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DIFFERENTIABILITY OF INVARIANT CIRCLES FOR STRONGLY INTEGRABLE CONVEX BILLIARDS

NOBUHIRO INNAMI

ABSTRACT. Let C be a closed convex curve of class C^2 in the plane. We consider the domain bounded by C a billiard table. Assume that the convex billiard of C is integrable and satisfies a certain property. The property is that the limiting leaves are either closed curves or discrete points in the phase space. Then the set of points with irrational slopes make invariant circles of class C^1. If the sets of points with rational slopes do not make invariant circles, then they contains two invariant circles such that they are of class C^1 except at finitely many points in C.

1. Introduction

Let C be a simple closed and strictly convex curve of class C^k, $k \geq 2$, with length L in the Euclidean plane $\mathbf{E}$ and let $c : \mathbf{R} \to \mathbf{E}$ be its representation with respect to the arclength, namely $|\dot{c}(s)| = 1$ for all $s \in \mathbf{R}$ where $\mathbf{R}$ is the set of all real numbers. Let $x = (x_j)_{j\in\mathbf{Z}}$ be a sequence of points in C where $\mathbf{Z}$ is the set of all integers. We say that $\cup_{j=-\infty}^{\infty} T(x_{j-1}, x_j)$, or briefly x, is a *billiard ball trajectory* if the angle between the tangent vector v to C at x_i and the oriented segment $T(x_{i-1}, x_i)$ from x_{i-1} to x_i is equal to the one between v and $T(x_i, x_{i+1})$ for all $i \in \mathbf{Z}$.

A billiard ball trajectory $x = (x_j)_{j\in\mathbf{Z}}$ in C is represented by a sequence $s = (s_j)_{j\in\mathbf{Z}}$ of real numbers such that $x_j = c(s_j)$ and $s_j < s_{j+1} < s_j + L$ for all $j \in \mathbf{Z}$ and the sequence $s = (s_j)_{j\in\mathbf{Z}}$ will be considered a configuration $\{(j, s_j)\}_{j\in\mathbf{Z}}$ in the configuration space $\mathbf{X} = \mathbf{Z} \times \mathbf{R} \subset \mathbf{R}^2$. Every configuration $\{(j, s_j)\}_{j\in\mathbf{Z}}$ is identified with a broken line passing through $\{(j, s_j)\}_{j\in\mathbf{Z}}$ in $\mathbf{R}^2$. For convenience, s_j is considered to be (j, s_j) in $\mathbf{X}$. A configuration $s = (s_j)_{j\in\mathbf{Z}}$ for x is determined uniquely up to the difference pL ($p \in \mathbf{Z}$).

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We call $\Omega = C \times (-1,1)$ the *phase space* which is the set of all pairs (x,u) for $x \in C$ and $u \in (-1,1)$. Let $x_0, x_1, x_2 \in C$ and (x_0,x_1,x_2) the billiard ball trajectory. Let θ_0 (resp., θ_1) be the angle between the segment $T(x_0,x_1)$ from x_0 to x_1 (resp., $T(x_1,x_2)$) and the tangent vector to C at x_0 (resp., x_1). Set $u_0 = \cos\theta_0$ and $u_1 = \cos\theta_1$. Define a *billiard ball map* $\varphi : \Omega \to \Omega$ as $\varphi(x_0,u_0) = (x_1,u_1)$. The billiard ball map is an example of a monotone twist map ([14]). Let $\bar{x} = (x_0,u_0) \in \Omega$ and $\varphi^j(\bar{x}) = (x_j,u_j)$ for all $j \in \mathbf{Z}$. Then, the sequence $x = (x_j)_{j\in\mathbf{Z}}$ is a billiard ball trajectory. Any billiard ball trajectory is given in this way.

A convex billiard is said to be *integrable* if a subset of full measure of the phase space is foliated by closed curves invariant under the billiard ball map φ. The billiards in circles and ellipses are integrable. Note that invariant closed curves under φ may not be connected, but a union of closed curves. G. Birkhoff's conjecture states that the only examples of integrable billiards are circular and elliptic billiards ([4]). M. Bialy ([3]) has given a partial answer of the conjecture, proving that C is a circle if Ω is foliated by φ-invariant continuous closed curves not null-homotopic in Ω. M. Wojtkowski ([15]) proved that C is a circle if the domain bounded by C is foliated by smooth caustics to which almost every billiard ball trajectories are tangent. A system of caustics has been found by V. F. Lazutkin ([13]) for a plane convex domain with a sufficiently smooth boundary. These caustics are close to the boundary and occupy a set of positive measure. As was stated in [3], Bialy's theorem corresponds to a theorem of E. Hopf ([8]) which states that Riemannian metrics on tori without conjugate points are flat. N. Innami ([10]) extended Bialy's theorem to the higher dimensional case and the nonpositive curvature case as L. Green ([6]) did.

We say that a φ-invariant continuous closed curve f in Ω is an *invariant circle* if it is not null-homotopic. Since the billiard ball map φ becomes an orientation preserving homeomorphism of the invariant circle, the set of all configurations of all points in the invariant circle f makes a foliation of $\mathbf{X}$ by straight lines which is invariant under all translations on $\mathbf{X}$, and vice versa. If the billiard table is of class C^2, then the map φ in Ω is an area preserving twist map of class C^1, and Birkhoff's theorem ensures only that the invariant circles are Lipshitz and any invariant circle is the graph of a Lipshitz function, $\{G(s) = (c(s), u(s)) \,:\, 0 \le s \le L\}$ ([7], [14]). E. Gutkin and A. Katok ([7]) mention some examples of invariant circles and caustics. N. Innami ([9]) gives an example of convex billiards with invariant circle consisting of points with period $(3,1)$. N. Innami ([11]) discussed the differentiability of invariant circles by using the geometry of geodesics due to H. Busumann ([5]) which was reconstructed in the configuration space $\mathbf{X}$ by V. Bangert ([1], [2]).

In this note we apply his results to an integrable convex billiard and we note the differentiability of invariant circles.

We assume that the billiard of C is integrable. Let $\mathcal{F}$ be the foliation of the subset of full measure of Ω by closed curves invariant under the billiard ball map φ. We notice that a φ-invariant curve may not be connected, but a union of simple closed curves. We consider all connected components of φ-invariant curves in $\mathcal{F}$ to be elements of $\mathcal{F}$. Namely, if $g \in \mathcal{F}$, then there exists a union $f \subset \mathcal{F}$ of curves such that $\varphi(f) = f$ and $g \subset f$.

We extend the foliation $\mathcal{F}$. Let $\mathcal{U}$ denote the set of all closed subsets in the closure $\overline{\Omega}$ of Ω. Let o be a given point in Ω. The distance δ_o in $\mathcal{U}$ is defined by

$$\delta_o(M, N) = \sup_{\overline{x} \in \overline{\Omega}} |\, d(\overline{x}, M) - d(\overline{x}, N)\,|\, e^{-d(\overline{x}, o)}$$

for any closed subsets M and N in $\overline{\Omega}$ (cf. [5] p11). Since $\overline{\Omega}$ is compact, the set $\mathcal{U}$ is compact also (see [5] p.14, (3.15) Theorem). Let $\overline{\mathcal{F}}$ denote the closure of $\mathcal{F}$ in $\mathcal{U}$. Then, $\overline{\mathcal{F}}$ covers $\overline{\Omega}$. Moreover, all elements of $\overline{\mathcal{F}}$ are contained in a φ-invariant subset. Therefore, for every point $\overline{x} \in \Omega$ there exists a φ-invariant set $f \subset \overline{\mathcal{F}}$ containing $\overline{x}$. In general, f may not be a union of simple closed curves.

