Nihonkai Math. J.
Vol.27(2016), 125–133

NOTE ON DUNKL-WILLIAMS INEQUALITY WITH n ELEMENTS

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Abstract. Recently, Pečarić and Rajić established a generalization of the Dunkl-Williams inequality for n elements in a Banach space. In this note we show a refinement of this inequality.

1. Introduction

Let X be a Banach space. For nonzero elements $x, y \in X$ the *angular distance* $\alpha[x, y]$ between x and y is defined by

$$\alpha[x,y] = \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\|$$

(Clarkson [1]). Then the well-known Dunkl-Williams inequality [2] states that for any two nonzero elements x, y,

$$\alpha[x,y] \leq \frac{4\|x-y\|}{\|x\|+\|y\|}. \tag{1}$$

It has been treated by many authors (e.g., [4, 10, 11], see also [3, 7, 8, 9, 12]). Particularly, the following sharp Dunkl-Williams inequality and its reverse one were obtained by Maligranda [5] and Mercer [6], respectively.

Theorem A' ([5, 6])**.** For any two nonzero elements x, y in a Banach space X,

$$\frac{\|x-y\| - |\|x\| - \|y\||}{\min\{\|x\|, \|y\|\}} \leq \alpha[x,y] \leq \frac{\|x-y\| + |\|x\| - \|y\||}{\max\{\|x\|, \|y\|\}}.$$

Moreover, Pečarić and Rajić [11] showed that the following sharp Dunkl-Williams inequality and its reverse one with n elements in a Banach space.

2010 *Mathematics Subject Classification.* Primary 46B20.
Key words and phrases. Dunkl-Williams inequality, triangle inequality.

Theorem A ([11])**.** For all nonzero elements $x_1, x_2, \cdots, x_n$ in a Banach space X,

$$\max_{1\le i\le n}\left\{\frac{1}{\|x_i\|}\left(\left\|\sum_{j=1}^n x_j\right\| - \sum_{j=1}^n \big|\|x_j\| - \|x_i\|\big|\right)\right\}$$

$$\le \left\|\sum_{j=1}^n \frac{x_j}{\|x_j\|}\right\| \tag{2}$$

$$\le \min_{1\le i\le n}\left\{\frac{1}{\|x_i\|}\left(\left\|\sum_{j=1}^n x_j\right\| + \sum_{j=1}^n \big|\|x_j\| - \|x_i\|\big|\right)\right\}. \tag{3}$$

In this note we consider a refinement of these inequalities. In [7], some norm inequalities on intermediate values of the triangle inequality were presented. For positive integer $n \ge 2$, let $M_n([0,1])$ be the set of all n by n matrices whose all elements belong to the interval $[0,1]$ and L_n denote the set of all lower triangular matrices of $M_n([0,1])$; i.e.,

$$L_n = \left\{a = (a_{ij}) \in M_n([0,1]) \;\middle|\; a_{ij} = 0 \quad (i < j)\ \right\}.$$

Let $1 \le m \le n$. For each $a = (a_{ij})$ in L_n, we set

$$\ell^a_{mj}(m) = a_{mj} \quad (1 \le j \le m)$$

and if $2 \le n$, then, for each m with $2 \le m \le n$, we put

$$\ell^a_{ij}(m) = a_{ij}\prod_{k=i+1}^{m}(1 - a_{kj}) \quad (1 \le i \le m-1,\ 1 \le j \le m).$$

Theorem 1.1 ([7])**.** *Let $n \ge 2$. With the above notation, take any $a = (a_{ij})$ in L_n. For all elements $x_1, x_2, \cdots, x_n$ in a Banach space X, the following inequality holds*

$$\sum_{i=1}^n\left(\sum_{j=1}^i \|\ell^a_{ij}(n)x_j\| - \left\|\sum_{j=1}^i \ell^a_{ij}(n)x_j\right\|\right) \le \sum_{j=1}^n \|x_j\| - \left\|\sum_{j=1}^n x_j\right\|.$$

Let $2 \le m \le n$. For each $a = (a_{ij}) \in L_n$ with $a_{im} \ne 0$, put

$$r^a_{ij}(m) = \begin{cases} \ell^a_{ij}(m-1)\left(\frac{1}{a_{mj}} - 1\right) & (1 \le j \le m-1) \\ \frac{1}{a_{mj}} & (i = m,\ 1 \le j \le m) \end{cases}.$$

Then $r^a_{nj}(n) \ge 1$ for each j with $1 \le j \le n$ and the reverse of the above inequality was given in [13].

