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THE CONSTANTS RELATED TO ISOSCELES ORTHOGONALITY IN NORMED SPACES AND ITS DUAL

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Abstract. We consider isosceles orthogonality and Birkhoff orthogonality, which are the most used notions of generalized orthogonality. In 2006, Ji and Wu introduced a geometric constant $D(X)$ to give a quantitative characterization of the difference between these two orthogonality types. From their results, we have that $D(X) = D(X^*)$ holds for any symmetric Minkowski plane. On the other hand, for the James constant $J(X)$, Saito, Sato and Tanaka recently showed that if the norm of a two-dimensional space X is absolute and symmetric then $J(X) = J(X^*)$ holds. In this paper, we consider the constant $D(X, \lambda)$ such that $D(X) = \inf_{\lambda\in\mathbb{R}} D(X, \lambda)$ and obtain that in the same situation $D(X, \lambda) = D(X^*, \lambda)$ holds for any $\lambda \in (0, 1)$.

1. Introduction

We denote by X a real normed space with the norm $\|\cdot\|$, the unit ball B_X and the unit sphere S_X. Throughout this paper, we assume that the dimension of X is at least two. In the case of that X is an inner product space, an element $x \in X$ is said to be orthogonal to $y \in X$ (denoted by $x \perp y$) if the inner product $\langle x, y\rangle$ is equal to zero. In the general setting of normed spaces, many notions of orthogonality have been introduced by means of equivalent propositions to the usual orthogonality in inner product spaces. For example, Birkhoff [2] introduced *Birkhoff orthogonality* : x is said to be Birkhoff orthogonal to y (denoted by $x \perp_B y$) if

$$\forall\lambda \in \mathbb{R}, \quad \|x + \lambda y\| \geq \|x\|.$$

James [6] introduced *isosceles orthogonality* : x is said to be isosceles orthogonal to y (denoted by $x \perp_I y$) if

$$\|x + y\| = \|x - y\|.$$

These generalized orthogonality types have been studied in a lot of papers ([1, 7] and so on).

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Recently, quantitative studies of the difference between two orthogonality types have been performed (see [5, 8, 13, 14]). Among of them, the constant

$$D(X) = \inf\left\{ \inf_{\lambda\in\mathbb{R}} \|x+\lambda y\| : x, y \in S_X, x \perp_I y \right\}$$

is introduced and studied by D. Ji and S. Wu [8]. This constant measures the difference between Birkhoff and isosceles orthogonalities in the unit sphere of X. In the paper [8], they obtained a result on a symmetric Minkowski plane.

Let X be a Minkowski plane. If there exists $e_1, e_2 \in S_X$ such that

$$\|e_1 + te_2\| = \|e_1 - te_2\| = \|e_2 + te_1\| = \|e_2 - te_1\|$$

for any $t \in \mathbb{R}$, then we call X a symmetric Minkowski plane and $\{e_1, e_2\}$ a pair of axes of X. Ji and Wu obtained the following

Theorem 1.1 ([8]). *Let X be a symmetric Minkowski plane on $\mathbb{R}^2$ and $e_1 = (1, 0)$, $e_2 = (0, 1)$ be a pair of axes of X. Then*

$$D(X) = \inf_{t\in\mathbb{R}} \frac{1+t^2}{\|(t,1)\|\|(t,1)\|_*},$$

where $\|\cdot\|_*$ denotes the norm of the dual space of X.

On the other hand, Saito, Sato and Tanaka obtained a results on the James constant of the space $\mathbb{R}^2$ and its dual space. A normed space X is said to be uniformly non-square if there exists a number $\delta > 0$ such that $\|x+y\| > 2(1-\delta)$ and $x, y \in S_X$ implies $\|x-y\| < 2(1-\delta)$. The James constant

$$J(X) = \sup\{\min\{\|x+y\|, \|x-y\|\} : x, y \in S_X\}$$

was defined by Gao and Lau [4]. It is known that X is uniformly non-square if and only if $J(X) < 2$. On the other hand, in general, the equality $J(X) = J(X^*)$ does not necessarily hold. A norm $\|\cdot\|$ on $\mathbb{R}^2$ is said to be symmetric if $\|(x, y)\| = \|(y, x)\|$ for all (x, y) and absolute if $\|(x, y)\| = \|(|x|, |y|)\|$ for all (x, y). Saito, Sato and Tanaka obtained the following

Theorem 1.2 ([16]). *Let X be a two-dimensional real normed space $\mathbb{R}^2$ equipped with a symmetric absolute norm. Then $J(X) = J(X^*)$.*

We consider a function

$$D(X, \lambda) = \inf\{\|x+\lambda y\| : x, y \in S_X, x \perp_I y\}.$$

It is easy to see that $D(X) = \inf_{\lambda\in\mathbb{R}} D(X, \lambda)$.

In the paper [4], Gao and Lau also introduce Schäffer constant

$$S(X) = \inf\{\max\{\|x+y\|, \|x-y\|\} : x, y \in S_X\}.$$

The equality $J(X)S(X) = 2$ holds for any normed space. Gao and Lau showed that the James constant $J(X)$ and the Schäffer constant $S(X)$ are reformulated in

$$J(X) = \sup\{\|x + y\| : x, y \in S_X, x \perp_I y\}$$
$$S(X) = \inf\{\|x + y\| : x, y \in S_X, x \perp_I y\},$$

respectively. Hence one has $D(X, 1) = S(X) = 2/J(X)$. Thus, for the space $X = \mathbb{R}^2$ with a symmetric absolute norm, from the above theorems, we have

$$\inf_{\lambda\in\mathbb{R}} D(X, \lambda) = \inf_{\lambda\in\mathbb{R}} D(X^*, \lambda) \quad \text{and} \quad D(X, 1) = D(X^*, 1).$$

Meanwhile, for $\lambda = 0$, we clearly have $D(X, 0) = 1 = D(X^*, 0)$. Therefore, our aim in this paper is to obtain that if X is the space $\mathbb{R}^2$ with a symmetric absolute norm, then $D(X, \lambda) = D(X^*, \lambda)$ for any $\lambda \in (0, 1)$

2. Preliminaries

Recall that a norm $\|\cdot\|$ on $\mathbb{R}^2$ is said to be absolute if

$$\|(x, y)\| = \|(|x|, |y|)\|$$

for all $(x, y) \in \mathbb{R}^2$, and normalized if $\|(1, 0)\| = \|(0, 1)\| = 1$. The family of all absolute normalized norms on $\mathbb{R}^2$ is denoted by AN_2. As in Bonsall and Duncan [3], AN_2 is in a one-to-one correspondence with the family Ψ_2 of all convex functions ψ on $[0, 1]$ with $\max\{1 - t, t\} \le \psi(t) \le 1$ for all $0 \le t \le 1$. Indeed, for any $\|\cdot\| \in AN_2$ we put $\psi(t) = \|(1 - t, t)\|$. Then $\psi \in \Psi_2$. Conversely, for all $\psi \in \Psi_2$ let

$$\|(x, y)\|_\psi = \begin{cases} (|x| + |y|)\psi\left(\dfrac{|y|}{|x| + |y|}\right) & \text{if} \quad (x, y) \neq (0, 0), \\ 0 & \text{if} \quad (x, y) = (0, 0). \end{cases}$$

Then $\|\cdot\|_\psi \in AN_2$, and $\psi(t) = \|(1 - t, t)\|_\psi$ (cf. [15]).

We also recall *symmetry.* A norm $\|\cdot\|$ on $\mathbb{R}^2$ is said to be symmetric if $\|(x, y)\| = \|(y, x)\|$ for all $(x, y) \in \mathbb{R}^2$. A function $\psi \in \Psi_2$ is said to be symmetric if $\psi(1 - t) = \psi(t)$ for all $t \in [0, 1]$. We note that a symmetric norm $\|\cdot\| \in AN_2$ is associated with a symmetric function $\psi \in \Psi_2$. Let Ψ_2^S be the collection of all symmetric function in Ψ_2.

