Nihonkai Math. J.
Vol.28(2017), 55–64

REAL HYPERSUFRACES OF NON-FLAT COMPLEX HYPERBOLIC PLANES WHOSE JACOBI STRUCTURE OPERATOR SATISFIES A GENERALIZED COMMUTATIVE CONDITION

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Abstract. Real hypersurfaces satisfying the condition $\phi l = l\phi$, $(l = R(.,\xi)\xi)$, have been studied by many authors under at least one more condition, since the class of these hypersurfaces is quite tough to be classified. The aim of the present paper is the classification of real hypersurfaces in complex hyperbolic plane $\mathbb{C}H^2$ satisfying a generalization of $\phi l = l\phi$ under an additional restriction on a specific function.

0. Introduction

An n-dimensional Kaehlerian manifold of constant holomorphic sectional curvature c is called complex space form, which is denoted by $M_n(c)$. A complete and simply connected complex space form is a projective space $\mathbb{C}P^n$ if $c > 0$, a hyperbolic space $\mathbb{C}H^n$ if $c < 0$, or a Euclidean space $\mathbb{C}^n$ if $c = 0$. The induced almost contact metric structure of a real hypersurface M of $M_n(c)$ will be denoted by (ϕ, ξ, η, g).

Real hypersurfaces in $\mathbb{C}P^n$ which are homogeneous, were classified by R. Takagi [17]. The same author classified real hypersurfaces in $\mathbb{C}P^n$, with constant prinicipal curvatures in [18], but only when the number g of distinct principal curvatures satisfies $g = 3$. M. Kimura showed in [12] that if a Hopf real hypersurface M in $\mathbb{C}P^n$ has constant principal curvatures, then the number of distinct principal curvatures of M is 2, 3 or 5. J. Berndt gave the equivalent result for Hopf hypersurfaces in $\mathbb{C}H^n$ ([1]) where he divided real hypersurfaces into four model spaces, named A_0, A_1, A_2 and B. Real hypersurfaces of type A_1 and A_2 in $\mathbb{C}P^n$ and of type A_0, A_1 and A_2 in $\mathbb{C}H^n$ are said to be hypersurfaces of *type A* for simplicity. Another class of real hypersurfaces that appears quite often is the *Hopf hypersurfaces* where the structure vector field is a principal vector field. For more details and examples on real hypersurfaces of type A and Hopf, we refer to [14].

2010 *Mathematics Subject Classification.* Primary Primary 53B25; Secondary 53D15.
Key words and phrases. Almost contact manifold, Jacobi structure operator.

A Jacobi field along geodesics of a given Riemannian manifold $(M,\, g)$ plays an important role in the study of differential geometry. It satisfies a well known differential equation which inspires Jacobi operators. For any vector field X, the Jacobi operator is defined by R_X: $R_X(Y) = R(Y, X))X$, where R denotes the curvature tensor and Y is a vector field on M. R_X is a self-adjoint endomorphism in the tangent space of M, and is related to the Jacobi differential equation, which is given by $\nabla_{\acute{\gamma}}(\nabla_{\acute{\gamma}}Y) + R(Y, \acute{\gamma})\acute{\gamma} = 0$ along a geodesic γ on M, where $\acute{\gamma}$ denotes the velocity vector along γ on M.

In a real hypersurface M of a complex space form $M_n(c)$, $c \neq 0$, the Jacobi operator on M with respect to the structure vector field ξ, is called the structure Jacobi operator and is denoted by $R_\xi(X) = R(X, \xi)\xi = lX$.

Real hypersurfaces have been studied from many points of view. Certain authors have studied real hypersurfaces under conditions which include the operator l ([4], [6], [11], [16], [19]). Other authors have studied real hypersurfaces under the condition $\phi l = l\phi$, equipped with one or two additional conditions ([3], [7], [8], [9] [10], [20]), proving that these hypersurfaces are Hopf and classifying them as type A.