Theorem 1.2 ([13])**.** *Let $n \ge 2$ and take $a = (a_{ij}) \in L_n$ with $a_{in} \ne 0$ ($i \in \{1, \ldots, n\}$). For all elements $x_1, x_2, \cdots, x_n$ in a Banach space X, the following* *inequality holds*

$$\sum_{j=1}^{n}\|x_j\| - \left\|\sum_{j=1}^{n}x_j\right\| \tag{4}$$
$$\leq \left(\sum_{j=1}^{n}\|r_{nj}^a(n)x_j\| - \left\|\sum_{j=1}^{n}r_{nj}^a(n)x_j\right\|\right) - \sum_{i=2}^{n-1}\left(\sum_{j=1}^{i}\|r_{ij}^a(n)x_j\| - \left\|\sum_{j=1}^{i}r_{ij}^a(n)x_j\right\|\right).$$

These results will lead to new Dunkl-Williams inequalities with n elements which are sharper than the ones in Theorem A.

2. The results

We remark that the inequalities in Theorem A' are equivalent to the following:

$$\begin{aligned}&\frac{1}{\|x_2\|}\|x_1+x_2\| - \Big(\frac{1}{\|x_2\|}-\frac{1}{\|x_1\|}\Big)\|x_1\|\\ &\leq \left\|\frac{x_1}{\|x_1\|}+\frac{x_2}{\|x_2\|}\right\|\\ &\leq \frac{1}{\|x_1\|}\|x_1+x_2\| + \Big(\frac{1}{\|x_2\|}-\frac{1}{\|x_1\|}\Big)\|x_2\|,\end{aligned}$$

whenever $\|x_1\| \geq \|x_2\| > 0$. In view of these inequalites we obtain the following ones with n elements by using Theorem 1.1 and Theorem 1.2.

Theorem 2.1. *Let $n \geq 2$. For all nonzero elements $x_1, x_2, \cdots, x_n$ in a Banach space X,*

$$\frac{1}{\min\limits_{1\leq j\leq n}\|x_j\|}\left\|\sum_{j=1}^{n}x_j\right\| - \sum_{k=1}^{n-1}\left(\frac{1}{\|x_{k+1}^*\|}-\frac{1}{\|x_k^*\|}\right)\left\|\sum_{j=1}^{k}x_j^*\right\|$$
$$\leq \left\|\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}\right\| \tag{5}$$
$$\leq \frac{1}{\max\limits_{1\leq j\leq n}\|x_j\|}\left\|\sum_{j=1}^{n}x_j\right\| + \sum_{k=2}^{n}\left(\frac{1}{\|x_k^*\|}-\frac{1}{\|x_{k-1}^*\|}\right)\left\|\sum_{j=k}^{n}x_j^*\right\|, \tag{6}$$

where $(x_1^*, x_2^*, \cdots, x_n^*)$ is the rearrangement of $(x_1, x_2, \cdots, x_n)$ satisfying $\|x_1^*\| \geq \|x_2^*\| \geq \cdots \geq \|x_n^*\|$.

Remark 2.1. When $n = 2$, the inequalities in the previous theorem are equivalent to those in Theorem A'.