For $\psi \in \Psi_2$, the dual function ψ^* on $[0, 1]$ is defined by

$$\psi^*(s) = \sup\left\{\frac{(1 - t)(1 - s) + ts}{\psi(t)} : t \in [0, 1]\right\}$$

for $s \in [0, 1]$. It was proved that $\psi^* \in \Psi_2$ and that $\|\cdot\|_{\psi^*} \in AN_2$ is the dual norm of $\|\cdot\|_\psi$, that is, $(\mathbb{R}^2, \|\cdot\|_\psi)^*$ is identified with $(\mathbb{R}^2, \|\cdot\|_{\psi^*})$ (cf. [10, 11, 12]).

For simplicity, we write X_ψ for the space $\mathbb{R}^2$ with an absolute norm $\|\cdot\|_\psi$. Our aim can be turned into the following form:

Theorem 2.1. *Let $\psi \in \Psi_2^S$. Then $D(X_\psi, \lambda) = D(X_{\psi^*}, \lambda)$ for any $\lambda \in (0, 1)$.*

Let $\psi \in \Psi_2^S$ and $t \in [0, 1]$. It is known that for $x = (1 - t, t)/\psi(t) \in S_{X_\psi}$, an element $y = (-t, 1 - t)/\psi(t)$ is the unique one satisfying $x \perp_I y$ and $y \in S_{X_\psi}$, with the exception of $-y$ ([4, 9]). Letting

$$g_\lambda(t) = \begin{cases} \dfrac{\lambda + (1-\lambda)t}{1 + \lambda - 2\lambda t} & \text{if } 0 \le t \le \dfrac{1-\lambda}{2}, \\ \dfrac{1 - (1+\lambda)t}{1 + \lambda - 2\lambda t} & \text{if } \dfrac{1-\lambda}{2} \le t \le \dfrac{1}{2} \end{cases}$$

and

$$G_{\psi,\lambda}(t) = \frac{1 + \lambda - 2\lambda t}{\psi(t)} \psi(g_\lambda(t)),$$

we have $0 \le g_\lambda(t) \le 1/2$ and $\|x + \lambda y\|_\psi = G_{\psi,\lambda}(t)$. In addition, let

$$h_\lambda(t) = \begin{cases} \dfrac{\lambda - (1+\lambda)t}{1 + \lambda - 2t} & \text{if } 0 \le t \le \dfrac{\lambda}{1+\lambda}, \\ \dfrac{-\lambda + (1+\lambda)t}{1 - \lambda + 2\lambda t} & \text{if } \dfrac{\lambda}{1+\lambda} \le t \le \dfrac{1}{2} \end{cases}$$

and

$$H_{\psi,\lambda}(t) = \begin{cases} \dfrac{1 + \lambda - 2t}{\psi(t)} \psi(h_\lambda(t)) & \text{if } 0 \le t \le \dfrac{\lambda}{1+\lambda}, \\ \dfrac{1 - \lambda + 2\lambda t}{\psi(t)} \psi(h_\lambda(t)) & \text{if } \dfrac{\lambda}{1+\lambda} \le t \le \dfrac{1}{2}. \end{cases}$$

Then we have $0 \le h_\lambda(t) \le 1/2$ and $\|x - \lambda y\|_\psi = H_{\psi,\lambda}(t)$. Thus, the constant $D(X_\psi, \lambda)$ is given by

$$D(X_\psi, \lambda) = \min\left\{ \min_{0 \le t \le 1/2} G_{\psi,\lambda}(t), \min_{0 \le t \le 1/2} H_{\psi,\lambda}(t) \right\}.$$

The function g_λ maps $[0, 1/2]$ to $[\min\{(1-\lambda)/2, \lambda/(1+\lambda)\}, 1/2]$. In particular, one has $g_\lambda : [0, (1-\lambda)/2] \mapsto [\lambda/(1+\lambda), 1/2]$ and $g_\lambda : [(1-\lambda)/2, 1/2] \mapsto [(1-\lambda)/2, 1/2]$. On the other hand, the function h_λ maps $[0, 1/2]$ to $[0, \max\{(1-\lambda)/2, \lambda/(1+\lambda)\}]$. Especially, one has $h_\lambda : [0, \lambda/(1+\lambda)] \mapsto [0, \lambda/(1+\lambda)]$ and $h_\lambda : [\lambda/(1+\lambda), 1/2] \mapsto [0, (1-\lambda)/2]$. We obtain

Lemma 2.2. *Let $\psi \in \Psi_2^S$. Then the following hold:*

1. *If $t \in [0, (1-\lambda)/2]$, then $h_\lambda(g_\lambda(t)) = t$ and $H_{\psi,\lambda}(g_\lambda(t)) = (1+\lambda^2)/G_{\psi,\lambda}(t)$.*
2. *If $t \in [(1-\lambda)/2, 1/2]$, then $g_\lambda(g_\lambda(t)) = t$ and $G_{\psi,\lambda}(g_\lambda(t)) = (1+\lambda^2)/G_{\psi,\lambda}(t)$.*
3. *If $t \in [0, \lambda/(1+\lambda)]$, then $h_\lambda(h_\lambda(t)) = t$ and $H_{\psi,\lambda}(h_\lambda(t)) = (1+\lambda^2)/H_{\psi,\lambda}(t)$.*
4. *If $t \in [\lambda/(1+\lambda), 1/2]$, then $g_\lambda(h_\lambda(t)) = t$ and $G_{\psi,\lambda}(h_\lambda(t)) = (1+\lambda^2)/H_{\psi,\lambda}(t)$.*

Proof. For elements $x = (a, b)$ and $y = (-b, a)$, one has $x \perp_I y$ and $x + \lambda y = (a - \lambda b, b + \lambda a)$. Let $z = (-(b + \lambda a), a - \lambda b)$. Then we obtain $x + \lambda y \perp_I z$ and

$$x + \lambda y - \lambda z = (1 + \lambda^2)(a, b) = (1 + \lambda^2)x.$$

In addition, for $x - \lambda y = (a + \lambda b, b - \lambda a)$, letting $w = (-(b - \lambda a), a + \lambda b)$, we have $x - \lambda y \perp_I w$ and $x - \lambda y + \lambda w = (1 + \lambda^2)x$. Thus, we obtain this lemma. □

3. On the piecewise linear functions

In this section, we treat the piecewise linear functions. A finite sequence $(t_i)_{i=0}^n$ is said to be a partition of the interval $[0, 1/2]$ if $0 = t_0 < t_1 < \cdots < t_n = 1/2$. Any finite subset of $[0, 1/2]$ including 0 and 1/2 can be viewed as a partition of $[0, 1/2]$ by taking strictly increasing rearrangement. Thus one can identify a partition $(t_i)_{i=0}^n$ with a set $\{t_i : 0 \le i \le n\}$. A function ψ on $[0, 1/2]$ is said to be piecewise linear if its graph is broken line. That is, ψ is piecewise linear if there exists a partition $(t_i)_{i=0}^n$ of $[0, 1/2]$ and a sequence $(a_i)_{i=0}^n \subset [1/2, 1]$ such that

$$\psi(t) = \frac{a_i - a_{i-1}}{t_i - t_{i-1}} t + \frac{a_{i-1}t_i - a_i t_{i-1}}{t_i - t_{i-1}}$$

for each $t \in [t_{i-1}, t_i]$. It is easy to see that $\psi(t_i) = a_i$ for each $0 \le i \le n$. Letting

$$\alpha_i = \frac{a_i - a_{i-1}}{t_i - t_{i-1}} \quad \text{and} \quad \beta_i = \frac{a_{i-1}t_i - a_i t_{i-1}}{t_i - t_{i-1}},$$

we have $\psi(t) = \alpha_i t + \beta_i$ for each $t \in [t_{i-1}, t_i]$. A point t_i satisfying $0 < i < n$ and $\alpha_i < \alpha_{i+1}$ is called the corner point of ψ. We remark that every corner point is deduced by determining the intersection point of two distinct lines. A partition is said to be simplified if all the elements $t_1, t_2, \cdots, t_{n-1}$ are corner points. The piecewise linear function ψ on $[0, 1/2]$ extends to a piecewise linear function on the interval $[0, 1]$ by $\psi(1 - t) = \psi(t)$.

