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A REFINEMENT OF THE GRAND FURUTA INEQUALITY

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Abstract. A refinement of the Löwner–Heinz inequality has been discussed by Moslehian–Najafi. In the preceding paper, we improved it and extended to the Furuta inequality. In this note, we give a further extension for the grand Furuta inequality. We also discuss it for operator means. A refinement of the arithmetic-geometric mean inequality is obtained.

1. Introduction

For an operator A acting on a Hilbert space H, A is said to be positive, denoted by $A \geq 0$ if $(Ax, x) \geq 0$ for all $x \in H$. In particular, A is said to be strictly positive, denoted by $A > 0$ if A is positive and invertible, i.e., $A \geq m$ for some $m > 0$. Based on the strict positivity, we define the strict order $A > B$ for selfadjoint operators A and B if $A - B > 0$. Recently the Löwner–Heinz inequality was refined under the strict order by Moslehian–Najafi [9]:

Theorem A. *If $A - B \geq m > 0$ for $A > B \geq 0$, then*

$$A^r - B^r \geq \| A \|^r - (\| A \| - m)^r \quad \text{for } r \in [0, 1].$$

Very recently we discussed a generalization of Theorem A for operator monotone functions, and consequently gave it an improvement in [3], cf. [7].

If $A - B \geq m > 0$ for $A > B \geq 0$ and f is non-constant operator monotone on $[0, \infty)$, then

$$f(A) - f(B) \geq f(\| B \| + m) - f(\| B \|) > 0.$$

As a consequence, we obtained the following improvement of Theorem A:

Theorem B. *If $A - B \geq m > 0$ for $A,\ B \geq 0$ and $r \in [0, 1]$, then*

$$A^r - B^r \geq (\| B \| + m)^r - \| B \|^r > 0 \quad \text{for } r \in [0, 1].$$

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On the other hand, the Löwner–Heinz inequality has a beautiful extension, so-called the Furuta inequality. So another direction of generalizations is to discuss it for the Furuta inequality, [4]. We obtained that the Furuta inequality preserves the strict operator order $A > B > 0$.

In this note, we prove it for the grand Furuta inequality which is a further extension of the Furuta inequality.

2. The Furuta inequality

First of all, we cite the Furuta inequality (FI) established in [5], see also [2], [6], [8] and [10] for the best possibility of it.

The Furuta inequality. If $A \geq B \geq 0$, then for each $r \geq 0$,

$$A^{\frac{p+r}{q}} \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}}$$

holds for $p \geq 0,\ q \geq 1$ with

$$(1+r)q \geq p+r.$$

We here remark that the case $r = 0$ in (FI) is just the Löwner–Heinz inequality. As a matter of fact, we showed an extension of Theorem B in the form of Furuta inequality in our preceding work [4]. For convenience, we introduced a constant $k(b, m, p, q, r)$ for $b, m, p, q, r \geq 0$ by

$$k(b, m, p, q, r) = (b+m)^{\frac{p+r}{q}-r} - b^{\frac{p+r}{q}-r},$$

and denoted by $m_B = \| B^{-1} \|^{-1}$. We showed the following results:

Lemma 1. *Let A and B be invertible positive operators with $A - B \geq m > 0$. Then for $0 < r \leq 1$,*

$$A^{\frac{p+r}{q}} - (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}} \geq k(\| B \|, m, p, q, r) m_A^r$$

holds for $p \geq 0,\ q \geq 1$ with $(1+r)q \geq p+r \geq qr$.

Proof. For the reader's convenience, we cite a proof. We note that $q \geq 1$ and $(1+r)q \geq p+r \geq qr$ assure the exponent $\frac{p+r}{q} - r$ in the constant k belongs to $[0, 1]$.