Proof of Theorem 2.1. Without loss of generality we may assume that $\|x_1\| > \|x_2\| > \cdots > \|x_n\| > 0$. We first show the inequality (5). We set $a = (a_{ij})$ satisfying

$$\begin{cases} a_{ii} = 1 \qquad (1 \le i \le n) \\ a_{nj} = \dfrac{\|x_n\|}{\|x_j\|} \qquad (1 \le j \le n) \\ a_{ij} = \dfrac{\|x_j\|}{\|x_i\|} \cdot \dfrac{\|x_i\| - \|x_{i+1}\|}{\|x_j\| - \|x_{i+1}\|} \qquad (2 \le i \le n-1,\ 1 \le j \le n,\ i \ge j) \\ a_{ij} = 0 \qquad (i < j). \end{cases}$$

It is clear that $a \in L_n$. Put $\ell_{ij} = \ell_{ij}^a(n)$. Then

$$\begin{cases} \ell_{11} = a_{11} = 1 \\ \ell_{nj} = a_{nj} = \dfrac{\|x_n\|}{\|x_j\|} \qquad (1 \le j \le n) \\ \ell_{ij} = a_{ij} \displaystyle\prod_{k=i+1}^{n} (1 - a_{kj}) = \dfrac{\|x_n\|}{\|x_i\|\|x_{i+1}\|}\Big(\|x_i\| - \|x_{i+1}\|\Big) \\ \qquad = \Big(\dfrac{1}{\|x_{i+1}\|} - \dfrac{1}{\|x_i\|}\Big)\|x_n\| \qquad (2 \le i \le n-1,\ 1 \le j \le n,\ i \ge j). \end{cases} \tag{7}$$

Substituting (7) into the inequality in Theorem 1.1, we have

$$\begin{aligned} \sum_{j=1}^{n} \|x_j\| - \Big\|\sum_{j=1}^{n} x_j\Big\| &\ge \sum_{i=1}^{n}\Big(\sum_{j=1}^{i}\|\ell_{ij}x_j\| - \Big\|\sum_{j=1}^{i} \ell_{ij}x_j\Big\|\Big) \\ &= \sum_{i=1}^{n-1}\Big(\sum_{j=1}^{i}\|\ell_{ij}x_j\| - \Big\|\sum_{j=1}^{i} \ell_{ij}x_j\Big\|\Big) + \Big(\sum_{j=1}^{n}\|\ell_{nj}x_j\| - \Big\|\sum_{j=1}^{n} \ell_{nj}x_j\Big\|\Big) \\ &= \sum_{i=1}^{n-1}\Big\{\|x_n\|\Big(\frac{1}{\|x_{i+1}\|} - \frac{1}{\|x_i\|}\Big)\Big(\sum_{j=1}^{i}\|x_j\| - \Big\|\sum_{j=1}^{i} x_j\Big\|\Big)\Big\} \\ &\qquad + \|x_n\|\Big(\sum_{j=1}^{n}\frac{\|x_j\|}{\|x_j\|} - \Big\|\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}\Big\|\Big). \end{aligned}$$

Hence

$$\begin{aligned} \frac{1}{\|x_n\|}\Big(\sum_{j=1}^{n}\|x_j\| - \Big\|\sum_{j=1}^{n} x_j\Big\|\Big) &\ge \sum_{i=1}^{n-1}\Big(\frac{1}{\|x_{i+1}\|} - \frac{1}{\|x_i\|}\Big)\sum_{j=1}^{i}\|x_j\| \\ &\quad - \sum_{i=1}^{n-1}\Big(\frac{1}{\|x_{i+1}\|} - \frac{1}{\|x_i\|}\Big)\Big\|\sum_{j=1}^{i} x_j\Big\| + n - \Big\|\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}\Big\| \end{aligned}$$

and so

$$\frac{1}{\|x_n\|}\Big\|\sum_{j=1}^{n} x_j\Big\| - \sum_{i=1}^{n-1}\Big(\frac{1}{\|x_{i+1}\|} - \frac{1}{\|x_i\|}\Big)\Big\|\sum_{j=1}^{i} x_j\Big\|$$

$$\leq \frac{1}{\|x_n\|}\sum_{j=1}^{n}\|x_j\| - \sum_{i=1}^{n-1}\Big(\frac{1}{\|x_{i+1}\|} - \frac{1}{\|x_i\|}\Big)\sum_{j=1}^{i}\|x_j\| - n + \Big\|\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}\Big\|.$$