For a piecewise linear function $\psi \in \Psi_2^S$, we obtain the following

Lemma 3.1. *Let $\psi \in \Psi_2^S$ be a piecewise linear function with a partition $(t_i)_{i=0}^n$. Suppose that there exists n_1, n_2 such that $0 \le n_1, n_2 \le n$, $t_{n_1} = (1 - \lambda)/2$ and $t_{n_2} = \lambda/(1 + \lambda)$. Then*

$$\begin{aligned} D(X_\psi, \lambda) = \min\{ &\min_{0 \le i \le n} G_{\psi,\lambda}(t_i), \min_{0 \le i \le n} H_{\psi,\lambda}(t_i), \min_{n_1 \le i \le n} G_{\psi,\lambda}(g_\lambda(t_i)), \\ &\min_{n_2 \le i \le n} G_{\psi,\lambda}(h_\lambda(t_i)), \min_{0 \le i \le n_1} H_{\psi,\lambda}(g_\lambda(t_i)), \min_{0 \le i \le n_2} H_{\psi,\lambda}(h_\lambda(t_i))\}. \end{aligned}$$

Proof. For i, let $I_i = [t_{i-1}, t_i]$,

$$J_i = \begin{cases} [g_\lambda(t_{i-1}), g_\lambda(t_i)] & \text{if } 1 \le i \le n_1 \\ [g_\lambda(t_i), g_\lambda(t_{i-1})] & \text{if } n_1 < i \le n \end{cases}$$

and

$$K_i = \begin{cases} [h_\lambda(t_i), h_\lambda(t_{i-1})] & \text{if } 1 \le i \le n_2, \\ [h_\lambda(t_{i-1}), h_\lambda(t_i)] & \text{if } n_2 < i \le n. \end{cases}$$

One has that $(1-\lambda)/2 \le \lambda/(1+\lambda)$ if and only if $\sqrt{2}-1 \le \lambda$. We first suppose $\sqrt{2}-1 \le \lambda \le 1$. Then $(1-\lambda)/2 \le \lambda/(1+\lambda)$ and hence $n_1 \le n_2$.

Let $t \in [0, (1-\lambda)/2]$. Take i, j, k such that $t \in I_i$, $g_\lambda(t) \in I_j$ and $h_\lambda(t) \in I_k$. Then one has $0 \le i \le n_1$, $n_2 \le j \le n$ and $0 \le k \le n_2$. From Lemma 2.2, we have $t \in K_j \cap K_k$. Then the functions $G_{\psi,\lambda}$ and $H_{\psi,\lambda}$ are given by

$$\begin{aligned} G_{\psi,\lambda}(t) = \frac{1+\lambda-2\lambda t}{\psi(t)} \psi(g_\lambda(t)) &= \frac{1+\lambda-2\lambda t}{\alpha_i t + \beta_i} \left\{ \alpha_j \left(\frac{\lambda + (1-\lambda)t}{1+\lambda-2\lambda t} \right) + \beta_j \right\} \\ &= \frac{\{(1-\lambda)\alpha_j - 2\lambda\beta_j\}t + \lambda\alpha_j + (1+\lambda)\beta_j}{\alpha_i t + \beta_i} \end{aligned}$$

and

$$\begin{aligned} H_{\psi,\lambda}(t) = \frac{1+\lambda-2t}{\psi(t)} \psi(h_\lambda(t)) &= \frac{1+\lambda-2t}{\alpha_i t + \beta_i} \left\{ \alpha_k \left(\frac{\lambda - (1+\lambda)t}{1+\lambda-2t} \right) + \beta_k \right\} \\ &= \frac{-\{(1+\lambda)\alpha_k + 2\beta_k\}t + \lambda\alpha_k + (1+\lambda)\beta_k}{\alpha_i t + \beta_i}. \end{aligned}$$

Since $G_{\psi,\lambda}$ and $H_{\psi,\lambda}$ are monotone on $I_i \cap K_j$ and $I_i \cap K_k$ respectively, we have

$$G_{\psi,\lambda}(t) \ge \min\{G_{\psi,\lambda}(t_i), G_{\psi,\lambda}(t_{i-1}), G_{\psi,\lambda}(h_\lambda(t_j)), G_{\psi,\lambda}(h_\lambda(t_{j-1}))\}$$

and

$$H_{\psi,\lambda}(t) \ge \min\{H_{\psi,\lambda}(t_i), H_{\psi,\lambda}(t_{i-1}), H_{\psi,\lambda}(h_\lambda(t_k)), H_{\psi,\lambda}(h_\lambda(t_{k-1}))\}.$$

Let $t \in [(1-\lambda)/2, \lambda/(1+\lambda)]$. We take i, j, k such that $t \in I_i$, $g_\lambda(t) \in I_j$ and $h_\lambda(t) \in I_k$ also in this case. Then one has $n_1 \le i \le n_2$, $n_1 \le j \le n$ and $0 \le k \le n_2$. From Lemma 2.2, we have $t \in J_j \cap K_k$. Then the function $G_{\psi,\lambda}$ is given by

$$\begin{aligned} G_{\psi,\lambda}(t) = \frac{1+\lambda-2\lambda t}{\psi(t)} \psi(g_\lambda(t)) &= \frac{1+\lambda-2\lambda t}{\alpha_i t + \beta_i} \left\{ \alpha_j \left(\frac{1 - (1+\lambda)t}{1+\lambda-2\lambda t} \right) + \beta_j \right\} \\ &= \frac{-\{(1+\lambda)\alpha_j + 2\lambda\beta_j\}t + \alpha_j + (1+\lambda)\beta_j}{\alpha_i t + \beta_i}. \end{aligned}$$

On the other hand, the function $H_{\psi,\lambda}$ is given by the formula in the above case. Since $G_{\psi,\lambda}$ and $H_{\psi,\lambda}$ are monotone on $I_i \cap J_j$ and $I_i \cap K_k$ respectively, we have

$$G_{\psi,\lambda}(t) \ge \min\{G_{\psi,\lambda}(t_i), G_{\psi,\lambda}(t_{i-1}), G_{\psi,\lambda}(g_\lambda(t_j)), G_{\psi,\lambda}(g_\lambda(t_{j-1}))\}$$

and

$$H_{\psi,\lambda}(t) \ge \min\{H_{\psi,\lambda}(t_i), H_{\psi,\lambda}(t_{i-1}), H_{\psi,\lambda}(h_\lambda(t_k)), H_{\psi,\lambda}(h_\lambda(t_{k-1}))\}.$$

Let $t \in [\lambda/(1+\lambda), 1/2]$. Take i, j, k such that $t \in I_i$, $g_\lambda(t) \in I_j$ and $h_\lambda(t) \in I_k$. Then one has $n_2 \le i \le n$, $n_1 \le j \le n$ and $0 \le k \le n_1$. From Lemma 2.2, we have

$t \in J_j \cap J_k$. Then the functions $G_{\psi,\lambda}$ is given by the formula in the above paragraph. On the other hand, the function $H_{\psi,\lambda}$ is given by

$$\begin{aligned} H_{\psi,\lambda}(t) = \frac{1-\lambda+2\lambda t}{\psi(t)}\psi(h_\lambda(t)) &= \frac{1-\lambda+2\lambda t}{\alpha_i t+\beta_i}\left\{\alpha_k\left(\frac{-\lambda+(1+\lambda)t}{1-\lambda+2\lambda t}\right)+\beta_k\right\} \\ &= \frac{\{(1+\lambda)\alpha_k+2\lambda\beta_k\}t-\lambda\alpha_k+(1-\lambda)\beta_k}{\alpha_i t+\beta_i}. \end{aligned}$$