In the present note we assume that the following condition is satisfied.

(P) Every $f \in \overline{\mathcal{F}}$ is either a simple close curve or a point in $\overline{\Omega}$ and the set $\{f \in \overline{\mathcal{F}} \,|\, f \text{ is a point.}\}$ has no accumulation point in Ω.

Under the condition (P) we will study the structure of invariant circles and their differentiability. We say that a convex billiard is *strongly integrable* if it is integrable and (P) is satisfied.

The notion of slope is usful to classify the invariant circles. Let $x = (x_j)_{j \in \mathbf{Z}}$ be a billiard ball trajectory and let $a(x_j, x_{j+1})$ be the arclength of the subarc of C from x_j to x_{j+1} measured with the positive orientation of C. We define the *slope* $\alpha(x)$ of x as

$$\alpha(x) = \liminf_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} a(x_j, x_{j+1}) = \liminf_{n \to \infty} \frac{s_n}{n}.$$

where $s = (s_j)_{j \in \mathbf{Z}}$ is a configuration for x. Let $\alpha(\bar{x})$ denote the slope of the billiard ball trajectory determined by $\bar{x}$ for $\bar{x} \in \Omega$. Set

$$\Omega(a) = \{\bar{x} \in \Omega \,|\, \alpha(\bar{x}) = aL\}.$$

If f is an invariant circle in Ω, then $\alpha(\bar{x})$ are constant for all $\bar{x} \in f$, and, therefore, $f \subset \Omega(a)$ for some a with $0 < a < 1$. We say that a closed curve f in Ω not null-homotoic is a *circle with constant slope* if $\alpha(\bar{x})$ are constant for all $\bar{x} \in f$. We write the constant by $\alpha(f)$, also. We call aL an *irrational* (resp., *rational*) slope if a is irrational (resp., rational). An invariant circle is a circle with constant slope. However, the reverse is not true, in general.

Theorem 1.1. *Let C be a simple closed convex curve of class C^k, $k \geq 2$, with positive curvature κ and length L. If f is an invariant circle with irrational slope aL, then the graph $G_f(s)$ of f is of class C^1. In particular, if the convex billiard of C is strongly integrable, then $\Omega(a)$ is the invariant circle of class C^1 with slope aL in Ω for all irrational numbers a with $0 < a < 1$.*

If an invariant circle f is of class C^1, then the caustic K made from f is a continuous curve in the domain bounded by C. Here we say that a closed continuous curve K is a *caustic* if K has the following property. Let x_0 be an arbitrary point in C and let $T(x_0, x_1)$ be a segment tangent to K. If $x = (x_j)_{j \in \mathbf{Z}}$ is the billiard ball trajectory determined by $T(x_0, x_1)$, then $T(x_j, x_{j+1})$ are segments tangent to K for all $j \in \mathbf{Z}$. Without C^2 differentiability condition on C the caustics are not continuous, in general. More general and precise definition of caustics is seen in [7].

Next we treat the case that a is rational. Let $\overline{r}_j$ and $\underline{r}_j$ be sequences of irrational numbers such that $\overline{r}_j \to a+0$ and $\underline{r}_j \to a-0$ as $j \to \infty$. If the convex billiard of C is strongly integrable, then $\Omega(\overline{r}_j)$ and $\Omega(\underline{r}_j)$ are invariant circles because of Theorem 1.1. If $L(\overline{r}_j)$ (resp., $U(\underline{r}_j)$) are the domains bounded below (resp., bounded above) by $\Omega(\overline{r}_j)$ (resp., $\Omega(\underline{r}_j)$) in Ω, then we have $\Omega(a) = \cap_{j=1}^{\infty}(L(\overline{r}_j) \cap U(\underline{r}_j))$. Here, $L(\overline{r}_j) \cap U(\underline{r}_j)$ is the strip between $\Omega(\overline{r}_j)$ and $\Omega(\underline{r}_j)$ in Ω. Moreover, if $\Omega(a+0) := \lim_{j\to\infty} \Omega(\overline{r}_j)$ and $\Omega(a-0) := \lim_{j\to\infty} \Omega(\underline{r}_j)$, then $\Omega(a)$ is contained in the domain bounded by $\Omega(a+0) \cup \Omega(a-0)$ (possibly, $\Omega(a) = \Omega(a+0) \cap \Omega(a-0)$).

Let $a = p/q$ where p and q are mutually prime integers. There exists a periodic straight line $s = (s_j)_{j\in\mathbf{Z}}$ with period (q,p), i.e., $s_{j+q} = s_j + pL$ for all $j \in \mathbf{Z}$. In Section 2 we will introduce the definitions of technical terms, such as a straight line, a ray, an asymptote, a parallel and so on, used in geometry of geodesics for convex billiards.

Let $A \subset \mathbf{R}$ be the set of those parameters s_0 such that $s = (s_j)_{j\in\mathbf{Z}}$ is a periodic straight line with period (q,p). Then, A is a closed set in $\mathbf{R}$. Let $B = \mathbf{R} \setminus A$. The set B is either an empty set or a union of open intervals (b^k, t^k), $k \in I$, where I is an index set.

Theorem 1.2. *Let C be a simple closed convex curve of class C^k, $k \geq 2$, with positive curvature κ and length L. Assume that the convex billiard of C is strongly integrable. Let $a = p/q$ be an arbitrary rational number with $0 < a < 1$ where p and q are mutually prime numbers. Then the following are true.*

(1) *If $\Omega(a+0) = \Omega(a-0)$, then $A = \mathbf{R}$ and $f := \Omega(a)$ is the invariant circle with slope aL in Ω. The graph $G_f(s)$ of f is of class C^1.*
(2) *If $\Omega(a+0) \neq \Omega(a-0)$, we then have the following.*
   (a) *The number of connected components of $c(A)$ is finite.*

(b) $\Omega(a+0)$ *and* $\Omega(a-0)$ *are the invariant circles with slope* aL *in* Ω*, say* f*. The graph* $G_f(s)$ *of each invariant circle* f *is of class* C^1 *except at the points* $s \in A$ *such that* s *is isolated point in* A*.*

(c) *Assume that* $c(A)$ *consists of* q *points. Then there exist two closed curves* $G_1(s) = (c(s), u_1(s))$ *and* $G_2(s) = (c(s), u_2(s))$*,* $0 \leq s \leq i(q)L$*, of class* C^1 *with slope* aL *which are not null-homotopic, where* $i(q) = 2$ *if* q *is odd and* $i(q) = 1$ *if* q *is even.*
Moreover, the graphs $G_t(s) = (c(s), \max\{u_1(s), u_2(s)\})$ *and* $G_b(s) = (c(s), \min\{u_1(s), u_2(s)\})$*,* $0 \leq s \leq L$*, are the invariant circles with slope* aL *which are* $\Omega(a-0)$ *and* $\Omega(a+0)$*, respectively.*

The results in this note would be direct consequences if Birkohoff's conjecture were solved affirmatively. We wish our research would help to solve the conjecture. It is natural to ask whether the only examples of strongly integrable billiards are circular and elliptic billiards.

2. Foliation by asymptotes and parallels

Let C be a simple closed convex curve of class C^k, $k \geq 2$, with positive curvature κ and length L.

The contents in this section are based on the results in [1], [2] and [11]. We work in the configuration space $\mathbf{X}$ and apply geometry of geodesics for convex billiards.

