Nihonkai Math. J.
Vol.28(2017), 99–104

A NOTE ON GALOIS EMBEDDING AND ITS APPLICATION TO $\mathbb{P}^n$

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ABSTRACT. We show a condition that a Galois covering $\pi : V \longrightarrow \mathbb{P}^n$ is induced by a Galois embedding. Then we consider the Galois embedding for $\mathbb{P}^n$. If the Galois group G is abelian, then $G \cong \bigoplus^n Z_d$ and the projection π can be expressed as $\pi(X_0 : X_1 : \cdots : X_n) = ({X_0}^d : {X_1}^d : \cdots : {X_n}^d)$.

1. Introduction

This is a continuation of our previous paper [5]. First we recall the definition and some results of Galois embeddings. Let k be the ground field of our discussion, we assume it to be the field of complex numbers, however most results hold also for an algebraically closed field of characteristic zero. Let V be a nonsingular projective algebraic variety of dimension n with a very ample divisor D; we denote this by a pair (V, D). Let $f = f_D : V \hookrightarrow \mathbb{P}^N$ be the embedding of V associated with the complete linear system $|D|$, where $N + 1 = \dim H^0(V, \mathcal{O}(D))$. Suppose that W is a linear subvariety of $\mathbb{P}^N$ satisfying $\dim W = N - n - 1$ and $W \cap f(V) = \emptyset$. Consider the projection π_W with the center W, $\pi_W : \mathbb{P}^N \dashrightarrow W_0$, where W_0 is an n-dimensional linear subvariety not meeting W. The composition $\pi = \pi_W \circ f$ is a surjective morphism from V to $W_0 \cong \mathbb{P}^n$.

Let $K = k(V)$ and $K_0 = k(W_0)$ be the function fields of V and W_0 respectively. The covering map π induces a finite extension of fields $\pi^* : K_0 \hookrightarrow K$, the degree of which is $\deg f(V) = D^n$: the self-intersection number of D. It is easy to see that the structure of this extension does not depend on the choice of W_0 but only on W, hence we denote by K_W the Galois closure of this extension and by $G_W = Gal(K_W/K_0)$ the Galois group of K_W/K_0. Note that G_W is isomorphic to the monodromy group of the covering $\pi : V \longrightarrow W_0$, see [2].

2010 *Mathematics Subject Classification.* Primary 14N05, 14N10; Secondary 14J99.

Key words and phrases. Galois embedding, Galois group, projective space.

This work is based on research 15K04813 supported by Grant-in-Aid for Scientific Research (C) from Japan Society for the Promotion of Science.

Definition 1.1. In the above situation we call G_W the Galois group at W. If the extension K/K_0 is Galois, then we call f and W a Galois embedding and a Galois subspace for the embedding respectively.

Definition 1.2. A nonsingular projective algebraic variety V is said to have a Galois embedding if there exist a very ample divisor D satisfying that the embedding associated with $|D|$ has a Galois subspace. In this case the pair (V, D) is said to define a Galois embedding.

Hereafter we use the following notation and convention:

· $\mathrm{Aut}(V)$: the automorphism group of a variety V
· $\langle a_1, \cdots, a_m \rangle$: the subgroup generated by $a_1, \cdots, a_m$
· Z_m : the cyclic group of order m
· D_{2m} : the dihedral group of order $2m$
· $|G|$: the order of a group G

By definition, if W is the Galois subspace, then each element σ of G_W is an automorphism of $K = K_W$ over K_0. Therefore it induces a birational transformation of V over V_0. This implies that G_W can be viewed as a subgroup of $\mathrm{Bir}(V/W_0)$: the group of birational transformations of V over W_0. Further we can say the following:

Representation 1. ([5]) *Each element of G_W turns out to be regular on V, hence we have the representation*

$$\alpha : G_W \hookrightarrow \mathrm{Aut}(V).$$

Therefore, if the order of $\mathrm{Aut}(V)$ is small, then V cannot have a Galois embedding. On the other hand, we have examples such that there exist infinitely many distinct Galois embeddings, see Example 4.1 in [5].