Since $G_{\psi,\lambda}$ and $H_{\psi,\lambda}$ are monotone on $I_i \cap J_j$ and $I_i \cap J_k$ respectively, we have

$$G_{\psi,\lambda}(t) \geq \min\{G_{\psi,\lambda}(t_i), G_{\psi,\lambda}(t_{i-1}), G_{\psi,\lambda}(g_\lambda(t_j)), G_{\psi,\lambda}(g_\lambda(t_{j-1}))\}$$

and

$$H_{\psi,\lambda}(t) \geq \min\{H_{\psi,\lambda}(t_i), H_{\psi,\lambda}(t_{i-1}), H_{\psi,\lambda}(g_\lambda(t_k)), H_{\psi,\lambda}(g_\lambda(t_{k-1}))\}.$$

Suppose that $0 \leq \lambda \leq \sqrt{2}-1$. Then $\lambda/(1+\lambda) \leq (1-\lambda)/2$ and hence $n_2 \leq n_1$. In a similar way, we have the following:

Let $t \in [0, \lambda/(1+\lambda)]$. Then we have

$$G_{\psi,\lambda}(t) \geq \min\{G_{\psi,\lambda}(t_i), G_{\psi,\lambda}(t_{i-1}), G_{\psi,\lambda}(h_\lambda(t_j)), G_{\psi,\lambda}(h_\lambda(t_{j-1}))\}$$

and

$$H_{\psi,\lambda}(t) \geq \min\{H_{\psi,\lambda}(t_i), H_{\psi,\lambda}(t_{i-1}), H_{\psi,\lambda}(h_\lambda(t_k)), H_{\psi,\lambda}(h_\lambda(t_{k-1}))\},$$

where $0 \leq i \leq n_2$, $n_2 \leq j \leq n$ and $0 \leq k \leq n_2$.

Let $t \in [\lambda/(1+\lambda), (1-\lambda)/2]$. Then we have

$$G_{\psi,\lambda}(t) \geq \min\{G_{\psi,\lambda}(t_i), G_{\psi,\lambda}(t_{i-1}), G_{\psi,\lambda}(h_\lambda(t_j)), G_{\psi,\lambda}(h_\lambda(t_{j-1}))\}$$

and

$$H_{\psi,\lambda}(t) \geq \min\{H_{\psi,\lambda}(t_i), H_{\psi,\lambda}(t_{i-1}), H_{\psi,\lambda}(g_\lambda(t_k)), H_{\psi,\lambda}(g_\lambda(t_{k-1}))\},$$

where $n_2 \leq i \leq n_1$, $n_2 \leq j \leq n$ and $0 \leq k \leq n_1$.

Let $t \in [(1-\lambda)/2, 1/2]$. Then we have

$$G_{\psi,\lambda}(t) \geq \min\{G_{\psi,\lambda}(t_i), G_{\psi,\lambda}(t_{i-1}), G_{\psi,\lambda}(g_\lambda(t_j)), G_{\psi,\lambda}(g_\lambda(t_{j-1}))\}$$

and

$$H_{\psi,\lambda}(t) \geq \min\{H_{\psi,\lambda}(t_i), H_{\psi,\lambda}(t_{i-1}), H_{\psi,\lambda}(g_\lambda(t_k)), H_{\psi,\lambda}(g_\lambda(t_{k-1}))\},$$

where $n_1 \leq i \leq n$, $n_1 \leq j \leq n$ and $0 \leq k \leq n_1$. □

Combining the above lemma with Lemma 2.2 again, we have

Lemma 3.2. *Let $\psi \in \Psi_2^S$ be a piecewise linear function with a partition $(t_i)_{i=0}^n$. Suppose that there exists n_1, n_2 such that $0 \le n_1, n_2 \le n$, $t_{n_1} = (1-\lambda)/2$ and $t_{n_2} = \lambda/(1+\lambda)$. Then*

$$\begin{aligned} &D(X_\psi, \lambda) \\ &= \min\left\{ \min_{0\le i\le n} G_{\psi,\lambda}(t_i), \min_{0\le i\le n} H_{\psi,\lambda}(t_i), \frac{1+\lambda^2}{\max_{0\le i\le n} G_{\psi,\lambda}(t_i)}, \frac{1+\lambda^2}{\max_{0\le i\le n} H_{\psi,\lambda}(t_i)} \right\}. \end{aligned}$$

Proof. From Lemma 2.2, we have

$$\min_{n_1\le i\le n} G_{\psi,\lambda}(g_\lambda(t_i)) = \min_{n_1\le i\le n} \frac{1+\lambda^2}{G_{\psi,\lambda}(t_i)},$$

$$\min_{n_2\le i\le n} G_{\psi,\lambda}(h_\lambda(t_i)) = \min_{n_2\le i\le n} \frac{1+\lambda^2}{H_{\psi,\lambda}(t_i)},$$

$$\min_{0\le i\le n_1} H_{\psi,\lambda}(g_\lambda(t_i)) = \min_{0\le i\le n_1} \frac{1+\lambda^2}{G_{\psi,\lambda}(t_i)}$$

and

$$\min_{0\le i\le n_2} H_{\psi,\lambda}(h_\lambda(t_i)) = \min_{0\le i\le n_2} \frac{1+\lambda^2}{H_{\psi,\lambda}(t_i)}.$$

Thus, we obtain this lemma. □

4. The proof of main result

To prove Theorem 2.1, we first recall several results obtained in [16]:

Lemma 4.1 ([16], Lemma 4.5.)**.** *Let (ψ_n) be a monotone sequence of functions in Ψ_2^S. If $\|\psi_n - \psi\|_\infty \to 0$, then $\|\psi_n^* - \psi^*\|_\infty \to 0$.*

Lemma 4.2 ([16], Lemma 4.6.)**.** *Let $\psi \in \Psi_2^S$. Then there exists a sequence (ψ_n) of strictly convex functions in Ψ_2^S such that $\|\psi_n - \psi\|_\infty \to 0$ and $\|\psi_n^* - \psi^*\|_\infty \to 0$ as $n \to \infty$.*

To use these results, we need the following

Lemma 4.3. *Let $0 < \lambda < 1$. Then the function $\psi \mapsto D(X_\psi, \lambda)$ is continuous.*

Proof. Let (ψ_n) be a sequence in Ψ_2^S that converges uniformly to an element $\psi \in \Psi_2^S$. For any $t \in [0, 1/2]$, one has

$$\begin{aligned} |G_{\psi,\lambda}(t) - G_{\psi_n,\lambda}(t)| &= (1+\lambda-2\lambda t)\left|\frac{\psi(g_\lambda(t))}{\psi(t)} - \frac{\psi_n(g_\lambda(t))}{\psi_n(t)}\right| \\ &= (1+\lambda-2\lambda t)\frac{|\psi(g_\lambda(t))\psi_n(t) - \psi_n(g_\lambda(t))\psi(t)|}{\psi(t)\psi_n(t)} \\ &\le 8(1+\lambda-2\lambda t)\|\psi - \psi_n\|_\infty \\ &\le 16\|\psi - \psi_n\|_\infty \end{aligned}$$

and hence $\|G_{\psi,\lambda} - G_{\psi_n,\lambda}\|_\infty \le 16\|\psi - \psi_n\|_\infty$. Thus we have

$$\min_{0\le t\le 1/2} G_{\psi,\lambda}(t) = \lim_{n\to\infty}(\min_{0\le t\le 1/2} G_{\psi_n,\lambda}(t)).$$

For any $t \in [0, \lambda/(1+\lambda)]$, we have

$$\begin{aligned} |H_{\psi,\lambda}(t) - H_{\psi_n,\lambda}(t)| &= (1+\lambda-2t)\left|\frac{\psi(h_\lambda(t))}{\psi(t)} - \frac{\psi_n(h_\lambda(t))}{\psi_n(t)}\right| \\ &\le 8(1+\lambda-2t)\|\psi - \psi_n\|_\infty \\ &\le 16\|\psi - \psi_n\|_\infty. \end{aligned}$$

For any $t \in [\lambda/(1+\lambda), 1/2]$, one has $|H_{\psi,\lambda}(t) - H_{\psi_n,\lambda}(t)| \le 16\|\psi - \psi_n\|_\infty$, too. So we have $\|H_{\psi,\lambda} - H_{\psi_n,\lambda}\|_\infty \le 16\|\psi - \psi_n\|_\infty$ and hence

$$\min_{0\le t\le 1/2} H_{\psi,\lambda}(t) = \lim_{n\to\infty}(\min_{0\le t\le 1/2} H_{\psi_n,\lambda}(t)).$$

