.** If $t = 1$, then $j = 8$ and Claim 4.8.2 implies that $V \setminus \mathrm{Supp}(C+D) \cong \mathbb{C}^2$. Suppose that $t \geq 2$. The divisor $F := D_4 + D_{6+t} + 2D_{7+t} + 3C$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F|} : V \to \mathbb{P}^1$, D_3 and D_{5+t} become sections of Φ and $D - (D_3 + D_{5+t})$ is contained in fibers of Φ. Then D_0 is a fiber component of Φ. This contradicts Lemma 2.8 because $D_0^2 = -(t + 3) \leq -5$.

Subcase 5: $i = 4$ **and** $j = 8$**.** As seen from the argument as in Subcase 4, we may assume that $t \geq 2$. Then the divisor $F := D_4 + D_9 + 2D_8 + 3C$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F|} : V \to \mathbb{P}^1$. Then D_0, that is a $(-t - 3)$-curve, becomes a fiber component of Φ. This contradicts Lemma 2.8.

The proof of Claim 4.8.4 is thus completed. □

Theorefore, X contains $\mathbb{C}^2$ as a Zariski open subset.

4.9. Case (20)

In this subsection, we treat the case where the weighted dual graph of D is (20) in Theorem 1.1. Let $D = \sum_{i=0}^{6+t} D_i$ be the decomposition of D into irreducible components such that the weighted dual graph of D is given as in Figure 4.9, where $D_0^2 = -(t+3)$ and the weight of the vertex corresponding to a (-2)-curve is omitted. In this case, $\rho(V) = 8 + t$.

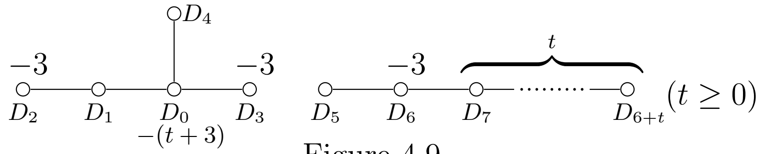


Figure 4.9.

Let α_i $(i = 0, 1, \ldots, 6+t)$ be the coefficient of D_i in $D^{\#}$. Then

$$\alpha_0 = \frac{30t+46}{30t+47}, \quad \alpha_1 = \frac{24t+37}{30t+47}, \quad \alpha_2 = \frac{18t+28}{30t+47}, \quad \alpha_3 = \frac{20t+31}{30t+47},$$

$$\alpha_4 = \frac{15t+23}{30t+47}, \quad \alpha_5 = \frac{t+1}{3t+5}, \quad \alpha_{6+i} = \frac{2(t+1-i)}{3t+5} \quad (i = 0, 1, \ldots, t).$$

Let $C \in \mathrm{MV}(V, D)$. By Lemmas 4.1 and 4.2, $CD_0 = 0$ and $CD^{(1)} = 1$.

Claim 4.9.1. $CD^{(2)} = 1$.

Proof. The assertion can be proved by using the same argument as in the proof of Claim 4.5.1 □

We take $i \in \{1, 2, 3, 4\}$ and $j \in \{5, 6, \ldots, 6+t\}$ such that $CD_i = CD_j = 1$.

Claim 4.9.2. If $j = 5$, then $i = 3$ and $V \setminus \mathrm{Supp}(C + D) \cong \mathbb{C}^2$.

Proof. Assume that $i = 3$. Then there exists a positive integer n and an effective divisor Δ such that $\mathrm{Supp}\Delta = \mathrm{Supp}(D - D_2)$, $nC + \Delta$ defines a $\mathbb{P}^1$-fibration $\Phi_{|nC+\Delta|} : V \to \mathbb{P}^1$ and D_2 becomes a section of $\Phi_{|nC+\Delta|}$. (We can wright down the divisor $nC + \Delta$ explicitly; we omit the description.) It is then clear that $V \setminus \mathrm{Supp}(C + D) \cong \mathbb{C}^2$. (See the arguments as in Section 3.)

Suppose that $i \neq 3$. We consider the following subcases separately.

Subcase 1: $i = 1$ or 4. The divisor $D_i + D_5 + 2C$ defines a $\mathbb{P}^1$-fibration $\Phi_{|D_i+D_5+2C|} : V \to \mathbb{P}^1$. Then D_3, that is a (-3)-curve, is a fiber component of $\Phi_{|D_i+D_5+2C|}$. This contradicts Lemma 2.7. Therefore, this subcase does not take place.

Subcase 2: $i = 2$. The divisor $F_1 := D_1 + D_6 + 2D_2 + 3D_5 + 5C$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F_1|} : V \to \mathbb{P}^1$ and D_0 becomes a section of Φ. Moreover, if $t \geq 1$, then D_7 becomes a section of Φ and $D - (D_0 + D_7)$ is contained in fibers of Φ. Let F_2 (resp. F_3) be the fiber of Φ containing D_3 (resp. D_4). Then F_1, F_2 and F_3 exhaust the singular fibers of Φ. Indeed, if G is a singular fiber of Φ other than F_1, F_2 and F_3, then the component of G meeting D_0 is a (-1)-curve. This contradicts Lemma 4.1. We know that $\mathrm{Supp}F_3$ consists only of (-1)-curves and (-2)-curves. So we infer from Lemma 2.6 (2) that $\#F_3 = 3$.

Suppose that $\mathrm{Supp}F_3$ contains a component of $D^{(2)}$. Then $\mathrm{Supp}F_2$ contains no components of $D^{(2)}$ and so $F_2 = E_{2,1} + E_{2,2} + E_{2,3} + D_3$, where $E_{2,1}$, $E_{2,2}$ and $E_{2,3}$ are (-1)-curves and $E_{2,1}D_3 = E_{2,2}D_3 = E_{2,3}D_3 = 1$. Since D_7 is a section of Φ and $D^{(2)} - D_7$ is contained in fibers of Φ, at least two of $E_{2,1}$, $E_{2,2}$ and $E_{2,3}$ do not meet $D^{(2)}$. We may assume that $E_{2,1}D^{(2)} = 0$. Then $E_{2,1}D = E_{2,1}D_3 = 1$ and so the intersection matrix of $E_{2,1} + D$ is negative definite. This contradicts Lemma 2.3.

Hence, $\mathrm{Supp}F_3$ contains no components of $D^{(2)}$. Then $F_3 = E_{3,1} + D_4 + E_{3,2}$, where $E_{3,1}$ and $E_{3,2}$ are (-1)-curves and $E_{3,1}D_4 = E_{3,2}D_4 = 1$. Then, by using the same argument as in the previous paragraph, we derive a contradiction. Therefore, this subcase does not take place.

The proof of Claim 4.9.2 is thus completed. □

Claim 4.9.3. The case $j = 6$ does not take place.