When (V, D) defines a Galois embedding, we identify $f(V)$ with V. Let H be a hyperplane of $\mathbb{P}^N$ containing W and D' the intersection divisor of V and H. Since D' is linearly equivalent to D and $\sigma^*(D') = D'$ for any $\sigma \in G_W$, we see that σ induces an automorphism of $H^0(V, \mathcal{O}(D))$.

Representation 2. ([5]) *We have the second representation*

$$\beta : G_W \hookrightarrow \mathrm{PGL}(N, \mathbb{C}).$$

In the case where W is a Galois subspace we identify $\sigma \in G_W$ with $\beta(\sigma) \in \mathrm{PGL}(N, \mathbb{C})$ hereafter. Since G_W is a finite subgroup of $\mathrm{Aut}(V)$, we can consider the quotient V/G_W and let π_G be the quotient morphism, $\pi_G : V \longrightarrow V/G_W$.

Proposition 1.1. ([5]) *If (V, D) defines a Galois embedding with the Galois subspace W such that the projection is $\pi_W : \mathbb{P}^N \dashrightarrow W_0$, then there exists an isomorphism $g : V/G_W \longrightarrow W_0$ satisfying $g \circ \pi_G = \pi$. Hence the projection π is a finite morphism and the fixed loci of G_W consists of only divisors.*

Therefore, π turns out to be a Galois covering in the sense of Namba [3], the definition is as follows:

Definition 1.3. A branched covering $\pi : X \longrightarrow M$ is called a Galois covering if the covering transformation group acts transitively on every fiber of π.

Now we present the criterion that (V, D) defines a Galois embedding.

Theorem 1.1. ([5]) *The pair (V, D) defines a Galois embedding if and only if the following conditions hold*:

(1) *There exists a subgroup G of* $\mathrm{Aut}(V)$ *satisfying that $|G| = D^n$.*
(2) *There exists a G-invariant linear subspace $\mathcal{L}$ of $H^0(V, \mathcal{O}(D))$ of dimension $n+1$ such that, for any $\sigma \in G$, the restriction $\sigma^*|_{\mathcal{L}}$ is a multiple of the identity.*
(3) *The linear system $\mathcal{L}$ has no base points.*

We have several results and problems for the Galois embedding, see the website; `http://hyoshihara.web.fc2.com/`

In this article we consider the condition that Galois covering is induced by Galois embedding. Next we consider the Galois embedding for $\mathbb{P}^n$. In the case where $n = 1$, each group of the Galois covering $\mathbb{P}^1 \longrightarrow \mathbb{P}^1$ can appear as the group of Galois embedding, see [4]. It is well-known that the Galois group is isomorphic to Z_d, D_{2m}, A_4, S_4 or A_5.

Here we notice that there exist Galois subspaces with distinct Galois groups for some Galois embedding.

Remark 1.1. The 12-uple embedding of $\mathbb{P}^1$: $f(X_0, X_1) = ({X_0}^{12} : {X_0}^{11}X_1 : \cdots : {X_1}^{12})$ has three Galois subspaces such that the Galois groups are Z_{12}, D_{12} and A_4. Indeed, let $(Y_0 : Y_1 : \cdots : Y_{12})$ be coordinates on $\mathbb{P}^{12}$. Since the Galois covering corresponding to the Galois groups are given by $({X_0}^{12} : {X_1}^{12})$, $({X_0}^6{X_1}^6 : ({X_0}^6 - {X_1}^6)^2)$ and $(({X_0}^4 - 2\sqrt{3}{X_0}^2{X_1}^2 - {X_1}^4)^3 : ({X_0}^4 + 2\sqrt{3}{X_0}^2{X_1}^2 - {X_1}^4)^3)$ respectively ([4]), the Galois subspaces are given by $Y_0 = Y_{12} = 0$, $Y_6 = Y_0 - 2Y_6 + Y_{12} = 0$ and $Y_{12} + 33Y_8 - 33Y_4 - Y_0 = Y_{10} + 2Y_6 + Y_2 = 0$, respectivelly.