In the present paper we classify real hypersurfaces of complex hyperbolic planes, satisfying

$$(\phi l - l\phi)X = \psi(X)lX, \tag{0.1}$$

restricted in the subspace $\mathbb{D} = \ker(\eta)$ of T_pM for every point $p \in M$, where $\ker(\eta)$ consists of all vector fields orthogonal to the Reeb flow vector field ξ and the form ψ is assumed non-linear with respect to scalar product. If ψ is linear, then by replacing X with $2X$, we obtain $(\phi l - l\phi)X = 2\psi(X)lX$ which implies $\psi(X)lX = 0$. So (0.1) takes the simpler form $\phi l = l\phi$.

Since this class is rather difficult to classify, a second condition is imposed. However it is not a condition acting in vector fields, but only in the function $\alpha = g(A\xi, \xi)$: $\nabla_\xi\xi \cdot \alpha = 0$, where A is the shape operator. Geometrically speaking, we demand the function α to be constant in the direction of the integral curves of ξ (from now on we will write $(\nabla_\xi\xi\alpha)$ instead of $\nabla_\xi\xi \cdot \alpha$). Namely we prove:

Main Theorem. *A real hypersurface M of a complex hyperbolic plane $\mathbb{C}H^2$, satisfying $(\phi l - l\phi)X = \psi(X)lX$, $\forall X \in \mathbb{D}$ (ψ is non linear) and $\nabla_\xi\xi \cdot \alpha = 0$ is Hopf. Furtermore, if $\alpha = g(A\xi, \xi) \neq 0$ then M locally congruent to a model space of type A and $\psi(X)lX = 0$ for any vector fields X on M.*

We mention that for a Hopf hypersurface in $\mathbb{C}H^n$ $(n > 2)$, it is known that the associated principal curvature of ξ never vanishes [1]. However, in $\mathbb{C}H^2$ there exists a Hopf hypersurface with $A\xi = 0$ [5].

1. Preliminaries

Let M_n be a Kaehlerian manifold of real dimension $2n$, equipped with an almost complex structure J and a Hermitian metric tensor G. Then for any vector fields X and Y on $M_n(c)$, the following relations hold:

$$J^2X = -X, \qquad G(JX, JY) = G(X, Y), \qquad \widetilde{\nabla} J = 0,$$

where $\widetilde{\nabla}$ denotes the Riemannian connection of G of M_n.

Now, let M_{2n-1} be a real $(2n-1)$-dimensional hypersurface of $M_n(c)$, and denote by N a unit normal vector field on a neighborhood of a point in M_{2n-1} (from now on we shall write M instead of M_{2n-1}). For any vector field X tangent to M we have $JX = \phi X + \eta(X)N$, where ϕX is the tangent component of JX, $\eta(X)N$ is the normal component, and

$$\xi = -JN, \qquad \eta(X) = g(X, \xi), \qquad g = G|_M.$$

By properties of the almost complex structure J, and the definitions of η and g, the following relations hold ([2]):

$$\phi^2 = -I + \eta \otimes \xi, \qquad \eta \circ \phi = 0, \qquad \phi\xi = 0, \quad \eta(\xi) = 1. \tag{1.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \qquad g(X, \phi Y) = -g(\phi X, Y). \tag{1.2}$$

The above relations define an *almost contact metric structure* on M which is denoted by (ϕ, ξ, g, η). By virtue of this structure, we can define a local orthonormal basis $\{e_1, e_2, ...e_{n-1}, \phi e_1, \phi e_2, ...\phi e_{n-1}, \xi\}$, called a ϕ-basis. Furthermore, let A be the shape operator in the direction of N, and denote by ∇ the Riemannian connection of g on M. Then A is symmetric and the following equations are satisfied:

$$\nabla_X \xi = \phi AX, \qquad (\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi. \tag{1.3}$$