Let s_i and s_k be points in $\mathbf{X}$. For any configuration $t = (t_j)_{i \leq j \leq k}$ such that $t_i = s_i$, $t_k = s_k$ and $t_j < t_{j+1} < t_j + L$, set

$$H(s_j, s_k; t) = -\sum_{j=i}^{k-1} |c(t_{j+1}) - c(t_j)|.$$

We consider the variational problem for the functional $H(s_i, s_k; t)$. Then, $s = (s_j)_{i \leq j \leq k}$ is the configuration of a billiard ball trajectory $x = (x_j)_{i \leq j \leq k}$ if and only if it is a critical configuration of $H(s_j, s_k; t)$.

We say that $s = (s_j)_{i \leq j \leq k}$ is a *segment* from s_i to s_k in $\mathbf{X}$ if

$$\begin{aligned} H(s_i, s_k; s) &= -\sum_{j=i}^{k-1} |c(s_{j+1}) - c(s_j)| \\ &= \min_t \left\{ -\sum_{j=i}^{k-1} |c(t_{j+1}) - c(t_j)| \right\} \end{aligned}$$

where $t = (t_j)_{j \in \mathbf{Z}}$ is any configuration such that $t_i = s_i$, $t_k = s_k$ and $t_j < t_{j+1} < t_j + L$. The most important property of segments is that if two different segments have two points in common, then they are the endpoints of both segments. In particular, if any segment connecting an endpoint and an interior point of a segment s is unique and it is a subsegment of s.

We say that $s = (s_j)_{j \in \mathbf{Z}}$ is a *straight line* in $\mathbf{X}$ if the restriction of s to the interval $i < j < k$ in $\mathbf{Z}$ is a segment for every $i < k \in \mathbf{Z}$. We say that a straight line s is (*positively*) *asymptotic* to a straight line t if the sequence of segments from s_i to t_k converges to the sub-ray $s = (s_j)_{j \geq i}$ of s as $k \to \infty$ for every $i \in \mathbf{Z}$.

We say that a straight line s is a *parallel* to a straight line t if the sequences of segments from s_i to t_k converge to the sub-ray $s = (s_j)_{j \geq i}$ and $s = (s_j)_{j \leq i}$ of s as $k \to \infty$ and $k \to -\infty$, respectively, for every $i \in \mathbf{Z}$. In general, the asymptote and parallel relations are not symmetric.

A simple modification of the arguments in the proof of [11] Lemma 6.8 proves the following. The proof will be reviewed in Section 8 for convenience and completeness.

Lemma 2.1. *Let f be a continuous curve in Ω with its graph $G_f(s) = (c(s), u(s))$, $s \in [a, b]$. Assume that the configurations $s(\bar{x})$ for all $\bar{x} \in f$ are straight lines and they are parallels to each other. Then, the graph $G_f(s)$ is of class C^1.*

The continuity of the curvature of C plays an important role in the proof of Lemma 2.1 as was seen in [11] Lemma 6.4.

As was seen in [11] Lemma 4.15, an invariant circle f in Ω yields a foliation of $\mathbf{X}$ by straight lines, namely $s(\bar{x})$ is a straight line in $\mathbf{X}$ for every $\bar{x} \in f$. The situation in Lemma 2.1 appears in the case of irrational slopes (cf. [11] Lemma 5.17).

Lemma 2.2. *Let a be an irrational number with $0 < a < 1$. Let f be an invariant circle in Ω with slope aL. Let $s(\bar{x})$ be the configuration corresponding to $\bar{x} \in f$. Then, all $s(\bar{x})$ are parallels to each other, and, therefore, f is of class C^1.*

Note that without the condition of positiveness on curvature κ of C in Theorem 1.1 there exists no invariant circle in Ω.

3. Null-homotopic invariant closed curves

We assume that the convex billiard of C is strongly integrable.

Let $\overline{\mathcal{F}}_0$ be the subset of all elements $f \in \overline{\mathcal{F}}$ which are null homotopic in Ω. Let $f \in \overline{\mathcal{F}}_0$. Then, f is either a point or a simple closed curve which is null homotopic. If f is a point, we then set $D(f) = f$. Otherwise, let $D(f)$ denote the closed domain bounded by f in Ω.

We define a partial order $\leq$ in $\overline{\mathcal{F}}_0$ as follows: Let f and g be elements in $\overline{\mathcal{F}}_0$. Then, $f \leq g$ if and only if $D(f) \subset D(g)$. Let $\mathcal{M}(f)$ be a maximal totally ordered subset of $\overline{\mathcal{F}}_0$ containing f. We do not know whether it is uniquely determined or not. Each $\mathcal{M}(f)$ has the maximum and minimum elements. The minimum elements are points because of the strong integrability. We do not know whether $\mathcal{M}(f)$ is connected or not in $\overline{\mathcal{F}}_0$.

Let $\mathcal{C}(f)$ be the (pathwise) connected component containing f in $\overline{\mathcal{F}}_0$. Since the convex billiard of C is strongly integrable, we have $\mathcal{C}(f) \subset \mathcal{M}(f)$. Moreover, $\mathcal{C}(f)$ has the maximum and minimum elements, say $\bar{f}$ and $\underline{f}$, respectively. Set $\overline{D}(f) = D(\bar{f}) \setminus D(\underline{f})$ if $\underline{f}$ is a simple closed curve, and set $\overline{D}(f) = D(\bar{f})$ if $\underline{f}$ is a point. Thus, $\overline{D}(f)$ is an annulus if $\underline{f}$ is a simple closed curve, and it is a disk if $\underline{f}$ is a point. In particular, the domain $\overline{D}(f)$ is foliated by simple closed curves in $\overline{\mathcal{F}}_0$, namely $\overline{D}(f) = \cup_{g \in \mathcal{C}(f)} g$.

Lemma 3.1. *Assume that the convex billiard of C is strongly integrable. Let $f \in \overline{\mathcal{F}}_0$. Then the following are true.*

(1) *There exists the unique maximum element $\bar{f} \in \overline{\mathcal{F}}_0$ such that $\bar{f} \in \mathcal{M}(f)$ and $D(f) \subset D(\bar{f})$.*
(2) *$\mathcal{M}(f)$ contains a point belonging to $\overline{\mathcal{F}}_0$. There exist at most finitely many elements $g \in \overline{\mathcal{F}}_0$ such that g is a point and $g \in D(f)$.*
(3) *Let $\bar{f} \in \overline{\mathcal{F}}_0$ be a maximum in $\mathcal{M}(f)$ and $\underline{f}$ the minimum of $\mathcal{C}(\bar{f})$. Then there exist finitely many elements $g_1, \dots, g_n$ in $\overline{\mathcal{F}}_0$ such that $D(\underline{f}) = \cup_{i=1}^{n} D(g_i)$. In particular, $\mathcal{M}(g_i)$ can be extended to a totally ordered set $\mathcal{M}(\bar{f})$ and $D(\bar{f})$ is the union of $\overline{D}(f)$ and $\cup_{i=1}^{n} D(g_i)$.*

Proof. We prove (1). Let $N(f) = \{g \in \overline{\mathcal{F}}_0 \,|\, D(f) \subset D(g)\}$. Note that either $D(g) \subset D(h)$ or $D(g) \supset D(h)$ holds for all elements g and h in $N(f)$. Set $D = \cup_{g \in N(f)} D(g)$. Let $g_k \in N(f)$ be a sequence such that $D(g_k) \to D$. Then, g_k converges to an element $\bar{f} \in \overline{\mathcal{F}}_0$ which is the maximum and satisfies $D(f) \subset D(\bar{f})$.