Therefore we obtain

$$\begin{aligned} D(X_\psi, \lambda) &= \min\left\{\min_{0\le t\le 1/2} G_{\psi,\lambda}(t), \min_{0\le t\le 1/2} H_{\psi,\lambda}(t)\right\} \\ &= \lim_{n\to\infty} \min\left\{\min_{0\le t\le 1/2} G_{\psi_n,\lambda}(t), \min_{0\le t\le 1/2} H_{\psi_n,\lambda}(t)\right\} = \lim_{n\to\infty} D(X_{\psi_n}, \lambda), \end{aligned}$$

which implies that the function $\psi \mapsto D(X_\psi, \lambda)$ is continuous. $\square$

Lemma 4.4. *Suppose that $\psi \in \Psi_2^S$ is a piecewise linear function with a simplified partition $(t_i^{(0)})_{i=0}^{n_0}$. Let $(t_i)_{i=0}^{n} = (t_i^{(0)})_{i=0}^{n_0} \cup \{(1-\lambda)/2, \lambda/(1+\lambda)\}$, $t_{n_1} = (1-\lambda)/2$ and $t_{n_2} = \lambda/(1+\lambda)$. Then*

$$\begin{aligned} D(X_\psi, \lambda) = \min\{&\min_{0\le i\le n_0} G_{\psi,\lambda}(t_i^{(0)}), \min_{0\le i\le n_0} H_{\psi,\lambda}(t_i^{(0)}), \\ &\frac{1+\lambda^2}{\max_{0\le i\le n} G_{\psi,\lambda}(t_i)}, \frac{1+\lambda^2}{\max_{0\le i\le n} H_{\psi,\lambda}(t_i)}\} \end{aligned}$$

Proof. From the definitions, one has

$$G_{\psi,\lambda}(t_{n_1}) = \frac{1+\lambda-2\lambda t_{n_1}}{\psi(t_{n_1})}\psi(g_\lambda(t_{n_1})) = \frac{1+\lambda^2}{\psi(t_{n_1})}\psi(1/2) = \frac{1+\lambda^2}{G_{\psi,\lambda}(1/2)} = \frac{1+\lambda^2}{G_{\psi,\lambda}(t_n)}$$

and

$$H_{\psi,\lambda}(t_{n_2}) = \frac{1-\lambda+2\lambda t_{n_2}}{\psi(t_{n_2})}\psi(h_\lambda(t_{n_2})) = \frac{1+\lambda^2}{(1+\lambda)\psi(t_{n_2})}\psi(0) = \frac{1+\lambda^2}{H_{\psi,\lambda}(0)} = \frac{1+\lambda^2}{H_{\psi,\lambda}(t_0)}.$$

On the other hand, from the fact that both t_{n_1} and t_{n_2} are not a corner point,

$$\psi(t_{n_l}) = \frac{\psi(t_{n_l+1})-\psi(t_{n_l-1})}{t_{n_l+1}-t_{n_l-1}}t_{n_l} + \frac{\psi(t_{n_l-1})t_{n_l+1}-\psi(t_{n_l+1})t_{n_l-1}}{t_{n_l+1}-t_{n_l-1}} \quad (l=1,2).$$

As in the proof of Lemma 3.1, for j,k satisfying $g_\lambda(t_{n_2}) \in [t_{j-1},t_j]$, $h_\lambda(t_{n_1}) \in [t_{k-1},t_k]$, we have

$$G_{\psi,\lambda}(t_{n_2}) \geq \min\{G_{\psi,\lambda}(t_{n_2+1}), G_{\psi,\lambda}(t_{n_2-1}), G_{\psi,\lambda}(h_\lambda(t_j)), G_{\psi,\lambda}(h_\lambda(t_{j-1}))\},$$

$$H_{\psi,\lambda}(t_{n_1}) \geq \min\{H_{\psi,\lambda}(t_{n_1+1}), H_{\psi,\lambda}(t_{n_1-1}), H_{\psi,\lambda}(g_\lambda(t_k)), H_{\psi,\lambda}(g_\lambda(t_{k-1}))\}$$

when $0 \leq \lambda \leq \sqrt{2}-1$ and

$$G_{\psi,\lambda}(t_{n_2}) \geq \min\{G_{\psi,\lambda}(t_{n_2+1}), G_{\psi,\lambda}(t_{n_2-1}), G_{\psi,\lambda}(g_\lambda(t_j)), G_{\psi,\lambda}(g_\lambda(t_{j-1}))\},$$

$$H_{\psi,\lambda}(t_{n_1}) \geq \min\{H_{\psi,\lambda}(t_{n_1+1}), H_{\psi,\lambda}(t_{n_1-1}), H_{\psi,\lambda}(h_\lambda(t_k)), H_{\psi,\lambda}(h_\lambda(t_{k-1}))\}$$

when $\sqrt{2}-1 \leq \lambda \leq 1$.

Hence we have

$$\min_{0\leq i\leq n} G_{\psi,\lambda}(t_i) = \min_{0\leq i\leq n, i\neq n_1,n_2} G_{\psi,\lambda}(t_i) = \min_{0\leq i\leq n_0} G_{\psi,\lambda}(t_i^{(0)})$$

and

$$\min_{0\leq i\leq n} H_{\psi,\lambda}(t_i) = \min_{0\leq i\leq n, i\neq n_1,n_2} H_{\psi,\lambda}(t_i) = \min_{0\leq i\leq n_0} H_{\psi,\lambda}(t_i^{(0)})$$

Thus, from Lemma 3.2, we obtain

$$D(X_\psi,\lambda) = \min\{\min_{0\leq i\leq n_0} G_{X,\lambda}(t_i^{(0)}), \min_{0\leq i\leq n_0} H_{X,\lambda}(t_i^{(0)}), \frac{1+\lambda^2}{\max_{0\leq i\leq n} G_{X,\lambda}(t_i)}, \frac{1+\lambda^2}{\max_{0\leq i\leq n} H_{X,\lambda}(t_i)}\}$$

as desired. □

Now, we are ready to prove Theorem 2.1.

Proof of Theorem 2.1. Suppose that $\psi \in \Psi_2^S$ is a piecewise linear function with a simplified partition $(t_i^{(0)})_{i=0}^{n_0}$. One has that ψ^* is also piecewise linear function and hence there exist two sequences $(s_j)_{j=0}^m$ and $(b_j)_{j=0}^m$ such that

$$\psi^*(s) = \frac{b_j - b_{j-1}}{s_j - s_{j-1}}s + \frac{b_{j-1}s_j - b_j s_{j-1}}{s_j - s_{j-1}}$$

for any $s \in [s_{j-1}, s_j]$. We may consider that $(s_j)_{j=0}^m$ is also simplified.

For each pair (i, j) with $i \neq j$, let

$$\gamma_{i,j} = \frac{b_i - b_j}{s_i - s_j} \quad \text{and} \quad \delta_{i,j} = \frac{b_j s_i - b_i s_j}{s_i - s_j},$$

respectively. We note that $\psi^*(s) = \gamma_{j,(j-1)} s + \delta_{j,(j-1)}$ for any $s \in [s_{j-1}, s_j]$.

From the fact that $\psi^{**} = \psi$ ([12]), we have that

$$\psi(t) = \max_{0 \le s \le 1/2} \frac{(1-t)(1-s) + st}{\psi^*(s)} = \max_{1 \le j \le m} \max_{s \in [s_{j-1}, s_j]} \frac{(2t-1)s + 1 - t}{\gamma_{j,(j-1)} s + \delta_{j,(j-1)}}$$

$$= \max_{0 \le j \le m} \frac{(2s_j - 1)t + 1 - s_j}{b_j}$$

for any $t \in [0, 1/2]$. Let ℓ_j be the line given by

$$\ell_j(t) = \frac{(2s_j - 1)t + 1 - s_j}{b_j}$$

for each j, and let $I_k = \{t \in [0, 1/2] : \psi(t) = \ell_k(t)\}$ for each k.