As the ambient space $M_n(c)$ is of constant holomorphic sectional curvature c, the equations of Gauss and Codazzi are respectively given by:

$$\begin{gathered} R(X,Y)Z = \\ \frac{c}{4}[g(Y,Z)X - g(X,Z)Y + g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y - 2g(\phi X,Y)\phi Z] + \\ g(AY,Z)AX - g(AX,Z)AY, \end{gathered} \tag{1.4}$$

$$(\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}[\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi]. \tag{1.5}$$

The tangent space T_pM, for every point $p \in M$, is decomposed as following:

$$T_pM = \ker(\eta)^{\perp} \oplus \ker(\eta),$$

where $\ker(\eta)^{\perp} = \mathrm{span}\{\xi\}$ and $\ker(\eta)$ is defined as following:

$$\ker(\eta) = \{X \in T_pM : \eta(X) = 0\}.$$

Based on the above decomposition, by virtue of (1.3), we decompose the vector field $A\xi$ in the following way:

$$A\xi = \alpha\xi + \beta U, \tag{1.6}$$

where $\beta = |\phi\nabla_\xi\xi|$ and $U = -\frac{1}{\beta}\phi\nabla_\xi\xi \in \ker(\eta)$, provided that $\beta \neq 0$.

As stated before, if the vector field ξ is a principal vector field, the real hypersurface is called a *Hopf* hypersurface. In this case the vector field $A\xi$ is expressed as $A\xi = \alpha\xi$, $\alpha = g(A\xi, \xi)$.

Finally, differentiation of a function f along a vector field X will be denoted by (Xf). All manifolds of this paper are assumed to be connected and of class C^∞.

2. Auxiliary relations

In the study of real hypersurfaces of a complex space form $M_n(c)$, $c \neq 0$, it is a crucial condition that the structure vector field ξ is principal. The purpose of this paragraph is to establish relations that will help us prove this condition.

Let $\mathcal{N} = \{p \in M : \beta \neq 0$ in a neighborhood of $p\}$. If we had at least one point of $\mathcal{N}$ where $\alpha = 0$, then from (1.4) we would obtain $lU = (\frac{c}{4} - \beta^2)U$, $l\phi U = \frac{c}{4}\phi U$. Combining the last two equations with (0.1) we would take $\beta = 0$ which is a contradiction. Therefore $\alpha \neq 0$ in $\mathcal{N}$.

Lemma 2.1. *Let M be a real hypersurface of a complex hyperbolic plane $\mathbb{C}H^2$, satisfying (0.1). Then the following relations hold in $\mathcal{N}$.*

$$AU = \Big(\frac{\gamma}{\alpha} - \frac{c}{4\alpha} + \frac{\beta^2}{\alpha}\Big)U + \beta\xi, \qquad A\phi U = \Big(\frac{\gamma}{\alpha} - \frac{c}{4\alpha}\Big)\phi U, \tag{2.1}$$

$$\nabla_\xi\xi = \beta\phi U, \qquad \nabla_U\xi = \Big(\frac{\gamma}{\alpha} - \frac{c}{4\alpha} + \frac{\beta^2}{\alpha}\Big)\phi U, \qquad \nabla_{\phi U}\xi = \Big(\frac{c}{4\alpha} - \frac{\gamma}{\alpha}\Big)U, \tag{2.2}$$

$$\nabla_\xi U = \kappa_1\phi U, \qquad \nabla_U U = \kappa_2\phi U, \qquad \nabla_{\phi U}U = \kappa_3\phi U + \Big(\frac{\gamma}{\alpha} - \frac{c}{4\alpha}\Big)\xi, \tag{2.3}$$

$$\nabla_\xi\phi U = -\kappa_1 U - \beta\xi, \quad \nabla_U\phi U = -\kappa_2 U - \Big(\frac{\gamma}{\alpha} - \frac{c}{4\alpha} + \frac{\beta^2}{\alpha}\Big)\xi, \tag{2.4}$$
$$\nabla_{\phi U}\phi U = -\kappa_3 U,$$

where κ_1, κ_2, κ_3 are smooth functions in $\mathcal{N}$.