We prove (2). Each maximal totally ordered subset $\mathcal{M}(f)$ containing f has a minimum g. If g is not a point, we then have a simple closed curve in $D(g)$, contradicting that $\mathcal{M}(f)$ is maximal. Therefore, g is a point in $D(f)$. Since $D(f)$ is compact and the convex billiard of C is strongly integrable, the set $\{g \in \overline{\mathcal{F}}_0 \,|\, g \in D(f) \text{ is a point.}\}$ is a finite set.

We prove (3). For any g and h in $\overline{\mathcal{F}}_0$ we have $\mathcal{C}(g) = \mathcal{C}(h)$ or $\mathcal{C}(g) \cap \mathcal{C}(h) = \emptyset$, since $\mathcal{C}(g)$ and $\mathcal{C}(h)$ are connected components. Hence, we have

$$D(\underline{f}) = \cup_{k \in I} \overline{D}(g_k),$$

where I is the index set of those k such that $g_k \subset D(\underline{f})$ and $\mathcal{C}(g_k) \cap \mathcal{C}(g_{k'}) = \emptyset$ if $k \neq k'$. It follows from (2) that I is a finite set. ☐

Let $E \subset \mathcal{F}$ consists of null-homotopic closed curves such that E is invariant under φ and any proper subset of E is not invariant under φ. Since the billiard ball map φ preserves the measure of the domain bounded by a closed curve which is a connected component of E, the number of connected components of E is finite. Let $E = \cup_{i=1}^{q} f_i$

where $f_i \in \mathcal{F}$ is a simple closed curve for every $i = 1, \dots, q$. By Brouwer's fixed point theorem we have $q \geq 2$ because φ has no fixed point in Ω. The billiard ball map φ is considered a permutation of $\{f_1, \dots, f_q\}$. Therefore, the slope $\alpha(\overline{x})$ is constant which is rational for all $\overline{x} \in E$. Actually, $\alpha(E) = (p/q)L$ where p and q are mutually prime numbers, since, otherwise, there exists a φ-invariant proper subset of E.

Lemma 3.2. *Assume that the convex billiard of C is strongly integrable. Let $f \in \overline{\mathcal{F}}_0$. Let $\alpha(f) = (p/q)L$. Then the following are true.*

(1) *Then $\alpha(\bar{x}) = (p/q)L$ for all $\bar{x} \in D(f)$.*
(2) *If $g \in \overline{\mathcal{F}}_0$ and $D(f) \subset D(g)$, then $\alpha(g) = (p/q)L$.*
(3) *Every $g \in \overline{\mathcal{F}}$ which touches f has $\alpha(g) = (p/q)L$.*

Proof. We prove (1). Let $s(\bar{x}) = (s_j)_{j \in \mathbf{Z}}$ for $\bar{x} \in D(f)$. Since $\varphi^q(D(f)) = D(f)$ and $\alpha(f) = (p/q)L$ for some $p \in \mathbf{Z}$, we have $s_0 + kpL - L \leq s_{kq} \leq s_0 + kpL + L$ for $k \in \mathbf{Z}$. Hence, we have $\alpha(\bar{x}) = (p/q)L$. Since $\varphi^q(D(g)) = D(g)$, (2) is proved.

Take a point $\bar{x} \in g \cap f$. We then have $\alpha(g) = \alpha(\bar{x}) = \alpha(f)$. This shows (3). $\square$

4. Proof of Theorem 1.1

The first part of Theorem 1.1 is a direct consequence of Lemma 2.2.

To prove the second part of Theorem 1.1 we assume that the convex billiard of C is strongly integrable. Let $\overline{x}$ be a point in Ω whose slope aL is irrational. Since $\overline{\mathcal{F}}$ covers $\overline{\Omega}$, there exists an element $f \in \overline{\mathcal{F}}$ passing through $\overline{x}$. If f is a point, then $\{\varphi^q(f) \,|\, q \in \mathbf{Z}\}$ consists of infinitely many points and, hence, has an accumulation point in Ω, a contradiction. Thus, f is an invariant circle because of Lemma 3.2. As was seen before, $f \subset \Omega(a)$ and it yields the unique foliation of $\mathbf{X}$ by straight lines with slope aL (cf. [11] Theorem 4.16). Let $\overline{r}_j$ (resp., $\underline{r}_j$) be sequences of irrational numbers such that $\overline{r}_j \to a+0$ (resp., $\underline{r}_j \to a-0$). Let $\overline{f}_j \in \mathcal{F}$ (resp., $\underline{f}_j \in \mathcal{F}$) be a sequence of invariant circles with slope $\overline{r}_j L$ (resp., $\underline{r}_j L$). Let $\mathbf{X}(\overline{r}_j)$ (resp., $\mathbf{X}(\underline{r}_j)$) be the foliation of $\mathbf{X}$ by straight lines with slope $\overline{r}_j L$ (resp., $\underline{r}_j L$). They correspond to $\overline{f}_j$ and $\underline{f}_j$. Since both $\mathbf{X}(\overline{r}_j)$ and $\mathbf{X}(\underline{r}_j)$ converge to $\mathbf{X}(a)$ (cf. [11] Lemma 4.11), the sequences of invariant circles $\overline{f}_j$ and $\underline{f}_j$ converge to f. Since the slope is invariant under the billiard ball map φ, the set $\Omega(a)$ lies in the strip between $\overline{f}_j$ and $\underline{f}_j$ in Ω. Therefore, as the limiting situation, we have $f = \Omega(a)$. This completes the proof of Theorem 1.1.

5. Straight lines with rational slope

We treat the case that the slope aL is rational. Assume that the convex billiard of C is strongly integrable. Let $a = p/q$ where p and q are mutually prime integers. In this case there exists a periodic straight line $s = (s_j)_{j\in\mathbf{Z}}$ with period (q,p), i.e., $s_{j+q} = s_j + pL$ for all $j \in \mathbf{Z}$ (cf. [11] Proposition 4.4).

Let $A \subset \mathbf{R}$ be the set of those parameters s_0 such that $s = (s_j)_{j\in\mathbf{Z}}$ is a periodic straight line with period (q,p). Then, A is a closed set in $\mathbf{R}$. The following lemma holds from [11] Proposition 5.15 and Lemma 2.1.

Lemma 5.1. *All periodic straight lines with period (q,p) are parallels to each other. Let A be as above. Assume that the interior $\mathrm{Int}(A)$ of A is not empty. If $G(s) = (c(s), u(s))$ is the graph of the set of those points in Ω corresponding to periodic straight lines with period (q,p), then $G(s)$ is of class C^1 in $s \in \mathrm{Int}(A)$.*

Let $B = \mathbf{R} \setminus A$. The set B is either an empty set or a union of open intervals (b^k, t^k), $k \in I$, where I is an index set. We prove the following

Lemma 5.2. *Let $\pi : \Omega \to C$ be the natural projection. We then have $c(A) = \pi(\Omega(a+0) \cap \Omega(a-0))$. In particular, for every $\bar{x} \in \Omega(a+0) \cap \Omega(a-0)$, its configuration $s(\bar{x})$ is a periodic straight line with period (q,p) in $\mathbf{X}$.*

Proof. We have, from [11] Proposition 5.16, $c(A) \subset \pi(\Omega(a+0) \cap \Omega(a-0))$.