Let $(t_i)_{i=0}^n = (t_i^{(0)})_{i=0}^{n_0} \cup \{(1-\lambda)/2, \lambda/(1+\lambda)\}$, $t_{n_1} = (1-\lambda)/2$ and $t_{n_2} = \lambda/(1+\lambda)$. For $i = 0$, t_0 is just 0 and hence

$$G_{\psi,\lambda}(t_0) = H_{\psi,\lambda}(t_0) = (1+\lambda)\psi\left(\frac{\lambda}{1+\lambda}\right) = (1+\lambda)\psi(t_{n_2}).$$

Take j with $t_{n_2} \in I_j$. Then we have

$$\psi(t_{n_2}) = \frac{(2s_j - 1)t_{n_2} + 1 - s_j}{b_j} = \frac{(2s_j - 1)\lambda + (1+\lambda)(1 - s_j)}{(1+\lambda)b_j} = \frac{-(1-\lambda)s_j + 1}{(1+\lambda)b_j}$$

and so

$$(1+\lambda)\psi(t_{n_2}) = \frac{-(1-\lambda)s_j + 1}{b_j}.$$

If $s_j \le t_{n_2}$, then we have

$$h_\lambda(s_j) = \frac{\lambda - (1+\lambda)s_j}{1 + \lambda - 2s_j}$$

and

$$(1 + \lambda - 2s_j)\psi^*(h_\lambda(s_j)) \ge (1 + \lambda - 2s_j)(1 - h_\lambda(s_j)) = -(1-\lambda)s_j + 1.$$

Suppose that $t_{n_2} \le s_j$. Then one has

$$h_\lambda(s_j) = \frac{-\lambda + (1+\lambda)s_j}{1 - \lambda + 2\lambda s_j}$$

and

$$(1 - \lambda + 2\lambda s_j)\psi^*(h_\lambda(s_j)) \ge (1 - \lambda + 2\lambda s_j)(1 - h_\lambda(s_j)) = -(1-\lambda)s_j + 1.$$

It follows from $\psi^*(s_j) = b_j$ that

$$G_{\psi,\lambda}(t_0) = H_{\psi,\lambda}(t_0) = (1+\lambda)\psi(t_{n_2}) \le H_{\psi^*,\lambda}(s_j).$$

For $i = n$, we easily have

$$G_{\psi,\lambda}(t_n) = H_{\psi,\lambda}(t_n) = \frac{\psi(t_{n_1})}{\psi(t_n)}.$$

Take k with $t_{n_1} \in I_k$. Then one has

$$\psi(t_{n_1}) = \frac{(2s_k - 1)t_{n_1} + 1 - s_k}{b_k} = \frac{-2\lambda s_k + 1 + \lambda}{2b_k}$$

and hence

$$\frac{\psi(t_{n_1})}{\psi(t_n)} = \frac{-2\lambda s_k + 1 + \lambda}{2\psi(t_n) b_k}.$$

On the other hand,

$$\psi^*(s) \geq \frac{(1-s) \cdot 1/2 + s \cdot 1/2}{\psi(1/2)} = \frac{1}{2\psi(t_n)}$$

holds for any $s \in [0, 1/2]$. Thus for $g_\lambda(s_k)$, we have

$$(-2\lambda s_k + 1 + \lambda)\psi^*(g_\lambda(s_k)) \geq \frac{-2\lambda s_k + 1 + \lambda}{2\psi(t_n)}.$$

Since $\psi^*(s_k) = b_k$, we obtain

$$G_{\psi,\lambda}(t_n) = H_{\psi,\lambda}(t_n) = \frac{\psi(t_{n_1})}{\psi(t_n)} \leq G_{\psi^*,\lambda}(s_k).$$

We also have

$$G_{\psi,\lambda}(t_{n_1}) = \frac{1 + \lambda^2}{\psi(t_{n_1})}\psi(1/2).$$

From $\psi^*(1/2) = 1/(2\psi(1/2))$ and

$$\psi^*(t_{n_1}) \geq \frac{(1 - t_{n_1})^2 + t_{n_1}^2}{\psi(t_{n_1})} = \frac{1 + \lambda^2}{2\psi(t_{n_1})},$$

one has

$$G_{\psi,\lambda}(t_{n_1}) \leq \frac{\psi^*(t_{n_1})}{\psi^*(1/2)} = G_{\psi^*,\lambda}(s_m) = H_{\psi^*,\lambda}(s_m).$$

One can check $H_{\psi,\lambda}(t_{n_2}) \leq G_{\psi^*,\lambda}(s_0) = H_{\psi^*,\lambda}(s_0)$, too.

Take arbitrary $i \in \{1, 2, \cdots, n-1\}$ with $i \neq n_1, n_2$. Since t_i is a corner point, there exist two distinct lines ℓ_i and ℓ_j such that $\psi(t_i) = \ell_i(t_i) = \ell_j(t_i)$. It follows that

$$t_i = \frac{\gamma_{i,j} + \delta_{i,j}}{\gamma_{i,j} + 2\delta_{i,j}} \quad \text{and} \quad \psi(t_i) = \frac{1}{\gamma_{i,j} + 2\delta_{i,j}}.$$

Hence, for any $s \in [0, 1]$, we have

$$\psi^*(s) \geq \frac{(1-s)(1-t_i) + st_i}{\psi(t_i)} = \delta_{i,j}(1-s) + (\gamma_{i,j} + \delta_{i,j})s.$$

We pick the index k satisfying $g_\lambda(t_i) \in I_k$. If $i < n_1$, then

$$g_\lambda(t_i) = \frac{\lambda + (1-\lambda)t_i}{1+\lambda-2\lambda t_i} = \frac{\gamma_{i,j} + (1+\lambda)\delta_{i,j}}{(1-\lambda)\gamma_{i,j} + 2\delta_{i,j}}.$$

Thus we have

$$\psi(g_\lambda(t_i)) = \frac{(2s_k - 1)g_\lambda(t_i) + 1 - s_k}{b_k} = \frac{\{(1+\lambda)\gamma_{i,j} + 2\lambda\delta_{i,j}\}s_k - \lambda\gamma_{i,j} + (1-\lambda)\delta_{i,j}}{b_k\{(1-\lambda)\gamma_{i,j} + 2\delta_{i,j}\}}$$

and hence

$$G_{\psi,\lambda}(t_i) = \frac{1+\lambda-2\lambda t_i}{\psi(t_i)}\psi(g_\lambda(t_i)) = \frac{\{(1+\lambda)\gamma_{i,j} + 2\lambda\delta_{i,j}\}s_k - \lambda\gamma_{i,j} + (1-\lambda)\delta_{i,j}}{b_k}.$$

When $s_k \le t_{n_2}$, one has

$$h_\lambda(s_k) = \frac{\lambda - (1+\lambda)s_k}{1+\lambda-2s_k}.$$

For this $h_\lambda(s_k)$, it follows from $\lambda - (1+\lambda)s_k > 0$ that

$$\begin{aligned}
(1+\lambda-2s_k)\psi^*(h_\lambda(s_k)) &\ge \{1-(1-\lambda)s_k\}\delta_{i,j} + \{\lambda-(1+\lambda)s_k\}(\gamma_{i,j}+\delta_{i,j}) \\
&\ge \{1-(1-\lambda)s_k\}\delta_{i,j} + \{-\lambda+(1+\lambda)s_k\}(\gamma_{i,j}+\delta_{i,j}) \\
&= \{(1+\lambda)\gamma_{i,j} + 2\lambda\delta_{i,j}\}s_k - \lambda\gamma_{i,j} + (1-\lambda)\delta_{i,j}.
\end{aligned}$$

From $\psi^*(s_k) = b_k$ we have that

$$G_{\psi,\lambda}(t_i) \le \frac{1+\lambda-2s_k}{\psi^*(s_k)}\psi^*(h_\lambda(s_k)) = H_{\psi^*,\lambda}(s_k).$$

In the situation of $t_{n_2} \le s_k$, we have

$$h_\lambda(s_k) = \frac{-\lambda + (1+\lambda)s_k}{1-\lambda+2\lambda s_k},$$

and

$$\begin{aligned}
(1-\lambda+2\lambda s_k)\psi^*(h_\lambda(s_k)) &\ge \{1-(1-\lambda)s_k\}\delta_{i,j} + \{-\lambda+(1+\lambda)s_k\}(\gamma_{i,j}+\delta_{i,j}) \\
&= \{(1+\lambda)\gamma_{i,j} + 2\lambda\delta_{i,j}\}s_k - \lambda\gamma_{i,j} + (1-\lambda)\delta_{i,j}.
\end{aligned}$$

Thus, the inequality

$$G_{\psi,\lambda}(t_i) \le \frac{1-\lambda+2\lambda s_k}{\psi^*(s_k)}\psi^*(h_\lambda(s_k)) = H_{\psi^*,\lambda}(s_k)$$

holds.