Since $0 \leq r \leq 1$, it follows from Theorem B that

$$\begin{aligned} A^{\frac{p+r}{q}} - (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}} &= A^{\frac{p+r}{q}} - A^{\frac{r}{2}} B^{\frac{p}{2}} (B^{\frac{p}{2}} A^r B^{\frac{p}{2}})^{\frac{1}{q}-1} B^{\frac{p}{2}} A^{\frac{r}{2}} \\ &= A^{\frac{p+r}{q}} - A^{\frac{r}{2}} B^{\frac{p}{2}} (B^{\frac{-p}{2}} A^{-r} B^{\frac{-p}{2}})^{1-\frac{1}{q}} B^{\frac{p}{2}} A^{\frac{r}{2}} \\ &\geq A^{\frac{p+r}{q}} - A^{\frac{r}{2}} B^{\frac{p}{2}} (B^{-\frac{p}{2}} B^{-r} B^{-\frac{p}{2}})^{1-\frac{1}{q}} B^{\frac{p}{2}} A^{\frac{r}{2}} \\ &= A^{\frac{p+r}{q}} - A^{\frac{r}{2}} B^{p-(p+r)(1-\frac{1}{q})} A^{\frac{r}{2}} \\ &= A^{\frac{r}{2}} (A^{\frac{p+r}{q}-r} - B^{\frac{p+r}{q}-r}) A^{\frac{r}{2}} \\ &\geq k(\| B \|, m, p, q, r) A^r \\ &\geq k(\| B \|, m, p, q, r) m_A^r. \quad \square \end{aligned}$$

$\square$

In the Furuta inequality, the optimal case where $p \geq 1$ and $(1+r)q = p+r$ is the most important, for which a beautiful mean theoretic expression is presented by Kamei [8] as follows:

A satellite of (FI) *If $A \geq B \geq 0$, then*

$$A^{-r} \#_{\frac{1+r}{p+r}} B^p \leq B (\leq A)$$

holds for $p \geq 1$ and $r \geq 0$, where $A \#_s B = A^{\frac{1}{2}} (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^s A^{\frac{1}{2}}$ for $A > 0$.

By the use of it, we have the following estimation for the optimal case of the Furuta inequality.

Theorem 2. *Let A and B be invertible positive operators with $A - B \geq m > 0$. Then*

$$A^{1+r} - (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1+r}{p+r}} \geq m m_A^r$$

holds for $p \geq 1$ and $r \geq 0$.

Proof. Taking $r = 1$ in Lemma 1, we have

$$A^2 - (A^{\frac{1}{2}} B^p A^{\frac{1}{2}})^{\frac{2}{p+1}} \geq m m_A := m_1,$$

where $m_B = \| B^{-1} \|^{-1}$. Put $A_1 = A^2$, $B_1 = (A^{\frac{1}{2}} B^p A^{\frac{1}{2}})^{\frac{2}{p+1}}$, $C = A^{\frac{1}{2}} B^p A^{\frac{1}{2}}$ and $p_1 = \frac{p+1}{2}$. Then the satellite assures that

$$A_1^{-s} \#_{\frac{1+s}{p_1+s}} B_1^{p_1} \leq B_1,$$

that is,

$$A^{-2s} \#_{\frac{2(1+s)}{p+1+2s}} C \leq B_1, \text{ i.e., } (A^s C A^s)^{\frac{2(1+s)}{p+1+2s}} \leq A^s B_1 A^s$$

for $s \geq 0$ and $p \geq 1$. Hence it follows that for $r = 2s+1$ with $s \geq 0$,

$$A^{1+r} - (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1+r}{p+r}} = A^{2(1+s)} - (A^s C A^s)^{\frac{2(1+s)}{p+1+2s}}$$
$$\geq A^s(A^2 - B_1)A^s \geq m_1 A^{2s} \geq m m_A^{2s+1} = m m_A^r.$$

Hence the conclusion is obtained. □

An estimation of the Furuta inequality for a general case is given as an application of Theorem 2 and Theorem A.

Theorem 3. *Let A and B be invertible positive operators with $A - B \geq m > 0$. Then*

$$A^{\frac{p+r}{q}} - (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}} \geq \| A \|^{\frac{p+r}{q}} - \| A^{1+r} - m m_A^r \|^{\frac{p+r}{(1+r)q}}$$

holds for $p, r \geq 0,\ q \geq 1$ with $(1+r)q \geq p+r$.