Proof. Suppose to the contrary that $j = 6$. Since $CD^{\#} = \alpha_i + \alpha_6 < 1$, $i = 2$ or 4. If $i = 2$, then $t = 0$ since $CD^{\#} = \alpha_2 + \alpha_6 < 1$. The intersection matrix of $C + D$ is then negative definite, which contradicts Lemma 2.3.

Suppose that $i = 4$. Since $CD^{\#} = \alpha_4 + \alpha_6 < 1$, $t \leq 1$. We consider the following subcases separately.

Subcase 1: $t = 0$. The divisor $F_1 := D_0 + D_5 + 2D_6 + 3D_4 + 5C$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F_1|} : V \to \mathbb{P}^1$, D_1 and D_3 become sections of Φ and $D - (D_1 + D_3)$ is contained in fibers of Φ. Let F_2 be the fiber of Φ containing D_2. Then $\#F_2 \geq 4$ because $D_2^2 = -3$. We have

$$8 = \rho(V) \geq 2 + (\#F_1 - 1) + (\#F_2 - 1) = 5 + \#F_2 \geq 9,$$

a contradiction. Therefore, this subcase does not take place.

Subcase 2: $t = 1$. The divisor $F_1 := D_5 + D_7 + 2(D_4 + D_6) + 4C$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F_1|} : V \to \mathbb{P}^1$, D_0 becomes a 2-sections and $D - D_0$ is contained in fibers of Φ. Let F_2 be the fiber of Φ containing D_3. Since $D_3^2 = -3$, $\#F_2 \geq 4$. Since

$$9 = \rho(V) \geq 2 + (\#F_1 - 1) + (\#F_2 - 2) = 5 + \#F_2 \geq 9,$$

we know that $\#F_2 = 4$ and that F_1 and F_2 exhaust the singular fibers of Φ. In particular, $D_1, D_2 \subset \mathrm{Supp} F_2$, which implies that $\mathrm{Supp} F_2$ consists only of D_1, D_2, D_3 and a (-1)-curve, say E_2. Then $E_2 D_3 = E_2(D_1 + D_2) = 1$. This is a contradiction because $E_2 D^{\#} \geq \alpha_2 + \alpha_3 > 1$. Therefore, this subcase does not take place.

The proof of Claim 4.9.3 is thus completed. □

Claim 4.9.4. If $j \geq 7$, then $t = 1$ and $V \setminus \mathrm{Supp}(C + D) \cong \mathbb{C}^2$.

Proof. Suppose that $j \geq 7$. Then D_j is a (-2)-curve and $t \geq 1$. We consider the following subcases separately.

Subase 1: $i = 1$ or 4. Then D_i is a (-2)-curve. By using the same argument as in Subcase 1 in the proof of Claim 4.9.2, we know that this subcase does not take place.

Subcase 2: $t \geq 3$ and $8 \leq j \leq 5 + t$. The divisor $F_1 := D_{j-1} + D_{j+1} + 2(C + D_j)$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F_1|} : V \to \mathbb{P}^1$. Then D_0, that is a $(-t-3)$-curve, is a fiber component of Φ since $CD_0 = 0$ by Lemma 4.1. This contradicts Lemma 2.7. Therefore, this subcase does not take place.

Subcase 3: $i = 2$**.** By the argument as in Subcase 2, we may assume that $j = 7$ or $6 + t$. By Claim 4.9.2, we know that $t \geq 2$. If $j = 7$ (resp. $j = 6 + t$), then the divisor $F := D_2 + D_8 + 2D_7 + 3C$ (resp. $F := D_2 + D_{5+t} + 2D_{6+t} + 3C$) defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F|} : V \to \mathbb{P}^1$. Then D_0, that is a $(-t-3)$-curve, becomes a fiber component of Φ. This contradicts Lemma 2.8 since $t + 3 \geq 5$. Therefore, this subcase does not take place.

Subcase 4: $i = 3$ **and** $j = 6 + t$**.** If $t = 1$, then $j = 7$ and $V \setminus \mathrm{Supp}(C + D) \cong \mathbb{C}^2$ by Claim 4.9.2. So we may assume that $t \geq 2$. Then the divisor $F_1 := D_3 + D_{5+t} + 2D_{6+t} + 3C$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F_1|} : V \to \mathbb{P}^1$, D_0 and D_{4+t} become sections of Φ and $D - (D_0 + D_{4+t})$ is contained in fibers of Φ. Let F_2 (resp. F_3) be the fiber of Φ containing $D_1 + D_2$ (resp. D_4). By using the same argument as in Subcase 2 in the proof of Claim 4.9.2, we know that F_1, F_2 and F_3 exhaust the singular fibers of Φ. Further, at least one of $\mathrm{Supp}F_2$ and $\mathrm{Supp}F_3$ contains no components of $D^{(2)}$.

Suppose that $\mathrm{Supp}F_2$ contains no components of $D^{(2)}$. Then $\mathrm{Supp}F_2$ consists only of D_1, D_2 and some (-1)-curves and so $F_2 = E_{2,1} + D_1 + D_2 + E_{2,2} + E_{2,3}$, where $E_{2,1}$, $E_{2,2}$ and $E_{2,3}$ are (-1)-curves and $E_{2,1}D_1 = E_{2,2}D_2 = E_{2,3}D_2 = 1$. Since D_{4+t} is a section of Φ and $D^{(2)} - D_{4+t}$ is contained in fibers of Φ, two of $E_{2,1}$, $E_{2,2}$ and $E_{2,3}$ do not meet $D^{(2)}$. So we may assume that $E_{2,2}$ does not meet $D^{(2)}$. Then $E_{2,2}D = E_{2,2}D_2 = 1$ and so the intersection matrix of $E_{2,2} + D$ is negative definite. This contradicts Lemma 2.3.

Suppose that $\mathrm{Supp}F_3$ contains no components of $D^{(2)}$. Then $\mathrm{Supp}F_3$ consists only of D_4 and some (-1)-curves and so $F_3 = E_{3,1} + D_4 + E_{3,2}$, where $E_{3,1}$ and $E_{3,2}$ are (-1)-curves and $E_{3,1}D_4 = E_{3,2}D_4 = 1$. By using the same argument as in the preceding paragraph, we derive a contradiction.

Therefore, this subcase does not take place.

Subcase 5: $i = 3$ **and** $j = 7$**.** By the argument as in Subcase 3, we may assume that $t \geq 2$. Then the divisor $F_1 := D_3 + D_8 + 2D_7 + 3C$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F_1|} : V \to \mathbb{P}^1$, D_6 becomes a 2-section of Φ and D_0 becomes a section of Φ. Moreover, if $t \geq 3$, then D_9 becomes a section of Φ and $D - (D_0 + D_9)$ is contained in fibers of Φ. Let F_2 (resp. F_3) be the fiber of Φ containing $D_1 + D_2$ (resp. D_4). By using the same argument as in Subcase 2 in the proof of Claim 4.9.2, we know that F_1, F_2 and F_3 exhaust the singular fibers of Φ. Since $\mathrm{Supp}F_3$ consists only of (-1)-curves and (-2)-curves, we infer from Lemma 2.6 (2) that $\#F_3 = 3$. If $D_5 \subset \mathrm{Supp}F_3$, then $F_3 = D_4 + D_5 + 2E_3$, where E_3 is a (-1)-curve and $E_3D_4 = E_3D_5 = 1$. Then $2 = F_3D_6 = 1 + 2E_3D_6$, a contradiction.