From

$$\frac{1}{\|x_n\|}\sum_{j=1}^{n}\|x_j\| - \sum_{i=1}^{n-1}\Big(\frac{1}{\|x_{i+1}\|} - \frac{1}{\|x_i\|}\Big)\sum_{j=1}^{i}\|x_j\|$$

$$= \frac{1}{\|x_n\|}\Big(\sum_{j=1}^{n}\|x_j\| - \sum_{j=1}^{n-1}\|x_j\|\Big) + \frac{1}{\|x_{n-1}\|}\Big(\sum_{j=1}^{n-1}\|x_j\| - \sum_{j=1}^{n-2}\|x_j\|\Big) + \cdots + \frac{1}{\|x_1\|}\|x_1\|$$

$$= \sum_{i=1}^{n}\frac{\|x_i\|}{\|x_i\|} = n, \tag{8}$$

we obtain the inequality (5). We next prove the inequality (6). For each j, put $y_{n+1-j} = x_j$. It is clear that $\|y_1\| < \|y_2\| < \cdots < \|y_n\|$. We set $a = (a_{ij})$ satisfying

$$\begin{cases} a_{nj} = \dfrac{\|y_j\|}{\|y_n\|} & (1 \leq j \leq n) \\ a_{ij} = \dfrac{\|y_j\|}{\|y_i\|}\cdot\dfrac{\|y_{i+1}\| - \|y_i\|}{\|y_{i+1}\| - \|y_j\|} & (1 \leq j \leq i \leq n-1) \\ a_{ij} = 0 & (i < j). \end{cases}$$

It is clear that $a \in L_n$. Put $r_{ij} = r_{ij}^a(n)$. Then

$$\begin{cases} r_{nj} = \dfrac{1}{a_{nj}} = \dfrac{\|y_n\|}{\|y_j\|} & (1 \leq j \leq n) \\ r_{ij} = \ell_{ij}^a(n-1)\Big(\dfrac{1}{a_{nj}} - 1\Big) = \|y_n\|\Big(\dfrac{1}{\|y_i\|} - \dfrac{1}{\|y_{i+1}\|}\Big) & (1 \leq j \leq i \leq n-1). \end{cases} \tag{9}$$

Substituting (9) into the inequality in Theorem 1.2 we have

$$\sum_{j=1}^{n}\|y_j\| - \Big\|\sum_{j=1}^{n} y_j\Big\| \leq \Big(\sum_{j=1}^{n}\|r_{nj}y_j\| - \Big\|\sum_{j=1}^{n} r_{nj}y_j\Big\|\Big) - \sum_{i=2}^{n-1}\Big(\sum_{j=1}^{i}\|r_{ij}y_j\| - \Big\|\sum_{j=1}^{i} r_{ij}y_j\Big\|\Big)$$

$$= \Big(\sum_{j=1}^{n}\|r_{nj}y_j\| - \Big\|\sum_{j=1}^{n} r_{nj}y_j\Big\|\Big) - \sum_{i=1}^{n-1}\Big(\sum_{j=1}^{i}\|r_{ij}y_j\| - \Big\|\sum_{j=1}^{i} r_{ij}y_j\Big\|\Big)$$

$$= \|y_n\|\Big(\sum_{j=1}^{n}\frac{\|y_j\|}{\|y_j\|} - \Big\|\sum_{j=1}^{n}\frac{y_j}{\|y_j\|}\Big\|\Big)$$

$$-\sum_{i=1}^{n-1}\Big\{\|y_n\|\Big(\frac{1}{\|y_i\|}-\frac{1}{\|y_{i+1}\|}\Big)\Big(\sum_{j=1}^{i}\|y_j\|-\Big\|\sum_{j=1}^{i}y_j\Big\|\Big)\Big\}.$$