We prove that $c(B) \cap \pi(\Omega(a+0) \cap \Omega(a-0)) = \emptyset$, meaning that $c(A) \supset \pi(\Omega(a+0) \cap \Omega(a-0))$. Let $u^k = ({u^k}_j)_{j\in\mathbf{Z}}$ and $v^k = ({v^k}_j)_{j\in\mathbf{Z}}$ be periodic straight lines with period (q,p) such that ${u^k}_0 = b^k$ and ${v^k}_0 = t^k$. They are parallels to each other (Lemma 5.1).

For every $s_0 \in (b^k, t^k) \subset B$, we show that there exist just two straight lines with slope aL. Let $\overline{s}$ be the positive asymptote to v^k through s_0 and let $\underline{s}$ be the positive asymptote to u^k through s_0. In fact, $\overline{s}$ (resp., $\underline{s}$) is given as the straight line to which a sequence of straight lines with irrational slope $\overline{r}_jL$ greater than aL (resp., $\underline{r}_jL$ less than aL) converges (cf. [11] Lemma 4.11). This is possible becouse of the second part of Theorem 1.1. It follows from the construction that $\overline{s}$ and $\underline{s}$ are the negative asymptotes to u^k and v^k through s_0, respectively.

Moreover, they satisfy that $|\overline{s}_j - {v^k}_j| \to 0$ and $|\underline{s}_j - {u^k}_j| \to 0$ as $j \to \infty$. In fact, if this is not true, then we can find a straight line t through some $w \in (b^k, t^k)$ with period (q,p) such that t is invariant under the translation τ and τ^{-1} given by $\tau((j, x_j)) = (j-q, x_j - pL)$ for all $(j, x_j) \in \mathbf{X}$, using τ for $\overline{s}$ or $\underline{s}$ repeatedly. Actually, $t = \lim_{j\to\infty} \tau^j(\overline{s})$ or $t = \lim_{j\to\infty} \tau^j(\underline{s})$. Then, [11] Lemma 4.6 shows that t is a periodeic straight line with period (q,p), contradicting that $(b^k, t^k) \cap A = \emptyset$.

Since $\overline{s} \in \Omega(a+0)$, $\underline{s} \in \Omega(a-0)$ and $\overline{s} \neq \underline{s}$, we have $c(s_0) \notin \pi(\Omega(a+0) \cap \Omega(a-0))$. $\square$

Since p and q are mutually prime integers, we have at least q initial parameters s_0 ($\mod L$) of periodic straight lines $s = (s_j)_{j\in\mathbf{Z}}$ with period (q, p).

Assume that A is discrete. Then, we may assume that $t^k = b^{k+1}$ for all $k \in I$. Let $S(u^k, v^k) \subset \mathbf{X}$ be the strip bounded by two straight lines u^k and v^k which were defined before. We have two foliations $\overline{F}_k = \{\overline{s} \mid s_0 \in (b^k, t^k)\}$ and $\underline{F}_k = \{\underline{s} \mid s_0 \in (b^k, t^k)\}$ of the interior of the strip $S(u^k, v^k)$ in $\mathbf{R}^2$ by parallels for each $k \in I$ (cf. [11] Lemma 5.10), since any two straight lines $\overline{s}^1$ and $\overline{s}^2$ (resp., $\underline{s}^1$ and $\underline{s}^2$) satisfy that $|\overline{s}^1{}_j - \overline{s}^2{}_j|$ (resp., $|\underline{s}^1{}_j - \underline{s}^2{}_j|$) converges to 0 as $|j|$ goes to ∞, as was seen in the proof of Lemma 5.2.

Recall that $S(u^{k+1}, v^{k+1})$ is adjacent to $S(u^k, v^k)$. Since both u^{k+1} and v^k are periodic straight lines with period (q, p) and passing through $b^{k+1} = t^k$, we have $u^{k+1} = v^k$ (cf. [11] Theorem 4.12). Let F_0 be the set of all periodic straight lines with period (q, p) through all $s_0 \in A$. Then, each set of straight lines $F_1 = \cdots \cup \underline{F}_{k-1} \cup \overline{F}_k \cup \underline{F}_{k+1} \cup \cdots \cup F_0$ and $F_2 = \cdots \cup \overline{F}_{k-1} \cup \underline{F}_k \cup \overline{F}_{k+1} \cup \cdots \cup F_0$ gives a foliation of $\mathbf{X}$ by paralleles to each other in the interior of each strip $S(u^k, v^k)$. Moreover, all straight lines in $\overline{F}_k \cup \underline{F}_{k+1}$ (resp., $\underline{F}_k \cup \overline{F}_{k+1}$) are asymptotic to the positive v^k (resp., the negative v^k) for all $k \in I$. These foliations correspond to closed curves, say f, not null-homotopic in Ω of class C^1 because of Lemma 2.1.

Here we should note that there exists no foliation of $S(u^k, v^k)$ by configurations of billiard ball trajectories other than $\overline{F}^k$ and $\underline{F}^k$. In fact, if we have another foliation, we then have a straight line t in $S(u^k, v^k)$ which does not belong to either $\overline{F}^k$ or $\underline{F}^k$ (cf. [11] Proposition 2.9). The straight line t does not approach positively to either u^k or v^k. Therefore, as was seen in the proof of Lemma 5.2, this shows that there exists a periodic straight line with period (q, p) in $S(u^k, v^k)$, a contradiction.

Lemma 5.3. *Assume $\Omega(a + 0) \cap \Omega(a - 0)$ consists of q points. Let f be a closed curve in Ω made from the foliation F_1 or F_2 of $\mathbf{X}$ as above. Then the projection of the curves f to C cover C in Ω twice (resp., once) if q is odd (resp., even).*

Proof. From Lemma 5.2, A is discrete because $c(A)$ consists of q points. Suppose that q is odd (resp., even). If the k-th component of the foliations is $\overline{F}_k$, then the $(k + q)$-th component is $\underline{F}_{k+q}$ (resp., $\overline{F}_{k+q}$). Therefore, the foliations F_1 and F_2 of $\mathbf{X}$ have the period $2L$ (resp., L) for initial parameters $s_0 \in \mathbf{R}$. □

Under the assumption of Lemma 5.3, we prove that there exists no invariant circle in Ω with slope aL other than $\Omega(a + 0)$ and $\Omega(a - 0)$. Let F be a foliation of $\mathbf{X}$ by straight lines with slope aL. Then, F contains all periodic straight lines with period (q, p) as its leaves because of [11] Proposition 5.16. As was seen just before Lemma 5.3, the leaves of F in $S(u^k, v^k)$ must be either the positive asymptotes to v^k or the positive asymptotes to u^k. The foliation F is determined by a foliation of only one $S(u^k, v^k)$, since F corresponds to an invariant circle in Ω and the periodic straight lines with period (q,p) corresponds to $\Omega(a+0)\cap\Omega(a-0)$ which is the trajectory of one point in it by the billiard ball map φ. In other words, the foliation F is the set of all straight lines which are translated from all leaves in $S(u^k,v^k)$. Therefore, F is the foliation corresponding to either $\Omega(a+0)$ or $\Omega(a-0)$. We have proved the following.