Proof. From (1.4) we get

$$lX = \frac{c}{4}[X - \eta(X)\xi] + \alpha AX - g(AX, \xi)A\xi \tag{2.5}$$

which, for $X = U$ and $X = \phi U$ yields

$$\text{(i)}\ \ lU = \frac{c}{4}U + \alpha AU - \beta A\xi, \quad \text{(ii)}\ \ l\phi U = \frac{c}{4}\phi U + \alpha A\phi U. \tag{2.6}$$

The scalar products of (2.6.i) with U and ϕU yield respectively

$$g(AU, U) = \frac{\gamma}{\alpha} - \frac{c}{4\alpha} + \frac{\beta^2}{\alpha}, \tag{2.7}$$

$$g(AU, \phi U) = g(A\phi U, U) = \frac{\delta}{\alpha}, \tag{2.8}$$

where $\gamma = g(lU, U)$, $\delta = g(lU, \phi U)$. From (2.7), (2.8) and $g(AU, \xi) = g(A\xi, U) = \beta$ we obtain $AU = (\frac{\gamma}{\alpha} + \frac{\beta^2}{\alpha} - \frac{c}{4\alpha})U + \beta\xi + \frac{\delta}{\alpha}\phi U$. From (2.8), $g(A\phi U, \xi) = g(A\xi, \phi U) = 0$, if we put $\epsilon = g(l\phi U, \phi U)$, then we obtain $A\phi U = (\frac{\epsilon}{\alpha} - \frac{c}{4\alpha})\phi U + \frac{\delta}{\alpha}U$. In order to prove (2.1) we need to show that $\gamma = \epsilon$, $\delta = 0$. Combining the analysis of AU, $A\phi U$ with (1.6) and (2.6) we obtain $lU = \gamma U + \delta\phi U$, $l\phi U = \delta U + \epsilon\phi U$. The last two equations and $\phi lU - l\phi U = \psi(U)lU$, which holds due to (0.1), yield

$$\text{(i)}\ \ \gamma - \epsilon = \psi(U)\delta, \quad \text{(ii)}\ \ -2\delta = \psi(U)\gamma. \tag{2.9}$$

Moreover, the decompositions of lU, $l\phi U$ combined with $\phi l\phi U + lU = \psi(\phi U)l\phi U$, which holds due to (0.1), (1.1), yield

$$\text{(i)}\ \ \gamma - \epsilon = \psi(\phi U)\delta, \quad \text{(ii)}\ \ 2\delta = \psi(\phi U)\epsilon. \tag{2.10}$$

Let's assume that $\delta \neq 0$ in a neighborhood of a point in $\mathcal{N}$. Then (2.9.i) and (2.10.i) give $\psi(U) = \psi(\phi U)$. Apparently $\psi(U)\gamma \neq 0$ otherwise (2.9.ii) would yield $\delta = 0$. As a result, (2.9) and (2.10) lead respectively to $-\gamma(\gamma - \epsilon) = 2\delta^2$, $\epsilon(\gamma - \epsilon) = 2\delta^2$. The last two relations are added and result to $(\gamma - \epsilon)^2 = -4\delta^2$ which is a contradiction. This means $\delta = 0$ holds, and (2.9), (2.10) imply $\gamma = \epsilon$.

(2.2) is obtained from equation (2.1) and relation (1.3) for $X = \xi$, $X = U$, $X = \phi U$. Next we recall the rule