Hence

$$\Big\|\sum_{j=1}^{n}\frac{y_j}{\|y_j\|}\Big\|\le\frac{1}{\|y_n\|}\Big\|\sum_{j=1}^{n}y_j\Big\|+n-\frac{1}{\|y_n\|}\sum_{j=1}^{n}\|y_j\|$$
$$-\sum_{i=1}^{n-1}\Big(\frac{1}{\|y_i\|}-\frac{1}{\|y_{i+1}\|}\Big)\sum_{j=1}^{i}\|y_j\|+\sum_{i=1}^{n-1}\Big(\frac{1}{\|y_i\|}-\frac{1}{\|y_{i+1}\|}\Big)\Big\|\sum_{j=1}^{i}y_j\Big\|.$$

As in (8) we obtain

$$\frac{1}{\|y_n\|}\sum_{j=1}^{n}\|y_j\|+\sum_{i=1}^{n-1}\Big(\frac{1}{\|y_i\|}-\frac{1}{\|y_{i+1}\|}\Big)\sum_{j=1}^{i}\|y_j\|=n.$$

Thus

$$\Big\|\sum_{j=1}^{n}\frac{y_j}{\|y_j\|}\Big\|\le\frac{1}{\|y_n\|}\Big\|\sum_{j=1}^{n}y_j\Big\|+\sum_{i=1}^{n-1}\Big(\frac{1}{\|y_i\|}-\frac{1}{\|y_{i+1}\|}\Big)\Big\|\sum_{j=1}^{i}y_j\Big\|.$$

By $x_{n+1-j}=y_j$,

$$\Big\|\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}\Big\|\le\frac{1}{\|x_1\|}\Big\|\sum_{j=1}^{n}x_j\Big\|+\sum_{i=1}^{n-1}\Big(\frac{1}{\|x_{n+1-i}\|}-\frac{1}{\|x_{n-i}\|}\Big)\Big\|\sum_{j=1}^{i}x_{n+1-j}\Big\|$$
$$=\frac{1}{\|x_1\|}\Big\|\sum_{j=1}^{n}x_j\Big\|+\sum_{k=2}^{n}\Big(\frac{1}{\|x_k\|}-\frac{1}{\|x_{k-1}\|}\Big)\Big\|\sum_{j=k}^{n}x_j\Big\|.$$

Thus the inequality (6) holds. □

In the following, we shall show that the inequalities in Theorem 2.1 are sharper than those in Theorem A.

Theorem 2.2. *The inequality (6) and the inequality (5) in Theorem 2.1 are sharper than the inequality (3) and the inequality (2) in Theorem A, respectively.*

Proof. Without loss of generality we may assume that $\|x_1\|\ge\|x_2\|\ge\cdots\ge\|x_n\|>0$. We put P and Q as the right side of the inequality (6) in Theorem 2.1 and the right side of the inequality (3) in Theorem A, respectively. Let us show the inequality $P\le Q$. For each k with $1\le k\le n$ we put $u_k=\sum_{j=k}^{n}x_j$ and $v_k=\sum_{j=1}^{k}x_j$. Moreover, for each k with $2\le k\le n$ we put $\alpha_k=1/\|x_k\|-1/\|x_{k-1}\|$. Fix i with $2\le i\le n-1$. Since

$$\frac{1}{\max\limits_{1\le j\le n}\|x_j\|}=\frac{1}{\|x_1\|}=\frac{1}{\|x_i\|}-\sum_{k=2}^{i}\alpha_k$$

and $\|u_1\| = \|u_k + v_{k-1}\| \geq \|u_k\| - \|v_{k-1}\|$ for each k with $2 \leq k \leq n$, it follows that

$$\begin{aligned} P &= \frac{\|u_1\|}{\|x_i\|} - \sum_{k=2}^{i} \alpha_k \|u_1\| + \sum_{k=2}^{n} \alpha_k \|u_k\| \\ &\leq \frac{\|u_1\|}{\|x_i\|} - \sum_{k=2}^{i} \alpha_k (\|u_k\| - \|v_{k-1}\|) + \sum_{k=2}^{n} \alpha_k \|u_k\| \\ &= \frac{\|u_1\|}{\|x_i\|} + \sum_{k=i+1}^{n} \alpha_k \|u_k\| + \sum_{k=2}^{i} \alpha_k \|v_{k-1}\| \\ &\leq \frac{\|u_1\|}{\|x_i\|} + \sum_{k=i+1}^{n} \alpha_k \sum_{j=k}^{n} \|x_j\| + \sum_{k=2}^{i} \alpha_k \sum_{j=1}^{k-1} \|x_j\|. \end{aligned}$$