So $D_5 \subset \mathrm{Supp}F_2$. By Lemma 2.8, $\mathrm{Supp}F_2$ consists only of the (-3)-curve D_2, some (-1)-curves and some (-2)-curves. It follows from [14, Lemma 1.6] that $\mathrm{Supp}F_2$

has one of the configurations (i)∼(v) in [14, Picture (2) in Lemma 1.6]. However, this is impossible.

Therefore, this subcase does not take place.

The proof of Claim 4.9.4 is thus completed. □

Theorefore, X contains $\mathbb{C}^2$ as a Zariski open subset.

4.10. Case (17)

In this subsection, we treat the case where the weighted dual graph of D is (17) in Theorem 1.1. Let $D = \sum_{i=0}^{7+t} D_i$ be the decomposition of D into irreducible components such that the weighted dual graph of D is given as in Figure 4.10, where $D_0^2 = -(t+3)$ and the weight of the vertex corresponding to a (-2)-curve is omitted. In this case, $\rho(V) = 9 + t$.

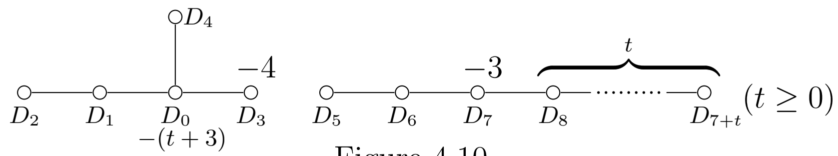


Figure 4.10.

Let α_i $(i = 0, 1, \ldots, 7+t)$ be the coefficient of D_i in $D^{\#}$. Then

$$\alpha_0 = \frac{12t+18}{12t+19}, \quad \alpha_1 = \frac{8t+12}{12t+19}, \quad \alpha_2 = \frac{4t+6}{12t+19}, \quad \alpha_3 = \frac{9t+14}{12t+19},$$

$$\alpha_4 = \frac{6t+9}{12t+19}, \quad \alpha_5 = \frac{t+1}{4t+7}, \quad \alpha_6 = \frac{2(t+1)}{4t+7},$$

$$\alpha_{7+i} = \frac{3(t+1-i)}{4t+7} \quad (i = 0, 1, \ldots, t).$$

Let $C \in \mathrm{MV}(V, D)$. By Lemmas 4.1 and 4.2, $CD_0 = 0$ and $CD^{(1)} = 1$.

Claim 4.10.1. $CD^{(2)} = 1$.

Proof. The assertion can be proved by using the same argument as in the proof of Claim 4.5.1. □

We take $i \in \{1, 2, 3, 4\}$ and $j \in \{5, 6, \ldots, 7+t\}$ such that $CD_i = CD_j = 1$.

Claim 4.10.2. If $j = 5$, then $i = 3$ and $V \setminus \mathrm{Supp}(C + D) \cong \mathbb{C}^2$.

Proof. Assume that $i = 3$. Then there exists a positive integer n and an effective divisor Δ such that $\mathrm{Supp}\Delta = \mathrm{Supp}(D - D_2)$, $nC + \Delta$ defines a $\mathbb{P}^1$-fibration $\Phi_{|nC+\Delta|} : V \to \mathbb{P}^1$ and D_2 becomes a section of $\Phi_{|nC+\Delta|}$. (We can wright down the divisor $nC + \Delta$ explicitly; we omit the description.) It is then clear that $V \setminus \mathrm{Supp}(C + D) \cong \mathbb{C}^2$. (See the arguments as in Section 3.)

Suppose that $i \neq 3$. Then D_i is a (-2)-curve and so the divisor $D_i + D_5 + 2C$ defines a $\mathbb{P}^1$-fibration $\Phi_{|D_i+D_5+2C|} : V \to \mathbb{P}^1$. Then D_7, that is a (-3)-curve, is a fiber component of $\Phi_{|D_i+D_5+2C|}$. This contradicts Lemma 2.7. □

Claim 4.10.3. The case $j = 6$ does not take place.

Proof. Suppose to the contrary that $j = 6$. Since $CD^{\#} = \alpha_i + \alpha_6 < 1$, $i \neq 3$. Then D_i is a (-2)-curve and so the divisor $D_i + D_6 + 2C$ defines a $\mathbb{P}^1$-fibration $\Phi_{|D_i+D_6+2C|} : V \to \mathbb{P}^1$. Then D_3, that is a (-4)-curve, is a fiber component of $\Phi_{|D_i+D_6+2C|}$. This contradicts Lemma 2.7. This proves Claim 4.10.3. □

Claim 4.10.4. The case $j = 7$ does not take place.

Proof. Suppose to the contrary that $j = 7$. Since $CD^{\#} = \alpha_i + \alpha_7 < 1$, $i = 2$ or 4. We consider the following subcases separately.

Subcase 1: $i = 4$. Since $CD^{\#} = \alpha_4 + \alpha_7 < 1$, $t = 0$. The divisor $F_1 := D_0 + D_6 + 2D_7 + 3D_4 + 5C$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F_1|} : V \to \mathbb{P}^1$, D_1, D_3 and D_5 become sections of Φ and $D - (D_1 + D_3 + D_4)$ is contained in fibers of Φ. Let F_2 be the fiber of Φ containing D_2. Since $\mathrm{Supp}F_2$ consists only of D_2 and some (-1)-curves, we infer from Lemma 2.6 (2) that $F_2 = E_{2,1} + D_2 + E_{2,2}$, where $E_{2,1}$ and $E_{2,2}$ are (-1)-curves and $E_{2,1}D_2 = E_{2,2}D_2 = 1$. Since D_3 is a section of Φ, we may assume that $E_{2,1}D_3 = 1$. Then $E_{2,1}D^{\#} \geq \alpha_2 + \alpha_3 > 1$, a contradiction. Therefore, this subcase does not take place.

Subcase 2: $i = 2$. The divisor $F := D_1 + D_7 + 2D_2 + 3C$ defines a $\mathbb{P}^1$-fibration $\Phi_{|F|} : V \to \mathbb{P}^1$. Then D_3, that is a (-4)-curve, becomes a fiber component of $\Phi_{|F|}$. This contradicts Lemma 2.8. Therefore, this subcase does not take place.

The proof of Claim 4.10.4 is thus completed. □

Claim 4.10.5. If $j \geq 8$, then $t = 2$, $j = 9$ and $V \setminus \mathrm{Supp}(C + D) \cong \mathbb{C}^2$.