$$Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z). \tag{2.11}$$

By virtue of (2.11) for $X = Z = \xi$, $Y = U$ and for $X = \xi$, $Y = Z = U$, it is shown respectively $\nabla_\xi U \perp \xi$ and $\nabla_\xi U \perp U$, which means $\nabla_\xi U = \kappa_1 \phi U$. In a similar way, equation (2.11) for $X = Y = Z = U$ and $X = Z = U$, $Y = \xi$ yields respectively $\nabla_U U \perp U$ and $\nabla_U U \perp \xi$. So $\nabla_U U = \kappa_2 \phi U$ holds. Finally, (2.11) for $X = \phi U$, $Y = Z = U$ and $X = \phi U$, $Y = U$, $Z = \xi$ (with the aid of (2.2)) yields respectively $\nabla_{\phi U} U \perp U$ and $g(\nabla_{\phi U} U, \xi) = \frac{\gamma}{\alpha} - \frac{c}{4\alpha}$. Therefore we have $\nabla_{\phi U} U = \kappa_3 \phi U + (\frac{\gamma}{\alpha} - \frac{c}{4\alpha})\xi$ and (2.3) has been proved. In order to prove (2.4) we use the second of (1.3) with the following combinations: i) $X = \xi$, $Y = U$, ii) $X = Y = U$, iii) $X = \phi U$, $Y = U$, and make use of (1.6), (2.1), (2.3). $\square$

Lemma 2.2. *Let M be a real hypersurface of a complex hyperbolic plane $\mathbb{C}H^2$, satisfying (0.1). Then in $\mathcal{N}$ we have* $\big(\phi U(\frac{\gamma}{\alpha}-\frac{c}{4\alpha})\big)=\frac{3\beta}{\alpha}\big[(\frac{\gamma}{\alpha}-\frac{c}{4\alpha})^2-\frac{c}{4}\big]$.

Proof. Putting $X=U$, $Y=\xi$ in (1.5), we obtain $(\nabla_U A)\xi-(\nabla_\xi A)U=-\frac{c}{4}\phi U$. Combining the last equation with (1.6) and Lemma 2.1, it follows:

$$\Big[(U\alpha)-(\xi\beta)\Big]\xi+\Big[(U\beta)-\Big(\xi\Big(\frac{\gamma}{\alpha}-\frac{c}{4\alpha}+\frac{\beta^2}{\alpha}\Big)\Big)\Big]U+$$

$$\Big[\gamma-\frac{c}{4}+\kappa_2\beta-\Big(\frac{\gamma}{\alpha}-\frac{c}{4\alpha}\Big)\Big(\frac{\gamma}{\alpha}-\frac{c}{4\alpha}+\frac{\beta^2}{\alpha}\Big)-\kappa_1\frac{\beta^2}{\alpha}\Big]\phi U=-\frac{c}{4}\phi U.$$

The last equation because of the linear independency of U, ϕU and ξ, yields

$$(U\alpha)=(\xi\beta), \tag{2.12}$$

$$(U\beta)=\Big(\xi\Big(\frac{\gamma}{\alpha}-\frac{c}{4\alpha}+\frac{\beta^2}{\alpha}\Big)\Big), \tag{2.13}$$

$$\gamma+\kappa_2\beta-\Big(\frac{\gamma}{\alpha}-\frac{c}{4\alpha}\Big)\Big(\frac{\gamma}{\alpha}-\frac{c}{4\alpha}+\frac{\beta^2}{\alpha}\Big)-\kappa_1\frac{\beta^2}{\alpha}=0. \tag{2.14}$$

In the same way, putting $X=\phi U$, $Y=\xi$ in (1.5) we obtain $(\nabla_{\phi U}A)\xi-(\nabla_\xi A)\phi U=\frac{c}{4}U$. Combining the last equation with (1.6) and Lemma 2.1, we have

$$(\phi U\beta)+\Big(\frac{\gamma}{\alpha}-\frac{c}{4\alpha}\Big)\Big(\frac{\gamma}{\alpha}-\frac{c}{4\alpha}+\frac{\beta^2}{\alpha}\Big)-\kappa_1\frac{\beta^2}{\alpha}-\beta^2-\gamma=0, \tag{2.15}$$

$$\kappa_3\beta=\xi\Big(\frac{\gamma}{\alpha}-\frac{c}{4\alpha}\Big), \tag{2.16}$$

$$(\phi U\alpha)+3\beta\Big(\frac{\gamma}{\alpha}-\frac{c}{4\alpha}\Big)-\kappa_1\beta-\alpha\beta=0. \tag{2.17}$$