Suppose that $n_1 < i$. Then $g_\lambda(t_i)$ is given by

$$g_\lambda(t_i) = \frac{1-(1+\lambda)t_i}{1+\lambda-2\lambda t_i} = 1 - \frac{\lambda+(1-\lambda)t_i}{1+\lambda-2\lambda t_i} = \frac{-\lambda\gamma_{i,j} + (1-\lambda)\delta_{i,j}}{(1-\lambda)\gamma_{i,j} + 2\delta_{i,j}}.$$

Hence we have

$$\psi(g_\lambda(t_i)) = \frac{(2s_k-1)g_\lambda(t_i)+1-s_k}{b_k} = \frac{-\{(1+\lambda)\gamma_{i,j}+2\lambda\delta_{i,j}\}s_k+\gamma_{i,j}+(1+\lambda)\delta_{i,j}}{b_k\{(1-\lambda)\gamma_{i,j}+2\delta_{i,j}\}}$$

and

$$G_{\psi,\lambda}(t_i) = \frac{1+\lambda-2\lambda t_i}{\psi(t_i)}\psi(g_\lambda(t_i)) = \frac{-\{(1+\lambda)\gamma_{i,j}+2\lambda\delta_{i,j}\}s_k+\gamma_{i,j}+(1+\lambda)\delta_{i,j}}{b_k}.$$

In the case of $s_k \le t_{n_1}$, for

$$g_\lambda(s_k) = \frac{\lambda+(1-\lambda)s_k}{1+\lambda-2\lambda s_k},$$

we have the inequality

$$\begin{aligned}(1+\lambda-2\lambda s_k)\psi^*(g_\lambda(s_k)) &\ge (1+\lambda-2\lambda s_k)\{g_\lambda(s_k)\delta_{i,j}+(1-g_\lambda(s_k))(\gamma_{i,j}+\delta_{i,j})\}\\ &= \{\lambda+(1-\lambda)s_k\}\delta_{i,j}+\{1-(1+\lambda)s_k\}(\gamma_{i,j}+\delta_{i,j}).\end{aligned}$$

When $t_{n_1} \le s_k$, we have

$$g_\lambda(s_k) = \frac{1-(1+\lambda)s_k}{1+\lambda-2\lambda s_k}$$

and

$$\begin{aligned}(1+\lambda-2\lambda s_k)\psi^*(g_\lambda(s_k)) &\ge (1+\lambda-2\lambda s_k)\{(1-g_\lambda(s_k))\delta_{i,j}+g_\lambda(s_k)(\gamma_{i,j}+\delta_{i,j})\}\\ &= \{\lambda+(1-\lambda)s_k\}\delta_{i,j}+\{1-(1+\lambda)s_k\}(\gamma_{i,j}+\delta_{i,j}).\end{aligned}$$

Thus, the inequality

$$\begin{aligned}(1+\lambda-2\lambda s_k)\psi^*(g_\lambda(s_k)) &\ge \{\lambda+(1-\lambda)s_k\}\delta_{i,j}+\{1-(1+\lambda)s_k\}(\gamma_{i,j}+\delta_{i,j})\\ &= -\{(1+\lambda)\gamma_{i,j}+2\lambda\delta_{i,j}\}s_k+\gamma_{i,j}+(1+\lambda)\delta_{i,j}\end{aligned}$$

holds. From $\psi^*(s_k) = b_k$, we obtain

$$G_{\psi,\lambda}(t_i) \le \frac{1+\lambda-2\lambda s_k}{\psi^*(s_k)}\psi^*(g_\lambda(s_k)) = G_{\psi^*,\lambda}(s_k).$$

We pick the index p satisfying $h_\lambda(t_i) \in I_p$, too. Assume that $i < n_2$. Then one has

$$h_\lambda(t_i) = \frac{\lambda-(1+\lambda)t_i}{1+\lambda-2t_i} = \frac{-\gamma_{i,j}-(1-\lambda)\delta_{i,j}}{-(1-\lambda)\gamma_{i,j}+2\lambda\delta_{i,j}}.$$

Thus we have

$$\psi(h_\lambda(t_i)) = \frac{(2s_p-1)h_\lambda(t_i)+1-s_p}{b_p} = \frac{-\{(1+\lambda)\gamma_{i,j}+2\delta_{i,j}\}s_p+\lambda\gamma_{i,j}+(1+\lambda)\delta_{i,j}}{b_p\{-(1-\lambda)\gamma_{i,j}+2\lambda\delta_{i,j}\}}$$

and

$$H_{\psi,\lambda}(t_i) = \frac{1+\lambda-2t_i}{\psi(t_i)}\psi(h_\lambda(t_i)) = \frac{-\{(1+\lambda)\gamma_{i,j}+2\delta_{i,j}\}s_p+\lambda\gamma_{i,j}+(1+\lambda)\delta_{i,j}}{b_p}.$$

In the situation of $s_p \leq t_{n_2}$, one has

$$h_\lambda(s_p) = \frac{\lambda - (1+\lambda)s_p}{1+\lambda-2s_p}.$$

For this $h_\lambda(s_p)$, we obtain

$$\begin{aligned}(1+\lambda-2s_p)\psi^*(h_\lambda(s_p)) &\geq \{1-(1-\lambda)s_p\}\delta_{i,j} + \{\lambda-(1+\lambda)s_p\}(\gamma_{i,j}+\delta_{i,j}) \\ &= -\{(1+\lambda)\gamma_{i,j}+2\delta_{i,j}\}s_p + \lambda\gamma_{i,j} + (1+\lambda)\delta_{i,j}\end{aligned}$$

holds. Since $\psi^*(s_p) = b_p$, we obtain

$$H_{\psi,\lambda}(t_i) \leq \frac{1+\lambda-2s_p}{\psi^*(s_p)}\psi^*(h_\lambda(s_p)) = H_{\psi^*,\lambda}(s_p).$$

When $t_{n_2} \leq s_p$, for

$$h_\lambda(s_p) = \frac{-\lambda+(1+\lambda)s_p}{1-\lambda+2\lambda s_p},$$

it follow from $-\lambda+(1+\lambda)s_p \geq 0$ that

$$\begin{aligned}(1-\lambda+2\lambda s_p)\psi^*(h_\lambda(s_p)) &\geq \{1-(1-\lambda)s_p\}\delta_{i,j} + \{-\lambda+(1+\lambda)s_p\}(\gamma_{i,j}+\delta_{i,j}) \\ &\geq \{1-(1-\lambda)s_p\}\delta_{i,j} + \{\lambda-(1+\lambda)s_p\}(\gamma_{i,j}+\delta_{i,j}) \\ &= -\{(1+\lambda)\gamma_{i,j}+2\delta_{i,j}\}s_p + \lambda\gamma_{i,j} + (1+\lambda)\delta_{i,j}\end{aligned}$$

Hence,

$$H_{\psi,\lambda}(t_i) \leq \frac{1+\lambda-2s_p}{\psi^*(s_p)}\psi^*(h_\lambda(s_p)) = H_{\psi^*,\lambda}(s_p)$$

holds.