Lemma 5.4. *Let $a=p/q$ be a rational number with $0<a<1$ where p and q are mutually prime integers. Let $s(\bar{x})$ be the configuration in $\mathbf{X}$ corresponding to $\bar{x}\in\Omega$. If $s(\bar{x})$ are periodic straight lines with period (q,p) for all $\bar{x}\in\Omega(a)$, then $\Omega(a)$ is an invariant circle of class C^1. Otherwise, $\Omega(a+0)$ and $\Omega(a-0)$ are invariant circles, say f, with slope aL in Ω. The graph $G_f(s)$ of each invariant circle f is of class C^1 in $s\in B\cup\mathrm{Int}(A)$. Moreover, if $\Omega(a+0)\cap\Omega(a-0)$ consists of q points, then there is no invariant circles with slope aL other than $\Omega(a+0)$ and $\Omega(a-0)$.*

6. Proof of Theorem 1.2

Lemmas 2.1, 5.1, 5.2 and 5.4 show (1).

Suppose $\Omega(a)$ is not an invariant circle. Let f be a simple closed curve contained in $\Omega(a+0)\cup\Omega(a-0)$. Since $\varphi^q(f)=f$, by applying the argument in the proof of Lemma 3.1, we notice that $D(f)$ contains a point which is an element in $\overline{\mathcal{F}}_0$. Since the set of those points has no accumulation, the number of simple closed curve in $\Omega(a+0)\cup\Omega(a-0)$ is finite. This shows (2) (a).

Let $\overline{r}_j$ (resp., $\underline{r}_j$) be a sequence of irrational numbers with $\overline{r}_j>a$ (resp., $\underline{r}_j<a$) converging to a. Then, $\Omega(\overline{r}_j)$ and $\Omega(\underline{r}_j)$ converge to subsets G_b and G_t contained in $\Omega(a)$ which are invariant circles. More precisely, $G_b\cup G_t$ is the boundary of $\Omega(a)$ and the configurations $s(\bar{x})$ in $\mathbf{X}$ corresponding to $\bar{x}\in G_b\cap G_t$ are periodic straight lines with period (q,p). Lemma 5.3 and 5.4 complete the proof of Theorem 1.2.

7. A remark on islands

The following proposition states what the neighborhood of $\Omega(a+0)\cup\Omega(a-0)$.

Proposition 7.1. *Assume that the convex billiard of C is strongly integrable. Let $a\in(-1,1)$ be rational and $\Omega(a+0)\neq\Omega(a-0)$. Let f be a simple closed curve in $\Omega(a+0)\cup\Omega(a-0)$. Then f is a maximum element in $\overline{\mathcal{F}}_0$. Moreover, $\overline{D}(f)$ has the nonempty interior, namely $\bar{f}\neq\underline{f}$. The annulus bounded by $\bar{f}=f$ and $\underline{f}$ in Ω is foliated by simple closed curves in $\overline{\mathcal{F}}_0$ homotopic to f.*

Proof. We claim that each simple closed curve f contained in $\Omega(a+0)\cup\Omega(a-0)$ is an element of $\overline{\mathcal{F}}_0$. Since $f\subset\Omega(a+0)\cup\Omega(a-0)$, every point in f corresponds to a straight line in $\mathbf{X}$. Let $(c(b^k),u_1)$ and $(c(t^k),u_2)$ be points in $f\cap\Omega(a+0)\cap\Omega(a-0)$.

Then, they correspond to adjacent periodic straight lines with period (q, p). Here, there is no periodic straight line with period (q, p) through $s_0 \in (b^k, t^k) \subset B$. Recall that there exist finitely many elements $g_1, \ldots, g_n$ in $\overline{\mathcal{F}}_0$ such that $D(f) = \cup_{i=1}^n D(g_i)$, as was seen in the proof of Lemma 3.1.

We prove that $f = g_j$ for some j, implying the claim. Since $f \cap \Omega(a+0) \subset \cup_{i=1}^n D(g_i)$, there exists the element g_j such that $(c(t^k), u_2) \in g_j$. Let $(c(s_1), u)$ be the endpoint of the arc $g_j \cap \Omega(a+0)$ other than $(c(t^k), u_2)$. Since $\varphi^{kq}(g_j) = g_j$ for some $k \in \mathbf{Z}$ and $\varphi^q(f \cap \Omega(a+0)) = (f \cap \Omega(a+0))$, we have $\varphi^{kq}((c(s_1), u)) = ((c(s_1), u))$. Hence, because of [11] Lemma 4.6, the configuration determined by $(c(s_1), u)$ is a periodic straight line with period (q, p), since (kq, kp) is a period of any periodic straight line with period (q, p) for $k \in \mathbf{Z}$. Therefore, we have $s_1 \notin (b^k, t^k) \subset B$, namely $s_1 = b^k$. In particular, we have $f \cap \Omega(a+0) = g_j \cap \Omega(a+0)$. Set $g^k = \varphi^m(g_j \setminus g_j \cap \Omega(a+0))$ for some $m \in \mathbf{Z}$ where $\pi(g^k) = c([b^k, t^k])$. Then, the union of those curves g^k and $\Omega(a+0) \cap \Omega(a-0)$ make an invariant circle in Ω. Since all points in an invariant circle are straight lines, they are with period (q, p) if they are not asymptotes to u^k which was used before in the definition of $S(u^k, v^k)$. This implies that $g^k \subset \Omega(a-0)$. Therefore we conclude that $f = g_j$ for some j.

Since f touches an invariant circle, f is a maximum in $\overline{\mathcal{F}}_0$.

The argument above shows that $\overline{D}(f)$ has the nonempty interior as well. $\square$

8. Appendix : Proof of Lemma 2.1

In this section we review the proof of Lemma 2.1 for convenience and completeness. We can see it in [11].

Let C be a simple closed strictly convex curve in the plane $\mathbf{E}$ of class C^2 with length L. Let $c : (-\infty, \infty) \to \mathbf{E}$ be the representation of C by arclength and $\kappa(s)$ the curvature of C at $c(s)$. Let B be the closed domain bounded by C.

Let $x = (x_j)_{j \in \mathbf{Z}}$ be a billiard ball trajectory in C and let $\gamma : (-\infty, \infty) \to B$ be the unit speed broken line such that $\gamma(t_j) = x_j$ for all $j \in \mathbf{Z}$. Let $Q = Q_j$ be the reflection with respect to the tangent line to C at $\gamma(t_j)$ which is by definition

$$Q(X) = X - 2 < X, N > N$$

where X is any vector at $\gamma(t_j)$ and N is the inward unit normal vector to C. Then, $\dot\gamma(t_j + 0) = Q(\dot\gamma(t_j - 0))$. Let the angle between $\dot c(t_j)$ and $T(x_j, x_{j+1})$ be θ_j for any $j \in \mathbf{Z}$. We say that $Y(t)$, $-\infty < t < \infty$, is a *perpendicular Jacobi vector field along* γ if Y satisfies the following (see [10]).

(1) Y is of class C^∞, $Y''(t) = 0$ and $\langle \dot\gamma(t), Y(t) \rangle = 0$ in each interval $[t_j, t_{j+1}]$.
(2) $Y(t_j + 0) = Q(Y(t_j - 0))$ for any $j \in \mathbf{Z}$.

(3) $Q(Y'(t_j-0))-Y'(t_j+0)=(2\kappa(t_j)/\sin\theta_j)Y(t_j+0)$ where $\kappa(t_j)$ is the geodesic curvature of C at $\gamma(t_j)$ with respect to N.