Proof. Put $A_1 = A^{1+r},\ B_1 = (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1+r}{p+r}}$ and $m_1 = m m_A^r$. Then Theorem 2 says that $A_1 - B_1 \geq m_1$. So we apply Theorem A to $A_1 > B_1$ and $r_1 = \frac{p+r}{(1+r)q} \in (0,1]$. Namely we have

$$A_1^{r_1} - B_1^{r_1} \geq \| A_1 \|^{r_1} - \| A_1 - m_1 \|^{r_1} .$$

□

3. Grand Furuta inequality

First of all, we mention the Ando-Hiai inequality, [1]:

(AH) If $A \#_\alpha B \leq I$ for $A,\ B \geq 0$, then $A^r \#_\alpha B^r \leq I$ for $r \geq 1$.

To compare with (AH) and (FI), we arrange (AH) as a Furuta type operator inequality. Now the assumption of (AH) $A \#_\alpha B \leq I$ is equivalent to the inequality

$$B_1 = (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^\alpha \leq A^{-1} = A_1.$$

Similarly, the conclusion $A^r \#_\alpha B^r \leq I$ is equivalent to

$$A^{-r} \geq [A^{-\frac{r}{2}} B^r A^{-\frac{r}{2}}]^\alpha = [A^{-\frac{r}{2}} (A^{\frac{1}{2}} (A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) A^{\frac{1}{2}})^r A^{-\frac{r}{2}}]^\alpha.$$

Replacing $p = \alpha^{-1}$, (AH) is reformulated that

$$A_1 \geq B_1 > 0 \implies A_1^r \geq (A_1^{\frac{r}{2}} (A_1^{-\frac{1}{2}} B_1^p A_1^{-\frac{1}{2}})^r A_1^{\frac{r}{2}})^{\frac{1}{p}} \qquad (\dagger)$$

for $r \geq 1$ and $p \geq 1$.

Moreover, to make a simultaneous extension of both (FI) and (AH), Furuta added another variable $t \in [0,1]$ as in the case of an extension of (LH) to (FI).

Grand Furuta inequality (GFI) *If $A \geq B > 0$ and $t \in [0,1]$, then*

$$[A^{\frac{r}{2}} (A^{-\frac{t}{2}} B^p A^{-\frac{t}{2}})^s A^{\frac{r}{2}}]^{\frac{1-t+r}{(p-t)s+r}} \leq A^{1-t+r}$$

holds for $r \geq t$ and $p, s \geq 1$.

As a matter of fact, (GFI) interpolates (FI) with (AH), i.e.,

$$\text{(GFI) for } t = 1,\ r = s \iff \text{(AH)}$$
$$\text{(GFI) for } t = 0,\ (s = 1) \iff \text{(FI)}.$$

In this section, we discuss the strict positivity for the grand Furuta inequality. For convenience, for $s \notin [0, 1]$, we denote by $A \natural_s B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^s A^{\frac{1}{2}}$ for $A > 0$.

Theorem 4. *If $A - B \geq m$ for some $m > 0$ and $t \in [0, 1]$, then*

$$A^{1-t+r} - [A^{\frac{r}{2}}(A^{-\frac{t}{2}}B^p A^{-\frac{t}{2}})^s A^{\frac{r}{2}}]^{\frac{1-t+r}{(p-t)s+r}} \geq m m_A^{r-t}$$

for $p,\ s \geq 1$ and $r \geq t$.

Proof. It is known that $B_1 = (A^t \natural_s B^p)^{\frac{1}{p_1}} \leq B$, where $p_1 = (p - t)s + t$. Actually, if $1 \leq s \leq 2$, then

$$A^t \natural_s B^p \leq B^{p_1} \quad \text{and so } B_1 \leq B.$$

Next, if $B_1 = (A^t \natural_s B^p)^{\frac{1}{p_1}} \leq B$ holds for some $s \geq 1$, then, for $s_1 \in [1, 2]$ and $p_2 = (p - t)ss_1 + t$,

$$B_2 = (A^t \natural_{ss_1} B^p)^{\frac{1}{p_2}} = (A^t \natural_{s_1} (A^t \natural_s B^p))^{\frac{1}{p_2}} = (A^t \natural_{s_1} B_1^{p_1})^{\frac{1}{p_2}} \leq B_1 \leq B$$

by $s_1 \in [1, 2]$ and $(p_1 - t)s_1 + t = p_2$. Hence we make it sure.