Similarly, putting $X=U$, $Y=\phi U$ in (1.5), we get $(\nabla_U A)\phi U-(\nabla_{\phi U}A)U=-\frac{c}{2}\xi$, which, by use of (1.6) and Lemma 2.1, implies:

$$-\kappa_2\frac{\beta^2}{\alpha}-3\beta\Big(\frac{\gamma}{\alpha}-\frac{c}{4\alpha}\Big)-\frac{\beta^3}{\alpha}+\Big(\phi U\Big(\frac{\gamma}{\alpha}-\frac{c}{4\alpha}+\frac{\beta^2}{\alpha}\Big)\Big)=0, \tag{2.18}$$

$$U\Big(\frac{\gamma}{\alpha}-\frac{c}{4\alpha}\Big)=\kappa_3\frac{\beta^2}{\alpha}. \tag{2.19}$$

We expand (2.18) and then replace the terms κ_2, $(\phi U\beta)$, $(\phi U\alpha)$ from (2.14), (2.15) and (2.17) respectively. The final equation is

$$\Big(\phi U\Big(\frac{\gamma}{\alpha}-\frac{c}{4\alpha}\Big)\Big)=\frac{3\beta}{\alpha}\Big[\Big(\frac{\gamma}{\alpha}-\frac{c}{4\alpha}\Big)^2-\frac{c}{4}\Big]. \tag{2.20}$$

□

Lemma 2.3. *Let M be a real hypersurface of a complex hyperbolic plane $\mathbb{C}H^2$, satisfying (0.1). Then, $\kappa_3=0$ holds in $\mathcal{N}$.*

Proof. Because of (2.3), (2.4), (2.16), (2.19) and Lemma 2.2, the well known relation $[U,\phi U]=\nabla_U\phi U-\nabla_{\phi U}U$ takes the form

$$[U,\phi U]\Big(\frac{\gamma}{\alpha}-\frac{c}{4\alpha}\Big)=$$

$$-\frac{\kappa_2\kappa_3\beta^2}{\alpha}-\kappa_3\beta\Big(\frac{\gamma}{\alpha}-\frac{c}{4\alpha}+\frac{\beta^2}{\alpha}\Big)-\frac{3\beta\kappa_3}{\alpha}\left[\Big(\frac{\gamma}{\alpha}-\frac{c}{4\alpha}\Big)^2-\frac{c}{4}\right]-\kappa_3\beta\,\Big(\frac{\gamma}{\alpha}-\frac{c}{4\alpha}\Big).$$

On the other hand (2.15), (2.17), (2.19) and Lemma 2.2 yield

$$[U,\phi U]\Big(\frac{\gamma}{\alpha}-\frac{c}{4\alpha}\Big)=U\left(\phi U\Big(\frac{\gamma}{\alpha}-\frac{c}{4\alpha}\Big)\right)-\phi U\left(U\Big(\frac{\gamma}{\alpha}-\frac{c}{4\alpha}\Big)\right)=$$

$$\frac{3(U\beta)}{\alpha}\left[\Big(\frac{\gamma}{\alpha}-\frac{c}{4\alpha}\Big)^2-\frac{c}{4}\right]-\frac{3\beta(U\alpha)}{\alpha^2}\left[\Big(\frac{\gamma}{\alpha}-\frac{c}{4\alpha}\Big)^2-\frac{c}{4}\right]$$

$$+\frac{6\kappa_3\beta^3}{\alpha^2}\Big(\frac{\gamma}{\alpha}-\frac{c}{4\alpha}\Big)-\frac{\beta^2}{\alpha}\left(\phi U(\kappa_3)\right)+\frac{2\kappa_3\beta}{\alpha}\Big(\frac{\gamma}{\alpha}-\frac{c}{4\alpha}\Big)\Big(\frac{\gamma}{\alpha}-\frac{c}{4\alpha}+\frac{\beta^2}{\alpha}\Big)$$

$$-\frac{2\kappa_3\beta\gamma}{\alpha}-\frac{\kappa_1\kappa_3\beta^3}{\alpha^2}-\frac{\kappa_3\beta^3}{\alpha}-\frac{3\kappa_3\beta^3\gamma}{\alpha^3}+\frac{3\kappa_3c\beta^3}{4\alpha^3}.$$