Proof. Suppose that $j \geq 8$. Then D_j is a (-2)-curve and $t \geq 1$. We consider the following subcases separately.

Subase 1: $i \neq 3$. By using the same argument as in the proof of Claim 4.10.3, we know that this subcase does not take place.

Subcase 2: $i = 3$, $t \geq 3$ and $9 \leq j \leq 6 + t$. The divisor $F := D_{j-1} + D_{j+1} + 2(C + D_j)$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F|} : V \to \mathbb{P}^1$. Then D_0, that is a $(-t-3)$-curve, is a fiber component of Φ. This contradicts Lemma 2.7 because $t + 3 \geq 6$. Therefore, this subcase does not take place.

Subcase 3: $i = 3$ and $j = 7 + t$. Since $CD^{\#} = \alpha_3 + \alpha_{7+t} < 1$, $t \geq 2$. If $t = 2$, then Claim 4.10.2 implies that $V \setminus \mathrm{Supp}(C + D) \cong \mathbb{C}^2$. Suppose that $t \geq 3$. Then the divisor $F_1 := D_3 + D_{5+t} + 2D_{6+t} + 3D_{7+t} + 4C$ defines a $\mathbb{P}^1$-fibration

$\Phi := \Phi_{|F_1|} : V \to \mathbb{P}^1$, D_0 and D_{4+t} become sections of Φ and $D - (D_0 + D_4)$ is contained in fibers of Φ. Let F_2 (resp. F_3) be the fiber of Φ containing $D_1 + D_2$ (resp. D_4). Then F_1, F_2 and F_3 exhaust the singular fibers of Φ. Indeed, if G is a singular fiber of Φ other than F_1, F_2 and F_3, then the component of G meeting D_0 is a (-1)-curve. This contradicts Lemma 4.1. At least one of SuppF_2 and SuppF_3 contains no components of $D^{(2)}$.

Suppose that SuppF_2 contains no components of $D^{(2)}$. Since SuppF_2 then consists only of D_1, D_2 and some (-1)-curves, we infer from Lemma 2.6 (2) that $F_2 = E_{2,1} + D_1 + D_2 + E_{2,2}$, where $E_{2,1}$ and $E_{2,2}$ are (-1)-curves and $E_{2,1}D_1 = E_{2,2}D_2 = 1$. Since D_{4+t} is a section of Φ and $D^{(2)} - D_{4+t}$ is contained in fibers of Φ, either $E_{2,1}$ or $E_{2,2}$ does not meet $D^{(2)}$. So either $E_{2,1} + D^{(1)}$ or $E_{2,2} + D^{(1)}$ has negative definite intersection matrix. This contradicts Lemma 2.3.

Suppose that SuppF_3 contains no components of Supp$D^{(2)}$. By using the same argument as in the previous paragraph, we derive a contradiction. Indeed, F_3 is expressed as $F_3 = E_{3,1} + D_4 + E_{3,2}$, where $E_{3,1}$ and $E_{3,2}$ are (-1)-curves and $E_{3,1}D_4 = E_{3,2}D_4 = 1$. Then $E_{3,1} + D$ or $E_{3,2} + D$ has negative definite intersection matrix, which contradicts Lemma 2.3.

Therefore, we see that $t = 2$ and $V \setminus \mathrm{Supp}(C + D) \cong \mathbb{C}^2$.

Subcase 4: $i = 3$ **and** $j = 8$**.** Since $t \geq 1$, $CD^{\#} = \alpha_3 + \alpha_8 > 1$. This is a contradiction. Therefore, this subcase does not take place.

The proof of Claim 4.10.5 is thus completed. □

Therefore, X contains $\mathbb{C}^2$ as a Zariski open subset.

4.11. Case (22)

In this subsection, we treat the case where the weighted dual graph of D is (22) in Theorem 1.1. Let $D = \sum_{i=0}^{8+t} D_i$ be the decomposition of D into irreducible components such that the weighted dual graph of D is given as in Figure 4.11, where $D_0^2 = -(t+3)$ and the weight of the vertex corresponding to a (-2)-curve is omitted. In this case, $\rho(V) = 10 + t$.

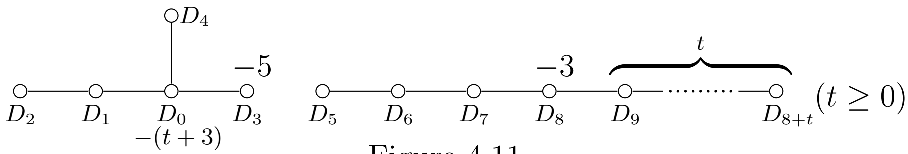


Figure 4.11.

Let α_i $(i = 0, 1, \ldots, 8+t)$ be the coefficient of D_i in $D^{\#}$. Then

$$\alpha_0 = \frac{30t+48}{30t+49}, \quad \alpha_1 = \frac{20t+32}{30t+49}, \quad \alpha_2 = \frac{10t+16}{30t+49}, \quad \alpha_3 = \frac{24t+39}{30t+49},$$

$$\alpha_4 = \frac{15t+24}{30t+49}, \quad \alpha_5 = \frac{t+1}{5t+9}, \quad \alpha_6 = \frac{2(t+1)}{5t+9}, \quad \alpha_7 = \frac{3(t+1)}{5t+9},$$
$$\alpha_{8+i} = \frac{4(t+1-i)}{5t+9} \quad (i = 0, 1, \ldots, t).$$

Let $C \in \mathrm{MV}(V, D)$. By Lemmas 4.1 and 4.2, $CD_0 = 0$ and $CD^{(1)} = 1$.

Claim 4.11.1. $CD^{(2)} = 1$.

Proof. The assertion can be proved by using the same argument as in the proof of Claim 4.5.1. □

We take $i \in \{1, 2, 3, 4\}$ and $j \in \{5, 6, \ldots, 8+t\}$ such that $CD_i = CD_j = 1$.

Claim 4.11.2. If $j = 5$, then $i = 3$ and $V \setminus \mathrm{Supp}(C + D) \cong \mathbb{C}^2$.

Proof. The assertion can be proved by using the same argument as in the proof of Claim 4.10.2. □

Claim 4.11.3. The case $j = 6$ does not take place.

Proof. The assertion can be proved by using the same argument as in Subcase 2 in the proof of Claim 4.11.6 given below. □

Claim 4.11.4. The case $j = 7$ does not take place.

Proof. Suppose to the contrary that $j = 7$. Since $CD^{\#} = \alpha_i + \alpha_7 < 1$, $i \neq 3$. The divisor $F := D_5 + D_8 + 2D_6 + 3(C + D_7)$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F|} : V \to \mathbb{P}^1$. Since $i \neq 3$, D_3, that is a (-5)-curve, becomes a fiber component of Φ. This contradicts Lemma 2.8. □

Claim 4.11.5. The case $j = 8$ does not take place.