Let $\gamma_u:(-\infty,\infty)\longrightarrow B$ be a variation through billiard ball trajectories with unit speed such that $\gamma_0(t)=\gamma(t)$ for any $t\in(-\infty,\infty)$. Let

$$Y(t)=\left.\frac{\partial\gamma_u}{\partial u}\right|_{u=0}(t)$$

for any $t\in(-\infty,\infty)$. If $\langle Y(a),\gamma'(a)\rangle=0$ for some $a\in\mathbf{R}$, then $Y(t)$ is a perpendicular Jacobi vector field along γ. Any perpendicular Jacobi vector field is given in this way. Let K be the envelope of a variation γ_u through billiard ball trajectories with unit speed and let γ be tangent to K at a_λ, $\lambda\in\Lambda$. Then, $\gamma(a_\lambda)$ are conjugate points to each other along γ, since the perpendicular component of the variation vector field Y is a nontrivial perpendicular Jacobi vector field with $Y(a_\lambda)=0$ for any $\lambda\in\Lambda$. We sometimes call such points *focal points to* C along γ.

We say that the conjugate points of a nontrivial perpendicular Jacobi vector field $Y(t)$, $-\infty<t<\infty$, along γ *separate the boundary* if there exists a sequence $\{a_j\}_{j\in\mathbf{Z}}$ such that $\gamma(a_j)$ lie in $T(x_j,x_{j+1})$ and $Y(a_j)=0$ for any $j\in\mathbf{Z}$. Let $\gamma_u:(-\infty,\infty)\to B$ be a variation through billiard ball trajectories such that the straight lines in $\mathbf{X}$ corresponding to all γ_u are asymptotes to the straight line in $\mathbf{X}$ corresponding to $\gamma=\gamma_0$. Then, $T(x(u)_j,x(u)_{j+1})$ intersects $T(x_j,x_{j+1})$ for any $j\in\mathbf{Z}$ where $x(u)_j=\gamma_u(t_j)$. From this it follows that there exists a nontrivial perpendicular Jacobi vector field along γ whose conjugate points separate the boundary.

Let J be the set of all perpendicular Jacobi vector fields along γ whose conjugate points separate the boundary. We think that J contains the trivial Jacobi vector field along γ. We prove the following.

Lemma 8.1. *Let $\gamma:(-\infty,\infty)\to B$ be a billiard ball trajectory which corresponds to a straight line in $\mathbf{X}$. Then, $J\neq\{0\}$.*

Proof. Suppose for indirect proof that $J=\{0\}$. Then, we have a perpendicular Jacobi vector field $Y(t)$, $t\in(-\infty,\infty)$, along γ such that there exist i_0 and $j_0\geq i_0+2$ with $Y(t_{i_0})=0$ and $Y(t)\neq 0$ for all $t\in(t_{j_0},t_{j_0+1}]$ where $\gamma(t_j)\in C$. Let $\gamma_u:(-\infty,\infty)\to B$ be the variation through billiard ball trajectories such that $\gamma_0=\gamma$, $\gamma_u(t_{j_0}(u))=c(s_{j_0}(u))\in C$, $\gamma_u(t_0(u))=c(u)$, $\gamma_u(t_{i_0}(u))=\gamma(t_{i_0})$ and its variation vector field is Y. Then, $\gamma_u((t_{j_0}(u),t_{j_0+1}(u)])$ do not cross to one another for sufficiently small $|u|$. Let $\theta(u)$ be the angle between $\dot c(s_{j_0}(u))$ and the oriented segment $T(\gamma_u(t_{j_0}(u)),\gamma_u(t_{j_0+1}(u)))$ and let $\theta_1(u)$ be the angle between $\dot c(s_{j_0}(u))$ and the oriented segment $T(\gamma_u(t_{j_0}(u)),\gamma(t_{j_0+1}))$. We may suppose without loss of generality that ${s_{j_0}}'(0)>0$. Since the neighborhood of $T(\gamma(t_{j_0}),\gamma(t_{j_0+1}))$ is foliated by segments $T(\gamma_u(t_{j_0}(u)),\ \gamma_u(t_{j_0+1}(u)))$, we see that $\theta(u)>\theta_1(u)$ for any $u<0$ and

$\theta(u) < \theta_1(u)$ for any $u > 0$. Hence, from the first variation formula at $t_{j_0}(u)$, there exists a u_0 such that

$$\sum_{i=i_0}^{j_0} H(\gamma(t_j), \gamma(t_{j+1})) > \sum_{i=i_0}^{j_0-1} H(\gamma_{u_0}(t_j(u_0)), \gamma_{u_0}(t_{j+1}(u_0))) \\ + H(\gamma_{u_0}(t_{j_0}(u_0)), \gamma(t_{j_0+1})),$$

contradicting the straightness of γ. $\square$

Assume that $J \neq \{0\}$ and $Y \in J$. Let $\{a_j\}_{j \in \mathbf{Z}}$ be the set of all parameters such that $\gamma(a_j)$ is the point conjugate $\gamma(t_{j-1})$, $t_j < a_j < t_{j+1}$, for any $j \in \mathbf{Z}$. Let Y_m be a perpendicular Jacobi vector field along γ such that $Y_m(t_0 + 0) \perp \gamma'(t_0 + 0)$, $\|Y_m(t_0 + 0)\| = 1$ and $Y_m(t_m) = 0$. Let $S_m = \{b(m)_j \,|\, Y_m(b(m)_j) = 0, t_j < b(m)_j < t_{j+1}\}$.

The following lemma is a generalization of the Sturm comparison theorem which is seen in [12].

Lemma 8.2. (Separation property) *Suppose $\gamma(b)$ is the first point conjugate to $\gamma(a)$ with $a < b$. Any nontrivial perpendicular Jacobi vector field Y along γ with $Y(a) \neq 0$ or $Y(b) \neq 0$ has a unique zero point $\gamma(t_0)$ at $t_0 \in (a, b)$.*

Proof. Let $e(t)$, $t \in \mathbf{R}$, be a vector field along γ such that $\langle \dot\gamma(t), e(t) \rangle = 0$ and $\|e(t)\| = 1$ for each interval $[t_j, t_{j+1}]$ and $e(t_j + 0) = Q(e(t_j - 0))$. Any perpendicular Jacobi vector field Y along γ is denoted by $Y(t) = y(t)e(t)$ for any $t \in \mathbf{R}$. Then, $y(t)$ is continuous for $t \in \mathbf{R}$ and it satisfies

$$y'(t_j + 0) = y'(t_j - 0) - \frac{2\kappa(t_j)}{\sin\theta_j} y(t_j)$$

for all $j \in \mathbf{Z}$. Thus, if Y and Z are perpendicular Jacobi vector fields along γ, then $f(t) = y'(t)z(t) - y(t)z'(t)$ is constant for all $t \in \mathbf{R}$. In fact, we have $f'(t) = 0$ for $t \neq t_j$ and

$$\begin{aligned} f(t_j - 0) &= y'(t_j - 0)z(t_j) - y(t_j)z'(t_j - 0) \\ &= \left(y'(t_j + 0) + \frac{2\kappa(t_j)}{\sin\theta_j} y(t_j) \right) z(t_j) - y(t_j) \left(z'(t_j + 0) + \frac{2\kappa(t_j)}{\sin\theta_j} z(t_j) \right) \\ &= y'(t_j + 0)z(t_j) - y(t_j)z'(t_j + 0) = f(t_j + 0). \end{aligned}$$

By the assumption there exists a nontrivial perpendicular Jacobi vector field Y along γ such that $y(a) = y(b) = 0$, $y'(a) = 1$ and $y(t) > 0$ for any $t \in (a, b)$. Since $\gamma(b)$ is the first conjugate point to $\gamma(a)$, we have $y'(b) < 0$. Let Z be a nontrivial perpendicular Jacobi vector field along γ with $Z(a) \neq 0$, say $z(a) > 0$. Since $y'(b)z(b) = z(a)$, we have $z(b) < 0$. Therefore, there exists a $t_0 \in (a, b)$ such that $z(t_0) = 0$, proving the existence of zeros.