The last equations using (2.12), (2.13) and (2.16) yield

$$\frac{3}{\alpha}\left[\Big(\frac{\gamma}{\alpha}-\frac{c}{4\alpha}\Big)^2-\frac{c}{4}\right](\xi\beta)-\frac{3\beta}{\alpha^2}\left[\Big(\frac{\gamma}{\alpha}-\frac{c}{4\alpha}\Big)^2-\frac{c}{4}\right](\xi\alpha)-\beta(\phi U\kappa_3)= \tag{2.21}$$

$$\left[2c-\beta\kappa_2+\frac{\beta^2}{\alpha}\kappa_1-8\Big(\frac{\gamma}{\alpha}-\frac{c}{4\alpha}\Big)^2-\frac{5\beta^2}{\alpha}\Big(\frac{\gamma}{\alpha}-\frac{c}{4\alpha}\Big)\right]\kappa_3.$$

Following a similar way, we calculate $[\xi,\phi U](\frac{\gamma}{\alpha}-\frac{c}{4\alpha})=(\nabla_\xi\phi U-\nabla_{\phi U}\xi)(\frac{\gamma}{\alpha}-\frac{c}{4\alpha})$ and then $[\xi,\phi U](\frac{\gamma}{\alpha}-\frac{c}{4\alpha})=\xi\big(\phi U(\frac{\gamma}{\alpha}-\frac{c}{4\alpha})\big)-\phi U\big(\xi(\frac{\gamma}{\alpha}-\frac{c}{4\alpha})\big)$. By equalizing the results we obtain

$$\frac{3}{\alpha}\left[\Big(\frac{\gamma}{\alpha}-\frac{c}{4\alpha}\Big)^2-\frac{c}{4}\right](\xi\beta)-\frac{3\beta}{\alpha^2}\left[\Big(\frac{\gamma}{\alpha}-\frac{c}{4\alpha}\Big)^2-\frac{c}{4}\right](\xi\alpha)-\beta(\phi U\kappa_3)= \tag{2.22}$$

$$\left[\gamma-\Big(\frac{\gamma}{\alpha}-\frac{c}{4\alpha}\Big)^2-\frac{6\beta^2}{\alpha}\Big(\frac{\gamma}{\alpha}-\frac{c}{4\alpha}\Big)\right]\kappa_3.$$

Comparing (2.21) with (2.22) and by making use of (2.14) we obtain

$$\kappa_3\left[\Big(\frac{\gamma}{\alpha}-\frac{c}{4\alpha}\Big)^2-\frac{c}{4}\right]=0.$$

The last equation and $c<0$ give $\kappa_3=0$ in $\mathcal{N}$. □

3. Proof of Main Theorem

We first prove the following proposition.

Proposition 3.1. *Let M be a real hypersurface of a complex hyperbolic plane $\mathbb{C}H^2$ $(c \neq 0)$, satisfying (0.1) and $(\nabla_\xi \xi\alpha) = 0$. Then M is Hopf.*

Proof. We keep working in $\mathcal{N}$. By virtue of Lemma 2.3 and equations (1.6), (2.16), (2.19), we obtain $[A\xi, \xi](\frac{\gamma}{\alpha} - \frac{c}{4\alpha}) = A\xi\big(\xi(\frac{\gamma}{\alpha} - \frac{c}{4\alpha})\big) - \xi\big(A\xi(\frac{\gamma}{\alpha} - \frac{c}{4\alpha})\big) = 0$. However, from Lemmas 2.1, 2.2, 2.3 and (1.6) we calculate $[A\xi, \xi](\frac{\gamma}{\alpha} - \frac{c}{4\alpha}) = \frac{3\beta}{\alpha}(\nabla_{A\xi}\xi - \nabla_\xi A\xi)(\frac{\gamma}{\alpha} - \frac{c}{4\alpha}) = (\frac{\gamma}{\alpha} - \frac{c}{4\alpha} + \frac{\beta^2}{\alpha} - \kappa_1)((\frac{\gamma}{\alpha} - \frac{c}{4\alpha})^2 - \frac{c}{4})$. The two expressions of $[A\xi, \xi](\frac{\gamma}{\alpha} - \frac{c}{4\alpha})$ yield