Proof. Suppose to the contrary that $i = 8$. Since $CD^{\#} = \alpha_i + \alpha_8 < 1$, $i = 2$ or 4. We consider the following subcases separately.

Subcase 1: $i = 4$. Since $CD^{\#} = \alpha_4 + \alpha_8 < 1$, $t = 0$. The divisor $F_1 := D_0 + D_7 + 2D_8 + 3D_4 + 5C$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F_1|} : V \to \mathbb{P}^1$, D_1, D_3 and D_6 become sections of Φ and $D - (D_1 + D_3 + D_6)$ is contained in fibers of Φ. Let F_2 be the fiber of Φ containing D_2. Since $\mathrm{Supp} F_2$ consists only of some (-1)-curves and some (-2)-curves, we infer from Lemma 2.6 (2) that either $F_2 = D_2 + E_2 + D_5$, where E_2 is a (-1)-curve and $E_2 D_2 = E_2 D_5 = 1$, or $F_2 = E_{2,1} + D_2 + E_{2,2}$, where $E_{2,1}$ and $E_{2,2}$ are (-1)-curves and $E_{2,1} D_2 = E_{2,2} D_2 = 1$. If $F_2 = D_2 + D_5 + 2E_2$, then $1 = D_3 F_2 = 2 D_3 E_2$, a contradiction. Suppose that $F_2 = E_{2,1} + D_2 + E_{2,2}$. We may assume that $E_{2,1} D_3 = 1$ since D_3 is a section of Φ. Then

$$E_{2,1} D^{\#} \geq \alpha_2 + \alpha_3 > 1,$$

a contradiction. Therefore, this subcase does not take place.

Subcase 2: $i = 2$**.** The divisor $F := D_1 + D_8 + 2D_2 + 3C$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F|} : V \to \mathbb{P}^1$. Then D_3, that is a (-5)-curve, is a fiber component of Φ. This contradicts Lemma 2.8. Therefore, this subcase does not take place.

The proof of Claim 4.11.5 is thus completed. $\square$

Claim 4.11.6. If $j \geq 9$, then $t = 3$, $j = 11$ and $V \setminus \mathrm{Supp}(C + D) \cong \mathbb{C}^2$.

Proof. The assertion can be proved by using the same argument as in Claim 4.10.5. By the reader's convenience, we reproduce the proof.

Suppose that $j \geq 9$. Then D_j is a (-2)-curve and $t \geq 1$. We consider the following subcases separately.

Subase 1: $i \neq 3$**.** Then D_i is a (-2)-curve and so the divisor $D_i + D_j + 2C$ defines a $\mathbb{P}^1$-fibration $\Phi_{|D_i+D_j+2C|} : V \to \mathbb{P}^1$. Then D_3, that is a (-5)-curve, is a fiber component of $\Phi_{|D_i+D_j+2C|}$. This contradicts Lemma 2.7. Therefore, this subcase does not take place.

Subcase 2: $i = 3$**,** $t \geq 3$ **and** $10 \leq j \leq 7 + t$**.** The divisor $D_{j-1} + D_{j+1} + 2(C + D_j)$ defines a $\mathbb{P}^1$-fibration $\Phi_{|D_{j-1}+D_{j+1}+2(C+D_j)|} : V \to \mathbb{P}^1$. Then D_0, that is a $(-t-3)$-curve, is a fiber component of $\Phi_{|D_{j-1}+D_{j+1}+2(C+D_j)|}$. This contradicts Lemma 2.7 because $t + 3 \geq 6$. Therefore, this subcase does not take place.

Subcase 3: $i = 3$ **and** $j = 8 + t$**.** Since $CD^{\#} = \alpha_3 + \alpha_{8+t} < 1$, $t \geq 3$. If $t = 3$, then Claim 4.11.2 implies that $V \setminus \mathrm{Supp}(C + D) \cong \mathbb{C}^2$.

Suppose that $t \geq 4$. Then the divisor $F_1 := D_3 + D_{5+t} + 2D_{6+t} + 3D_{7+t} + 4D_{8+t} + 5C$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F_1|} : V \to \mathbb{P}^1$, D_0 and D_{4+t} become sections of Φ and $D - (D_0 + D_4)$ is contained in fibers of Φ. Let F_2 (resp. F_3) be the fiber of Φ containing $D_1 + D_2$ (resp. D_4). Then F_1, F_2 and F_3 exhaust the singular fibers of Φ. Indeed, if G is a singular fiber of Φ other than F_1, F_2 and F_3, then the component of G meeting D_0 is a (-1)-curve. This contradicts Lemma 4.1. At least one of $\mathrm{Supp}F_2$ and $\mathrm{Supp}F_3$ contains no components of $D^{(2)}$.

Suppose that $\mathrm{Supp}F_2$ contains no components of $D^{(2)}$. Since $\mathrm{Supp}F_2$ then consists only of D_1, D_2 and some (-1)-curves, we infer from Lemma 2.6 (2) that $F_2 = E_{2,1} + D_1 + D_2 + E_{2,2}$, where $E_{2,1}$ and $E_{2,2}$ are (-1)-curves and $E_{2,1}D_1 = E_{2,2}D_2 = 1$. Since D_{4+t} is a section of Φ and $D^{(2)} - D_{4+t}$ is contained in fibers of Φ, either $E_{2,1}$ or $E_{2,2}$ does not meet $D^{(2)}$. So either $E_{2,1} + D^{(1)}$ or $E_{2,2} + D^{(1)}$ has negative definite intersection matrix. This contradicts Lemma 2.3.

Suppose that $\mathrm{Supp}F_3$ contains no components of $\mathrm{Supp}D^{(2)}$. By using the same argument as in the previous paragraph, we derive a contradiction.

Subcase 4: $i = 3$ **and** $j = 9$**.** Since $t \geq 1$, $CD^{\#} = \alpha_3 + \alpha_9 > 1$. This is a contradiction. Therefore, this subcase does not take place.

Therefore, we see that $t = 3$, $j = 11$ and $V \setminus \mathrm{Supp}(C + D) \cong \mathbb{C}^2$. □

Theorefore, X contains $\mathbb{C}^2$ as a Zariski open subset.

4.12. Case (12)

In this subsection, we treat the case where the weighted dual graph of D is (12) in Theorem 1.1. Let $D = \sum_{i=0}^{5+t} D_i$ be the decomposition of D into irreducible components such that the weighted dual graph of D is given as in Figure 4.12, where $D_0^2 = -(t+3)$ and the weight of the vertex corresponding to a (-2)-curve is omitted. In this case, $\rho(V) = 7 + t$.