2. Statement of results

Galois embedding $f : V \hookrightarrow \mathbb{P}^N$ induces the Galois covering $\pi : V \longrightarrow \mathbb{P}^n$ by definition, but the converse assertion does not hold true. The simplest case is the double covering $\pi : C \longrightarrow \mathbb{P}^1$, where C is a hyperelliptic curve. However, for some case the converse holds true.

Theorem 2.1. *For a Galois covering $\pi : V \longrightarrow \mathbb{P}^n$ if $D = \pi^*(H)$ is a very ample divisor, where H is a hyperplane of $\mathbb{P}^n$, then (V, D) defines a Galois embedding $f : V \hookrightarrow \mathbb{P}^N$ such that $\pi = \pi_W \circ f$, where W is a linear subspace of $\mathbb{P}^N$ and $f(V) \cap W = \emptyset$.*

The following is clear from Theorem 2.1.

Corollary 2.1. *Each Galois covering $\pi : \mathbb{P}^n \longrightarrow \mathbb{P}^n$ is induced by a Galois embedding by $(\mathbb{P}^n, \pi^*(H))$, where H is a hyperplane of $\mathbb{P}^n$.*

Theorem 2.2. *If f is a Galois embedding of $\mathbb{P}^n$ by a divisor of degree d and the Galois group G is abelian, then G is isomorphic to $\bigoplus^{n} Z_d$ and the projection $\pi = \pi_W \circ f : \mathbb{P}^n \longrightarrow \mathbb{P}^n$ can be expressed as*

$$\pi\ (X_0 : X_1 : \cdots : X_n) = ({X_0}^d : {X_1}^d : \cdots : {X_n}^d)$$

by taking a suitable coordinates $(X_0 : X_1 : \cdots : X_n)$ on $\mathbb{P}^n$.

Corollary 2.2. *If f is a Galois embedding of $\mathbb{P}^2$ by a divisor of degree d such that d is a prime number p, then $G \cong Z_p \oplus Z_p$ and $\pi\ (X_0 : X_1 : X_2) = ({X_0}^p : {X_1}^p : {X_2}^p)$.*

3. Proof

First we prove Theorem 2.1. Let $\pi : V \longrightarrow \mathbb{P}^n$ be the Galois covering with the Galois group G. Then we have $|G| = D^n$. Let $f = f_D : V \longrightarrow \mathbb{P}^N$ be the embedding associated with $|D|$ and take $\mathcal{L} = H^0(V,\ \pi^*\mathcal{O}_{\mathbb{P}^n}(1))$. Then $\mathcal{L}$ satisfies the conditions (2) and (3) of Theorem 1.1.

The proof of Corollary 2.1 is clear, because $\pi^*(H)$ is a very ample divisor of $\mathbb{P}^n$.

The proof of Theorem 2.2 is as follows. Let D be the divisor and $f = f_D : \mathbb{P}^n \longrightarrow X \subset \mathbb{P}^N$ be the embedding associated with the complete linear systme $|D|$, i.e., it is the d-uple embedding of $\mathbb{P}^n$ in $\mathbb{P}^N$. Suppose this is a Galois embedding. Then, by definition $p = \pi_W|_X$ is a Galois covering of degree $D^n = d^n$. Hence $|D| = d^n$.

$$\begin{tikzcd}
\mathbb{P}^n \arrow[r, "f"] \arrow[dr, "\pi"'] & X \subset \mathbb{P}^N \arrow[d, shift right, "p"'] \arrow[d, shift left, dotted, "\pi_W"] \\
& \mathbb{P}^n
\end{tikzcd}$$

Since $\sigma \in G$ induces a projective transformation of $\mathbb{P}^N$ and $\sigma(X) = X$, σ can be regarded as a projective transformation of $X \cong \mathbb{P}^n$ and hence $G \subset \mathrm{PGL}(n,k)$. Since X and $\mathbb{P}^n$ are smooth, each ramification locus of p has codimension one. Denote by H an irreducible component of the ramification divisor of p.