Suppose there exists another zero point of Z. Let $\gamma(t_1)$ be the first point conjugate to $\gamma(t_0)$ with $t_0 < t_1 \leq b$. Since $z'(t_0)z'(t_1) < 0$ and $y(t_0)z'(t_0) = y(t_1)z'(t_1)$, we have $y(t_0)y(t_1) < 0$, proving that there exists a zero point of Y between t_0 and t_1, a contradiction. This proves that Z does not have more than one zero in (a, b). □

The separation property of conjugate points shows that $t_j < a_j < b(m)_j < t_{j+1}$ for all m with $j \leq m - 2$. The sequence $\{b(m)_0\}_{m>2}$ is monotone decreasing and bounded. Let Y_m be the perpendicular Jacobi vector field along γ such that $Y_m(t_0) = e(t_0 + 0)$ and $Y_m(b(m)_0) = 0$.

Then, $Y_f = \lim_{m\to\infty} Y_m$ exists and $Y_f \in J$. In the same manner, $Y_b = \lim_{m\to-\infty} Y_m$ exists and $Y_b \in J$. Let $\{\underline{b}_j\}_{j\in\mathbf{Z}}$ and $\{\bar{b}_j\}_{j\in\mathbf{Z}}$ be the sequence of parameters such that $Y_b(\underline{b}_j) = 0$, $Y_f(\bar{b}_j) = 0$ and $t_j < \underline{b}_j \leq \bar{b}_j < t_{j+1}$ for any $j \in \mathbf{Z}$.

We notice the following.

Lemma 8.3. *If Y is a non-trivial perpendicular Jacobi vector field along γ such that $Y(a) = 0$ for some $a \in [\underline{b}_j, \bar{b}_j]$, then Y has the unique zero in the interval $[t_j, t_{j+1}]$ for every $j \in \mathbf{Z}$.*

Lemma 8.4. *Let $J \neq \{0\}$. Then, J is a one-dimensional vector space if and only if $Y_f = Y_b$.*

Let $s = (s_j)_{j\in\mathbf{Z}}$ be a straight line in $\mathbf{X}$. We define the Busemann function of s as

$$B_s(0, t_j) = \lim_{n\to\infty} \{H(t_j, s_n) - H(s_0, s_n; s)\}$$

for $t_j = (j, t_j)$ where $H(t_j, s_n)$ is the H-length of a segment connecting t_j and $s_n = (n, s_n)$ in $\mathbf{X}$.

The following is a condition that $Y_f = Y_b$.

Lemma 8.5. *Let $x = (x_j)_{j\in\mathbf{Z}}$ be a billiard ball trajectory in C which corresponds to γ in $\mathbf{E}$ and to a straight line $s = (s_j)_{j\in\mathbf{Z}}$ in $\mathbf{X}$. Suppose there exists a variation through parallels $x(u) = (x(u)_j)_{j\in\mathbf{Z}}$ to x in $\mathbf{X}$ such that $x(0) = x$. Then, J is a one-dimensional vector space.*

Proof. Suppose for indirect proof that $\bar{b}_0 > \underline{b}_0$, meaning that $Y_f \neq Y_b$. Let $\gamma_u : (-\infty, \infty) \to B$ be billiard ball trajectories corresponding to $x(u)$ with $x(u)_0 = \gamma_u(0)$, $t_0 = 0$. Let $0 < \epsilon < \min\{\underline{b}_0, \bar{b}_0 - \underline{\mathrm{b}}_0\}$ and S the $\epsilon/2$-ball around $\gamma(\epsilon)$. For a sufficient small ε we may assume that S is foliated by $\gamma_u([t_0(u), t_1(u)])$. We define a function $F_{\pm s}$ on S as

$$F_{\pm s}(\gamma_u(t)) = B_{\pm s}(0, s_0(u)) \mp t$$

where $s(u) = (s_j(u))_{j\in\mathbf{Z}}$ is parallels to $s = (s(0)_j)_{j\in\mathbf{Z}}$ in $\mathbf{X}$ corresponding to $x(u)$. The functions $F_{\pm s}$ are of class C^1 in S such that the gradient vector field is $\mp\dot{\gamma}_u$

because S is foliated by γ_u which correspond to parallels $x(u)$ to x. In particular, $F_s + F_{-s}$ is constant on S.

Choose constants k_f and k_b such that $k_f Y_f(\epsilon) = e$ and $k_b Y_b(\epsilon) = e$ where e is the unit vector perpendicular to $\dot{\gamma}_0(\epsilon)$. Set $Y_1 = k_f Y_f$ and $Y_2 = k_b Y_b$. Then, $Y_1{}'(\epsilon) \neq Y_2{}'(\epsilon)$. If $Y_1(t) = y_1(t)e$, $Y_2(t) = y_2(t)e$, then $y_2{}'(\epsilon) < y_1{}'(\epsilon) < 0$ because $\underline{b}_0 - \epsilon < \bar{b}_0 - \epsilon$. However, this means that $F_s + F_{-s}$ is not constant, since $F_{\pm s}{}^{-1}(F_{\pm s}(\gamma(\epsilon)))$ are the limit sets of spheres through $\gamma(\epsilon)$ with center $c(s_{\pm n})$ measured by H as $n \to \pm\infty$ and $y_1{}'(\epsilon)$, $y_2{}'(\epsilon)$ are their curvatures at $\gamma(\epsilon)$. Thus, we have a contradiction. $\square$

Proof of Lemma 2.1 : Let $\bar{x}(s) = (c(s), u(s))$ for every $s \in [a, b]$. Let $x(s) = (x(s)_j)_{j \in \mathbf{Z}}$ be the billiard ball trajectory corresponding to $\bar{x}(s)$ and $r(s) = (r(s)_j)_{j \in \mathbf{Z}}$ the configuration for $\bar{x}(s)$ with $r(s)_0 = s$. We note that $r(s)_1$ is continuous in $s \in [a, b]$ because $F_{\pm s}$ is of class C^1.

We prove that $r(s)_1$ is of class C^1. Let $\gamma_s : (-\infty, \infty) \to B$ be the unit speed broken line such that $\gamma_s(t(s)_j) = x(s)_j$ for all $j \in \mathbf{Z}$. Let $Y_f{}^s$ be the unique nontrivial perpendicular Jacobi vector field along γ_s whose conjugate points separate the boundary for $s \in [a, b]$. Then, we can assume that $Y_f{}^s$ is continuous for $s \in [a, b]$ and $\langle Y_f{}^s(t(s)_0), \dot{c}(s)\rangle = 1$ because of Lemma 8.5. We then have

$$r(s)_1 = \int_a^s \langle Y_f{}^s(t(s)_1 + 0), \dot{c}(r(s)_1)\rangle \, ds + r(a)_1.$$

This shows that $r(s)_1$ is of class C^1. In particular, we have proved that if $(c_1(s), u_1(s)) = \varphi(c(s), u(s))$, then $c_1(s) = c(r(s)_1)$ is of class C^1. Since

$$u(s) = \frac{\langle c_1(s) - c(s), \dot{c}(s)\rangle}{\|c_1(s) - c(s)\|},$$

we conclude that $u(s)$ is also of class C^1. $\square$

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(N. Innami) Department of Mathematics, Faculty of Science, Niigata University, Niigata, 950-2181, JAPAN

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