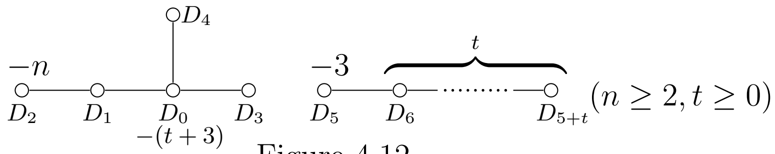


Figure 4.12.

Let α_i $(i = 0, 1, \ldots, 5+t)$ be the coefficient of D_i in $D^{\#}$. Then

$$\alpha_0 = \frac{(2n-1)t + 3n - 3}{(2n-1)t + 3n - 2}, \quad \alpha_1 = \frac{(2n-2)t + 3n - 4}{(2n-1)t + 3n - 2}, \quad \alpha_2 = \frac{(2n-3)t + 3n - 5}{(2n-1)t + 3n - 2},$$

$$\alpha_3 = \alpha_4 = \frac{(2n-1)t + 3n - 2}{2\{(2n-1)t + 3n - 2\}}, \quad \alpha_{5+i} = \frac{t + 1 - i}{2t + 3} \quad (i = 0, 1, \ldots, t).$$

Let $C \in \mathrm{MV}(V, D)$. By Lemmas 4.1 and 4.2, $CD_0 = 0$ and $CD^{(1)} = 1$.

Claim 4.12.1. $CD^{(2)} = 1$.

Proof. Suppose to the contrary that $CD^{(2)} = 0$. Then $CD^{(1)} = CD_i = 1$ for some $i \in \{1, 2, 3, 4\}$. By Lemma 2.3, we know that $i = 1$, $n = 2$ and $t = 0$. Then $(D_3 + D_4 + 2(D_0 + D_2) + 4(C + D_1))^2 = 0$ and so the intersection matrix of $C + D^{(1)}$ is negative semidefinite, which contradicts Lemma 2.3. This proves the claim. □

We take $i \in \{1, 2, 3, 4\}$ and $j \in \{5, 6, \ldots, 5+t\}$ such that $CD_i = CD_j = 1$. By the dual graph of $D^{(1)}$, we may assume that $i \leq 3$.

Claim 4.12.2. If $j = 5$, then $i = 3$ and $V \setminus \mathrm{Supp}(C + D) \cong \mathbb{C}^2$.

Proof. Assume that $i = 3$. Then there exists a positive integer n and an effective divisor Δ such that $\mathrm{Supp}\Delta = \mathrm{Supp}(D - D_2)$, $mC + \Delta$ defines a $\mathbb{P}^1$-fibration $\Phi_{|mC+\Delta|} : V \to \mathbb{P}^1$ and D_2 becomes a section of $\Phi_{|nC+\Delta|}$. (We can wright down the divisor $mC + \Delta$ explicitly; we omit the description.) It is then clear that $V \setminus \mathrm{Supp}(C + D) \cong \mathbb{C}^2$. (See the arguments as in Section 3.)

Suppose that $i \neq 3$. Then $i = 1$ or 2. We consider the following subcases separately.

Subcase 1: $i = 1$. Since $CD^{\#} = \alpha_1 + \alpha_5 < 1$, $n = 2$. Then the divisor $F_1 := D_2 + D_5 + 2D_1 + 3C$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F_1|} : V \to \mathbb{P}^1$ and D_0 becomes a 2-section of Φ. Moreover, if $t \geq 1$, then D_6 becomes a section of Φ and $D-(D_0+D_6)$ is contained in fibers of Φ. Let F_2 be the fiber of Φ containing D_3. Since $\mathrm{Supp}F_2$ consists only of some (-1)-curves and some (-2)-curves, we infer from Lemma 2.6 (2) that $\#F_2 = 3$.

Suppose that $D_4 \subset \mathrm{Supp}F_2$. Then $F_2 = D_3 + D_4 + 2E_2$, where E_2 is a (-1)-curve and $E_2 D_3 = E_2 D_4 = 1$. If $t \geq 1$, then $1 = D_6 F_2 = 2D_6 E_2$, a contradiction. So $t = 0$. Since

$$7 = \rho(V) \geq 2 + (\#F_1 - 1) + (\#F_2 - 1) = 7,$$

F_1 and F_2 exhaust the singular fibers of Φ. Let $\nu : V \to W$ be the contraction of E_2, D_4, C, D_1 and D_2. Then W is a Hirzebruch surface, $\nu(D_0)$ is a smooth rational curve with $\nu(D_0)^2 = -3 + 1 + 2 = 0$ and $\nu(D_0)$ is a 2-section of the ruling $\Phi \circ \nu^{-1} : W \to \mathbb{P}^1$. Let M (resp. ℓ) be a minimal section (resp. a fiber) of the ruling $\Phi \circ \nu^{-1} : W \to \mathbb{P}^1$. Then $\nu(D_0) \sim 2M + \alpha\ell$ for some integer α. Since $\nu(D_0)^2 = 0$, $\alpha = -M^2$. Then $\nu(D_0)(\nu(D_0) + K_W) = -4$, which is a contradiction.

Suppose that $D_4 \not\subset \mathrm{Supp}F_2$. Let F_3 be the fiber of Φ containing D_4. Then at least one of $\mathrm{Supp}F_2$ and $\mathrm{Supp}F_3$ contains no components of $D^{(2)}$. So we may assume that $\mathrm{Supp}F_2$ contains no components of $D^{(2)}$. Then $F_2 = E_{2,1} + D_3 + E_{2,2}$, where $E_{2,1}$ and $E_{2,2}$ are (-1)-curves and $E_{2,1}D_3 = E_{2,2}D_3 = 1$. If $t \geq 1$, then we may assume that $E_{2,2}D_6 = 1$ since D_6 is a section of Φ. Then $E_{2,1}D = E_{2,1}D_3 = 1$ and so the intersection matrix of $E_{2,1} + D$ is negative definite. This contradicts Lemma 2.3.

Therefore, this subcase does not take place.

Subcase 2: $i = 2$. Since $CD^{\#} = \alpha_2 + \alpha_5 < 1$, $n \leq 3$. If $n = 3$, then $CD^{\#} = \alpha_2 + \alpha_5 < 1$ implies that $t \leq 1$. So the intersection matrix of $C + D$ is negative definite, which contradicts Lemma 2.3.

Suppose that $n = 2$. Then the divisor $F_1 := D_1 + D_5 + 2D_2 + 3C$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F_1|} : V \to \mathbb{P}^1$ and D_0 becomes a section of Φ. Furthermore, if $t \geq 1$, then D_6 becomes a section of Φ and $D-(D_0+D_6)$ is contained in fibers of Φ. Let F_2 (resp. F_3) be the fiber of Φ containing D_3 (resp. D_4). Then F_1, F_2 and F_3 exhaust the singular fibers of Φ. Indeed, if G is a singular fiber of Φ other than F_1, F_2 and F_3, then the component of G meeting D_0 is a (-1)-curve. This contradicts Lemma 4.1. Since $D^{(2)} - (D_5 + D_6)$ is contained in a fiber of Φ provided $t \geq 2$, at least one of $\mathrm{Supp}F_2$ and $\mathrm{Supp}F_3$ contains no components onf $D^{(2)}$. We may assume that $\mathrm{Supp}F_2$ contains no components of $D^{(2)}$. Since $\mathrm{Supp}F_2$ consists only of D_3 and some (-1)-curves, we infer from Lemma 2.6 (2) that $F_2 = E_{2,1} + D_3 + E_{2,2}$, where $E_{2,1}$ and $E_{2,2}$ are (-1)-curves and $E_{2,1}D_3 = E_{2,2}D_3 = 1$. By using the same argument as in the third paragraph of Subcase 1, we derive a contradiction. Therefore, this subcase does not take place.