$$\Big(\frac{\gamma}{\alpha} - \frac{c}{4\alpha} + \frac{\beta^2}{\alpha} - \kappa_1\Big)\Big(\Big(\frac{\gamma}{\alpha} - \frac{c}{4\alpha}\Big)^2 - \frac{c}{4}\Big) = 0.$$

Since the curvature is negative, the above relation gives

$$\kappa_1 = \frac{\gamma}{\alpha} - \frac{c}{4\alpha} + \frac{\beta^2}{\alpha}. \tag{3.1}$$

The condition $(\nabla_\xi \xi\alpha) = 0$ is equivalent to $(\phi U\alpha) = 0$. The last relation, (3.1) and (2.17) yield

$$\gamma - \frac{c}{4} = \frac{\alpha^2 + \beta^2}{2}. \tag{3.2}$$

By virtue of (3.1) and (3.2) we simplify (2.15) and obtain

$$\phi U\beta = \frac{3\beta^4}{4\alpha^2} + \frac{\beta^2}{2} - \frac{\alpha^2}{4} + \gamma. \tag{3.3}$$

Next, we differentiate (3.2) along ϕU, with the aid of Lemma 2.2, (3.2), (3.3) and $(\phi U\alpha) = 0$ to acquire $\gamma + \frac{3c}{4} = \alpha^2 + \beta^2$. The last relation and (3.2) imply $\gamma = \frac{5c}{4}$ which violates (2.20). Therefore we have a contradiction in $\mathcal{N}$, hence $\mathcal{N} = \emptyset$ and M is Hopf. □

From Proposition 3.1 we have on M:

$$A\xi = \alpha\xi, \qquad \alpha = g(A\xi, \xi) \tag{3.4}$$

and α is a constant ([14]). We consider a ϕ-basis $\{e, \phi e, \xi\}$ which satisfies

$$Ae = \lambda_1 e, \quad A\phi e = \lambda_2 \phi e, \quad A\xi = \alpha\xi. \tag{3.5}$$

From (1.4) and (3.5) we obtain

$$le = \frac{c}{4}e + \alpha\lambda_1 e, \qquad l\phi e = \frac{c}{4}\phi e + \alpha\lambda_2 \phi e. \tag{3.6}$$

By making use of (0.1) with $X = e$, in combination with (3.6), we result to

$$\alpha(\lambda_1 - \lambda_2) = 0.$$

If $\alpha \neq 0$ then $\lambda_1 = \lambda_2 = \lambda$ and λ is the root of the quadratic $t^2 - \alpha t - \frac{c}{4}$ ([14]) and consequently a constant. The classification follows from [1]. By (3.6),

$(\phi l - l\phi)e = (\phi l - l\phi)\phi e = (\phi l - l\phi)\xi = 0$. Thus, by virtue of (0.1), $\psi(X)lX = 0$ for any vector fields X on M.

Concerning the case $\alpha = 0$ we state the following. For a Hopf hypersurface in $\mathbb{C}H^n$ $(n > 2)$, it is known that the associated principal curvature of ξ never vanishes ([1]). However, in $\mathbb{C}H^2$ there exists a Hopf hypersurface with $A\xi = 0$, which was constructed using moving frames by T. Ivey and P. Ryan. For more details in the construction of this hypersurface, we refer to their work in [5].

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Received March 19, 2016
Revised June 28, 2016