Here we clearly have

$$\sum_{k=i+1}^{n} \alpha_k \sum_{j=k}^{n} \|x_j\| = \frac{1}{\|x_i\|} \sum_{k=i+1}^{n} (\|x_i\| - \|x_k\|)$$

and

$$\sum_{k=2}^{i} \alpha_k \sum_{j=1}^{k-1} \|x_j\| = \frac{1}{\|x_i\|} \sum_{k=1}^{i-1} (\|x_k\| - \|x_i\|).$$

Noting $\|x_1\| \geq \|x_2\| \geq \cdots \geq \|x_n\| > 0$ we have the inequality

$$P \leq \frac{1}{\|x_i\|} \left(\left\| \sum_{j=1}^{n} x_j \right\| + \sum_{k=1}^{n} \big| \|x_i\| - \|x_k\| \big| \right).$$

Thus we obtain $P \leq Q$.

We put R and S as the left side of the inequality (5) in Theorem 2.1 and the left side of the inequality (2) in Theorem A, respectively. Let us show the inequality $R \geq S$. Fix i with $2 \leq i \leq n-1$. Since

$$\frac{1}{\min\limits_{1 \leq j \leq n} \|x_j\|} = \frac{1}{\|x_n\|} = \frac{1}{\|x_i\|} + \sum_{k=i}^{n-1} \alpha_{k+1}$$

and $\|v_n\| = \|v_k + u_{k+1}\| \geq \|v_k\| - \|u_{k+1}\|$ for each k with $1 \leq k \leq n-1$, it follows that

$$\begin{aligned} R &= \frac{\|v_n\|}{\|x_i\|} + \sum_{k=i}^{n-1} \alpha_{k+1} \|v_n\| - \sum_{k=1}^{n-1} \alpha_{k+1} \|v_k\| \\ &\geq \frac{\|v_n\|}{\|x_i\|} + \sum_{k=i}^{n-1} \alpha_{k+1} (\|v_k\| - \|u_{k+1}\|) - \sum_{k=1}^{n-1} \alpha_{k+1} \|v_k\| \\ &= \frac{\|v_n\|}{\|x_i\|} - \sum_{k=1}^{i-1} \alpha_{k+1} \|u_k\| - \sum_{k=i}^{n-1} \alpha_{k+1} \|u_{k+1}\| \end{aligned}$$

$$\geq \frac{\|u_n\|}{\|x_i\|} - \sum_{k=1}^{i-1} \alpha_{k+1} \sum_{j=1}^{k} \|x_j\| - \sum_{k=i}^{n-1} \alpha_{k+1} \sum_{j=k}^{n} \|x_j\|.$$

Here we clearly have

$$\sum_{k=1}^{i-1} \alpha_{k+1} \sum_{j=1}^{k} \|x_j\| = \frac{1}{\|x_i\|} \sum_{k=1}^{i-1} (\|x_k\| - \|x_i\|)$$

and

$$\sum_{k=i}^{n-1} \alpha_{k+1} \sum_{j=k+1}^{n} \|x_j\| = \frac{1}{\|x_i\|} \sum_{k=i+1}^{n} (\|x_i\| - \|x_k\|).$$

Noting $\|x_1\| \geq \|x_2\| \geq \cdots \geq \|x_n\| > 0$ we have the inequality

$$R \geq \frac{1}{\|x_i\|} \left(\left\| \sum_{j=1}^{n} x_j \right\| - \sum_{k=1}^{n} \big| \|x_i\| - \|x_k\| \big| \right).$$

Thus we obtain $R \geq S$. □

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Received January 27, 2016
Revised March 23, 2016