The proof of Claim 4.12.2 is thus completed. □

Claim 4.12.3. The case $j \geq 6$ does not take place.

Proof. Suppose to the contrary that $j \geq 6$. Then $t \geq 1$ and D_j is a (-2)-curve. We consider the following subcases separately.

Subcase 1: $i = 2$ and $n = 2$. The divisor $D_2 + D_j + 2C$ defines a $\mathbb{P}^1$-fibration $\Phi_{|D_2+D_j+2C|} : V \to \mathbb{P}^1$. Then D_0, that is a $(-t-3)$-curve, is a fiber component of $\Phi_{|D_2+D_j+2C|}$. This contradicts Lemma 2.7. Therefore, this subcase does not take place.

Subcase 2: $t \geq 3$ and $7 \leq j \leq 4+t$. The divisor $D_{j-1} + D_{j+1} + 2(C + D_j)$ defines a $\mathbb{P}^1$-fibration $\Phi_{|D_{j-1}+D_{j+1}+2(C+D_j)|} : V \to \mathbb{P}^1$. Then D_0 is a fiber component of $\Phi_{|D_{j-1}+D_{j+1}+2(C+D_j)|}$ because $CD_0 = 0$ by Lemma 4.1. This contradicts Lemma 2.7 because $D_0^2 = -(t+3) \leq -3$. Therefore, this subcase does not take place.

Subcase 3: $t \geq 2$ and $j = 5+t$. By Subcase 1, we may assume that $i \neq 2$ or $n \geq 3$. If $i = 1$ or 3, then the divisor $F := D_i + D_{5+t} + 2C$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F|} : V \to \mathbb{P}^1$. Then D_5, that is a (-3)-curve, is a fiber component of Φ. This contradicts Lemma 2.7. Hence, $i = 2$ and $n \geq 3$. Lemma 2.3 implies that $n \leq t+2$.

(3-1) Suppose further that $n = t+2$. Then the divisor $F_1 := D_1 + D_5 + 2D_2 + 3D_6 + 5D_7 + \cdots + (2t+1)D_{5+t} + (2t+3)C$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F_1|} : V \to \mathbb{P}^1$, D_0 becomes a section of Φ and $D - D_0$ is contained in fibers of Φ. Here we note that $\#F_1 = t+4$. Let F_2 (resp. F_3) be the fiber of Φ containing D_3 (resp. D_4). By the same argument as in the second paragraph of Subcase 2 in the proof of Claim 4.12.2, we know that F_1, F_2 and F_3 exhaust the singular fibers of Φ. Since $\#F_2$, $\#F_3 \geq 3$, we have

$$7 + t = \rho(V) = 2 + (\#F_1 - 1) + (\#F_2 - 1) + (\#F_3 - 1) \geq 9 + t,$$

which is a contradiction.

(3-2) By the argument as in (3-1), we see that $n \leq t+1$. Then the divisor $G_1 := D_2 + D_{7+t-n} + 2D_{8+t-n} + \cdots + (n-1)D_{5+t} + nC$ defines a $\mathbb{P}^1$-fibration $\Psi := \Phi_{|G_1|} : V \to \mathbb{P}^1$, D_1 and D_{6+t-n} become sections of Ψ and $D - (D_1 + D_{6+t-n})$ is contained in fibers of Ψ. Let G_2 be the fiber of Ψ containing $D_0 + D_3 + D_4$. If SuppG_2 contains no components of $D^{(2)}$, then the component $E_{2,1}$ of SuppG_2 meeting D_{6+t-n} is a (-1)-curve. Since D_{6+t-n} is a section of Ψ, SuppG_2 has a (-1)-curve $E_{2,2}$ other than $E_{2,1}$. Then $E_{2,2}D = E_{2,2}(D_0 + D_3 + D_4) = 1$ and so the intersection matrix of

$E_{2,2} + D$ is negative definite. This contradicts Lemma 2.3. Hence, SuppG_2 contains $D_5 + \cdots + D_{5+t-n}$.

Since SuppG_2 then consists only of $D_0, D_3, D_4, D_5, \ldots, D_{5+t-n}$ and some (-1)-curves, SuppG_2 has a (-1)-curve E_2 such that $E_2(D_0 + D_3 + D_4) = E_2(D_5 + \cdots + D_{5+t-n}) = 1$. Furthermore, since the intersection matrix of $D_k + E_2 + D_\ell$ is negative definite, where D_k ($k \in \{0, 3, 4\}$) and D_ℓ ($\ell \in \{5, \ldots, 5+t-n\}$) are curves meeting E_2, and by Lemma 4.1, we may assume that $E_2D_3 = E_2D_5 = 1$. Then the intersection matrix of $E_2 + D_0 + D_3 + D_4 + D_5 + \cdots + D_{5+t-n}$ is negative definite. So SuppG_2 contains a (-1)-curve E_2' other than E_2. Then we have $E_2'D = 1$. If $E_2'D^{(1)} = 1$, then $E_2'D^{(2)} = 0$ and so the intersection matrix of $E_2' + D$ is negative definite. This contradicts Lemma 2.3. If $E_2'D^{(2)} = 1$, then the intersection matrix of $D_3 + E_2 + D_5 + \cdots + D_{5+t-n} + E_2'$ is not negative definite, which contradicts Supp$(D_3 + E_2 + D_5 + \cdots + D_{5+t-n} + E_2') \subsetneq$ SuppF_2.

Therefore, this subcase does not take place.

Subcase 4: $j = 6$**.** If $t \geq 3$, then the divisor $F := D_5 + D_8 + 2D_7 + 3(C + D_6)$ defines a $\mathbb{P}^1$-fibration $\Phi_{|F|} : V \to \mathbb{P}^1$. Then D_0, that is a $(-t-3)$-curve, is a fiber component of $\Phi_{|F|}$. This contradicts Lemma 2.8 since $t + 3 \geq 6$. Hence, $t \leq 2$.