If $i > n_2$, then $h_\lambda(t_i)$ is given by

$$h_\lambda(t_i) = \frac{-\lambda+(1+\lambda)t_i}{1-\lambda+2\lambda t_i} = \frac{\gamma_{i,j}+(1-\lambda)\delta_{i,j}}{(1+\lambda)\gamma_{i,j}+2\delta_{i,j}}.$$

Hence we have

$$\psi(h_\lambda(t_i)) = \frac{(2s_p-1)h_\lambda(t_i)+1-s_p}{b_p} = \frac{\{(1-\lambda)\gamma_{i,j}-2\lambda\delta_{i,j}\}s_p + \lambda\gamma_{i,j}+(1+\lambda)\delta_{i,j}}{b_p\{(1+\lambda)\gamma_{i,j}+2\delta_{i,j}\}}$$

and

$$H_{\psi,\lambda}(t_i) = \frac{1-\lambda+2\lambda t_i}{\psi(t_i)}\psi(h_\lambda(t_i)) = \frac{\{(1-\lambda)\gamma_{i,j}-2\lambda\delta_{i,j}\}s_p + \lambda\gamma_{i,j}+(1+\lambda)\delta_{i,j}}{b_p}.$$

When $s_p \leq t_{n_1}$, one has

$$g_\lambda(s_p) = \frac{\lambda+(1-\lambda)s_p}{1+\lambda-2\lambda s_p}.$$

Hence, we obtain

$$\begin{aligned}(1+\lambda-2\lambda s_p)\psi^*(g_\lambda(s_p)) &\geq (1+\lambda-2\lambda s_p)\{(1-g_\lambda(s_p))\delta_{i,j}+g_\lambda(s_p)(\gamma_{i,j}+\delta_{i,j})\}\\ &= \{1-(1+\lambda)s_p\}\delta_{i,j}+\{\lambda+(1-\lambda)s_p\}(\gamma_{i,j}+\delta_{i,j}).\end{aligned}$$

In the case of $t_{n_1} \leq s_p$, for

$$g_\lambda(s_p)=\frac{1-(1+\lambda)s_p}{1+\lambda-2\lambda s_p},$$

the inequality

$$\begin{aligned}(1+\lambda-2\lambda s_p)\psi^*(g_\lambda(s_p)) &\geq (1+\lambda-2\lambda s_p)\{g_\lambda(s_p)\delta_{i,j}+(1-g_\lambda(s_p))(\gamma_{i,j}+\delta_{i,j})\}\\ &= \{1-(1+\lambda)s_p\}\delta_{i,j}+\{\lambda+(1-\lambda)s_p\}(\gamma_{i,j}+\delta_{i,j})\end{aligned}$$

holds. Thus, we obtain

$$\begin{aligned}(1+\lambda-2\lambda s_p)\psi^*(g_\lambda(s_p)) &\geq \{1-(1+\lambda)s_p\}\delta_{i,j}+\{\lambda+(1-\lambda)s_p\}(\gamma_{i,j}+\delta_{i,j})\\ &= \{(1-\lambda)\gamma_{i,j}-2\lambda\delta_{i,j}\}s_p+\lambda\gamma_{i,j}+(1+\lambda)\delta_{i,j}\end{aligned}$$

and

$$H_{\psi,\lambda}(t_i) \leq \frac{1+\lambda-2\lambda s_p}{\psi^*(s_p)}\psi^*(g_\lambda(s_p)) = G_{\psi^*,\lambda}(s_p).$$

Let $(r_j)_{j=0}^q = (s_j)_{j=0}^m \cup \{(1-\lambda)/2, \lambda/(1+\lambda)\}$, $r_{m_1}=(1-\lambda)/2$ and $r_{m_2}=\lambda/(1+\lambda)$. From the above paragraph, we obtain

$$\begin{aligned}\max\{\max_{0\leq i\leq n} G_{\psi,\lambda}(t_i), \max_{0\leq i\leq n} H_{\psi,\lambda}(t_i)\} &\leq \max\{\max_{0\leq j\leq m} G_{\psi^*,\lambda}(s_j), \max_{0\leq j\leq m} H_{\psi^*,\lambda}(s_j)\}\\ &\leq \max\{\max_{0\leq j\leq q} G_{\psi^*,\lambda}(r_j), \max_{0\leq j\leq q} H_{\psi^*,\lambda}(r_j)\}.\end{aligned}$$

On the other hand, considering about $(s_j)_{j=0}^m$ in the above paragraph, one can obtain the following:

(1) For i satisfying $g_\lambda(t_i) \in I_j$, if $i < n_1$ then $G_{\psi,\lambda}(t_i) \leq H_{\psi^*,\lambda}(s_j)$, in the situation of $n_1 < i$ the inequality $G_{\psi,\lambda}(t_i) \leq G_{\psi^*,\lambda}(s_j)$ holds.
(2) For i satisfying $h_\lambda(t_i) \in I_j$, if $i < n_2$ then $H_{\psi,\lambda}(t_i) \leq H_{\psi^*,\lambda}(s_j)$, in the case of $n_2 < i$ the inequality $H_{\psi,\lambda}(t_i) \leq G_{\psi^*,\lambda}(s_j)$ holds.

Thus we also obtain

$$\min\{\min_{0\leq i\leq n_0} G_{\psi,\lambda}(t_i^{(0)}), \min_{0\leq i\leq n_0} H_{\psi,\lambda}(t_i^{(0)})\} \leq \min\{\min_{0\leq j\leq m} G_{\psi^*,\lambda}(s_j), \min_{0\leq j\leq m} H_{\psi^*,\lambda}(s_j)\}$$

It follows from $\psi^{**}=\psi$ that

$$\min\{\min_{0\leq i\leq n_0} G_{\psi,\lambda}(t_i^{(0)}), \min_{0\leq i\leq n_0} H_{\psi,\lambda}(t_i^{(0)})\} = \min\{\min_{0\leq j\leq m} G_{\psi^*,\lambda}(s_j), \min_{0\leq j\leq m} H_{\psi^*,\lambda}(s_j)\}$$

and

$$\max\{\max_{0\le i\le n} G_{\psi,\lambda}(t_i), \max_{0\le i\le n} H_{\psi,\lambda}(t_i)\} = \max\{\max_{0\le j\le q} G_{\psi^*,\lambda}(r_j), \max_{0\le j\le q} H_{\psi^*,\lambda}(r_j)\}.$$

By the above lemma, $D(X_\psi, \lambda) = D(X_{\psi^*}, \lambda)$ holds.

Next we suppose that ψ is a strictly convex function in Ψ_2^S. For each $n \in \mathbb{N}$, let $(t_{(n)})_{i=0}^{k_n}$ be the dyadic partition $(k/2^n)_{k=0}^{2^{n-1}}$, and let $\psi_n \in \Psi_2^S$ be the piecewise linear function determined by the partition $(t_{(n)})_{i=0}^{k_n}$ and the values $(\psi(t_{(n)}))_{i=0}^{k_n}$. Then the sequence ψ_n is decreasing and $\|\psi_n - \psi\|_\infty \to 0$. From Lemma 4.1, we also have $\|\psi_n^* - \psi^*\|_\infty \to 0$. Since the partition $(t_{(n)})_{i=0}^{k_n}$ is simplified by the strict convexity of ψ, it follows that $D(X_{\psi_n}, \lambda) = D(X_{\psi_n^*}, \lambda)$. The continuity of $\psi \mapsto D(X_\psi, \lambda)$ implies that $D(X_\psi, \lambda) = D(X_{\psi^*}, \lambda)$.

Finally, let ψ be an arbitrary element in Ψ_2^S. By Lemma 4.2, we have a sequence (ψ_n) of strictly convex functions in Ψ_2^S such that $\|\psi_n - \psi\|_\infty \to 0$ and $\|\psi_n^* - \psi^*\|_\infty \to 0$ as $n \to \infty$. Thus, from Lemma 4.3, we obtain

$$D(X_\psi, \lambda) = \lim_{n\to\infty} D(X_{\psi_n}, \lambda) = \lim_{n\to\infty} D(X_{\psi_n^*}, \lambda) = D(X_{\psi^*}, \lambda),$$

as desired. $\square$

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