Lemma 3.1. *The H is a hyperplane of $\mathbb{P}^n$ and $\sigma(H) = H$ for each $\sigma \in G$.*

Proof. By Proposition 1.1 we can regard π as the quotient map $\mathbb{P}^n \longrightarrow \mathbb{P}^n/G$. If $P \in \mathbb{P}^n$ is a ramification point, then there exists $\tau \in G$ such that $\tau(P) = P$. Therefore, if H is the ramification divisor, then it is an eigen space for some $\tau \in G$, i.e., $\tau|_H = \mathrm{id}$ and $\tau \neq \mathrm{id}$. For each $x \in \sigma(H)$, there exists $a \in H$ such that $x = \sigma(a)$. Since G is assumed to be commutative, we have $\tau(x) = \tau(\sigma(a)) = \sigma(\tau(a)) = \sigma(a) = x$, i.e., $\tau|_{\sigma(H)} = \mathrm{id}$. Suppose there exists another ramification divisor H' of τ. Then, $\tau|_H = \mathrm{id}$ and $\tau|_{H'} = \mathrm{id}$. This implies $\tau = \mathrm{id}$, which is a contradiction. Hence the ramification divisor of τ is unique, so we have $\sigma(H) = H$. □

Fix the ramification divisor H. By Lemma 3.1, G acts on $\mathbb{P}^n \setminus H \cong \mathbb{C}^n$. Hence we can regard as $G \subset \mathrm{GL}(n,k)$. Since G is a finite abelian group, every element of G can be diagonalized simultaneously.

Lemma 3.2. ([1]) *For a vector space V a finite subgroup G of* $\mathrm{GL}(V)$ *of order prime to char(K) is a reflection group if and only if the algebra $S(V)^G$ of invariants in the symmetric algebra of V is isomorphic to a polynomial algebra.*

By definition we have $p(X \setminus H) = (X \setminus H)/G \cong \mathbb{C}^n$, hence by Lemma 3.2, we see G is generated by reflections. Taking the projective coordinates $(X_0 : X_1 : \cdots : X_n)$ such that H is defined by $X_0 = 0$ and put $x_i = X_i/X_0$ $(i = 1, \ldots, n)$. From the above consideration we infer that $G \subset \mathrm{GL}(n,k)$, $G = \langle \sigma_1, \ldots, \sigma_n \rangle$ and σ_i is a diagonal matrix such that (i,i)-component is α_i and the others are 1, where α_i is a root of 1. Thus each ramification divisor of $\mathbb{P}^n \setminus \{X_0 = 0\} \cong \mathbb{C}^n$ is contained in $x_i = 0$ for some i $(1 \leq i \leq n)$. Let $\mathcal{V}$ be the vector space consisting of the polynomial of $k[x_1, \ldots, x_n]$ with degree $\leq d$. Put

$$\mathcal{V}_0 = \{\ P \in \mathcal{V} \mid P^\sigma = P \text{ for each } \sigma \in G\}.$$

The P^{σ_r} can be expressed as

$$P^{\sigma_r} = \sum_{i_1 + \cdots + i_n \leq d} {\alpha_r}^{i_r} c_{i_1,\ldots i_n} {x_1}^{i_1} \cdots {x_n}^{i_n},$$

where

$$P = \sum_{i_1 + \cdots + i_n \leq d} c_{i_1,\ldots i_n} {x_1}^{i_1} \cdots {x_n}^{i_n}.$$

We make use of (2) and (3) of Theorem 1.1. Then the basis of $\mathcal{V}_0$ defines a surjective morphism $\mathbb{C}^n \longrightarrow \mathbb{C}^n$. Since $P^{\sigma_r} = P$, we have $\mathrm{ord}(\alpha_r) \leq d$ for each

$1 \leq r \leq n$. Since $|G| = d^n$, we infer that $\mathrm{ord}(\alpha_r) = d$ for all $1 \leq r \leq n$. Consequently we see $\mathcal{V}_0$ is generated by $\{{x_1}^d, \ldots, {x_n}^d\}$, this proves the theorem.

The proof of Corollary is easy. Since $|G| = p^2$, we have $G \cong Z_{p^2}$ or $Z_p \oplus Z_p$. Then G is abelian, hence only the latter case takes place by the theorem.

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Received September 22, 2016
Revised May 13, 2017