(4-1) We consider the case where $i = 3$. The divisor $F_1 := D_3 + D_6 + 2C$ defines a $\mathbb{P}^1$-fibration $\Phi_1 := \Phi_{|F_1|} : V \to \mathbb{P}^1$ and D_0 and D_5 become sections of Φ. Further, if $t = 2$, then D_7 becomes a section of Φ_1 and $D - (D_0 + D_5 + D_7)$ is contained in fibers of Φ_1. Let F_2 (resp. F_3) be the fiber of Φ_1 containing $D_1 + D_2$ (resp. D_4). By the argument as in Subcase 2 in the proof of Claim 4.12.2, we know that F_1, F_2 and F_3 exhaust the singular fibers of Φ_1. Since $t \leq 2$, SuppF_3 consists only of D_4 and some (-1)-curves. We infer from Lemmas 2.6 (2) that $F_3 = E_{3,1} + D_4 + E_{3,2}$, where $E_{3,1}$ and $E_{3,2}$ are (-1)-curves and $E_{3,1}D_4 = E_{3,2}D_4 = 1$. Since D_5 is a section of Φ, we may assume that $E_{3,1}D_5 = 1$. Then

$$-E_{3,1}(D^\# + K_V) \leq 1 - (\alpha_4 + \alpha_5) < 1 - (\alpha_3 + \alpha_6) = -C(D^\# + K_V),$$

which contradicts $C \in \mathrm{MV}(V, D)$.

(4-2) We consider the case where $i = 1$. The divisor $G_1 := D_1 + D_6 + 2C$ defines a $\mathbb{P}^1$-fibration $\Phi_2 := \Phi_{|G_1|} : V \to \mathbb{P}^1$ and D_0, D_2 and D_5 become sections of Φ_2. Further, if $t = 2$, then D_7 becomes section of Φ and $D - (D_0 + D_2 + D_5 + D_7)$ is contained in fibers of Φ_2. Let G_2 (resp. G_3) be the fiber of Φ containing D_3 (resp. D_4). By the argument as in Subcase 2 in the proof of Claim 4.12.2, we know that G_1, G_2 and G_3 exhaust the singular fibers of Φ_2. Since SuppG_2 and SuppG_3 contain no components of $D^{(2)}$, we infer from Lemma 2.6 (2) that $G_2 = E_{2,1} + D_3 + E_{2,2}$ and $G_3 = E_{3,1} + D_4 + E_{3,2}$, where $E_{2,1}$, $E_{2,2}$, $E_{3,1}$ and $E_{3,2}$ are (-1)-curves and

$E_{2,1}D_3 = E_{2,2}D_3 = E_{3,1}D_4 = E_{3,2}D_4 = 1$. Then

$$7 + t = \rho(V) = 2 + (\#G_1 - 1) + (\#G_2 - 1) + (\#G_3 - 1) = 8$$

and so $t = 1$. Since D_2 is a section of Φ_2, we may assume that $E_{2,1}D_2 = E_{3,1}D_2 = 1$. Then $E_{2,2}D_2 = E_{3,2}D_2 = 0$.

Let $\nu : V \to W$ be the contraction of C, D_6, $E_{2,1}$, $E_{2,2}$, $E_{3,1}$ and $E_{3,2}$. Then $W = \Sigma_4$ and $\nu(D_0)$ is the minimal section of Σ_4. Then $\nu(D_2)$ is the section of the ruling on W and

$$\nu(D_2)^2 = -n + 2 \le 0.$$

This is a contradiction.

(4-3) We consider the case where $i = 2$ and $t = 1$. By Lemma 2.3, we know that $n \le 3$. If $n = 2$, then the divisor $D_2 + D_6 + 2C$ defines a $\mathbb{P}^1$-fibration $\Phi_{|D_2+D_6+2C|} : V \to \mathbb{P}^1$. Then D_0, that is a (-4)-curve, is a fiber component of Φ. This contradicts Lemma 2.7.

If $n = 3$, then the divisor $H_1 := D_1 + D_5 + 2D_2 + 3D_6 + 5C$ defines a $\mathbb{P}^1$-fibration $\Phi_3 := \Phi_{|H_1|} : V \to \mathbb{P}^1$ and D_0 becomes a section of Φ. Let H_2 (resp. H_3) be the fiber of Φ_3 containing D_3 (resp. D_4). By the argument as in Subcase 2 in the proof of Claim 4.12.2, we know that H_1, H_2 and H_3 exhaust the singular fibers of Φ_3. Furthermore, $\#H_2, \#H_3 \ge 3$. Then we have

$$8 = \rho(V) = 2 + (\#H_1 - 1) + (\#H_2 - 1) + (\#H_3 - 1) \ge 10,$$

which is a contradiction.

(4-4) We consider the case where $i = 2$ and $t = 2$. The divisor $C + D^{(2)}$ can be contracted to a smooth point. Let $\mu : V \to V'$ be the contraction of $C + D^{(2)}$ to a smooth point and set $D' = \mu(D_0 + D_1 + D_3 + D_4) = \mu_*(D - D_2)$. Then $\rho(V') = 5 = 1 + \#D'$. Since $\overline{\kappa}(V \setminus \mathrm{Supp}D) = -\infty$ by [14, Remark 1.2 (2)], where $\overline{\kappa}(V \setminus \mathrm{Supp}D)$ denotes the logarithmic Kodaira dimension of $V \setminus \mathrm{Supp}D$ (cf. Introduction), we have $\overline{\kappa}(V \setminus \mathrm{Supp}(D - D_2)) = -\infty$. This implies that

$$\overline{\kappa}(V' \setminus \mathrm{Supp}D') = \overline{\kappa}(V \setminus \mathrm{Supp}(C + D - D_2)) = \overline{\kappa}(V \setminus \mathrm{Supp}(D - D_2)) = -\infty,$$

where the second equality follows from $C(D - D_2) = 1$. We infer from [14, Remark 1.2 (2)] that (V', D') is an LDP1-surface. On the other hand, the weighted dual graph of D' is given as in Figure 4.13. This weighted dual graph is not give in [7, Appendix A]. Therefore, this case does not take place.

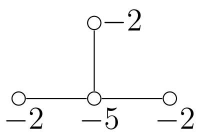


Figure 4.13.

Therefore, we know that Subcase 4 does not take place.

The proof of Claim 4.12.3 is thus completed. □

Therefore, X contains $\mathbb{C}^2$ as a Zariski open subset.

The proof of Theorem 1.1 is thus completed.

Acknowledgement. The authors would like to express their gratitude to the referee for giving useful comments and suggestions to improve the paper.

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(H. Kojima) Department of Mathematics, Faculty of Science, Niigata University, 8050 Ikarashininocho Nishi-ku, Niigata 950-2181, Japan.
E-mail address: kojima@math.sc.niigata-u.ac.jp

(T. Takahashi) Education Center for Engineering and Technology, Faculty of Engineering, Niigata University, 8050 Ikarashininocho Nishi-ku, Niigata 950-2181, Japan.
E-mail address: takeshi@eng.niigata-u.ac.jp

Received September 13, 2018
Revised December 7, 2018