So, applying Theorem 2 for $A - B_1 (\geq A - B) \geq m > 0$ and $p_1 = (p - t)s + t$, $r_1 = r - t$, we have

$$A^{1-t+r} - [A^{\frac{r}{2}}(A^{-\frac{t}{2}}B^p A^{-\frac{t}{2}})^s A^{\frac{r}{2}}]^{\frac{1-t+r}{(p-t)s+r}} = A^{1-t+r} - [A^{\frac{r-t}{2}} B_1^{p_1} A^{\frac{r-t}{2}}]^{\frac{1-t+r}{p_1-t+r}} \geq m m_A^{r-t}.$$

□

4. A refined arithmetic-geometric mean inequality

In the above, we discuss the strict positivity on the Furuta inequality and the grand Furuta inequality. In this section, we apply its idea to the arithmetic-geometric mean inequality and consequently obtian a refinement of the inequality. It says that if $A > B > 0$, then $A \nabla_t B > A \#_t B$. In other words, the arithmetic-geometric mean inequality preserves the strict positivity.

Theorem 5. *If $A - B \geq m > 0$ for $A,\ B > 0$, then for each $0 \leq t \leq 1$*

$$f_t(1 - \frac{m}{\| A \|}) m_A \leq A \nabla_t B - A \#_t B \leq f_t(\frac{m_B}{\| A \|}) \| A \|,$$

where $m_B = \| B^{-1} \|^{-1}$ and $f_t(x) = 1 - t + tx - x^t$.

Proof. If $A - B \geq m > 0$ for $A, B > 0$, then $H = A^{-\frac{1}{2}} B A^{-\frac{1}{2}}$ has bounds such that

$$\frac{m_B}{\| A \|} \leq H \leq 1 - \frac{m}{\| A \|} < 1.$$

Actually we have

$$H \geq m_B A^{-1} \geq m_B \| A \|^{-1}$$

and

$$H \leq A^{-1}(A - m) = 1 - mA^{-1} \leq 1 - m \| A \|^{-1} .$$

Now, for a fixed $t \in (0, 1)$, $f = f_t$ is decreasing and positive on $[0, 1)$ because

$$f'(x) = t(1 - x^{t-1}) < 0 \quad \text{and} \quad f(1) = 0.$$

Hence it follows that $0 < f(H) \leq m_H$ and so

$$f(H) \geq f(\| H \|) \geq f(1 - \frac{m}{\| A \|}); \; f(H) \leq f(m_H) \leq f(\frac{m_B}{\| A \|}).$$

Since $A \nabla_t B - A \#_t B = A^{\frac{1}{2}} f(H) A^{\frac{1}{2}}$, we have the conclusion. □

Finally we discuss the strict positivity of the geometric-harmonic mean inequality. For this, we cite the following lemma:

Lemma 6. *If $A - B \geq m$ for some $m > 0$, then $B^{-1} - A^{-1} \geq \frac{m}{(\|B\|+m)\|B\|} := m_1$.*

As a matter of fact, it is proved as

$$B^{-1} - A^{-1} \geq B^{-1} - (B + m)^{-1} \geq \| B \|^{-1} - \| B + m \|^{-1} = m_1.$$

Now, combining Lemma 6 with Theorem 5, we have

$$B^{-1} \nabla_{1-t} A^{-1} - B^{-1} \#_{1-t} A^{-1} \geq f_{1-t}(1 - \frac{m_1}{\| B^{-1} \|}) m_{B^{-1}} := m_2.$$

If we put $B_1 = (A \#_t B)^{-1} = B^{-1} \#_{1-t} A^{-1}$, then it follows from Lemma 6 again that

$$(B^{-1} \#_{1-t} A^{-1})^{-1} - (B^{-1} \nabla_{1-t} A^{-1})^{-1} \geq \frac{m_2}{(\| B_1 \| + m_2) \| B_1 \|}.$$

That is, we have the strict positivity of the geometric-harmonic mean inequality as follows:

Corollary 7. *Notation as in above. If $A - B \geq m$ for some $m > 0$ and $t \in (0, 1)$, then*

$$A \#_t B - A !_t B \geq \frac{m_2}{(\| B_1 \| + m_2) \| B_1 \|} \geq \frac{m_2}{(\| B^{-1} \| + m_2) \| B^{-1} \|}.$$

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